An Accessing Model

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ABSTRACT. A model of a storage system involving a disk and a buffer is analyzed under two access disciplines. The average stationary access rate is calculated for each discipline.

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1. *Introduction*

This paper contains an analysis of the efficiency of a strategy for requesting information stored on a so-called *paging drum* [1–4, 6]. An example of such a device is the IBM 2305 Fixed Head Storage Facility [5]. This disk has *rotational position sensing*: a specific rotational position of the disk surface, called a *sector*, can be searched for a record using an instruction which first specifies the sector number. The channel will be connected to the device only during the search of that sector leaving it otherwise free. The device is able to store a number of requests and overtaking is possible; that is, the device will service the request referring to the nearest sector first. We assume in this analysis

(1) a single disk is used,

(2) the channel is always available,

(3) a fixed length buffer holds b requests,

(4) the identity of the sector which contains a request is known by the program, and

(5) as each request is serviced a new request replaces it (during the transversal of that sector) so that the contents of the buffer remains fixed at b.

Under these assumptions we shall calculate the average number of requests satisfied in one complete rotational cycle of the disk.

2. *The Associated Markov Chain*

We denote the state of the buffer at the start of a cycle by the vector $\mathbf{x} = (x_1, x_2, \cdots, x_m)$ in which x_i is the number of requests in the buffer which reference the ith sector. Each x_i is a nonnegative integer and $x_1 + x_2 + \cdots + x_m = b$; B will denote the totality of such state vectors.

We begin by examining the effect of one complete rotation of the disk. Let $\mathcal{C}i$ $(0 \leq i < m)$ denote the contents of the buffer as the disk enters the $(i + 1)$th

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sector position. We have

$$p_i(\mathbf{x}/\mathbf{y}) = \Pr\{\mathcal{C}_j = \mathbf{x}/\mathcal{C}_i = \mathbf{y}\} = \begin{cases} \delta(\mathbf{x}, \mathbf{y}), & \text{if } \; y_{i+1} = 0, \\ 1/m, & \text{if } \; y_{i+1} > 0 \; \text{ and } \; \mathbf{x} = \mathbf{y} - \mathbf{u}_{i+1} + \mathbf{u}_t \; \text{ for some } \; t, \; 1 \le t \le m, \end{cases}$$

where

(1) $j = (i + 1)$ modulo m $(0 \le i < m)$,
(2) δ is the Kronecker delta, and
(3) $\mathbf{u}_k = ((0)_{k-1}, 1, (0)_{m-k})$ is the unit vector with *a* 1 in the kth coordinate.

We are assuming here that the replacement of service requests is made according to the uniform distribution; that is, the probability that the new request is for a reference to sector i is $1/m$ for each i $(1 \le i \le m)$.

Let $V(\mathbf{x}, \mathbf{y})$ be the collection of all sequences (of length $m + 1$) from B,

$$v\colon \mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_m,$$

satisfying $\mathbf{x} = v_0$ and $y = v_m$. Set

$$q(v) = \prod_{i=0}^{m-1} p_i(\mathbf{v}_{i+1}/\mathbf{v}_i), \qquad \mathbf{v} \in V(\mathbf{x}, \mathbf{y}),$$

and

$$p(\mathbf{y}/\mathbf{x}) = \sum_{v \in V(\mathbf{x},\mathbf{y})} q(v). \tag{1}$$

Equation (1) defines the transition function of the contents of the buffer for one complete revolution. The finite state Markov chain (with values in B) which p determines is irreducible and aperiodic so that a unique stationary probability distribution π on B exists. The generating function of π is the homogeneous polynomial (in $z_1, z_2, \cdots, z_m$) of degree b defined by

$$\theta(z_1, z_2, \cdots, z_m) = \sum_{\mathbf{x} \in B} \pi(x_1, x_2, \cdots, x_m) z_1^{x_1} z_2^{x_2} \cdots z_m^{x_m}.$$

If $\mathbf{X}^{(n)} = (X_1^{(n)}, X_2^{(n)}, \cdots, X_m^{(n)})$ denotes the contents of the buffer at the start of the nth cycle, then

$$\theta(z_1, z_2, \cdots, z_m) = \lim_{n \to \infty} E\{z_1^{X_1^{(n)}} z_2^{X_2^{(n)}} \cdots z_m^{X_m^{(n)}}\},$$

where E denotes the expected value.

Our object now is to derive an algebraic relation for this generating function by considering the effect of the rotation of the disk through the first sector. Let

$$(T_1\pi)(\mathbf{x}) = \sum_{\mathbf{y} \in B} p_0(\mathbf{x}/\mathbf{y}) \pi(\mathbf{y}). \tag{2}$$

$T_1\pi$ is a probability distribution on B; it is in fact the stationary distribution of the contents of the buffer just after rotation through the first sector. The important thing to observe is that the symmetry of the replacement process implies

$$(T_1\pi)(x_1, x_2, \cdots, x_m) = \pi(x_2, x_3, \cdots, x_m, x_1). \tag{3}$$

Denoting the generating function of $T_1\pi$ by $T_1\theta$ we have, from (3),

$$(T_1\theta)(z_1, z_2, \cdots, z_m) = \theta(z_2, z_2, \cdots, z_m, z_1). \tag{4}$$

Next we multiply (2) by $z_1^{x_1}z_2^{x_2}\cdots z_m^{x_m}$ and sum over $\mathbf{x} \in B$ obtaining $(T_1\theta)(z_1, z_2, \cdots, z_m)$ on the left-hand-side. Using the definition of p_0 we find for the right-hand-side

$$\sum_{\mathbf{x},\mathbf{y}\in B} z_1^{x_1}z_2^{x_2}\cdots z_m^{x_m}p_0(\mathbf{x}/\mathbf{y})\pi(\mathbf{y})$$

$$= \sum_{\substack{\mathbf{y}\in B\\ y_1=0}} \pi(\mathbf{y})z_1^{y_1}z_2^{y_2}\cdots z_m^{y_m} + \sum_{\substack{\mathbf{y}\in B\\ y_1>0}} \pi(\mathbf{y})z_1^{y_1}z_2^{y_2}\cdots z_m^{y_m}\,\frac{z_1 + z_2 + \cdots + z_m}{mz_1}. \tag{5}$$

The first term is $\theta(0, z_2, z_3, \cdots, z_m)$, while the second term is

$$(w/z_1)\{\theta(z_1, z_2, \cdots, z_m) - \theta(0, z_2, \cdots, z_m)\},$$

where $w = (z_1 + z_2 + \cdots + z_m)/m$. Note that the servicing of a request in the first sector corresponds to multiplication of the generating function by z_1^{-1} while the replacement corresponds to multiplication by w. Equations (4) and (5) together give

$$\begin{aligned}\theta(z_2, z_3, \cdots, z_m, z_1) &= (w/z_1)\theta(z_1, z_2, \cdots, z_m)\\ &\quad + (1 - w/z_1)\theta(0, z_2, \cdots, z_m).\end{aligned} \tag{6}$$

By permuting the indices in (6) we obtain the equivalent relations

$$\begin{aligned}&\theta(z_{r+1}, \cdots, z_m, z_1, \cdots, z_r)\\ &\quad = (w/z_r)\theta(z_r, \cdots, z_m, z_1, \cdots, z_{r-1})\\ &\qquad + (1 - w/z_r)\theta(0, z_{r+1}, \cdots, z_m, z_1, \cdots, z_{r-1}), \qquad 0 \le r \le m,\end{aligned} \tag{7}$$

with $z_{i+m} = z_i$. Starting with $r = m$ and working backwards, successive substitution yields

$$\begin{aligned}&\theta(z_1, z_2, \cdots, z_m)\{z_1z_2\cdots z_m - w^m\}\\ &\quad = \sum_{r=0}^{m-1} w^r(z_{m-r} - w)\left(\prod_{j=1}^{m-r-1} z_j\right)\theta(0, z_{m-r+1}, \cdots, z_m, z_1, \cdots, z_{m-r-1}).\end{aligned} \tag{8}$$

A request to reference the first sector is satisfied during a given cycle if and only if the buffer contains such a request for the first sector at the start of a cycle. The stationary probability of this event is

$$\sum_{\substack{\mathbf{x}\in B\\ x_1>0}} \pi(x_1, x_2, \cdots, x_m) = 1 - \theta(0, (1)_{m-1})$$

and accordingly we now turn to the calculation of $\theta(0, (1)_{m-1})$.

3. *The Expected Number of Accesses in a Cycle*

In eq. (8) we set

$$z_i = \begin{cases} z, & \text{if } i = 1,\\ 1, & \text{if } 1 < i \le m.\end{cases}$$

Then

$$\{z_1 z_2 \cdots z_m - w^m\} = z - [(z + m - 1)/m]^m = -(z - 1)^2 R(z),$$

where R is a polynomial of degree $m - 2$ with $R(1) = \frac{1}{2}(1 - 1/m)$. Likewise each of the m terms in the right-hand-side of (8) has a zero (at $z = 1$) but only of order 1. We have in fact

$$\theta(z, (1)_{m-1}) = \frac{1}{(z-1)R(z)} \left\{ \sum_{r=0}^{m-2} \frac{z}{m} \left(\frac{z + m - 1}{m} \right)^r \theta(0, (1)_r, z, (1)_{m-r-2}) - \frac{m-1}{m} \left(\frac{z + m - 1}{m} \right)^{m-1} \theta(0, (1)_{m-1} \right\}. \quad (9)$$

When $z \to 1$, the left-hand-side converges to 1 since π is a probability distribution. The right-hand-side is evaluated by differentiating numerator and denominator. There results

$$R(1) = \left\{ \frac{m-1}{2m} \theta(0, (1)_{m-1}) + \frac{1}{m} \sum_{r=0}^{m-2} \frac{d}{dz} \theta(0, (1)_r, z, (1)_{m-r-2}) \Big|_{z=1} \right\}. \quad (10)$$

Let $\mathbf{X} = (X_1, X_2, \cdots, X_m)$ be a random variable with values in B and distribution π. Then

$$\theta(0, (1)_r, z, (1)_{m-r-2}) = E\{z^{X_{r+2}} \chi_{\{X_1=0\}}\},$$

where χ_A is the indicator (or characteristic) function of (the set) A. Next

$$\frac{d}{dz} \theta(0, (1)_r, z, (1)_{m-r-2}) \Big|_{z=1} = E\{X_{r+2} \chi_{\{X_1=0\}}\}$$

and, when we sum (over r), we have

$$\sum_{r=0}^{m-2} \frac{d}{dz} \theta(0, (1)_r, z, (1)_{m-r-2}) \Big|_{z=1} = E\left\{ \sum_{k=2}^{m} X_k \chi_{\{X_1=0\}} \right\} = bE\{\chi_{\{X_1=0\}}\} = b\theta(0, (1)_{m-1}), \quad (11)$$

since $X_2 + \cdots + X_m = b$ whenever the event $\{X_1 = 0\}$. Equations (10) and (11) give

$$\theta(0, (1)_{m-1}) = (m - 1)/(2b + m - 1). \quad (12)$$

Finally let $\mathbf{R} = (R_1, R_2, \cdots, R_m)$ be the vector-valued random variable defined

$$= \begin{cases} 1 & \text{if a request for information stored in the } i\text{th sector is serviced during a comple cycle,} \\ 0 & \text{otherwise.} \end{cases}$$

Symmetry again dictates that $\Pr\{R_i = 1\}$ is independent of i and consequently given by $1 - \theta(0, (1)_{m-1})$. Thus the expected number of requests serviced in a complete cycle $E(R_1 + \cdots + R_m)$ is $2bm/(2b + m - 1)$.

A Variant of the Accessing Strategy

We conclude by describing a variant of this accessing strategy which is amenable to the same analysis. In this variant *all* requests referring to the ith sector can be serviced during the transversal of this sector and this number of requests are replaced during the transversal of that sector. We employ the same notation as in Sections 2 and 3. Then

$$p_0(\mathbf{x}/\mathbf{y}) = \binom{y_1}{i_1 i_2 \cdots i_m} m^{-y_1}$$

if $\mathbf{x} = \mathbf{y} - y_1\mathbf{u}_1 + (i_1\mathbf{u}_1 + \cdots + i_m\mathbf{u}_m)$ and zero otherwise. Here $i_j \geq 0$ $(1 \leq j \leq m)$, $i_1 + i_2 + \cdots + i_m = y_1$ and

$$\binom{m}{i_1 i_2 \cdots i_m}$$

denotes the usual multinomial coefficient. The analog of (6) is

$$\theta(z_2, z_3, \cdots, z_m, z_1) = \theta(z_1 + \cdots + z_m)/m, z_2, z_3, \cdots, z_m). \tag{13}$$

It is easy to verify that the solution of (13) is

$$\theta(z_1, z_2, \cdots, z_m) = \left\{\sum_{i=1}^{m} 2\frac{m - i + 1}{m(m + 1)} z_i\right\}^b. \tag{14}$$

If R_1 is the number of requests serviced in the ith sector during a complete cycle then (14) yields $E(R_i) = 2b/(m + 1)$.

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