Digital One-Third Octave Spectral Analysis

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Abstract. An economical approach is described for estimating power spectra from sampled data through the application of Z-transform theory. The method consists of computing a weighting sequence whose frequency characteristics approximate a one-third octave bandwidth filter, and applying these coefficients recursively to the digitized data record. Filtering is followed by a variance calculation and a division by the appropriate filter bandwidth. A specific example of power spectra computed in the usual manner (Fourier transformation of the autocorrelation function) and power spectra computed by the method in this paper demonstrates the practicability of the technique. The most significant advantage is the economical aspect. It is shown that owing to the variable bandwidth and the small number of filtering coefficients, the savings that may be realized by the employment of this technique, in lieu of the autocorrelation transformation approach, may be quite considerable, depending on the record length and the number of lag products.

1. *Motivation*

The statistical theory of spectral analysis for stationary random data considers questions of estimating frequency distributions for functions which theoretically endure for an infinite amount of time, but for which only a finite record length is available. While the theory plays a tremendously important role in providing "good" estimators for the most demanding theoretical studies, there are many physical situations in which only a rough, inexpensive frequency decomposition is desirable; in particular, for purposes of pre-emphasis, assessing vibration specification levels and providing a variable bandwidth amplitude smoothing effect (avoidance of unnecessary gibberish at the high frequencies and wide resolution at the low frequencies). These considerations led to the development of the following tech nique

2. *Conventional Analog Spectral Analysis*

One basic method employed by an analog frequency spectrometer in computing power spectra stems directly from the definition. Each power spectral density value is estimated by passing the data through a narrow band filter centered at a selected frequency, evaluating the mean square value and dividing by the appropriate bandwidth. By selecting different center frequencies for each filter, one may analyze the frequency range of interest. In this paper, the concept of the analog one-third octave analysis for continuous data is extended to digital data utilizing more flexible numerical recursive filters with almost linear phase shift and fairly flat gain characteristics.

3. *Numerical Filter Design*

The numerical filter is composed of two resonant second-order systems spaced at a selected amount in frequency to produce a filter bandwidth of one-third of an octave at the 3db point. The maximum steepness is of the order of 80db per octave, and the scanning frequency range may cover from f_0 to $f_s/2$, where f_0 is the center

The work reported here was developed while the author was associated with the Douglas Space Systems Center, Huntington Beach, California.

Journal of the Association for Computing Machinery, Vol. 13, No. 4 (October, 1966), pp. 605–614

frequency of the first filter and f_s is the sampling rate. The form of the filter equation in the S-domain is

$$G(S) = \frac{\omega_2^2 S^2}{(S^2 + 2\zeta\omega_1 S + \omega_1^2)(S^2 + 2\zeta\omega_2 S + \omega_2^2)}, \tag{1}$$

where $\omega_1 = 2\pi f_c (1 - \Delta f)$

$\omega_2 = 2\pi f_c(1 + \Delta f)$

$\zeta = 0.05$

$\Delta f = 0.085.$

The values of ζ and Δf were derived empirically. As a function of ω, the gain of (1) may be written as

$$G_1(\omega) = \frac{\omega_2^2 \omega^2}{\{[(\omega_1^2 - \omega^2)^2 + (2\zeta\omega_1 \omega)^2][(\omega_2^2 - \omega^2)^2 + (2\zeta\omega_2 \omega)^2]\}^{\frac{1}{2}}}.$$

A relative minimum occurs at $\omega \sim \sqrt{(\omega_1 \omega_2)}$, which is

$$G_1(\sqrt{\omega_1 \omega_2}) = \frac{\omega_2^2}{(\omega_1 - \omega_2)^2 + 4\zeta^2\omega_1 \omega_2},$$

$$G_1(\sqrt{\omega_1 \omega_2}) = \frac{0.25(1 + \Delta f)^2}{\Delta f^2 + \zeta^2(1 - \Delta f^2)}. \tag{2}$$

A relative maximum occurs at $\omega \sim \omega_1$, which is

$$G_1(\omega_1) = \frac{\omega_2^2}{2\zeta\,[(\omega_2^2 - \omega_1^2)^2 + 4\zeta^2\omega_2^2\omega_1^2]^{\frac{1}{2}}},$$

$$G_1(\omega_1) = \frac{(1 + \Delta f)^2}{2\zeta\,[(4\Delta f)^2 + 4\zeta^2(1 - \Delta f^2)^2]^{\frac{1}{2}}}. \tag{3}$$

$G_1(\omega)$ is normalized to unity by dividing by the geometric mean of (2) and (3), or by $P_1 = [G_1(\sqrt{\omega_1\omega_2}) \cdot G_1(\omega_1)]^{\frac{1}{2}}$, which gives the required transfer characteristics.

The Z-transform of eq. (1) may be derived by expanding into partial fractions and evaluating the Z-transform for each term of the expansion:

$$\frac{\omega_2^2 S^2}{(S^2 + 2\zeta\omega_1 S + \omega_1^2)(S^2 + 2\zeta\omega_2 S + \omega_2^2)} = \frac{A_1 S + B_1}{(S^2 + 2\zeta\omega_1 S + \omega_1^2)} + \frac{C_1 S + D_1}{(S^2 + 2\zeta\omega_2 S + \omega_2^2)}, \tag{4}$$

$$\frac{\omega_2^2 S^2}{(S^2 + 2\zeta\omega_1 S + \omega_1^2)(S^2 + 2\zeta\omega_2 S + \omega_2^2)} = \frac{A_1(S + a_1)}{(S + \alpha_1)^2 + \beta_1^2} + \frac{C_1(S + a_2)}{(S + \alpha_2)^2 + \beta_2^2},$$

where

$$a_1 = \frac{B_1}{A_1}, \quad \alpha_1 = \zeta\omega_1, \quad \beta_1 = \omega_1\sqrt{(1 - \zeta^2)},$$

and

$$a_2 = \frac{D_1}{C_1}, \quad \alpha_2 = \zeta\omega_2, \quad \beta_2 = \omega_2\sqrt{(1 - \zeta^2)}.$$

To simplify further calculations, let $L_1 = 2\zeta\omega_2$, $L_2 = \omega_2^2$, $K_1 = 2\zeta\omega_1$ and K_2

$= \omega_1^2$. To solve for the coefficients A, B, C and D, equate like powers in (4) and obtain

$$A_1 = \frac{(L_1 K_2 - L_2 K_1)\omega_2^2 P_2}{\Delta}, \qquad C_1 = \frac{-(L_1 K_2 - L_2 K_1)\omega_2^2 P_2}{\Delta},$$

$$B_1 = \frac{(K_2^2 - L_2 K_2)\omega_2^2 P_2}{\Delta}, \qquad D_1 = \frac{(L_2^2 - L_2 K_2)\omega_2^2 P_2}{\Delta},$$

where

$$\Delta = (L_1 - K_1)(L_1 K_2 - L_2 K_1) + (K_2 - L_2)^2,$$

$$P_2 = \frac{T}{P_1} = \text{the normalization factor},$$

$$T = \text{sampling interval},$$

$$P_1^2 = \frac{0.125(1 + \Delta f)^4}{[\Delta f^2 + \zeta^2(1 - \Delta f^2)]\zeta[16\Delta f^2 + 4\zeta^2(1 - \Delta f^2)^2]^{\frac{1}{2}}}.$$

The Z-transform for each of the two terms of (4) is computed by residues.

$$E^*(Z) = \sum_{\text{Roots of } B(p)} \text{Residue of } \frac{A(p)}{B(p)(1 - e^{T(p-s)})},$$

where

$$B(p) = B(S)\,|_{s=p}\,.$$

After many algebraic manipulations, the Z-transform of (4) results in

$$\frac{A_1 Z^2 - A_1 Z e^{-\alpha_1 T} \cos(\beta_1 T - \Phi_1) \sec \Phi_1}{Z^2 - 2Ze^{-\alpha_1 T} \cos \beta_1 T + e^{-2\alpha_1 T}} + \frac{C_1 Z^2 + C_1 Z e^{-\alpha_2 T} \cos(\beta_2 T - \Phi_2) \sec \Phi_2}{Z^2 - 2Ze^{-\alpha_2 T} \cos \beta_2 T + e^{-2\alpha_2 T}}, \tag{5}$$

where

$$\Phi_1 = \arctan \frac{\alpha_1 - a_1}{\beta_1},$$

$$\Phi_2 = \arctan \frac{\alpha_2 - \alpha_2}{\beta_2}.$$

Combining (5) into a rational polynomial ratio in Z, one obtains

$$E^*(Z) = \frac{\phi(Z)}{I(Z)} = \frac{BZ^{-1} + CZ^{-2} + DZ^{-3}}{1 + EZ^{-1} + FZ^{-2} + GZ^{-3} + HZ^{-4}}, \tag{6}$$

which is the pulse transfer function where

$A = K_1 + M_1 \equiv 0,$ (omitted in eq. (6))

$B = -K_2 - N_1K_1 - M_2 - L_1M_1$

$C = N_1K_2 + N_2K_1 + M_2L_1 + L_2M_1$

$D = -N_2K_2 + L_2M_2$

$E = -L_1 - N_1$

$F = L_1N_1 + L_2 + N_2$

$G = -L_2N_1 + L_1N_2$

$H = L_2N_2$

$K_1 = A_1$

$K_2 = A_1 \sec \Phi_1[e^{-\alpha_1 T} \cos(\beta_1 T - \Phi_1)]$

$M_1 = C_1$

$M_2 = C_1 \sec \Phi_2\,[e^{-\alpha_2 T} \cos(\beta_2 T - \Phi_2)].$

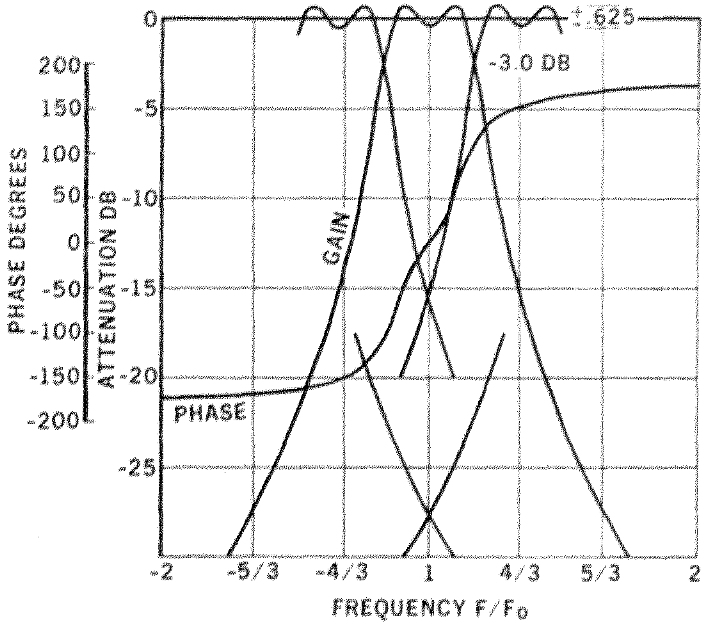

FIG. 1. Pulse transfer function

The normalized pulse transfer function (computed by replacing Z with e^{sT} in (6)) is plotted in Figure 1. As one may speculate, flatter and narrower filters are possible; the adopted width of one-third octave was strictly a convenient standard.

4. *Digital Scheme*

Reverting (5) to the time domain gives the recursive filter to be applied to the sampled data as

$$\phi_n = BI_{n-1} + CI_{n-2} + DI_{n-3} - E\phi_{n-1} - F\phi_{n-2} - G\phi_{n-3} - H\phi_{n-4}, \qquad (7)$$

where ϕ_n refers to the present output value and I_n refers to the present input value. Notice that the A-coefficient has been dropped out, since it is identical to zero.

5. *Uncertainty*

The uncertainty expression employed in the averaging process is based upon the number of independent statistical events present over the bandwidth B_L for a time interval T_R. The smallest bandwidth, B_L, and the total record length, T_R, determine the confidence bands for the power spectral density estimates. The degrees-of-freedom yielding a particular confidence are given by $N = 2B_LT_R$. For a variable filter bandwidth, $B_i = f_i/Q$, where f_i is the ith center frequency and Q is a constant. In other words, as the bandwidth becomes broader at each new frequency, the record length decreases, but the degrees-of-freedom remain constant. Corresponding confidence levels for a certain number of degrees-of-freedom may be obtained from any standard chi-square distribution table.

6. *Averaging Time*

With the averaging time T_{AV} specified as that time which is required to produce a certain number of degrees-of-freedom for the variance estimates, the BT product gives

$$B_L T_R = B_i T_{AV} = \text{const.},$$

where

$$B_i = \frac{f_i}{Q}, \quad Q = 4.332855, \quad f_i = f_0 2^{i/3},$$

and f_0 is equal to the center frequency of the first filter.

$$T_{AV} = n_i T = \frac{B_L B_R Q}{f_i}; \qquad n_i = \frac{B_L T_R Q}{f_i T} = \frac{B_L T_R Q}{f_0 2^{i/3} T}.$$

This is the number of points required for each variance estimate which yields a particular confidence interval. For simplicity, let $B_L T_R Q / f_0 T = K_1$; then $n_i = K_1 2^{-i/3}$. Let $i = 0$; then $K_1 = n_0 =$ total number of points needed in the variance calculation for the smallest bandwidth B_L.

7. *Cost Considerations*

Since the number of points needed in the averaging process is equal to $K_1 2^{-i/3}$ per filter, the total number of points to filter for the entire frequency range is:

$$S_L = \sum_{i=0}^{i=L} K_1 2^{-i/3},$$

where $i = 0, 1, 2, 3, \cdots, L$ and $L =$ number of filters minus 1. This sum is in geometric progression and may be computed from

$$S_L = \frac{K_1(R^L - 1)}{(R - 1)} = \frac{K_1(2^{-L/3} - 1)}{(2^{-1/3} - 1)},$$

where $R = 2^{-1/3}$ and $R^L = 2^{-L/3}$.

A rough estimate of the total IBM 7094 computer time may be obtained from

$$S_L \times 7 \cdot [2\,(\text{machine time for a single-precision floating ADD} + \text{machine time for a single-precision floating MULTIPLY})].$$

The factor of 2 is used to account for the double-precision instructions which are roughly twice the cycle time of a single-precision floating ADD and floating MULTIPLY.

Therefore, total computer time for the one-third octave spectral analysis is

$$\sim \frac{K_1(2^{-L/3} - 1)14[\quad]}{-0.2063}. \tag{8}$$

A similar equation may be derived for the amount of computing time required for the constant bandwidth autocorrelation technique (transformation of the autocorrelation function).

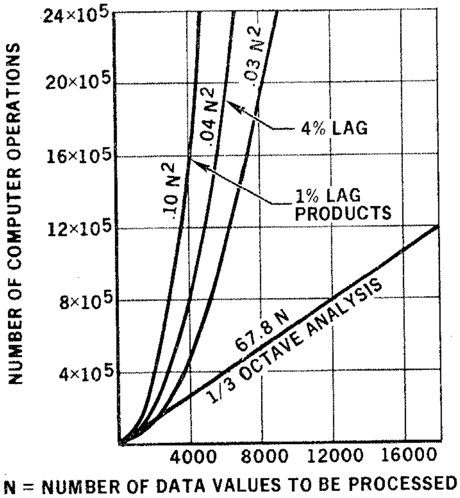

FIG. 2. Cost analysis in terms of record length

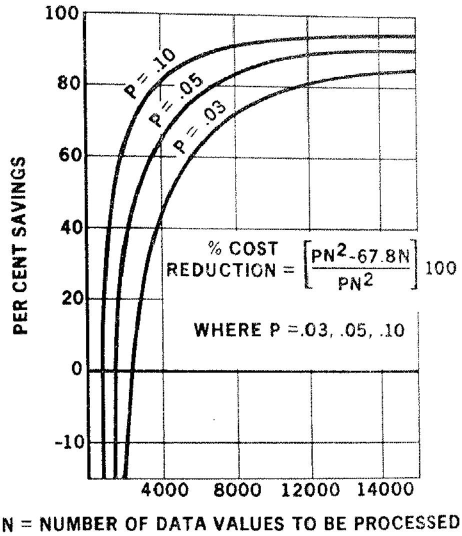

FIG. 3. Percent cost reduction

Total computer time for autocorrelation approach is

$$\sim \frac{(2n_0 - k)k}{2} \cdot \text{[machine time for a single-precision floating ADD} \\ + \text{machine time for a single-precision floating MULTIPLY]}, \tag{9}$$

where k equals number of lag products and varies depending on the accuracy and bandwidth of the analysis. It may be expressed as $k = Pn_0$, where P is given in percent.

In (8), we assume an infinite number of filters ($L = \infty$), so that we may write:

Total computer time for one-third octave spectral analysis

$$\sim \frac{14K_1[\quad]}{0.2063}$$

$$\sim 67.8n_0[\quad]. \tag{10}$$

The ratio of the computer time of the autocorrelation approach to the one-third approach ((9) divided by (10)), with k replaced by Pn_0, is

$$R = \frac{\left(\dfrac{2n_0 - Pn_0}{2}\right) Pn_0}{67.8n_0},$$

$$R \sim \frac{Pn_0}{67.8}. \tag{11}$$

This formula only takes into consideration the major portion of the operations and may vary slightly, depending on the cycle time of the computer and the efficiency of the subroutine. Savings of the one-third octave analysis over the correlation procedure increase linearly with an increase in data values. For example, with as few

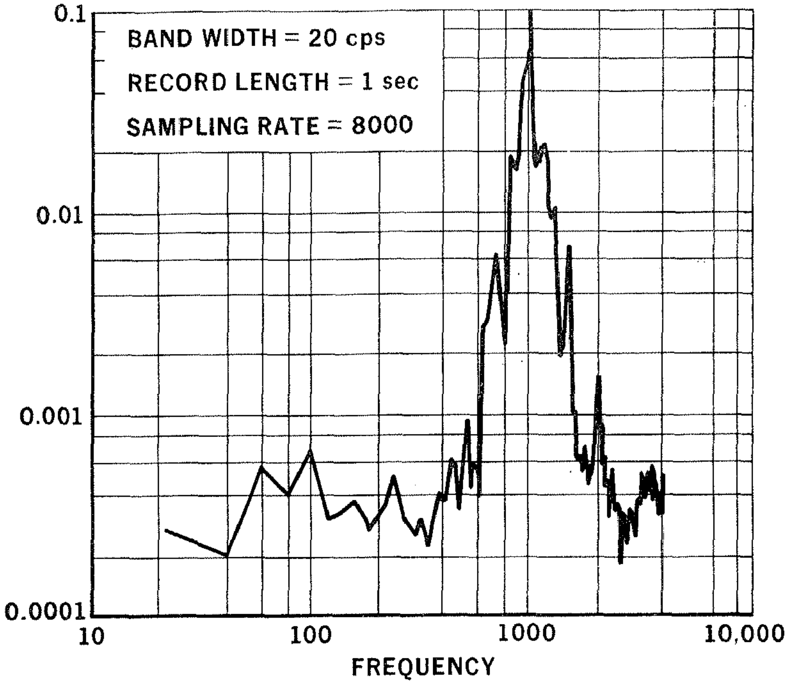

FIG. 4. Digital power spectra by Fourier transformation of the autocovariance function

data values as 2260 and $0.03n_0$ lag products, eq. (11) favors the correlation technique. It should be observed, however, that few data values for spectral analysis are of limited practical interest (see Figure 2). Note that a rapid rise in savings exists when P is large (0.02, 0.04, 0.1). Figure 3 shows the percentage savings for $0.03n_0$, $0.05n_0$, $0.1n_0$ lag products.

8. *An Actual Test Case*

Figures 4 and 5 are power spectral analyses resulting from the autocorrelation approach and the one-third octave method presented in this paper. In order to produce consistent confidence levels for both analyses, the center frequency on the one-third octave method was started at 86.65cps. This resulted in 40 degrees-of-freedom in accordance with the correlation techniques.

9. *Computer Implementation*

The recursive eq. (7) defines the filtering process by which the spectral estimates are to be computed. Figure 6 shows other parameters essential in the total analysis and represents a general computer subroutine flowchart which evaluates the power spectral estimates from the stored data record.

10. *Advantages*

1. If variable bandwidth spectral analysis is preferable, implementation of this approach circumvents the need to purchase or rent special equipment for the

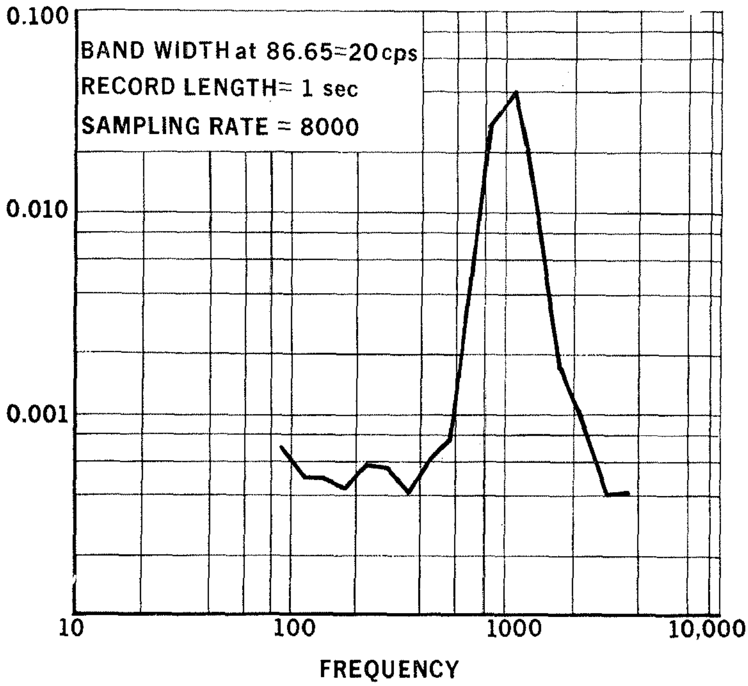

FIG. 5. Digital one-third octave spectral analysis

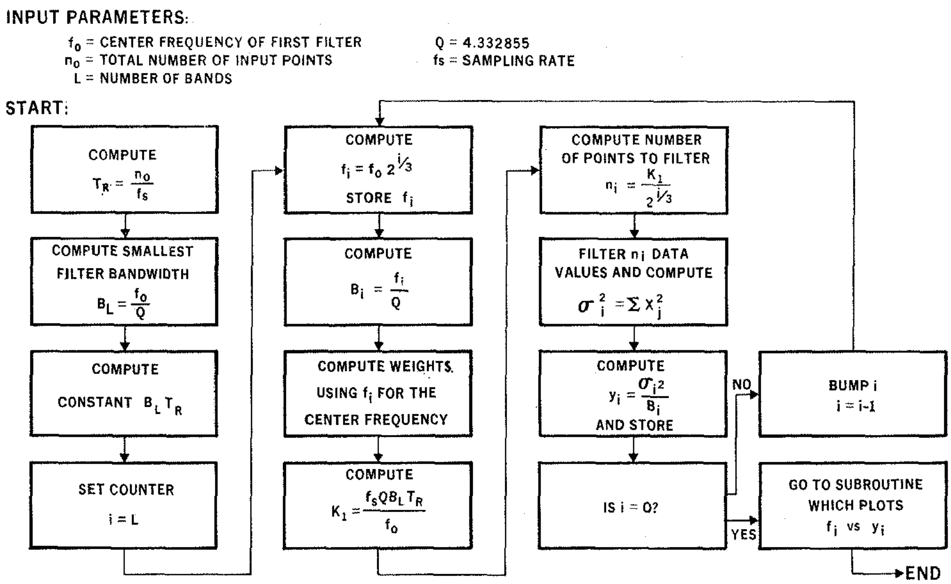

FIG. 6. Computer subroutine flowchart

frequency reduction of vibration data, provided a large-scale computer is available.

2. On many engineering applications, the digital one-third octave spectral analysis affords a very economical procedure over other well-known methods.

3. Owing to the constant percentage bandwidth flexibility of constructing the filters, digital one-third spectral analysis enables one to investigate low harmonic content at a very reasonable cost.

11. *Disadvantages*

One certain disadvantage in using an IBM 7094 computer is the need for the double-precision operations. It was found that due to the small magnitudes of the weights for high sampling rates (from 5000 to 10,000), the accuracy required could not be maintained in a 35-bit floating-point register. However, single-precision operations may be used if the sample rates do not exceed 5000 samples per second. If single-precision instructions are used for the case when the sampling rate is less than 5000, the cost ratio (eq. (11)) doubles. Generally, it is preferable to have fairly wide band spectral analysis capability, and this factor may outweigh the economics.

12. *Practical Observations*

One may realize significant improvements in the accuracy and cost of these filters by noting the following:

1. The selected one-third octave bands were strictly based on convenience; narrower filters may be constructed with steeper skirts and flatter gain characteristics.

2. Middle-range filtering may reduce the computing time considerably. The concept is based on the fact that if a sharp enough low-pass filter is applied at one-half the folding frequency (filtering from high to low frequencies), the required number of filtered values may be reduced by half. The uncertainty and the total averaging time remain constant, but the number of data values required for the variance calculations is halved by discarding every other point in the filtering operation. When this is done, the weighting sequence becomes a function of $2T$ instead of T.

13. *List of Symbols*

A, B, C, D, E, F, G, H	Weighting sequence
B_i	Bandwidth corresponding to the ith filter
B_L	Smallest bandwidth considered
$E^*(Z)$	Pulse transfer function
f_a	An arbitrary center frequency
f_1, f_2	Respective center frequencies of two second-order systems
f_i	ith center frequency
f_s	Sampling rate
$G(S)$	Continuous transfer function
$G_1(\omega)$	$\mid G(S) \mid$
I_n	Present input unfiltered data value
j	$\sqrt{-1}$, imaginary unit
k	Number of lag products considered in the autocorrelation computation
K_1	Constant equal to the total number of points in T_R
L	Number of filters -1
n_i	Filtered values required for the ith frequency band

N	Number of degrees-of-freedom in each spectral estimate
ϕ_n	Present filtered output data value
P	Percent
P_1	Normalization factor
Q	Constant of proportionality
R	Common ratio considered in a geometric progression
S	$j\omega$
S_L	Total number of points to filter for the entire analysis
T	The time interval, the reciprocal of f_s
T_R	Total record length in seconds
ω	Radian frequency
ω_1	$2\pi f_1$
ω_2	$2\pi f_2$
ζ	Damping ratio
Δf	Constant < 1

Received November, 1965; revised January, 1966

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