On a Continuous Method of Approximating Solutions of the Heat Equation

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ABSTRACT. Theoretical methods, based on a priori pointwise bounds, for approximating solutions of many elliptic and parabolic initial and/or boundary value problems, have been developed in recent years. These methods, however, are relatively unknown to potential users since applications of the methods have not appeared in the literature. In this paper their usefulness is illustrated by employing some of the author's theoretical results as a basis for the construction of a digital program to compute an approximate solution of an initial boundary value problem for the heat equation.

1. *Introduction*

In a recent paper [8] the author develops a procedure, based on a priori pointwise bounds, for approximating the solution of the first initial boundary value problem for a rather general second order parabolic equation. In this paper we examine the value of this procedure as a basis on which to develop a practical computational program for the numerical approximation of solutions of the above initial boundary value problem.

To avoid complicating the presentation with excessive detail, we focus our attention on a specific problem, viz., the first initial boundary value problem for the one-dimensional heat equation. We emphasize, however, that the general method of a priori bounds is applicable to a much wider class of problems which includes the Dirichlet, Neumann, and Robin boundary value problems for second order elliptic equations [1, 2, 3], the second initial boundary value problem for parabolic equations [7] and elliptic [5] and parabolic [6] integro-differential equations, as well as the initial boundary value problem treated herein. Furthermore, the method of a priori bounds, as developed in the above references, is valid in any finite number of dimensions and for regions of quite general shape.

2. *A Description of the Method*

In this section we give a detailed statement of the problem to be considered and illustrate how the a priori inequalities are used to yield the approximate solution and error bounds. The a priori inequalities are derived in Section 3.

The problem is defined in a rectangular region R_τ of the x, t-plane defined by $-l < x < l$, $0 < t < \tau$, where τ is an arbitrary (finite) positive constant, with the boundary of R_τ consisting of the four lines B, B_τ, $S_{\tau,-l}$, and $S_{\tau,l}$ defined as follows:

$$\begin{aligned}
&B: && -l < x < l, && t = 0,\\
&B_\tau: && -l < x < l, && t = \tau,\\
&S_{\tau,-l}: && 0 \le t < \tau, && x = -l,\\
&S_{\tau,+l}: && 0 \le t < \tau, && x = l.
\end{aligned}$$

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The notation $S_\tau = S_{\tau,-l} + S_{\tau,l}$ is frequently used. Then the first initial boundary value problem for the nonhomogeneous heat equation in R_τ is

$$Lu = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = f_1(x, t), \quad \text{in } R_\tau + B_\tau , \tag{1a}$$

$$u = f_2(x), \qquad \text{on } B, \tag{1b}$$

$$u = f_3(t), \qquad \text{on } S_\tau . \tag{1c}$$

(The notation $u = f_3(t)$ on S_τ is used as a shorthand notation for the more precise statement

$$u = f_3^-(t) \quad \text{on } S_{\tau,-l},$$

$$u = f_3^+(t) \quad \text{on } S_{\tau,+l},$$

where f_3^- and f_3^+ may be prescribed independently.)

Our approximating procedure is based on the following two inequalities which are valid for an arbitrary function u, piecewise C^2 with respect to x, and piecewise C^1 with respect to t in R_τ :

$$| u(P)|^2 \le K_1(P) \int_{R_\tau}\int u^2 \, dx \, dt + K_2(P) \int_{R_\tau}\int (Lu)^2 \, dx \, dt, \tag{2}$$

and

$$\int_{R_\tau}\int u^2 \, dx \, dt \le \alpha_1 \int_B u^2 \, dx + \alpha_2 \int_{S_\tau} u^2 \, dt + \alpha_3 \int_{R_\tau}\int (Lu)^2 \, dx \, dt, \tag{3}$$

where the point $P = (\xi, \tau)$ is on B_τ, $K_1(P)$, $K_2(P)$ are explicit constants for each fixed P, and the α_1, α_2, α_3 are explicit constants. The first bound is a pointwise bound; that is, it gives a bound for the value of the function u at the point P. The second, which is a bound on the integral over R_τ of the square of the function, is a norm bound. Combining (2) and (3), we obtain the computable pointwise bound

$$| u(P)|^2 \le C_1(P) \int_{R_\tau}\int (Lu)^2 dx \, dt + C_2(P) \int_B u^2 dx + C_3(P) \int_{S_\tau} u^2 \, dt. \tag{4}$$

So far u has been arbitrary except for certain differentiability conditions which must be imposed to derive inequalities (1) and (2). Now suppose that $u = v - \varphi$, where v is the solution of the boundary value problem and φ is any sufficiently differentiable approximating function. Substitution of this expression for u in (4) yields a pointwise bound for $v - \varphi$:

$$\begin{aligned} | v(P) - \varphi(P)|^2 \le C_1(P) \int_{R_\tau}\int (f_1 - L\varphi)^2 \, dx \, dt + C_2(P) \\ \cdot \int_B (f_2 - \varphi)^2 \, dx + C_3(P) \int_{S_\tau} (f_3 - \varphi)^2 \, dt. \end{aligned} \tag{5}$$

Denoting the right-hand side of (5) by E, we obtain upper and lower bounds for $v(P)$,

$$\varphi(P) - E^{\frac{1}{2}} \le v(P) \le \varphi(P) + E^{\frac{1}{2}}, \tag{6}$$

or, looked at another way, we obtain an approximation $\varphi(P)$ to $v(P)$ with an error bound on the approximation of $\pm E^1$. It follows from the nature of inequality (5) that if φ approximates the data of v sufficiently well, in a mean square sense, then $\varphi(P)$ will be a close approximation of $v(P)$.

In order to obtain a φ which will closely approximate the data of v we set

$$\varphi = \sum_{i=1}^{n} c_i \varphi_i , \tag{7}$$

the φ_i being a set of n linearly independent sufficiently smooth trial functions, and use the Rayleigh-Ritz procedure to determine the optimal c_i's. Hopefully, by a judicious choice of the φ_i, the right-hand side of (5) can be made small. This, of course, is not guaranteed unless we use a set of trial functions which is complete in the data norm, and such trial functions are not always easily found. Although it is desirable from a theoretical point of view to use trial functions from a complete set, from a purely practical point of view it makes little difference whether we do or do not use elements of a complete set. In either case the computational problem is the same.

3. *Derivation of the Inequalities*

In this section the a priori norm and pointwise bounds of Section 2 are derived. We must first derive an important auxiliary inequality.

It is well known [4] that for all piecewise C^1 functions u which vanish at $(-l, 0)$ and $(l, 0)$, the boundary of B,

$$\int_B u^2 \, dx \le \frac{1}{\lambda_1} \int_B \left(\frac{\partial u}{\partial x}\right)^2 dx,$$

where λ_1 is the first eigenvalue for the fixed string problem defined on B, i.e.,

$$\frac{d^2 v}{dx^2} + \lambda v = 0, \qquad -l < x < l,$$

$$v(-l) = v(l) = 0.$$

Thus for functions $w = w(x, t)$ which are zero on S_τ we have

$$\int_{B_t} w^2 \, dx \le \frac{1}{\lambda_1} \int_{B_t} \left(\frac{\partial w}{\partial x}\right)^2 dx, \qquad 0 \le t < \tau,$$

and integrating both sides with respect to t over the interval $0 \le t < \tau$ we obtain

$$\int_R\int w^2 \, dx \, dt \le \frac{1}{\lambda_1} \int_R\int \left(\frac{\partial w}{\partial x}\right)^2 dx \, dt. \tag{8}$$

(Here, and in the remainder of this section, the τ subscripts on S and R do not appear.)

A. *The Norm Bound.* In the following we assume that u is an arbitrary function, piecewise C^2 with respect to x, piecewise C^1 with respect to t. Now introduce the function w which satisfies

$$L^* w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial t} = u, \qquad \text{in } R,$$

$$w(x, \tau) = 0, \quad -l < x < l; \qquad w(-l, t) = w(l, t) = 0, \quad 0 < t \le \tau.$$

Then

$$\int_R\int u^2\,dx\,dt = \int_R\int u\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial t}\right)dx\,dt,$$

and this can be written as

$$\int_R\int u^2\,dx\,dt = \int_R\int w\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\right)dx\,dt + \int_S u\,\frac{\partial w}{\partial n}\,dt - \int_B uw\,dx,$$

using Green's second identity and the divergence theorem. The Schwarz inequality for vectors then gives

$$\begin{aligned}\int_R\int u^2\,dx\,dt \le &\left\{\int_S u^2\,dt\int_S\left(\frac{\partial w}{\partial n}\right)^2 dt\right\}^{\frac12}\\ &+\left\{\int_B w^2\,dx + \lambda_1\int_R\int w^2\,dx\,dt\right\}^{\frac12}\left\{\int_B u^2\,dx + \frac{1}{\lambda_1}\int_R\int (Lu)^2\,dx\,dt\right\}^{\frac12}.\end{aligned} \tag{9}$$

The object now is to obtain bounds on $\int_B w^2\,dx + \lambda_1\int_R\int w^2\,dx\,dt$, and $\int_S (\partial w/\partial n)^2\,dt$, in terms of $\int_R\int u^2\,dx\,dt$.

The bound on the first expression is easily found as follows. By the divergence theorem

$$-\int_B w^2\,dx = \int_B w^2 n_t\,dx = \int_R\int \frac{\partial(w^2)}{\partial t}\,dx\,dt = 2\int_R\int w\left(L^*w - w\,\frac{\partial^2 w}{\partial x^2}\right)dx\,dt,$$

which becomes

$$\int_B w^2\,dx + 2\int_R\int\left(\frac{\partial w}{\partial x}\right)^2 dx\,dt = -2\int_R\int wL^*w\,dx\,dt \tag{10}$$

upon using Green's first identity. We now write

$$\begin{aligned}\int_B w^2\,dx + \lambda_1\int_R\int w^2\,dx\,dt &= \int_B w^2\,dx + 2\lambda_1\int_R\int w^2\,dx\,dt - \lambda_1\int_R\int w^2\,dx\,dt\\ &\le \int_B w^2\,dx + 2\int_R\int\left(\frac{\partial w}{\partial x}\right)^2 dx\,dt - \lambda_1\int_R\int w^2\,dx\,dt\\ &= -2\int_R\int wL^*w\,dx\,dt - \lambda_1\int_R\int w^2\,dx\,dt\\ &\le \frac{1}{\lambda_1}\int_R\int (L^*w)^2\,dx\,dt = \frac{1}{\lambda_1}\int_R\int u^2\,dx\,dt,\end{aligned} \tag{11}$$

where the second line follows upon using (8), the third by using (10), and the fourth upon using the weighted arithmetic-geometric mean inequality with weight λ_1.

Furthermore, from (10),

$$\int_R\int\left(\frac{\partial w}{\partial x}\right)^2 dx\,dt \le -\int_R\int wL^*w\,dx\,dt,$$

and an application of the Schwarz inequality along with (8) yields

$$\int_R\int\left(\frac{\partial w}{\partial x}\right)^2 dx\,dt \le \frac{1}{\lambda_1}\int_R\int (L^*w)^2\,dx\,dt = \frac{1}{\lambda_1}\int_R\int u^2\,dx\,dt, \tag{12}$$

an inequality which will be useful shortly.

The bound on $\int_S (\partial w/\partial n)^2\, dt$ is a little more involved. We first introduce a continuously differentiable vector field whose components are (f_x, f_t). Then the following expression is an identity in R:

$$\left(f_x \frac{\partial w}{\partial x} + f_t \frac{\partial w}{\partial t}\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial t}\right) = \frac{\partial}{\partial x}\left\{f_x \left(\frac{\partial w}{\partial x}\right)^2 + f_t \frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\right\} - \frac{\partial f_x}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2$$
$$- \frac{\partial f_t}{dx}\frac{\partial w}{\partial x}\frac{\partial w}{\partial t} - \frac{1}{2}\frac{\partial}{\partial x}\left\{f_x\left(\frac{\partial w}{\partial x}\right)^2\right\}$$
$$= \frac{1}{2}\frac{\partial}{\partial t}\left\{f_t\left(\frac{\partial w}{\partial x}\right)^2\right\} + \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_t}{\partial t}\right)\left(\frac{\partial w}{\partial x}\right)^2 + \left(f_x \frac{\partial w}{\partial x} + f_t \frac{\partial w}{\partial t}\right)\frac{\partial w}{\partial t}.$$

Integrating this identity over R and using the divergence theorem gives

$$\int_R\int \left(f_x \frac{\partial w}{\partial x} + f_t \frac{\partial w}{\partial t}\right) L^* w\, dx\, dt = \int_S \left(f_x \frac{\partial w}{\partial x} + f_t \frac{\partial w}{\partial t}\right)\frac{\partial w}{\partial n}\, dt$$
$$- \int_R\int \left\{\frac{\partial f_x}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial f_t}{\partial x}\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\right\} dx\, dt$$
$$- \frac{1}{2}\int_S f_x \frac{\partial w}{\partial x}\frac{\partial w}{\partial n}\, \partial t - \frac{1}{2}\int_{B+B_\tau} n_t f_t \left(\frac{\partial w}{\partial x}\right)^2 dx$$
$$+ \frac{1}{2}\int_R\int \frac{\partial f_x}{\partial x} + \frac{\partial f_t}{\partial t}\left(\frac{\partial w}{\partial x}\right)^2 dx\, dt$$
$$+ \int_R\int \left(f_x \frac{\partial w}{\partial x} + f_t \frac{\partial w}{\partial t}\right)\frac{\partial w}{\partial t}\, dx\, dt.$$

This expression can be simplified as follows: (i) The derivatives $\partial w/\partial t$ on S, and $\partial w/\partial x$ on B_τ, are derivatives of w in tangential directions and since w is constant on $S+B_\tau$, these derivatives are zero there. Thus the term in the first integral containing $\partial w/\partial t$ drops out, as does the integral over B_τ. (ii) Since $w = 0$ on S, $\partial w/\partial x = n_x \partial w/\partial n$ there, and using this in the first and third integrals on the right allows us to combine them into the single expression

$$\frac{1}{2}\int_S f_x n_x \left(\frac{\partial w}{\partial n}\right)^2 dt.$$

This is just the expression which we wish to bound except for the appearance of $f_x n_x$. Now choose f_x such that the minimum of $f_x n_x$ on S is positive. One such choice is $f_x = x$ and for definiteness we use this definition of f_x throughout the remainder of this paper. As for f_t, we need only that f_t be positive, so we take $f_t = 1$. Using all of the above we can write

$$\frac{1}{2}\int_S \left(\frac{\partial w}{\partial n}\right)^2 dt + \frac{1}{2}\int_B \left(\frac{\partial w}{\partial x}\right)^2 dx = \int_R\int \left(x\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t}\right) L^* w\, dx\, dt$$
$$+ \frac{1}{2}\int_R\int \left(\frac{\partial w}{\partial x}\right)^2 dx\, dt - \int_R\int \left(x\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t}\right)\frac{\partial w}{\partial t}\, dx\, dt.$$

An application of the arithmetic-geometric mean inequality yields

$$\frac{1}{2}\int_S \left(\frac{\partial w}{\partial n}\right)^2 dt + \frac{1}{2}\int_B \left(\frac{\partial w}{\partial x}\right)^2 dx \le \frac{1}{2}\int_R\int x^2 \left(\frac{\partial w}{\partial x}\right)^2 dx\,dt$$
$$+ \frac{1}{2}\int_R\int \left(\frac{\partial w}{\partial t}\right)^2 dx\,dt + \int_R\int (L^*w)^2\,dx\,dt$$
$$+ \frac{1}{2}\int_R\int \left(\frac{\partial w}{\partial x}\right)^2 dx\,dt + \frac{1}{2}\int_R\int x^2 \left(\frac{\partial w}{\partial x}\right)^2 dx\,dt \quad (13)$$
$$+ \frac{1}{2}\int_R\int \left(\frac{\partial w}{\partial t}\right)^2 dx\,dt - \int_R\int \left(\frac{\partial w}{\partial t}\right)^2 dx\,dt$$
$$= \frac{1}{2}\int_R\int (1+2x^2)\left(\frac{\partial w}{\partial x}\right)^2 dx\,dt + \int_R\int (L^*w)^2\,dx\,dt.$$

Now the left-hand side of (13) is decreased if the positive term $\int_B (\partial w/\partial x)^2\,dx$ is dropped so that the bound becomes

$$\int_S \left(\frac{\partial w}{\partial n}\right)^2 dt \le 2\left\{\frac{1}{2}(1+2l^2)\int_R\int \left(\frac{\partial w}{\partial x}\right)^2 dx\,dt + \int_R\int (L^*w)^2\,dx\,dt\right\}$$
$$\le 2\left\{\frac{1}{2}\frac{(1+2l^2)}{\lambda_1} + 1\right\}\int_R\int (L^*w)^2\,dx\,dt = \frac{1}{\lambda_1}(1+2l^2+2\lambda_1)\int_R\int u^2\,dx\,dt,$$

using (12) and the fact that $\max_{x\in R} x^2 \le l^2$. This inequality along with (11) in (9) yields the norm bound

$$\left(\int_R\int u^2\,dx\,dt\right)^{\frac{1}{2}} \le \left\{\frac{(1+2l^2+2\lambda_1)}{\lambda_1}\right\}^{\frac{1}{2}}\left(\int_S u^2\,dt\right)^{\frac{1}{2}}$$
$$+ \left(\frac{1}{\lambda_1}\right)^{\frac{1}{2}}\left\{\int_B u^2\,dx + \frac{1}{\lambda_1}\int_R\int (Lu)^2\,dx\,dt\right\}^{\frac{1}{2}},$$

which can be written as

$$\int_R\int u^2\,dx\,dt \le 2\,\frac{(1+2l^2+2\lambda_1)}{\lambda_1}\int_S u^2\,dt + \frac{2}{\lambda_1}\left\{\int_B u^2\,dx + \frac{1}{\lambda_1}\int_R\int (Lu)^2\,dx\,dt\right\}, \quad (14)$$

using the arithmetic-geometric mean inequality.

B. *The Pointwise Bound.* Define two subregions R_1 and R_2 of R as follows:

$$\begin{aligned} R_1 &: \{(x,t) \mid (x-\xi)^2 + (\tau-t)^2 \le r_1,\ t < \tau\}, \\ R_2 &: \{(x,t) \mid (x-\xi)^2 + (\tau-t)^2 \le r_2,\ t < \tau\}, \end{aligned} \quad (15)$$

where $r_1 < r_2$ and r_2 is such that R_2 lies entirely in R.

Now consider the fundamental solution Γ^* of $L^*u = 0$:

$$\Gamma^*(x,t;\xi,\tau) = \frac{1}{2\pi^{\frac{1}{2}}(\tau-t)^{\frac{1}{2}}}\exp\left\{-\frac{(\xi-x)^2}{4(\tau-t)}\right\}, \qquad \tau > t. \quad (16)$$

The function Γ^* has the following properties:

(i) $L^*\Gamma^*(x,t;\xi,\tau) = 0$, for each fixed (ξ,τ), (17)

(ii) $\lim_{t\to\tau}\int_{B_t} \Gamma^*(x,t;\xi,\tau)f(\xi)\,d\xi = f(x)$, (18)

for every continuous function f in B_t.

Introduce a C^2 function $\eta = \eta(x, t)$ defined as follows:

$$\eta(x, t) = \begin{cases} 1, & (x, t) \in \bar{R}_1, \\ 0 \le \eta \le 1, & (x, t) \in R_2 - \bar{R}_1, \\ 0, & (x, t) \in R - R_2. \end{cases} \tag{19}$$

Then, for any function $u = u(x, t)$ with continuous second derivatives with respect to x and continuous first derivative with respect to t, we have, for any $P = (\xi, \tau) \in R_1$,

$$u(P) = u(\xi, \tau) = \int_R\int uL^*(\eta\Gamma_P^*)\, dx\, dt - \int_R\int \eta\Gamma_P^*\, Lu\, dx\, dt. \tag{20}$$

This equality is easily derived using (18) along with Green's second identity and the divergence theorem. Now since $L^*\Gamma_P^* = 0$ we have

$$L^*(\eta\Gamma_P^*) = \Gamma_P^* L^*\eta + 2\, \frac{\partial\Gamma_P^*}{\partial x}\, \frac{\partial\eta}{\partial x},$$

so that, in R_1, $L^*(\eta\Gamma^*) = 0$, since η is constant there. Notice that introduction of the function η does two things: (1) in the first integral in (20) it removes from consideration the singularity of Γ^*, and (2) in the derivation of (20) normal derivatives of u on S are removed since η is zero outside of R_2. *The introduction of η does, however, restrict the bound to interior points of R.*

Setting $L^*(\eta\Gamma^*) = H(x, t, \xi, \tau)$ we have

$$u(\xi, \tau) = \int_R\int uH\, dx\, dt - \int_R\int \eta\Gamma^* Lu\, dx\, dt.$$

An application of the Schwarz and arithmetic-geometric mean inequalities then yields

$$| u(P)|^2 = | u(\xi, \tau)|^2 \le K_1(P) \int_R\int u^2\, dx\, dt + K_2(P) \int_R\int (Lu)^2\, dx\, dt, \tag{21}$$

where

$$K_1(P) = 2 \int_R\int H^2\, dx\, dt \quad \text{and} \quad K_2(P) = 2 \int_R\int (\eta\Gamma^*)^2\, dx\, dt.$$

The singularity of Γ^* is square integrable over R since

$$\begin{aligned} \Gamma^{*2} &\le \frac{\text{constant}}{(\tau - t)} \cdot \exp\left\{-\frac{(\xi - x)^2}{2(\tau - t)}\right\} \\ &= \frac{\text{constant } [(\xi - x)^2/(\tau - t)]^\mu}{(\tau - t)^{1-\mu}\, (\xi - x)^{2\mu}} \cdot \exp\left\{-\frac{(\xi - x)^2}{2(\tau - t)}\right\} \\ &\le \frac{\text{constant}}{(\tau - t)^{1-\mu}(\xi - x)^{2\mu}} \cdot \exp\left\{-\frac{h(\xi - x)^2}{(\tau - t)}\right\}, \qquad 0 < \mu < \tfrac{1}{2}, \end{aligned}$$

where h is a positive constant.

4. *Numerical Example*

Here we give the results of applying our method to the calculation of approximate values, with error bounds, of the solution of the initial boundary value problem

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \qquad -\pi < x < \pi, \quad t > 0,$$

$$u = \cos x, \quad -\pi < x < \pi, \quad t = 0; \qquad u = e^{-t}, \quad x = \pm\pi, \quad t > 0,$$

using as trial functions the polynomials

$$\varphi_n(x, t) = \sum_{k=1}^{n} \frac{x^{2k-2} t^{n-k}}{(2k-2)!\,(n-k)!}, \qquad n = 1, 2, \cdots.$$

For this problem the exact solution can be found by inspection; it is

$$u(x, t) = e^{-t} \cos x.$$

The exact solution is used to give a comparison of actual approximation errors with computed error bounds.

The results of the calculations are given in Table 1 to seven significant figures. The following points bear mentioning:

(1) The approximate values are very close to the actual values; in many cases approximate and actual values agree to seven significant figures. Agreement between approximate and actual values is always much better than the error bounds indicate. Thus although the error bounds are generally good when looked at as a percentage of the approximating value, they are pessimistic when compared to actual errors.

(2) As pointed out in the preceding section, the error bounds worsen as the boundary is approached ($K_1(P)$ and $K_2(P)$ become unbounded as P approaches the boundary). This is noticeable at the larger values of t. Notice, however, that the approximations themselves are not so severely affected.

(3) For $t \geq 3.0$ the error bounds, and to a much smaller degree the approximations, become progressively worse. This happens because as time increases more trial functions are needed to construct the approximating function if a given error is to be maintained. Our program was written with provision for only ten trial functions.

With the exception of the values obtained for $t = 0.2$ and 0.4, all results in Table 1 were obtained using ten trial functions. At $t = 0.2$, eight functions were employed while nine were used for $t = 0.4$. This was necessary because the system formed in applying the Rayleigh-Ritz procedure tended to be slightly ill-conditioned, especially for the smaller values of t. Thus for the first two values of t, calculations employing ten trial functions yielded worse results than calculations employing fewer functions. This tendency toward ill-conditioned systems persisted throughout the calculations although it was not as pronounced at the larger t values. Because of this tendency, and also to avoid loss of significance due to the subtraction of two nearly equal numbers when mean square errors on the boundary were computed, all calculations were done using double precision arithmetic. It is imperative that highly accurate methods are used to determine the c_i's given by the Rayleigh-Ritz procedure since the error bounds, but not the approximations, are extremely sensitive to errors in these constants. Computation time on an IBM 7094 for the results given in the table was 1 minute, 30 seconds.

One last obvious observation concerns the selectivity of the method. That is, an approximate value can be calculated at a few points without the need to perform calculations at many additional points in which one has no interest. Thus if one is

TABLE 1*

x	t	*Approximate solution*	*Error bounds on approximation*	*Exact solution* $u = e^{-t} \cos x$
0	0.2	.8187307	.0000583	.8187307
$.2\pi$	0.2	.6623671	.0000583	.6623671
$.4\pi$	0.2	.2530017	.0000583	.2530017
$.6\pi$	0.2	−.2530017	.0000583	−.2530017
$.8\pi$	0.2	−.6623671	.0000583	−.6623671
0	0.4	.6703200	.0000146	.6703200
$.2\pi$	0.4	.5423003	.0000146	.5423003
$.4\pi$	0.4	.2071403	.0000146	.2071403
$.6\pi$	0.4	−.2071403	.0000146	−.2071403
$.8\pi$	0.4	−.5423003	.0000146	−.5423003
0	0.6	.5488116	.0000110	.5488116
$.2\pi$	0.6	.4439980	.0000110	.4439980
$.4\pi$	0.6	.1695921	.0000110	.1695921
$.6\pi$	0.6	−.1695921	.0000110	−.1695921
$.8\pi$	0.6	−.4439980	.0000110	−.4439980
0	0.8	.4493290	.0000065	.4493290
$.2\pi$	0.8	.3635148	.0000065	.3635148
$.4\pi$	0.8	.1388503	.0000065	.1388503
$.6\pi$	0.8	−.1388503	.0000065	−.1388503
$.8\pi$	0.8	−.3635148	.0000098	−.3635148
0	1.0	.3678794	.0000072	.3678794
$.2\pi$	1.0	.2976207	.0000072	.2976207
$.4\pi$	1.0	.1136810	.0000072	.1136810
$.6\pi$	1.0	−.1136810	.0000072	−.1136810
$.8\pi$	1.0	−.2976207	.0000142	−.2976207
0	1.25	.2865048	.0000115	.2865048
$.2\pi$	1.25	.2317872	.0000115	.2317873
$.4\pi$	1.25	$.8853485 \times 10^{-1}$	.0000115	$.8853485 \times 10^{-1}$
$.6\pi$	1.25	$-.8853487 \times 10^{-1}$	.0000115	$-.8853485 \times 10^{-1}$
$.8\pi$	1.25	−.2317872	.0000302	−.2317873
0	1.5	.2231302	.0000123	.2231302
$.2\pi$	1.5	.1805161	.0000123	.1805161
$.4\pi$	1.5	$.6895101 \times 10^{-1}$	.0000123	$.6895101 \times 10^{-1}$
$.6\pi$	1.5	$-.6895100 \times 10^{-1}$	.0000154	$-.6895101 \times 10^{-1}$
$.8\pi$	1.5	−.1805160	.0000407	−.1805161
0	1.75	.1737739	.0000013	.1737739
$.2\pi$	1.75	.1405861	.0000013	.1405861
$.4\pi$	1.75	$.5369910 \times 10^{-1}$	.0000013	$.5369910 \times 10^{-1}$
$.6\pi$	1.75	$-.5369910 \times 10^{-1}$	.0000020	$-.5369910 \times 10^{-1}$
$.8\pi$	1.75	−.1405861	.0000054	−.1405861
0	2.0	.1353353	.0000127	.1353353
$.2\pi$	2.0	.1094885	.0000127	.1094885
$.4\pi$	2.0	$.4182087 \times 10^{-1}$	.0000145	$.4182090 \times 10^{-1}$
$.6\pi$	2.0	$-.4182100 \times 10^{-1}$	.0000228	$-.4182090 \times 10^{-1}$
$.8\pi$	2.0	−.1094886	.0000620	−.1094885
0	2.5	$.8208500 \times 10^{-1}$	.0000253	$.8208500 \times 10^{-1}$
$.2\pi$	2.5	$.6640812 \times 10^{-1}$	.0000253	$.6640816 \times 10^{-1}$
$.4\pi$	2.5	$.2536559 \times 10^{-1}$	.0000374	$.2536566 \times 10^{-1}$
$.6\pi$	2.5	$-.2536571 \times 10^{-1}$	.0000590	$-.2536566 \times 10^{-1}$
$.8\pi$	2.5	$-.6640818 \times 10^{-1}$	.0001556	$-.6640816 \times 10^{-1}$
0	3.0	$.4978704 \times 10^{-1}$	.0000758	$.4978707 \times 10^{-1}$
$.2\pi$	3.0	$.4027856 \times 10^{-1}$	.0000934	$.4027859 \times 10^{-1}$
$.4\pi$	3.0	$.1538495 \times 10^{-1}$	.0001381	$.1538505 \times 10^{-1}$
$.6\pi$	3.0	$-.1538505 \times 10^{-1}$	.0002178	$-.1538505 \times 10^{-1}$
$.8\pi$	3.0	$-.4027803 \times 10^{-1}$	.0005743	$-.4027859 \times 10^{-1}$
0	4.0	$.1831575 \times 10^{-1}$	.0004035	$.1831564 \times 10^{-1}$
$.2\pi$	4.0	$.1481759 \times 10^{-1}$	.0004972	$.1481766 \times 10^{-1}$
$.4\pi$	4.0	$.5659724 \times 10^{-2}$	.0007355	$.5659844 \times 10^{-2}$
$.6\pi$	4.0	$-.5659134 \times 10^{-2}$	.0011600	$-.5659844 \times 10^{-2}$
$.8\pi$	4.0	$-.1481537 \times 10^{-1}$	.0030588	$-.1481766 \times 10^{-1}$

* Approximate values for negative x are obtained by reflecting those given across the t-axis since it is evident *from the initial and boundary data* that the solution is symmetric with respect to the t-axis.

interested in having an approximation at the point (x_1, t_1), the approximation can be computed immediately without "building up" the solution through a succession of calculations from $t = 0$ to $t = t_1$.

Remark. In applying our method with Rayleigh-Ritz improvement to the problem (1a), (1b), (1c), most of the computational effort is spent in solving n simultaneous equations in the n unknowns $c_1, c_2, \cdots, c_n$. These equations have the form

$$\sum_{i=1}^{n} c_i \left\{ A_1 K_1(P) \int_B \varphi_i(x, 0)\varphi_j(x, 0)\, dx + A_2 K_1(P) \int_{S_\tau} \varphi_i(\pm l, t)\varphi_j(\pm l, t)\, dt + A_3 K_2(P) \int_{R_\tau} L\varphi_i\, L\varphi_j\, dx\, dx \right\}$$

$$= A_1 K_1(P) \int_B f_2(x)\varphi(x, 0)\, dx + A_2 K_1(P) \cdot \int_{S_\tau} f_3(t)\varphi_j(\pm l, t)\, dt + A_3 K_2(P) \int_{R_\tau} f_1\, L\varphi_j\, dx\, dt \qquad (j = 1, 2, \cdots, n),$$

where the A_i $(i = 1, 2, 3)$ are constants and $P = (x, \tau)$ is the point at which the approximate value is being computed. Thus, although integrals over B need be evaluated only once *for all* P and the integrals over S_τ and R_τ need be evaluated only once *for each time value*, the dependence of K_1 and K_2 on P, and hence on x, would seem to force us to solve the above system at each point of the plane $t =$ const. as the x-coordinate varies from point to point of the plane. This situation can easily be overcome by making $K_1(P) = K_2(P)$ for each value of t independent of x, either by adjusting the function η by varying r_1 and r_2 (recall (15) and (19)), or, if one "constant" is always larger than the other, setting them both equal to the larger one. Either way the bound will be worsened slightly. However, what is lost here can be made up by using more trial functions. The time gained by making the Rayleigh-Ritz calculations independent of x will more than offset the small increase in computation time required because of the use of a few more trial functions.

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