A Mathematical Model for the Analysis of Contour-Line Data

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ABSTRACT. This paper describes a mathematical model for the study of contour-line data. Formal definitions are given for the various classes of contour lines found on a contour map. The concept of cliff lines is introduced and the properties of both contour lines and cliff lines are investigated. The objective of the paper is to lay a foundation for the development of algorithms that will facilitate the digital computer solution of problems involving contour-line data.

KEY WORDS AND PHRASES: contours, pattern recognition, maps, isotherms, isobars, equipotential lines, surface

CR CATEGORIES: 3.10, 3.14, 3.16, 3.19, 3.63

Introduction. Data that can be displayed by a contour map is found in many engineering applications. Any piecewise-continuous, single-valued function of two continuous independent variables can be represented in the form of a contour map. The most common example is a contour map representing elevation as a function of position in a two-dimensional geographic region. Other geographic-position–dependent variables that are commonly represented in the form of contour maps are temperature (isotherms) and pressure (isobars). However, the use of contour maps need not be restricted to dependent variables relating to geographic position. An example of a contour map used for a more general dependent variable is a plot of equal-loudness curves drawn as a function of the intensity and frequency of an audible tone.

In most of the applications of contour maps, the relationship between the dependent variable and the independent variables cannot be conveniently expressed by an equation. However, there are some applications in which contour maps are used even though an equation is readily at hand because the contour maps facilitate visualization of the data. An example of the latter is a plot of the equipotential lines around an electric dipole.

A formal method for analyzing contour maps through the use of a mathematical model is discussed in this paper. Various terms are defined to facilitate the discussion of contour maps. The objective of the paper is to lay a foundation for the development of algorithms that will facilitate the digital computer solution of problems involving contour-line data.

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The Contour-Line Model. The model described here can be used to represent certain piecewise-continuous, single-valued functions of two continuous independent variables. It is convenient to regard the independent variables as the position coordinates of points p on a (two-dimensional) surface and the dependent variable as an *elevation function*, E. The quantity $E(p)$ is called the *elevation* of p.

1. *Definition of Model*

A *semismooth domain* is a two-dimensional domain of points over which the elevation function is continuous and single-valued and the domain cannot be enlarged and still retain this property. A *semismooth subdomain* is a domain contained within a semismooth domain. A *boundary function* of a semismooth domain is defined for each boundary point of the semismooth domain to be the limiting value of the elevation function as the boundary point is approached providing such a limit exists. If, for arbitrarily small values of ϵ, every ϵ-neighborhood of a boundary point of a semismooth domain is divided by boundary lines into two or more semismooth subdomains of the same semismooth domain, the boundary function of the semismooth domain is defined with respect to each semismooth subdomain separately as the limiting value of the elevation function in that semismooth subdomain, providing such a limit exists. A *smooth domain* is a semismooth domain whose boundary function is defined (with respect to smooth subdomains if necessary) and continuous. A *smooth subdomain* is a domain contained within a smooth domain. The {*maximum*, *minimum*} *boundary function* of a smooth domain is defined for each boundary point of the smooth domain to be the value of the boundary function with respect to that smooth subdomain which {maximizes, minimizes} the value.[1]

A *cliff line* is a directed line[2] or line segment on which the elevation function at each point of the cliff line is multivalued and takes on all values between and including two bounds. The bounds form two single-valued functions of position along the cliff line. The function for the upper bound is called the *left limit function*, L, and the function for the lower bound is called the *right limit function*, R, of the cliff line. A *cliff point* is a cliff line that has degenerated into a point; such singular points are not considered in this model.

A *map* is a two-dimensional domain of points composed entirely of smooth domains and cliff lines such that the following conditions are satisfied:

1. Every boundary point of a smooth domain is on a cliff line or is a boundary point of the map.
2. For each point p_i on a cliff line, $\{L(p_i), R(p_i)\}$ equals the value of the {maximum, minimum} boundary function of the smooth domain(s) on the {left, right} of the cliff line at p_i.[2]

An example of an elevation function and its corresponding map is shown in Figure 1.

To eliminate certain pathological cases, the following restriction is included in the definition of a map:

Restriction. It is assumed that for every point p on the map, any finite-size neighborhood of p can be divided into a *finite* number of connected sets such that the

[1] A statement of the form: $\ldots\{a_1, a_2, \ldots, a_n\}\ldots\{b_1, b_2, \ldots, b_n\}\ldots\ldots\{m_1, m_2, \ldots, m_n\}\ldots$ is a shorthand notation indicating a set of n parallel statements. For example, the statement "$\{2, 5\} + \{3, 7\} = \{5, 12\}$" represents the statements "$2 + 3 = 5$" and "$5 + 7 = 12$."

[2] The word "line" when used in this paper does not necessarily imply a straight line.

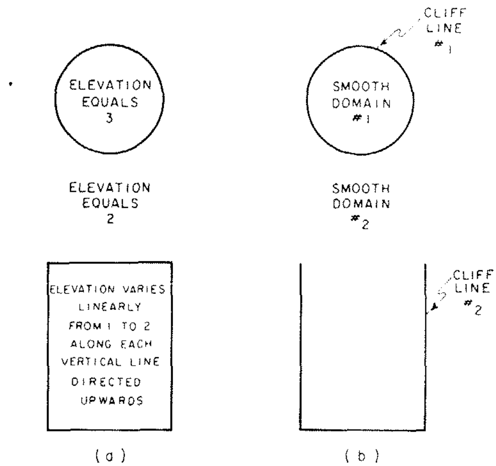


FIG. 1. (a) Example of an evaluation function; (b) its corresponding map

TABLE I. CLASSIFICATION OF NEIGHBORHOODS CENTERED AT POINT p

Neighborhood or elevation type	*Neighborhood contains only points (other than p) of each of the indicated elevations*		
	$<E(p)$	$=E(p)$	$>E(p)$
I	×		
II		×	
III			×
IV	×	×	
V		×	×
VI	×		×
VII	×	×	×

elevations of points in each set are in one of the following categories:

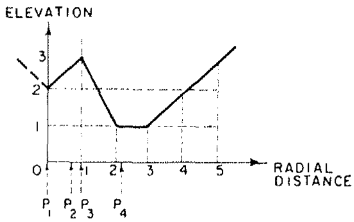
1. All elevations less than $E(p)$.
2. All elevations equal to $E(p)$.
3. All elevations greater than $E(p)$.

Also each connected set of points of elevation equal to $E(p)$ can be decomposed into a *finite* number of open connected sets and a finite number of lines.

2. *Elevation Types*

Contour lines will be defined by examining $E(p)$ for points p in the neighborhood of a given point on a map. This approach requires introduction of the concepts of neighborhood types and elevation types.

Select a point p on a given map. If p lies on a cliff line, $E(p)$ can be any number that satisfies the relation $R(p) \leq E(p) \leq L(p)$. Consider a neighborhood of p with a radius of δ. The points in this neighborhood could have elevations greater than, equal to, or less than $E(p)$. A table can be constructed in which a *neighborhood type* is assigned to every conceivable combination of these elevations, as shown in Table I. Not all the neighborhood types are possible as will be seen below. The



(a)

POINT	RADIAL DISTANCE	ELEVATION	ELEVATION TYPE	CRITICAL RADIUS	ELEVATION TYPE WITH RESPECT TO CIRCUMFERENCE THROUGH POINT	CRITICAL RADIUS WITH RESPECT TO CIRCUMFERENCE THROUGH POINT
P_1	0	2	III	$1\frac{1}{2}$	—	—
P_2	$\frac{3}{4}$	$2\frac{3}{4}$	VII	∞	VI	$\frac{1}{2}$
P_3	1	3	IV	4	I	4
P_4	$2\frac{1}{4}$	1	II	$\frac{1}{4}$	II	$\frac{1}{4}$

(b)

FIG. 2. Examples of elevation types and subcritical neighborhoods. The region has an elevation function that varies with radial distance as shown in (a). The elevation type and critical radius of selected points in the region are given in (b).

neighborhood type is a function of δ and p and is written as $T_N(p, \delta)$. If p is on a cliff line, the neighborhood type is also a function of the value of $E(p)$ chosen. There exists a maximum value, δ_0, such that for all $\delta < \delta_0$ the neighborhood type is not a function of δ. That neighborhood type is called the *elevation type* of p, written $T_E(p)$, and δ_0 is called the *critical radius* of p. Any neighborhood of p whose radius is less than δ_0 is called a *subcritical neighborhood* of p.

The neighborhood type of p with respect to line l_i, designated as $\tilde{T}_N(p)_i$, is defined as the neighborhood type obtained by considering only those points that are not on the line. The subscript i can be removed from $\tilde{T}_N(p)_i$ if there is no ambiguity as to which line is involved. The *elevation type with respect to a line*, $\tilde{T}_E(p)_i$, the *critical radius with respect to a line*, and a *subcritical neighborhood with respect to a line* are similarly defined. Examples of elevation types and subcritical neighborhoods are shown in Figure 2.

PROPERTY 1. *Neighborhood type VI is impossible.*

PROOF. Let p_0 be a point such that $T_N(p_0, \delta) = \text{VI}$. A line l can be drawn that lies entirely in the δ-neighborhood and connects a point whose elevation is less than $E(p_0)$ to a point whose elevation is greater than $E(p_0)$ and does not pass through p_0. As a consequence of the continuity of the elevation function in a smooth domain and the "bridging"[3] by cliff lines of the elevation function in adjacent smooth domains, l must pass through at least one point p_1 such that $E(p_1) = E(p_0)$. Thus $T_N(p_0, \delta) = \text{VII}$.

Note that neighborhood type VI with respect to a line is possible.

Some simple consequences that follow from the definition of elevation type II are the following:

1. If $T_E(p_0) = \text{II}$ and p_1 is a point in a subcritical neighborhood of p_0, then $E(p_0) = E(p_1)$.
2. No point on a cliff line can have elevation type II. It appears as though points on cliff lines cannot have elevation type I or III. This is not true as is shown in Figure 3.

[3] The bridging is a consequence of the fact that the elevation function at each point of the cliff line is multivalued and takes on all values between and including the values of the two limit functions.

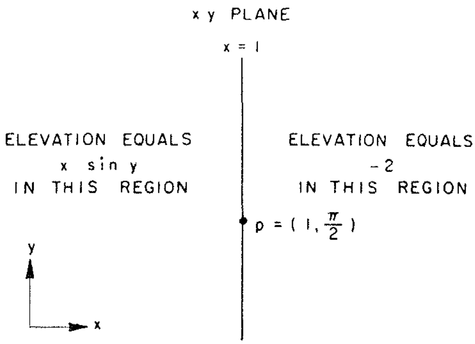


FIG. 3. Example of a point on a cliff line with elevation type I. Point p is on a cliff line. The left limit function at p is 1 and the right limit function at p is -2. For an assumed elevation of 1, $T_E(p) = \text{I}$.

PROPERTY 2. *Starting at a point p_0 for which $T_E(p_0) = \{II,IV,V,VII,VII\}$, a line segment can always be drawn such that for each point p_i on the line, $E(p_i) = E(p_0)$ and $T_E(p_i) = \{II,IV,V,IV \text{ or } VII, V \text{ or } VII\}$.*

PROOF. Only the last of the five parts of this property are proved. The other four proofs are similar. Let p_0 be a point such that $T_E(p_0) = \text{VII}$. Let N be a subcritical neighborhood of p_0. Lines can always be drawn through N that isolate those points of elevation greater than $E(p_0)$ from those points of elevation less than or equal to $E(p_0)$. These lines are composed entirely of points whose elevations are equal to $E(p_0)$. Every point on these lines has elevation type V or VII. Point p_0 lies on one of these lines. That line satisfies the requirements of this property.

COROLLARY 2.1. *A line segment can always be drawn through a point p_0 for which $T_E(p_0) = \{II,VII,VII\}$ such that for each point p_i on the line, $E(p_i) = E(p_0)$ and $T_E(p_i) = \{II,IV \text{ or } VII,V \text{ or } VII\}$.*

COROLLARY 2.2. *Starting at a point p_0 for which $T_E(p_0) = \{IV,V,VII,VII\}$, a line segment can always be drawn such that for each point p_i on the line and $p_i \neq p_0$, $E(p_i) = E(p_0)$ and $\tilde{T}_E(p_i)$ is constant and is equal to $\{I,III,IV,V\}$ or $\{IV,V,VI,VI\}$.*

COROLLARY 2.3. *A line segment can always be drawn through a point p_0 for which $T_E(p_0) = VII$ such that for each point p_i on the line and $p_i \neq p_0$, $E(p_i) = E(p_0)$ and $\tilde{T}_E(p_i) = \{IV \text{ or } VI,V \text{ or } VI\}$.*

PROPERTY 3. *If, for some point p, $\tilde{T}_E(p) = \{I,II,III,IV,V,VI,VII\}$, then $T_E(p) = \{I \text{ or } IV,II,III \text{ or } V,IV,V,VII,VII\}$.*

PROOF. Let N be a subcritical neighborhood of p. If a line l through p contains any points in N of elevations {greater, less} than $E(p)$, there must be other points in N but not on l of elevations {greater, less} than $E(p)$. Thus any difference between elevation type and elevation type with respect to a line must be due to the consideration of points of elevation equal to $E(p)$.

COROLLARY 3.1. *If p lies on a line and $T_E(p) = \{I,II,III,IV,V,VII\}$, then $\tilde{T}_E(p) = \{I,II,III,I \text{ or } IV,III \text{ or } V,VI \text{ or } VII\}$.*

PROPERTY 4. *If, for some point p, $\tilde{T}_E(p) = \{I,III\}$ and every point on the line has the same elevation,[4] then $T_E(p) = \{IV,V\}$.*

[4] If p is a point on the cliff line, then the requirement is that every point on the line have the same elevation as that elevation used for p in determining $T_E(p)$ and $\tilde{T}_E(p)$.

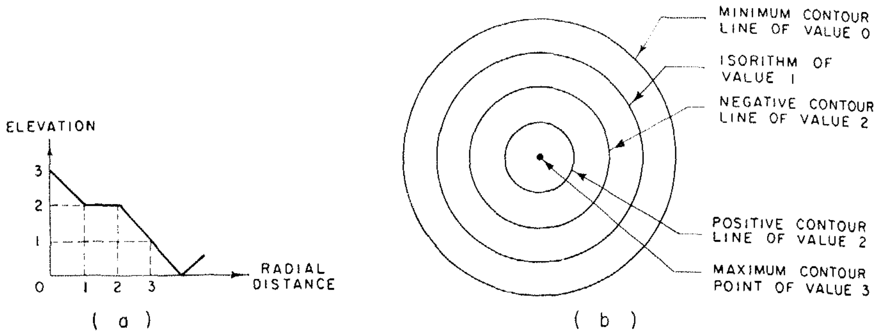


FIG. 4. Example of contour lines: (a) variation of elevation with radial distance; (b) some contour lines of the region

PROOF. The subcritical neighborhood and subcritical neighborhood with respect to a line differ by the inclusion of a set of points whose elevations equal $E(p)$. The property now follows directly from the definitions of elevation types as given in Table I.

3. *Contour Lines*

Contour lines are used as a graphical device for displaying the elevations of different points on a map [5]. There are a number of different classes of contour lines that can be drawn on a map. A line is a {*positive, negative, maximum, minimum*} *contour line* of *value e* if for all points p_i on the line, $E(p_i) = e$ and $\tilde{T}_E(p_i) = \{$V or VI or VII, IV or VI or VII,I,III$\}$. Note that if $\tilde{T}_E(p_i) =$ VI or VII for a segment of the line, that segment is part of both a positive contour line and a negative contour line. Such a segment is called an *isorithm*. If $E(p_0) = e$ and $T_E(p_0) = \{$I,III$\}$ for some point p_0, then p_0 is said to be a {*maximum, minimum*} *contour point* of *value e*. Maximum and minimum contour points are considered special cases of maximum and minimum contour lines.

Positive and negative contour lines are called *normal contour lines*. Maximum and minimum contour lines are called *degenerate contour lines*.[5] Examples of contour lines are shown in Figure 4.

In the preceding discussion contour lines are defined in terms of points on the lines. The following two properties establish the criteria for determining whether a point lies on a contour line.

PROPERTY 5. *A point p_0 lies on an isorithm of value $E(p_0)$ if $T_E(p_0) =$ VII.*

PROOF. Corollaries 2.3 and 3.1 state that a line segment can be drawn through p_0 such that for all points p_i on the line, $E(p_i) = E(p_0)$ and $\tilde{T}_E(p_i) =$ V,VI, or VII. This line satisfies the definition of a positive contour line of value $E(p_0)$. The existence of a negative contour line passing through p_0 can be similarly demonstrated.

PROPERTY 6. *If $T_E(p_0) = \{IV, V\}$, then p_0 lies on a {maximum, minimum} contour line of value $E(p_0)$. If no such line exists, p_0 is either the endpoint of a {maximum, minimum} contour line or p_0 lies on a {negative, positive} contour line of value $E(p_0)$.*

PROOF. Only the last of the two parts of this property are proved. The other

[5] A method of "normal-izing" degenerate contour lines is presented in the Appendix.

proof is similar. Let $T_E(p_0) = \text{V}$. If there exists a line l_1 through p_0 such that $\tilde{T}_E(p_0)_1 = \text{III}$, l_1 must consist entirely of points whose elevations are $E(p_0)$. (These are the very points that require that $T_E(p_0) = \text{V}$ rather than III.) This line, consisting of points p_i with $E(p_i) = E(p_0)$ and $\tilde{T}_E(p_i)_1 = \text{III}$, satisfies the definition of a minimum contour line. If no such line exists, there may be a line l_2 starting at p_0 for which $\tilde{T}_E(p_i)_2 = \text{III}$ for all points p_i on l_2 other than p_0. This line is a minimum contour line ending at p_0. If neither of these lines exist, Corollary 2.2 assures the existence of a line l_3 containing p_0 that consists entirely of points p_i such that $E(p_i) = E(p_0)$ and $\tilde{T}_E(p_i)_3 = \text{V}$. The line l_3 satisfies the definition of a positive contour line.

PROPERTY 7. *If p lies on a* {*positive, negative, maximum, minimum*} *contour line,* $T_E(p) = \{V, IV, I, III\}$ *or* $\{VII, VII, IV, V\}$.

PROOF. This property follows directly from Property 3 and the definitions of the different classes of contour lines.

PROPERTY 8. *Let* p_0 *be a point through which passes one and only one*[6] {*positive, negative*} *contour line of value e and through which passes no* {*minimum, maximum*} *contour line of value e. Every subcritical neighborhood of* p_0 *can be reduced to a smaller neighborhood that is divided by the contour line into two open sets of points—one consisting entirely of points whose elevations are* {*greater than, less than*} *e and the other containing no points of elevation* {*greater than, less than*} *e. These sets are called the* {*hi-set, lo-set*} *of* p_0 *and the* {*lo-set, hi-set*} *of* p_0, *respectively.*

PROOF. Let p_0 be a point through which passes only one positive contour line of value e. Draw a subcritical neighborhood N around p_0. In this neighborhood there must be at least one point p_1 such that $E(p_1) > e$. Let N_1 be the open set of points lying in N and on the same side of the contour line as p_1. Let N_2 be the open set of points lying in N but on the other side of the contour line. Assume there is at least one point in N_1 that has an elevation of e, and that N cannot be reduced to remove all such points. Then lines can be drawn through N_1 that isolate those points whose elevations are greater than e from those points whose elevations are not greater than e. These lines are positive or minimum contour lines of value e, and at least one of these lines, other than the given contour line, passes through p_0. Therefore p_0 is on a minimum contour line or is on more than one positive contour line of value e, and this is a contradiction. Thus there are no points of elevation e in N_1. Also, there are no points of elevation less than e in N_1 because that would require at least one point of elevation e in N_1. Thus the elevation of every point in N_1 is greater than e. If N_2 contains a point whose elevation is greater than e, every point in N_2 would have an elevation greater than e. The elevation type of p_0 with respect to the contour line would be III, but this contradicts the definition of a positive contour line. Thus every point in N_2 has an elevation less than or equal to e. A similar proof holds for negative contour lines.

If all points on a segment of a normal contour line have their hi-sets on the same side of the contour line, that segment is assigned a positive direction such that all hi-sets are on the left as the segment is traversed in its positive direction. The set of all points satisfying the definition of a normal contour line of value e can be de-

[6] This statement implicitly prohibits the possibility of the same normal contour line passing through p_0 more than once.

composed into a finite number of such directed line segments. These segments are defined to lie on the same contour line only if their positive directions are consistent. The *positive direction of a normal contour line* is the positive direction of its segments. Degenerate contour lines are undirected line segments.

PROPERTY 9. *Let p be a point on a normal contour line. Let S be any circle about p that does not intersect with the boundary of the domain of the map. If there is a point on the contour line that is outside S and in the {positive, negative} direction along the contour line from p, then there is a point on the contour line that is outside S and in the {negative, positive} direction along the contour line from p.*

PROOF. Let p be a point on a positive contour line of value e and let S be a circle about p. Draw lines through S that separate those points of elevation greater than e from those points of elevation less than or equal to e. Each of these lines are segments of positive contour lines of value e and every segment of a positive contour line of value e inside S is included in this collection of lines. Each line is either a closed curve inside S or intersects S twice. Thus if a positive contour line comes into S, it will go out of S. A similar proof is valid for negative contour lines.

COROLLARY 9.1. *Normal contour lines cannot start or end at any point that is not a boundary point of the map.*

COROLLARY 9.2. *Normal contour lines that do not intersect any boundaries of the map are closed curves.*

COROLLARY 9.3. *There are as many {positive, negative} contour line segments of a given value entering a point as there are {positive, negative} contour line segments of that value leaving the point.*

COROLLARY 9.4. *Between any two positive contour line segments of the same value {entering, leaving} a point there is a positive contour line segment of that value {leaving, entering} the point. This is also true for negative contour lines.*

A point that has more than one {positive, negative} contour line segment of a given value entering it is called a *saddle*. At points for which more than one segment of a normal contour line of a given value enters the point, there is an ambiguity as to which segment leaving the point is the continuation of which segment entering the point. The following rule resolves this ambiguity.

Rule 1. Each {positive, negative} contour line segment of value e leaving a point is considered as the continuation of the first {positive, negative} contour line segment of value e in the {counterclockwise, clockwise} direction entering the point.

4. *Properties of Cliff Lines*

Ambiguities may result unless some rules are established for the termination of cliff lines. Such rules can be formulated based on the behavior of the limit functions. The symbol $L(p^+)$ is used to represent $\lim_{p_1 \to p} L(p_1)$, where p_1 is another point on the cliff line and in the positive direction from p. Similar definitions apply to $L(p^-)$, $R(p^+)$, and $R(p^-)$.

PROPERTY 10. *Let p be any point on a cliff line that is not an endpoint of the cliff line. Then*

$$L(p) \geq max(L(p^+),\ L(p^-)), \qquad R(p) \leq min(R(p^+),\ R(p^-)).$$

PROOF. The limiting value of the elevation function at p^+ must be a limiting value of the elevation function at p because p^+ is in every neighborhood of p. Thus the value of the boundary function in the smooth domain on the left side of the cliff line at p^+ must be less than or equal to the maximum value of the boundary functions in the smooth domains on the left side of the cliff line at p. Thus $L(p^+) \le L(p)$. Similarly $L(p^-) \le L(p)$, $R(p^+) \ge R(p)$, and $R(p^-) \ge R(p)$.

COROLLARY 10.1. *If p is the {initial, terminal} point of a cliff line, $L(p) \ge \{L(p^+), L(p^-)\}$ and $R(p) \le \{R(p^+), R(p^-)\}$.*

PROPERTY 11. *Let p be a point such that there is at least one cliff line segment entering p and at least one cliff line segment leaving p. Let C be the line formed by these two segments. If $\{L, R\}$ is discontinuous at p, then there is another cliff line segment that contains p and lies on the {left, right} side of C.*

PROOF. Assume there is no cliff line segment that contains p and lies to the left of C. Then there is only one smooth domain to the left of C at p. A discontinuity in the left limit function of C at p must be accompanied by a discontinuity in the boundary function of the smooth domain on the left of the cliff line at p. But the boundary function of the smooth domain is continuous. This is a contradiction and hence the assumption is false.

COROLLARY 11.1. *The {left, right} limit function is continuous at all points of a cliff line that are not also on some other cliff line.*

PROPERTY 12. *Let p be a point that lies on one or more cliff lines. Let $C_0, C_1, \ldots, C_{n-1}$ be the cliff line segments entering or leaving p such that C_{i+1} is the next cliff line segment after C_i in the counterclockwise direction around p. Let ζ_i, ζ_i', and η_i be defined as follows:*

$$\zeta_i = \begin{cases} L & \text{if } C_i \text{ enters } p, \\ R & \text{if } C_i \text{ leaves } p; \end{cases} \qquad \zeta_i' = \begin{cases} R & \text{if } \zeta_i = L, \\ L & \text{if } \zeta_i = R; \end{cases} \qquad \eta_i = \begin{cases} - & \text{if } C_i \text{ enters } p, \\ + & \text{if } C_i \text{ leaves } p; \end{cases}$$

and all subscripts are taken modulo n. Let $L(p)_i$ and $R(p)_i$ represent the limit functions for cliff line C_i at point p. Then

$$\zeta_{i+1}(p^{\eta_{i+1}})_{i+1} = \zeta_i'(p^{\eta_i})_i, \qquad i = 0, 1, \ldots, n-1.$$

An example of this property is illustrated in Figure 5.

PROOF. Consider a neighborhood of p that is small enough so as not to contain any segments of cliff lines that do not pass through p and not to contain any other points that lie on more than one cliff line. The neighborhood is divided into sectors by all the cliff lines that pass through p. Let S_i be the sector bound by C_i and C_{i+1}

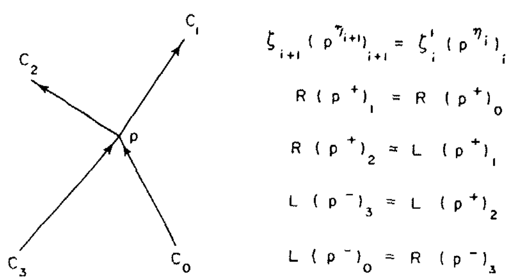


FIG. 5. Illustration of the relation between the limit functions of intersecting cliff lines

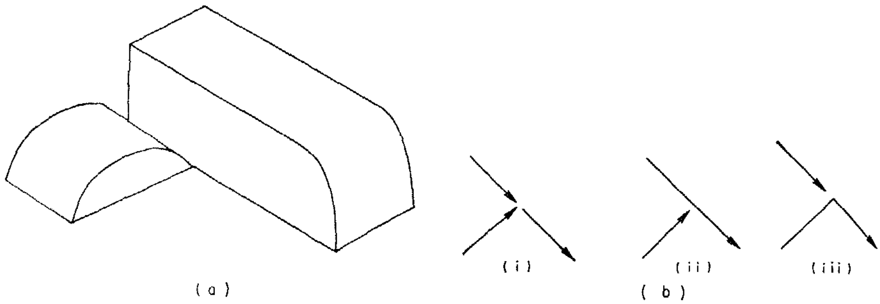


FIG. 6. Example of an ambiguity in selecting cliff lines: (a) surface representing elevation function; (b) three choices for cliff lines—choice (ii) is the correct choice.

Note that the interior of each sector is a smooth subdomain. Therefore, the limiting value of the boundary function of S_i along C_i as p is approached is equal to the limiting value of the boundary function along C_{i+1} as p is approached. But a limit function is equal to the boundary function on the corresponding side of the cliff line. Assume C_i and C_{i+1} are both directed away from p. Then $R(p^+)_{i+1} = L(p^+)_i$. Generalizing this equation for other combination of directions for C_i and C_{i+1} yields the equation stated in the property.

Rule 2. If at any point p on a cliff line either $L(p^+) \leq R(p^-)$, or $L(p^-) \leq R(p^+)$, or $L(p^+) = R(p^+)$, or $L(p^-) = R(p^-)$, then the cliff line is to be interpreted as two cliff lines with p being an endpoint of both.

Rule 3. If p is a point on a cliff line and more than one of the cliff line segments entering p do not have $L(p^-) = R(p^-)$ or more than one of the cliff line segments leaving p do not have $L(p^+)$ equal to $R(p^+)$, then p is considered an endpoint of all cliff lines containing p. This rule is illustrated in Figure 6.

PROPERTY 13. *Let C be a cliff line that cannot be subdivided into two or more cliff lines by Rule 2. If the left or right limit function of C is discontinuous at a point p, then p is an endpoint of all cliff lines containing p.*

PROOF. Let C_1 be the portion of C entering p and C_2 be the portion of C leaving p. By hypothesis $L(p^-)_1 \neq R(p^-)_1$ and $L(p^+)_2 \neq R(p^+)_2$. If L is discontinuous at p then $L(p^-)_1 \neq L(p^+)_2$. Then by Property 11 there exists a cliff line C_3 on the left of C such that p is contained on C_3. Assume C_3 is directed toward p and assume there is no other cliff line on the left of C that contains p. Then by Property 12, $L(p^-)_3 = L(p^+)_2$ and $R(p^-)_3 = L(p^-)_1$. Therefore $L(p^-)_3 \neq R(p^-)_3$. Thus there are two cliff line segments directed toward p for which $L(p) \neq R(p)$. But Rule 2 states that in such cases p is considered as an endpoint of all cliff lines containing p. If there is more than one cliff line that contains p and is on the left of C, an extension of this argument shows that for at least one of these cliff lines $L(p) \neq R(p)$.

COROLLARY 13.1. *The left and right limit functions of a cliff line are continuous along the cliff line.*

5. *Contour Lines in Cliff Lines*

A complete understanding of contour lines requires a knowledge of the behavior of contour lines at a cliff line. The following study of this behavior yields the result that the contour lines merging to form a cliff line do not cross each other.

PROPERTY 14. *Let p be a point on a cliff line C. Point p must also lie on the following contour lines:*

1. *Positive and negative contour lines of value e for all e such that* $R(p) < e < L(p)$;
2. *Negative or maximum contour line of value* $L(p)$;
3. *Positive or minimum contour line of value* $R(p)$.

oint p may also lie on the following contour lines:

4. *Positive contour line of value* $L(p)$;
5. *Negative contour line of value* $R(p)$.

oint p may not lie on any other contour line.

PROOF.

1. For any arbitrarily small neighborhood around p there must be a point in the ighborhood and on the left side of C whose elevation is greater than e and a point the neighborhood and on the right side of C whose elevation is less than e. If (p) is taken as e, then $T_E(p) = \text{VII}$. Therefore p lies on a positive and negative ntour line of value e as a result of Property 5.

2. For any arbitrarily small neighborhood around p there must be a point on e right side of C whose elevation is less than $L(p)$. If $E(p)$ is taken as $L(p)$, en $T_E(p) = \text{I}$, IV or VII. If $T_E(p) = \text{I}$, then p is a maximum contour point. $T_E(p) = \text{IV}$, then p lies on a maximum or negative contour line as a result of operty 6. If $T_E(p) = \text{VII}$, then p lies on a negative contour line as a result of operty 5.

3. Proved in like manner as 2.

4. In any arbitrarily small neighborhood around p there may be a point in the ghborhood and on the left side of C whose elevation is greater than $L(p)$. In s case if $E(p)$ is taken as $L(p)$, then $T_E(p) = \text{VII}$. Therefore p lies on a positive itour line as a result of Property 5.

5. Proved in like manner as 4.

Point p cannot lie on any contour line of value e for $e > L(p)$ or $e < R(p)$ be- se such an e would not be one of the possible elevations of p. Obviously p cannot on any degenerate contour lines of value e for $R(p) < e < L(p)$. The only naining contour lines for p to lie on are a maximum contour line of value $R(p)$ l a minimum contour line of value $L(p)$. If $E(p)$ is taken as $L(p)$, then $T_E(p) =$ V, or VII. Therefore, p cannot lie on a minimum contour line of value $L(p)$ a result of Property 7. Similarly, p cannot lie on a maximum contour line of ie $R(p)$.

'hus the contour lines mentioned in Property 14 must meet and travel along the line. When a contour line travels along a cliff line, there must be a point on the line at which the contour line joins the cliff line and another point at which the tour line separates from the cliff line. The contour line is said to "enter" and ve" the cliff line at these two points respectively.

ROPERTY 15. *When a normal contour line travels along a cliff line, the positive ction of the contour line is the same as the positive direction of the cliff line.*

ROOF. The cliff line has the higher elevation on its left side. But so does a nal contour line. Thus, the positive direction of a cliff line and the normal our lines that are traveling along it must be the same.

lthough a degenerate contour line is an undirected line, it is found convenient to define a positive direction to that portion of a degenerate contour line that travels along a cliff line. To be consistent with normal contour lines, the *positive direction* of that portion of a degenerate contour line that travels along a cliff line is defined to be the same as the positive direction of the cliff line. Using this definition, the word "normal" is not needed in Property 15.

PROPERTY 16.

1. *If there is a segment of a cliff line for which L is monotonically {increasing, decreasing} and* $L_{min} < e < L_{max}$, *then a positive and negative contour line of value e {entered, left} the cliff line segment from the left.*

2. *If there is a segment of a cliff line for which R is monotonically {increasing, decreasing} and* $R_{min} < e < R_{max}$, *then a positive and negative contour line of value e {left, entered} the cliff line segment from the right.*

3. *If there is a segment of a cliff line for which L is monotonically {increasing, decreasing} and* $L_{max} = e$, *then a negative or maximum contour line of value e {entered, left} the cliff line segment from the left.*

4. *If there is a segment of a cliff line for which R is monotonically {increasing, decreasing} and* $R_{min} = e$, *then a positive or minimum contour line of value e {left, entered} the cliff line segment from the right.*

5. *If there is a segment of a cliff line for which* $\{R, L\} = e$ *for all points on the segment, then a {negative, positive} contour line of value e may have entered or left the cliff line segment from the {right, left}.*

PROOF. Only the first half of the first part of this property will be proved here. The rest of the property is proved in a similar manner. At the beginning of the cliff line segment, there is a point p_1 such that $L(p_1) < e$. There are no contour lines of value e traveling in the cliff line at p_1 as a result of Property 14. At some point p_2 along the cliff line segment, $R(p_2) < e < L(p_2)$. Thus a positive and negative contour line of values e is traveling with the cliff line at p_2 as a consequence of Property 14. Therefore a positive and negative contour line have either entered or left the segment somewhere before p_2. Since the contour lines travel in the same direction as the cliff line (Property 15), the contour lines must have entered the segment. Since the left limit function is related to conditions on the left of the cliff line, the contour lines whose existence was predicted by examining the left limit function must have entered the cliff line from the left.

Notice that Property 16 provides the basis for a graphical method of determining where contour lines enter and leave cliff lines. The method was discussed in [5][7] and is illustrated in Figure 7.

PROPERTY 17.

1. *All contour lines that enter and leave a cliff line from the {left, right} leave on a last-in-first-out basis.*

2. *All contour lines that enter a cliff line from the {left, right} and leave towards the {right, left} do so on a first-in-first-out basis.*

3. *Each contour line that enters and leaves a cliff line from the {left, right} must either enter and leave before, or enter and leave after, any contour line that entered from the {right, left} can leave towards the {left, right}.*

[7] In [5] properties of contour lines are developed from an intuitive point of view. Contour lines are defined from the physical concept of level surfaces rather than the mathematical concept of elevation type.

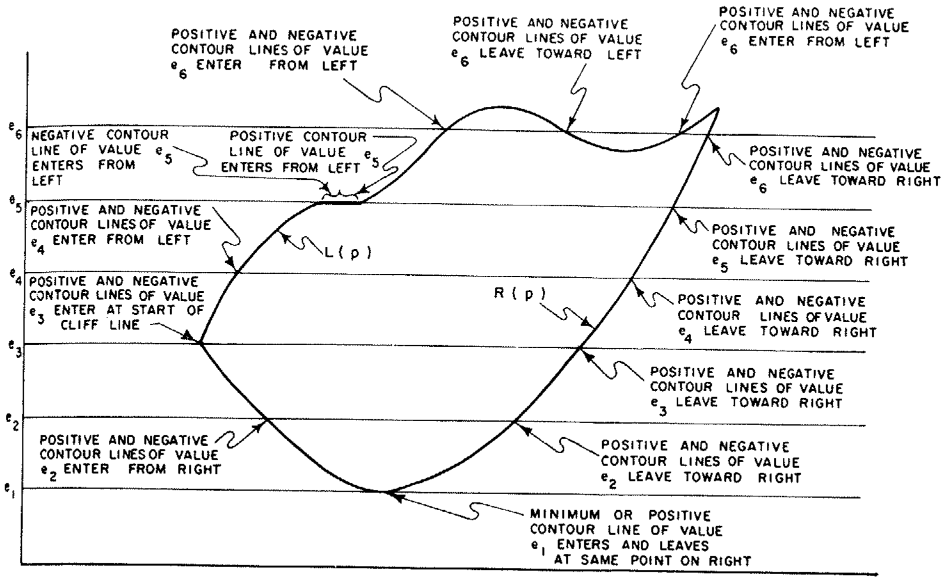


FIG. 7. Determining where contour lines enter and leave cliff line

4. *Each contour line that enters and leaves a cliff line from the {left, right} must either enter and leave before, or enter and leave after, any contour line that leaves towards the {right, left} can enter from the {left, right}.*

5. *If one contour line enters a cliff line from the left and leaves towards the right and another contour line enters the cliff line from the right and leaves towards the left, one contour line must both enter and leave before the other contour line can enter.*

PROOF. The proofs of these statements consist of simple arguments based on Property 16. For brevity, only the first half of the first statement is proved here. Let e_1 and e_2 be the values of two normal contour lines l_1 and l_2, and let $e_1 < e_2$. If the two contour lines are to enter from the left, then L must be less than e_1 at segments of the cliff line preceding the first point of entry. Contour line l_1 will have entered when L becomes greater than e_1 and l_2 will have entered when L becomes greater than e_2. Hence l_1 will enter before l_2. Similarly, l_2 will leave before l_1. Hence the contour lines leave on a last-in-first-out basis.

COROLLARY 17.1. *Contour lines do not cross each other when traveling along a cliff line.*

6. *Intersections of Contour Lines*

Contour lines can never cross each other but under certain conditions they may intersect. This is proved in what follows.

PROPERTY 18. *Two positive or two negative contour lines of the same value cannot cross.*

PROOF. This is a consequence of Corollary 9.4 and Rule 1.

PROPERTY 19. *A positive and a negative contour line of the same value cannot cross.*

PROOF. Let l be a positive contour line of value e passing through p. Let l_1 be the segment of l preceding p and l_2 be the segment of l following p. Assume a negative contour line segment l_3 of value e intersects with the right side of l at p. Further assume that l_3 is directed toward p. Let N be an arbitrarily small neighborhood of p. There are points of elevation less than e in N and to the right of both l and l_3. Thus there is a negative contour line segment l_4 directed away from p and to the right of both l and l_3. (l_4 may be superimposed on l; if so it is considered as being an infinitesimal distance to the right of l.) By Rule 1, l_4 is the continuation of l_3. Hence the positive contour line l_1l_2 and the negative contour line l_3l_4 do not cross.

Next assume a negative contour line segment l_5 intersects with the left side of l at p. Further assume that l_5 is directed toward p. By a similar argument there is a positive contour line segment l_6 directed toward p and to the left of both l and l_5. But, by Rule 1, l_2 is the continuation of l_6 and not l_1. This contradicts the hypothesis; thus this situation is impossible.

Notice that when a positive and negative contour line join to form an isorithm, although they are superimposed, the positive contour line should be considered as being an infinitesimal distance to the left of the negative contour line.

PROPERTY 20. *A {maximum, minimum} contour line and a {negative, positive} contour line of the same value cannot intersect.*

PROOF. Only the last part of this property is proved. The other proof is very similar.

Assume the intersection between a positive contour line l_1 and a minimum contour line l_2, both of value e, occurs at a single point p_0. Then $\tilde{T}_E(p_0)_2 = \text{III}$. Let N be a critical neighborhood of p with respect to l_2. Let p_1 be some point other than p_0 that is in N and lies on l_1. Point p_1 does not lie on l_2 and $E(p_1) = e$ because p_1 is on l_1. But since p_0 contains p_1 in its subcritical neighborhood with respect to l_2, $\tilde{T}_E(p_0)_2$ cannot be III. This is a contradiction; therefore a positive and minimum contour line cannot intersect at a single point. The intersection of the two contour lines cannot occur on a line because the elevation types of points on this line with respect to the line would have conflicting requirements as imposed by the definition of positive and of minimum contour lines.

PROPERTY 21. *A {maximum, maximum, minimum} contour line and a {minimum, positive, negative} contour line of the same value cannot intersect.*

PROOF. The proof for Property 20 is also valid here. However, an alternate proof is presented. Only the first of the three parts is proved. The other two proofs are similar. Property 7 says that all points on a maximum contour line have elevation types I or IV and all points on a minimum contour line have elevation types III or V. Hence, the two contour lines can have no points in common.

PROPERTY 22. *Two {maximum, minimum} contour lines of the same value cannot intersect.*

PROOF. The proof that they cannot intersect at a point is the same as the proof used in Property 20. If they intersect in a line there must be some point p for which they first join. But then the elevation type of p with respect to either of the degenerate contour lines cannot be I or III because every neighborhood of p contains a point that is on one of the contour lines and not the other. This leads to a contradiction.

PROPERTY 23. *Contour lines of different values cannot intersect at points that are inside a smooth domain.*

PROOF. Let p be the intersection point between a contour line of value e_1 and a contour line of value e_2. The elevation of p must be both e_1 and e_2. But this violates the single-valued elevation requirements of points lying in a smooth domain.

PROPERTY 24. *Contour lines do not cross each other when traveling along a cliff line.*

PROOF. See Corollary 17.1.

Properties 18 through 24 can be summarized by the following two properties.

PROPERTY 25. *Contour lines never cross.*

PROPERTY 26. *Two contour lines can intersect at a point of a smooth domain only if both are normal contour lines and of the same value.*

Conclusion. Contour maps representing different quantities can be studied in a unified manner by the use of a formal model. The contour map model presented in this paper uses the abstract concept of elevation type from which many of the intuitive notions about contour maps can be deduced. The results thus obtained are independent of the physical data represented by the contour map. One can apply the results obtained to any dependent variable providing the dependent variable satisfies the requirements set forth in this paper.

APPENDIX. Eliminating Degenerate Contour Lines

The concept of degenerate contour lines is a natural consequence of the theory based on elevation types. In certain applications, complications are caused by the presence of degenerate contour lines. In such cases a method is needed to eliminate degenerate contour lines. This can be accomplished by circumscribing in the {counterclockwise, clockwise} direction a {negative, positive} contour line of value

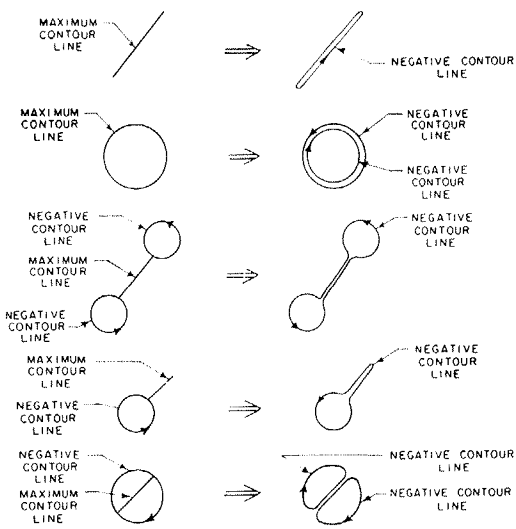


FIG. 8. Examples of transforming degenerate contour lines into normal contour lines

e around every {maximum, minimum} contour line of value e and, in addition, inscribing in the {clockwise, counterclockwise} direction a {negative, positive} contour line of value e inside every {maximum, minimum} contour line of value e that is a closed curve. These enveloping normal contour lines are shrunk such that the maximum distance from any point on the normal contour lines to its nearest point on the degenerate contour lines approaches zero. The degenerate contour lines can be replaced by these normal contour lines. Rule 1 is then applied to the intersections of these normal contour lines with themselves and with the other normal contour lines on the map. Examples of transforming degenerate contour lines into normal contour lines are shown in Figure 8. A contour map with no degenerate contour lines is called a *normal* contour map.

Note that the terms "inscribe" and "circumscribe" used in the preceding paragraph imply that a closed curve has an inside and an outside. If the domain of the map is a closed surface, the inside and outside of closed curve are undefined. However, the procedure outlined in the preceding paragraph gives the same result regardless of which side of the closed curve is selected as the inside.

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[*Note:* References [1–4] are not cited in the text.]

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