On the Interconnection of Asynchronous Control Structures

J. ROBERT JUMP

Rice University, Houston, Texas

AND

P. S. THIAGARAJAN

Massachusetts Institute of Technology, Cambridge, Massachusetts

ABSTRACT. The paper is concerned with a class of control systems which can be represented by a graphical model called an MG-control system (MGCS) In particular, the closure properties of this class are studied More precisely, this paper presents necessary and sufficient conditions for the composite system, obtained by interconnecting two of these systems, to be represented as an MGCS. These results are then extended to networks composed of several interconnected control systems. In solving this problem, it is shown that whenever the interconnection of two or more systems results in a system that is not representable as an MGCS, it is due to the presence of "deadlock" in the composite system. Hence the results of the paper provide a means of detecting deadlock in a network of control systems.

KEY WORDS AND PHRASES: asynchronous control structure, asynchronous control system, interconnected systems, control network, deadlock, marked graph

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1. *Introduction*

This paper is a continuation of the study of asynchronous control systems presented in [8]. As before, we view a control system as a device which enforces certain specified constraints on the order of occurrence of "events" where these occurrences are characterized as follows.

(1) An occurrence of an event is initiated by a control signal called a *ready signal*.
(2) Once initiated, an occurrence requires a finite but unbounded period of time.
(3) When an occurrence terminates, an *acknowledge signal* is generated.
(4) Each event may occur repeatedly, and several different events may occur concurrently.

A control system communicates with its environment through *links*, where there is one link for each event and each link contains a *ready terminal* and an *acknowledge terminal*. In the case of an *input link*, the corresponding event is initiated by the system's environment by sending a ready signal to the system through the link's ready terminal. When this event is terminated, an acknowledge signal is transmitted to the environment through the acknowledge terminal of the same link. Events are initiated by the con-

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Authors' addresses. J. R. Jump, Department of Electrical Engineering, Rice University, Houston, TX 77001; P. S. Thiagarajan, Project MAC, Massachusetts Institute of Technology, Cambridge, MA 02136

trol system when a ready signal is sent to the environment through the ready terminal of an *output link*. The environment signals the completion of this event by generating an acknowledge signal on the acknowledge terminal of that link.

The ready and acknowledge signals associated with a link will be referred to as *link signals*. If the last link signal transmitted through a link was an acknowledge signal, the link is said to be *idle*. A link is *active* if the last link signal was a ready signal. Initially, all links of a system are idle. It is assumed that neither the system nor its environment will ever try to activate an active link or deactivate an idle link. Hence the ready and acknowledge signals of a link must alternate.

Consider a control system which has reached a state in which all input links are active, all output links are idle, and the system is unable to produce any new output signals on any of its links. In this case, the environment must wait for an output signal from the system before it can generate a new input signal. Hence the system is incapable of any further activity. If it is possible for a control system to ever reach such a state, we say that the system contains *deadlock*. This is a condition analogous to the system deadlock or deadly embrace encountered in large multiprogramming systems [2].

When an output link is connected to an input link, the ready signals of the output link are transmitted directly to the ready terminal of the input link, and an acknowledge signal generated on the input link becomes an acknowledge signal for the output link. It has been shown that when links of control systems which are free of deadlock are connected in this way, it may result in a new system which does contain deadlock [1, 5]. Hence an important analysis problem is to determine when this can happen.

The purpose of this paper is to characterize the behavior of networks of control systems that communicate with each other through their links. A mathematical model called a marked graph is used for this purpose. In particular, it is shown how to construct a marked graph representation for the behavior of a network given such a representation of its component systems. Then a necessary and sufficient condition for a network to be free of deadlock is developed. Deadlock is characterized in terms of the familiar property of liveness in marked graphs. A major contribution of the paper is that it transforms the well-known mathematical characterization of liveness in marked graphs into a form that is more convenient for the analysis of control systems that are realized as a network of smaller control systems.

The effect of connecting a single pair of links is analyzed first. Section 3 considers the case where both links are on the same system. It is shown that if these links satisfy a condition called buffering, then their connection results in a well-formed system that is free of deadlock (Theorem 3.4). If, on the other hand, the links are not buffered, then connecting them will produce a system that will ultimately reach a deadlocked state (Theorem 3.5). The connection of two links on different systems is studied in Section 4. In this case, it is shown that such a connection will always result in a deadlock-free system (Theorems 4.1 and 4.5). In the process of characterizing the effect of connecting a pair of links, marked graph representations for the resulting systems are developed.

Any connection between two subsystems in a network can be viewed as either a connection between two distinct systems or one between two links on the same system. Hence the results of Sections 3 and 4 are used in Section 5 to analyze complex networks. Specifically, it is shown how deadlock in a network can be detected by analyzing its link connections, one at a time.

2. *MG-Control Systems*

This section consists of a brief introduction to the asynchronous control system model used in this paper. A more detailed and complete development of the basic properties of this model can be found in [8]. The model is based on a directed graph, called a marked graph [3, 6, 7], in which the state of the system is represented by placing markers on some of the edges. A change of state is then simulated by the movement of markers in the graph.

The model may be viewed as a special case of the P-net model developed to represent the behavior of control modules [4, 12]. Indeed, P-nets are based on a generalization of marked graphs called Petri nets [7, 14].

We view a (directed) *graph* as an ordered pair (T,P) where T is a countable set of *vertices* and $P \subseteq T \times T$ is the set of *edges*. A path of length n is a sequence $\pi = v_0, v_1, \cdots, v_n$ of vertices such that $(v_i, v_{i+1}) \in P$ for $i = 0, 1, \cdots, n-1$. If all of the inner vertices of a path are distinct, it is said to be *elementary*. Given an elementary path π, then $\pi(x,y)$ denotes a subpath of π which extends from x to y. If $\pi = v_0, v_1, \cdots, v_n$ and $\tau = u_0, u_1, \cdots, u_m$ are two paths with $v_n = u_0$, then the *composition* of π and τ is the path $\pi, \tau = v_0, v_1, \cdots, v_n, \cdots, u_m$. Given a vertex v, $I(v)$ denotes the set of edges directed into v and $O(v)$ the set of edges directed out of v.

A *marking* of a graph (T,P) is a function M from P into N_0, the set of nonnegative integers.[1] The interpretation of M is that edge e contains $M(e)$ *markers* under M. If π is a path in (T,P), then $\Sigma(M \mid \pi)$ denotes the number of markers on π under M.

A *marked graph* is a triple (T,P,M) where (T,P) is a graph in which $I(v)$ and $O(v)$ are finite sets for all $v \in T$ and M is a marking of (T,P) called the *initial marking*. If $\Sigma(M \mid \pi) = 0$ for a path π, then π is said to be *marker-free*.

Given a graph (T,P), a vertex $v \in T$ is *firable* under a marking M' if $M'(e) > 0$ for all $e \in I(v)$. Moreover, when v fires, a new marking M'' is produced, where M'' is defined by

$$M''(e) = \begin{cases} M'(e) - 1, & \text{if } \; e \in I(v) - O(v), \\ M'(e) + 1, & \text{if } \; e \in O(v) - I(v), \\ M'(e), & \text{otherwise.} \end{cases}$$

Hence, the operation of firing a vertex v can be represented by removing one marker from each edge in $I(v)$ and adding one to each edge in $O(v)$. Note that if e is a self-loop, then $M''(e) = M'(e)$. A sequence $v_0, v_1, v_2, \cdots, v_n$ of vertices in T is called a *firing sequence* of the marked graph (T,P,M) if there is a sequence $M_0, M_1, \cdots, M_{n+1}$ of markings such that $M_0 = M$, v_i is firable under M_i, and M_{i+1} is the marking produced when v_i fires, for $i = 1, 2, \cdots, n$.

A vertex in a marked graph G is said to be *live* if it appears in at least one firing sequence of G, otherwise it is *dead*. A marked graph is *live* if all of its vertices are live. It has been shown that a vertex v is dead iff there is an infinite, marker-free path directed into v [7]. Hence a finite marked graph is live iff every cycle contains at least one initial marker.

An edge of a marked graph is *safe* if it contains at most one marker under any marking reachable from the initial marking. It has been shown that an edge lying in a cycle that contains exactly one initial marker is safe [7]. Such cycles will be called *synchronizing loops*. A marked graph is said to be *safe* if all its edges are safe. It has also been shown that a finite and live marked graph is safe if and only if every edge is contained in a synchronizing loop [7].

The marked graph model will be used to represent the behavior of control systems by associating the links of the system with certain edges of a marked graph in the following way.

Definition. An *MG-control system* (MGCS) is a triple $C = (G,\alpha,L)$ where:

(1) $G = (T,P,M)$ is a finite marked graph that is live, safe, and strongly connected;

(2) L is a finite set of *links*;

(3) α is a partial function from P onto L such that

  (a) the set $\alpha^{-1}(q) = \{e \in P \mid \alpha(e) = q\}$ is contained in a synchronizing loop of G, for all $q \in L$,

  (b) no vertex in T is the endpoint of more than one edge in dom α,[2]

  (c) if $(x,y) \in$ dom α, then $|I(y)| = 1$,[3]

[1] In this paper, $Z = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$, $N = \{1, 2, 3, \cdots\}$, and $N_0 = \{0, 1, 2, \cdots\}$

[2] "Dom α" denotes the domain of the partial function α.

[3] $|I(y)|$ denotes the cardinality of the set $I(y)$.

(d) if p is an input link, then exactly one edge in $\alpha^{-1}(p)$ contains an initial marker; if p is an output link, then no edge in $\alpha^{-1}(p)$ contains an initial marker.

α is called a *valid link assignment.*

A valid link assignment α assigns edge e to link $\alpha(e)$. The number of edges assigned to link p is called the *multiplicity* of p and is denoted by the symbol $\mathbf{p}$.

We denote the ready signal on link p by r_p and the acknowledge signal by a_p. These signals are called *external* signals since they represent the interaction of the control system with its environment. Since a link assignment is not necessarily one-to-one, different occurrences of an external signal may be associated with different edges. In order to distinguish these different occurrences, we define an *internal signal* as one of the form r_p^j or a_p^j, where $p \in L$ and $0 \leq j < \mathbf{p}$.

In order to associate internal signals with vertices of G, we first order the edges in each of the sets $\alpha^{-1}(p)$ where $p \in L$. To this end, let π be the synchronizing loop containing $\alpha^{-1}(p)$ and $e_0 = (u,v)$ be the edge of π that contains an initial marker. If $e = (x,y)$ is an edge in $\alpha^{-1}(p)$, then we define the *loop order* of e to be j where j is the number of edges of $\alpha^{-1}(p)$ that lie on the subpath $\pi(v,x)$. Now the *signal assignment* for C is the function β, with domain T, defined as follows:

(1) $\beta(v) = t_v^0$ if v is not the endpoint of any edge in dom α. In this case, $\beta(v)$ is called an *internal vertex.*

(2) Let $e = (u,v)$ be an edge in dom α with $\alpha(e) = p$ and loop order j:

(a) $\beta(u) = r_p^j$ and $\beta(v) = a_p^j$ if p is an output link, and

(b) $\beta(u) = a_p^j$ and $\beta(v) = r_p^{R[j+1/\mathbf{p}]}$ if p is an input link.[4]

We now use the signal assignment of an MG-control system to get the following alternate representation for the behavior of the system.

Definition. Let $C = (G,\alpha,L)$ be an MGCS with $G = (T,P,M)$ and β its signal assignment. Then the *signal graph* for C is the marked graph $G_c = (T_c, P_c, M_c)$ where:

(1) $T_c = \beta(T) = \{\beta(v) \mid v \in T\}$,

(2) $P_c = \beta(P) = \{(\beta(u), \beta(v)) \mid (u,v) \in P\}$,

(3) $M_c = \beta(M) = \{(\beta(u),\beta(v)) \mid (u,v) \in M\}$.

We say that two MG-control systems $C1$ and $C2$ are *disjoint* if the vertex sets of the signal graphs G_{c1} and G_{c2} are disjoint. Note that disjoint MG-control systems will have disjoint link sets.

The activity of a control system is simulated by the movement of markers in the signal graph. When a vertex of the forms r_p^j or a_p^j fires, this is interpreted as the generation of the link signal r_p or a_p, respectively. Since the loop order has been used to assign the superscripts of internal signals, the firing of vertex x_p^j may be interpreted as the jth (modulo $\mathbf{p}$) occurrence of external signal x_p. Hence the condition that $\alpha^{-1}(p)$ be contained in a synchronizing loop will ensure that the ith occurrence of a signal precedes its $(i+1)$-th occurrence. Since every edge of a synchronizing loop is safe, the environment of a system is restricted to generating input signals so that ready signals alternate with acknowledge signals on each link. The condition that $\mid I(y) \mid = 1$ whenever $(x, y) \in \alpha^{-1}(p)$ guarantees that this is the only ordering constraint placed on the environment by the system.

The behavior of a control system is completely characterized by the set of all possible sequences of signals on its links. Due to the correspondence between the generation of link signals and the firing of vertices, we will define the behavior of an MGCS in terms of the firing sequences of its signal graph. To this end, let $\bar{u}$ be a firing sequence for a signal graph. Then the corresponding *reduced firing sequence* is the sequence of internal signals obtained by deleting all of the internal vertices from $\bar{u}$. The *signal sequence* corresponding to $\bar{u}$ is the sequence of external signals obtained by removing the superscripts from all of

[4] $R[j/\mathbf{p}]$ denotes the remainder and $Q[j/\mathbf{p}]$ the quotient obtained by dividing j by $\mathbf{p}$. Thus $j = Q[j/\mathbf{p}]\mathbf{p} + R[j/\mathbf{p}]$

the internal signals in the corresponding reduced firing sequence. We now define the *behavior* of an MGCS as the set of all possible signal sequences of its signal graph.

We will say that two MG-control systems are *equivalent* iff they have equal behaviors. Note that this definition imposes the constraint that two equivalent control systems have equal link sets.

The concept of system deadlock is formalized with the MGCS model in the following way.

Definition. Let G_c be the signal graph of an MGCS C. Then C is said to contain a *deadlock* if there is a reachable marking M' of G_c under which no internal signal (i.e. a vertex of the form $r_p{}^j$ or $a_p{}^j$) is firable. The marking M' is also referred to as a deadlock of C.

It has been shown that if a strongly connected, finite marked graph is live, then there is no upper bound on the number of times a vertex may fire. However, if such a marked graph contains at least one dead vertex, then the graph will reach a marking under which no vertex will fire [7]. It therefore follows that an MGCS is free of deadlocks.

We now introduce an alternative representation, called a behavior graph, for the behavior of an MGCS. To this end, we first introduce the following terminology. An elementary path in a signal graph is said to be *signal-free* if none of its inner vertices are internal signals. A path π from x to y is called *marker-minimal* if $\Sigma(M_c \mid \pi') \geq \Sigma(M_c \mid \pi)$ for all paths π' from x to y. Now let C be an MGCS and G_c its signal graph. Then the *constraint relation* of C is the ternary relation $\gamma_c \subseteq T_c \times T_c \times N_0$ defined by: $(x,y,m) \in \gamma_c$ iff x and y are internal signals and there is a marker-minimal, signal-free path π from x to y such that $\Sigma(M_c \mid \pi) = m$.

Definition. Let γ_c denote the constraint relation and S_E the set of external signals of an MGCS C. Then the *behavior graph* for C is the (infinite) marked graph $\tilde{G}_c = (\tilde{T}_c, \tilde{P}_c, \tilde{M}_c)$ where:

(1) $\tilde{T}_c = \{x_p{}^j \mid x_p \in S_E \text{ and } j \in Z\}$,
(2) $\tilde{P}_c = \{(x_p^{j+np}, y_q^{l(n+m)q}) \mid (x_p{}^j, y_q{}^l, m) \in \gamma_c \text{ and } n \in Z\}$,
(3) $\tilde{M}_c = \{x_p{}^j, y_q{}^l) \in \tilde{P}_c \mid j < 0 \text{ and } l \geq 0$.

We can associate signal sequences with firing sequences of $\tilde{G}_c$ in the same way it was done for signal graphs. It was shown in [8] that the set of signal sequences generated by $\tilde{G}_c$ is equal to the behavior of C. Hence if $C1$ and $C2$ are two MG-control systems, and $\tilde{G}_{c1} = \tilde{G}_{c2}$, then $C1$ and $C2$ are equivalent.

Some of the concepts introduced in this section are illustrated by the signal graph in Figure 1. This graph represents an MGCS with link set $L = \{1,2,3\}$ where links 2 and 3 are output links of multiplicity 1 and link 1 is an input link that has multiplicity 2. The initial markers are indicated by darkened circles on edges. Hence vertex $r_1{}^0$ is the only firable vertex under the initial marking and the path $\pi = r_1{}^0, t_4{}^0, r_2{}^0$ is signal-free. Vertex $t_4{}^0$ is the only internal vertex as all of the other vertices are labeled with internal signals. The synchronizing loop containing $\alpha^{-1}(2)$ is $t_4{}^0, r_2{}^0, a_2{}^0, t_4{}^0$ while the synchronizing loop $r_1{}^0, t_4{}^0, a_1{}^0, r_1{}^1, r_3{}^0, a_3{}^0, a_1{}^1, r_1{}^0$ contains both $\alpha^{-1}(1)$ and $\alpha^{-1}(3)$. Two examples of signal sequences are given by r_1, a_1, r_2, a_2, r_1 and r_1, r_2, a_2, a_1, r_1.

3. *Connection of Two Links on the Same System*

In this section we consider the effect of connecting a single output link to an input link on the same system. When this system is representable as an MGCS, we show that the

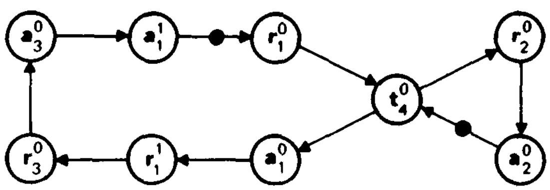

FIG. 1. Signal graph for the MGCS $C1$

following condition is necessary and sufficient for the resulting system, obtained by connecting the two links, to be represented by an MGCS.

Definition. Let p be an output link and q an input link of an MGCS C. Then (p,q) is a *buffered pair* of links if

(1) p and q have equal multiplicity (i.e. $\mathbf{p} = \mathbf{q}$), and

(2) every path from r_q^j to r_p^j and every path from a_p^j to a_q^j in the signal graph G_c contains at least one marker under M_c, for $0 \leq j < \mathbf{p}$.

In Subsection A below, we show that if (p,q) is a buffered pair of links, then connecting them produces a system that can be represented by an MGCS. In Subsection B we show that a system obtained by connecting two unbuffered links cannot be represented by a well-formed MGCS. Moreover, we show that such a system will contain a deadlock.

A. Sufficiency of the Buffering Condition. The operation of connecting a buffered pair of links is modeled in the following way.

Definition. Let (p,q) be a buffered pair of links of the MGCS C and let $G_c = (T_c, P_c, M_c)$ be the signal graph of C. Then G_{pq} is the marked graph (T_{pq}, P_{pq}, M_{pq}) where:

(1) $T_{pq} = T_c$,

(2) $P_{pq} = (P_c \cup \{(r_p^j, r_p^j), (a_q^j, a_p^j) \mid 0 \leq j < \mathbf{p}\}) - (\alpha^{-1}(p) \cup \alpha^{-1}(q))$,

(3) $M_{pq} = M_c \cap P_{pq}$.

To see that G_{pq} models the connection of links p and q, note that the physical connection of output link p to input link q establishes direct paths from the ready terminal of link p to the ready terminal of link q and from the acknowledge terminal of link q to the acknowledge terminal of link p. Hence the ready signal r_p is transmitted through this connection and becomes a ready signal for link q. Similarly, acknowledge signals are transmitted from link q to link p. The effect of this connection on the behavior of the system is to change the constraints on occurrences of link signals r_q and a_p. Recall that the edges in $\alpha^{-1}(p) \cup \alpha^{-1}(q)$ represent constraints that are enforced by a system's environment. Thus the only constraint on the jth occurrence of signal a_p is that it must be preceded by the jth occurrence of r_p. Similarly, the only restriction on occurrences of r_q is that the jth occurrence of a_q must precede the $(j + 1)$-th occurrence of r_q. When links p and q are connected, these constraints on r_q and a_p are replaced by new ones which ensure that the jth occurrences of a_p and r_q are preceded by the jth occurrences of a_q and r_p, respectively. Thus the new constraints on the signals r_q and a_p are represented by removing the edges in $\alpha^{-1}(p) \cup \alpha^{-1}(q)$ and adding the edges (r_p^j, r_q^j) and (a_q^j, a_p^j) for $j = 0, 1, 2, \cdots, \mathbf{p} - 1$.

If G_{pq} is to be a live marked graph, then the addition of edges to P_c in order to form P_{pq} must not create any marker-free cycles. Moreover, if G_{pq} is to be safe, then the deletion of the edges in $\alpha^{-1}(p)$ and $\alpha^{-1}(q)$ must not eliminate any necessary synchronizing loops. We now show that the buffering condition is sufficient to guarantee both liveness and safeness in G_{pq}.

Theorem 3.1. *Let (p, q) be a buffered pair of links for the MGCS C. Then G_{pq} is a live marked graph.*

Proof. Let μ be an elementary cycle in G_{pq}. If all of the edges of μ are in P_c, then μ is also a cycle of G_c so that $\Sigma(M_c \mid \mu) = \Sigma(M_{pq} \mid \mu) > 0$. We therefore assume that at least one edge of μ is in the set $P_{pq} - P_c$.

Let $X_{pq} = \{r_p^j, a_p^j, r_q^j, a_q^j \mid 0 \leq j < \mathbf{p}\}$. We order the elements of X_{pq} by means of the function $f : X_{pq} \to N$, which is defined as follows:

$$f(x_u^j) = \begin{cases} 4j & \text{if } x = r \text{ and } u = p, \\ 4j + 1 & \text{if } x = r \text{ and } u = q, \\ 4j + 2 & \text{if } x = a \text{ and } u = q, \\ 4j + 3 & \text{if } x = a \text{ and } u = p. \end{cases}$$

Note that if x_u^j and y_u^l (where $u = p$ or $u = q$) are two elements of X_{pq}, then they are both link signals for link u and therefore lie in the synchronizing loop π which contains $\alpha^{-1}(u)$. Moreover, if $f(x_u^j) < f(y_u^l)$, then the subpath $\pi(x_u^j, y_u^l)$ contains no markers under M_c.

Let Y_{pq} denote the set of all endpoints of those edges in $P_{pq} - P_c$ which are also edges of μ. Let x_u^j be the element of Y_{pq} such that $f(x_u^j) \leq f(z)$ for all z in Y_{pq}. Finally, let μ_1 be the subpath of μ such that:

(1) the terminal endpoint of μ_1 is x_u^j,
(2) y_v^l, the initial endpoint of μ_1, is an element of Y_{pq}, and
(3) no inner vertex of μ_1 is an element of Y_{pq}.

We will show that $\Sigma(M_{pq} \mid \mu_1) > 0$.

Case 1. $u = v$. Since $f(x_u^j) < f(y_v^l)$, there is a path μ_2 from x_u^j to y_v^l in G_c and $\Sigma(M_c \mid \mu_2) = 0$. Due to condition (3) above, μ_1 is also a path in G_c. Hence there is a cycle in G_c consisting of the two paths μ_1 and μ_2. Since $\Sigma(M_c \mid \mu_2) = 0$ and G_c is live, $\Sigma(M_c \mid \mu_1) = \Sigma(M_{pq} \mid \mu_1) > 0$.

Case 2. $u \neq v$. From the way that y_v^j was selected and the definition of f, (y_u^l, y_v^l) is an edge in $P_{pq} - P_c$ and $f(y_u^l) = f(y_v^l) - 1$. Hence $f(x_u^j) \leq f(y_u^l)$ so that either $x_u^j = y_u^l$ or there is a path μ_2 from x_u^j to y_u^l in G_c and $\Sigma(M_c \mid \mu_2) = 0$. But μ_1 is also a path in G_c so that the composition of μ_1 and μ_2 is a path from y_v^l to y_u^l. Hence $\Sigma(M_c \mid \mu_1) = \Sigma(M_{pq} \mid \mu_1)$ must be greater than 0 since links p and q are buffered. □

In order to prove that G_{pq} is safe, we first establish a relationship between paths in G_c and paths in G_{pq}.

LEMMA 3.1. *Let (p, q) be a buffered pair of links for the MGCS C and let π and τ denote synchronizing loops of G_c containing $\alpha^{-1}(p)$ and $\alpha^{-1}(q)$, respectively. Then π and τ have no edges in common.*

PROOF. Let $e = (u,v)$ be an edge on both π and τ. Since $\alpha^{-1}(p)$ and $\alpha^{-1}(q)$ are disjoint, there are three cases to consider.

Case 1. $e \notin \alpha^{-1}(p)$ and $e \notin \alpha^{-1}(q)$. Then there exist two integers i and j such that e is an edge on both the subpaths $\pi(a_p^j, r_p^{R[j+1/\mathbf{p}]})$ and $\tau(r_q^i, a_q^i)$. Note that the subpath $\tau(r_q^i, a_q^i)$ is marker-free. If $\Sigma(M_c \mid \pi(u, r_p^i)) = 0$, then $\tau(r_q^i, u)$, $\pi(u, r_p^i)$ extends from r_q^i to r_p^i and contains no markers under M_c. But this is a contradiction since links p and q are buffered. On the other hand, if $\Sigma(M_c \mid \pi(u, r_p^i)) = 1$, then $\Sigma(M_c \mid \pi(a_p^i, u)) = 0$ so that the path $\pi(a_p^i, u)$, $\tau(u, a_q^i)$ provides a contradiction.

Case 2. $e \in \alpha^{-1}(p)$. Then $e = (r_p^j, a_p^j)$ for some j, and there is an integer i such that e is on the subpath $\tau(r_q^i, a_q^i)$. As in case 1, it can be easily shown that either there is a marker-free path from r_q^i to r_p^i or one from a_p^i to a_q^i.

Case 3. $e \in \alpha^{-1}(q)$. Then $e = (a_q^i, r_q^{R[i+1/\mathbf{q}]})$ for some i, and e is on the subpath $\pi(a_p^j, r_p^{R[j+1/\mathbf{p}]})$ for some j. If $\Sigma(M_c \mid \pi(a_p^j, r_p^{R[j+1/\mathbf{p}]})) = 0$, then the proof that links p and q are not buffered is similar to case 1. If the subpath contains an initial marker, then $R[j + 1/\mathbf{p}] = 0$ and $j = \mathbf{p} - 1 = \mathbf{q} - 1$. Since the only marker on path τ is on edge $(a_q^{\mathbf{q}-1}, r_q^0)$, there is either a marker-free path from a_p^j to a_q^j or one from r_q^0 to r_p^0.

In each of the three cases, the assumption that π and τ share an edge leads to the conclusion that links p and q are not buffered. Hence they must not have any edge in common. □

THEOREM 3.2. *Let (p, q) be a buffered pair of links for the MGCS C. If there is a path μ_1 from node u to node v in the signal graph G_c, then there is a path μ_2 from u to v in G_{pq} and $\Sigma(M_c \mid \mu_1) = \Sigma(M_{pq} \mid \mu_2)$.*

PROOF. If μ_1 contains no edges in $\alpha^{-1}(p) \cup \alpha^{-1}(q)$, then $\mu_2 = \mu_1$ is the required path in G_{pq}. Assume that μ_1 contains an edge $e = (a_q^j, r_q^{R[j+1/\mathbf{q}]})$ in $\alpha^{-1}(q)$. If μ is the synchronizing loop containing $\alpha^{-1}(p)$, then $\mu(a_p^j, r_p^{R[j+1/\mathbf{p}]})$ is a path in G_c. Due to Lemma 3.1, it is also a path in G_{pq}. Hence the edge e can be replaced by the path (a_p^j, a_p^j), $\pi(a_p^j, r_p^{R[j+1/\mathbf{p}]})$, $(r_p^{R[j+1/\mathbf{p}]}, r_q^{R[j+1/\mathbf{q}]})$ in G_{pq}. Since the number of initial markers on $\pi(a_p^j, r_p^{R[j+1/\mathbf{p}]})$ equals the number on edge e, this replacement does not change the number of markers on the path. This procedure can be repeated until all of the edges of $\alpha^{-1}(p)$ have been removed. A similar procedure exists for replacing the edges in $\alpha^{-1}(q)$ with paths in G_{pq}. □

COROLLARY 3.1 *If (p,q) is a buffered pair of links for the MGCS C, then the set $P_{pq} - P_c = \{(r_p{}^j, r_q{}^j),\ (a_q{}^j, a_p{}^j) \mid 0 \leq j < \mathbf{p}\}$ is contained in a synchronizing loop of G_{pq}.*

PROOF. π, the synchronizing loop containing $\alpha^{-1}(p)$, contains all of the endpoints of edges in $P_{pq} - P_c$. The procedure for replacing edges of $\alpha^{-1}(p) \cup \alpha^{-1}(q)$ by paths in G_{pq} can be used to construct a cycle in G_{pq} that contains all of the edges in $P_{pq} - P_c$ and exactly one initial marker. Since G_{pq} is live, this cycle must be elementary and, therefore, a synchronizing loop. □

COROLLARY 3.2. *If (p,q) is a buffered pair of links for the MGCS C, then G_{pq} is strongly connected.*

THEOREM 3.3. *If (p,q) is a buffered pair of links for the MGCS C, then G_{pq} is a safe marked graph.*

PROOF. Let e be an edge of G_{pq}. We will show that there is a synchronizing loop in G_{pq} that contains e. If $e \notin P_c$, then the result follows at once from Corollary 3.1. Therefore, assume that $e \in P_c$. Since G_c is safe, there is a synchronizing loop in G_c which contains e. By Theorem 3.2, there is a cycle μ in G_{pq} which contains e and exactly one initial marker. Since G_{pq} is live, μ is elementary, and therefore a synchronizing loop of G_{pq}. □

We can now summarize this subsection with the following theorem.

THEOREM 3.4. *Let (p,q) be a buffered pair of links of the MGCS C. Then G_{pq} is the signal graph of an MGCS representing the system obtained by connecting links p and q.*

PROOF. Let $C = (G,\alpha,L)$ and consider the triple $C' = (G_{pq}, \alpha', L')$ where $L' = L - \{p,q\}$ and α' is α restricted to $P_c - (\alpha^{-1}(p) \cup \alpha^{-1}(q))$. It can be easily shown, using Theorems 3.1, 3.2, and 3.3, that C' is an MGCS and that its signal graph is G_{pq}. □

In order to illustrate the results presented in this subsection, consider the MGCS in Figure 2. Links 4 and 7 of this MGCS are buffered and can be connected. Links 5 and 6 are not buffered, nor are links 5 and 7. We show in the following subsection that if they were connected, the system would contain a deadlock.

B. NECESSITY OF THE BUFFERING CONDITION. Let p be an output link and q an input link of the MGCS C such that p and q are not buffered. Then in the signal graph G_c, either (1) $\mathbf{p} \neq \mathbf{q}$, or (2) there is a marker-free path from $r_q{}^j$ to $r_p{}^j$ or one from $a_p{}^j$ to $a_q{}^j$, for some j. In the second case, we could model the connection of links p and q with the marked graph G_{pq} as before In the first case, however, this technique will not work

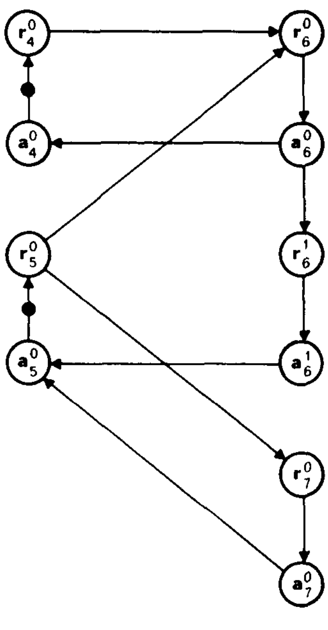

FIG. 2 Signal graph for the MGCS $C2$

since internal signals for links p and q do not match up. Moreover, it was shown in [8] that if two links have different multiplicities in one MGCS representation of a system, then they will be different in any MGCS representation of that system. Thus the effect of interconnecting links p and q cannot be modeled at the signal graph level using the method of Subsection A. The same argument used to justify the validity of the model G_{pq} can, however, be applied at the behavior graph level. This leads to the following model for the behavior of the system obtained by connecting links p and q.

Definition. Let $\bar{G}_c = (\bar{T}_c, \bar{P}_c, \bar{M}_c)$ be the behavior graph of an MGCS and let p be an output link and q an input link of C. Then $\bar{G}_{pq} = (\bar{T}_{pq}, \bar{P}_{pq}, \bar{M}_{pq})$ is the marked graph such that

(1) $\bar{T}_{pq} = \bar{T}_c$,
(2) $\bar{P}_{pq} = (\bar{P}_c - \{(a_q^j, r_q^{j+1}), (r_p^j, a_p^j) \mid j \geq 0\}) \cup \{(r_p^j, r_q^j), (a_q^j, a_p^j) \mid j \geq 0\}$, and
(3) $\bar{M}_{pq} = \bar{M}_c$.

Let $x_s^j \in \bar{T}_{pq}$ where $j \geq 0$. If this vertex is dead in $\bar{G}_{pq}$, then we can conclude that the link signal x_s does not occur more than $j - 1$ times. Furthermore, if we find that for every $s \in L - \{p,q\}$ there exists a $j \geq 0$ for which the vertex x_s^j is dead in $\bar{G}_{pq}$, then we can conclude that the system represented by $\bar{G}_{pq}$ contains deadlock and hence cannot be modeled as an MGCS. We now use these observations to show that connecting two links of the same MGCS always produces a system with deadlock when the links are not buffered.

LEMMA 3.2. *Let $\bar{G}_{pq}$ be the marked graph obtained from the behavior graph $\bar{G}_c$ as in the previous definition and let $j \geq 0$. Then*
(1) *either there is a path in $\bar{G}_{pq}$ from $a_q^j(r_p^j)$ to $r_q^{j+1}(a_p^j)$ or the vertex $r_q^{j+1}(a_p^j)$ is dead in $\bar{G}_{pq}$;*
(2) *either there is a path in $\bar{G}_{pq}$ from $a_p^j(r_q^j)$ to $r_p^{j+1}(a_q^j)$ or the vertex $r_p^{j+1}(a_q^j)$ is dead in $\bar{G}_{pq}$.*

PROOF. Let $K(0) = j$ and assume that there are no paths in $\bar{G}_{pq}$ from $a_q^{K(0)}$ to $r_q^{K(0)+1}$. Since the edges $(a_q^{K(0)}, a_p^{K(0)})$ and $(r_q^{K(0)+1}, r_q^{K(0)+1})$ are in $\bar{P}_{pq}$, we can conclude that there are no paths from $a_p^{K(0)}$ to $r_p^{K(0)+1}$ in $\bar{G}_{pq}$. But there is at least one such path in $\bar{G}_c$ [8]. Also, no path in $\bar{G}_c$ from $a_p^{K(0)}$ to $r_p^{K(0)+1}$ contains an edge of the form (r_p^l, a_p^l) since $\bar{G}_c$ is live. Hence, for some $K(1) \geq 0$, it must be the case that there is a path π_0 in $\bar{G}_c$ from $a_p^{K(0)}$ to $r_p^{K(0)+1}$ such that
(1) there is a subpath (edge) of π_0 from $a_q^{K(1)}$ to $r_q^{K(1)+1}$,
(2) there are no paths from $a_q^{K(1)}$ to $r_q^{K(1)+1}$ in $\bar{G}_{pq}$, and
(3) the subpath $\pi_0(r_q^{K(1)+1}, r_p^{K(0)+1})$ is also a path in $\bar{G}_{pq}$.

Due to the second condition above, the same argument can be used to show that for $i = 1, 2, 3, \cdots$, there is a path π_i and an integer $K(i + 1) \geq 0$ such that
(1) there is a subpath of π_i from $a_q^{K(i+1)}$ to $r_q^{K(i+1)+1}$ in $\bar{G}_c$,
(2) there are no paths from $a_q^{K(i+1)}$ to $r_q^{K(i+1)+1}$ in $\bar{G}_{pq}$, and
(3) the subpath $\pi_i(r_q^{K(i+1)+1}, r_p^{K(i)+1})$ is also a path in $\bar{G}_{pq}$.

Hence, for $i = 0, 1, 2, \cdots$, the sequence $\tau_i = \pi_i(r_q^{K(i+1)+1}, r_p^{K(i)+1}), (r_q^{K(i)+1}, r_p^{K(i)+1})$ is a path in $\bar{G}_{pq}$. Thus the infinite sequence $\cdots, \tau_i, \cdots, \tau_2, \tau_1, \tau_0$ is also a path of $\bar{G}_{pq}$ which terminates at $r_q^{K(0)+1} = r_q^{j+1}$. Since $K(i) \geq 0$ for all $i \geq 0$, this infinite path contains no initial markers. But this is a sufficient condition for the vertex r_q^{j+1} to be dead [7].

The dual statement for part (1) of the theorem, involving vertices r_p^j and a_p^j, can obviously be proved in the same way as above. Part (2) of the theorem follows as an immediate corollary to part (1) since vertex $r_p^{j+1}(a_q^j)$ lies on the infinite, marker-free path directed into $r_q^{j+1}(a_p^j)$. □

PROPOSITION 3.1. *Let x_s^j and y_u^l be two vertices in $\bar{G}_c$ with j, $l \geq 0$. If there is a path in $\bar{G}_c$ from x_s^j to y_u^l, then either there is a path from x_s^j to y_u^l in $\bar{G}_{pq}$ or the vertex y_u^l is dead.*

PROOF. Let π be a path in $\bar{G}_c$ from x_s^j to y_u^l and assume that there is no path in $\bar{G}_{pq}$ from x_s^j to y_u^l. Then for some $n \geq 0$, either (1) there is no path from a_q^n to r_q^{n+1} and there is a path from r_q^{n+1} to y_u^l in $\bar{G}_{pq}$, or (2) there is no path from r_p^n to a_p^n and there is a path from a_p^n to y_u^l in $\bar{G}_{pq}$. In the first case, r_q^{n+1} is dead and in the second case,

a_p^n is dead. In either case, y_u^l is dead since it lies on a marker-free path from a dead vertex. □

COROLLARY 3.3. *Let s be a link of the MGCS C. Then for any* $j \geq 0$, *either there is a path in* $\bar{G}_{pq}$ *from* $a_s^j(r_s^j)$ *to* $r_s^{j+1}(a_s^j)$ *or the vertex* $r_s^{j+1}(a_s^j)$ *is dead.*

PROOF. For any link s and $j \geq 0$, there is a path in $\bar{G}_c$ from $a_s^j(r_s^j)$ to $r_s^{j+1}(a_s^j)$ [8].

COROLLARY 3.4. *Let* x_s *be any link signal of the MGCS C and let* y_u^l *be a dead vertex in* $\bar{G}_{pq}$ *with* $l \geq 0$. *Then there is an integer k such that the vertex* x_s^k *is dead in* $\bar{G}_{pq}$.

PROOF. For some $k \geq 0$, there is a path from y_u^l to x_s^k in $\bar{G}_c$ [8]. □

We may therefore conclude that if $\bar{G}_{pq}$ contains a dead vertex, then the system it represents contains a deadlock.

LEMMA 3.3. *Let p be an output link and q an input link of the MGCS C. If p and q are not buffered, then* G_{pq} *contains a dead vertex.*

PROOF. First, assume that $\mathbf{p} \neq \mathbf{q}$ and, without loss of generality, that $\mathbf{p} < \mathbf{q}$. Let j be the least positive integer such that there is a path from r_q^0 to r_p^j in $\bar{G}_c$. Let n be the least positive integer such that $n(\mathbf{q} - \mathbf{p}) > j$ and set $m = n(\mathbf{q} - \mathbf{p}) - j$.

There is a path in $\bar{G}_c$ from r_p^j to r_p^{j+m} (Corollary 4.1. in [8]). Hence there is a path π_1 from $r_q^{n\mathbf{q}}$ to $r_p^{j+n\mathbf{p}}$ in $\bar{G}_c$ and also a path π_2 from $r_p^{j+n\mathbf{p}}$ to $r_p^{j+m+n\mathbf{p}}$ in $\bar{G}_c$ [8, Prop. 5.1]. But $j + m + n\mathbf{p} = j + n\mathbf{q} - n\mathbf{p} - j + n\mathbf{p} = n\mathbf{q}$. Hence there is a path $\pi = \pi_1, \pi_2$ from $r_p^{n\mathbf{q}}$, $r_q^{n\mathbf{q}}$ in $\bar{G}_c$.

If there is a path in $\bar{G}_{pq}$ from $r_q^{n\mathbf{q}}$ to $r_p^{n\mathbf{q}}$, then there is a marker-free cycle in $\bar{G}_{pq}$ since $(r_p^{n\mathbf{q}}, r_q^{n\mathbf{q}})$ is an edge of $\bar{G}_{pq}$. All of the vertices on this cycle are dead. If there are no paths in $\bar{G}_{pq}$ from $r_q^{n\mathbf{q}}$ to $r_p^{n\mathbf{q}}$, then $r_p^{n\mathbf{q}}$ is dead by Proposition 3.1.

Now assume that there is a marker-free path in G_c from r_q^j to r_p^j for some j. Then there is a path in $\bar{G}_c$ from r_q^j to r_q^j [8]. By Proposition 3.1 either there is a path from r_q^j to r_p^j in $\bar{G}_{pq}$ or r_p^j is dead in $\bar{G}_{pq}$. Since $\bar{G}_{pq}$ contains the edge (r_p^j, r_q^j), r_p^j is dead in either case. If there is a marker-free path in G_c from a_p^j to a_q^j for some j, then a similar argument shows that a_q^j is dead in $\bar{G}_{pq}$. □

Lemma 3.3 and Corollary 3.4 imply the following theorem, which describes the result of connecting an output link to an input link of the same MGCS when the two links are not buffered.

THEOREM 3.5. *Let p be an output link, and q an input link of the same MGCS such that p and q are not buffered. Then the connection links p and q result in a system which contains a deadlock and, therefore, cannot be represented by an MGCS.*

4. *Connection of Two Links on Different Systems*

We now consider connecting an output link on one control system to an input link on a different control system. Since the ready and acknowledge signals of a link alternate, one would not expect such a connection to introduce a deadlock into the system. Indeed, it can easily be shown that if the two component systems are represented by MG-control systems $C1$ and $C2$ in which the two links have equal multiplicities, then an MGCS representation for the composite system can be constructed from $C1$ and $C2$. This construction is given in the following theorem.

THEOREM 4.1. *Let p be an output link and q an input link of MG-control systems C1 and C2, respectively. Let* $G_{C1} = (T_{C1}, P_{C1}, M_{C1})$ *and* $G_{C2} = (T_{C2}, P_{C2}, M_{C2})$ *be the signal graphs for C1 and C2. If* $\mathbf{p} = \mathbf{q}$, *then the marked graph* (T', P', M') *where*

(1) $T' = T_{C1} \cup T_{C2}$,

(2) $P' = (P_{C1} \cup P_{C2} \cup \{(r_p^j, r_q^j), (a_q^j, a_p^j) \mid 0 \leq j < \mathbf{p}\}) - (\alpha^{-1}(p) \cup \alpha^{-1}(q))$, and

(3) $M' = (M_{C1} \cup M_{C2}) \cap P'$

is the signal graph of an MGCS representing the system obtained by connecting links p and q.

PROOF. Since links p and q trivially satisfy the buffering condition, the theorem follows immediately from Theorem 3.4. □

Theorem 4.1 is illustrated by Figure 3, which shows the signal graph obtained by connecting link 6 on the MGCS in Figure 2 to link 1 on the MGCS in Figure 1.

It is less trivial to construct an MGCS representation for a composite system from the

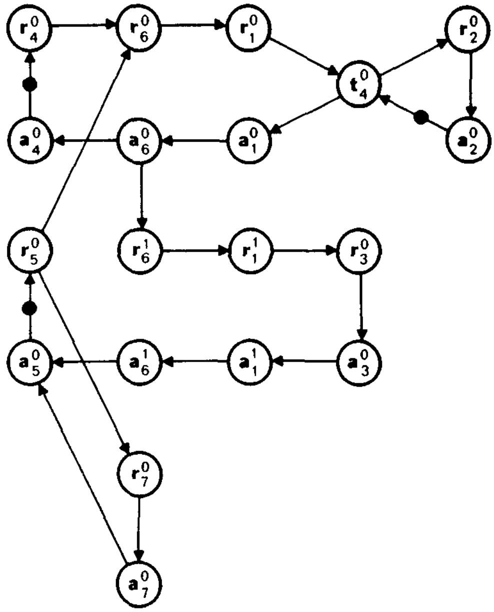

FIG. 3. Signal graph for the interconnection of MGCS $C1$ and MGCS $C2$

MGCS representations for its two component systems if the connecting links have different multiplicities in these representations. Our approach to this problem is as follows. Given an MGCS C and a positive integer n, we show how to construct an MGCS nC which is equivalent to C and therefore has the same set of links. This MGCS has the additional property that the multiplicity of link p, relative to nC, is n times the multiplicity of link p, relative to C. Now, given an output link p of an MGCS $C1$ and an input link q of a different MGCS $C2$, let m denote the least common multiple of $\mathbf{p}$ and $\mathbf{q}$, the multiplicities of links p and q relative to C_1 and C_2. Then the multiplicity of link p relative to $(m/\mathbf{p})C1$ and the multiplicity of link q relative to $(m/\mathbf{q})C2$ are both equal to m. Hence the construction in Theorem 4.1 can be used to find an MGCS representation for the system obtained by interconnecting $(m/\mathbf{p})C1$ and $(m/\mathbf{q})C2$ through links p and q. Since $(m/\mathbf{p})C1$ and $(m/\mathbf{q})C2$ are equivalent to $C1$ and $C2$ respectively, this new MGCS is also a valid representation for the interconnection of $C1$ and $C2$ through links p and q.

Rather than constructing nC directly, we will construct its signal graph G_{nc} by expanding the signal graph of C. The following function is used for this purpose.

Definition. Let T_c be the set of vertices of the signal graph G_c and $N_0 = \{0, 1, 2, \cdots, \}$. Then for $n > 0$, ψ_n is the function, with domain $T_c \times N_0$, defined by $\psi_n(x_p{}^j, u) = x_{j+p}^{R[u/n]\mathbf{p}}$ for all $(x_p{}^j, u)$ in $T_c \times N_0$.[5]

ψ_n is extended to paths of G_c in the following way. Let π be a path of length m in G_c and $u \in N_0$. Then $\psi_n(\pi, u)$ denotes the sequence of $m + 1$ elements defined inductively as follows:

$m = 1$. Then π has the form $x_p{}^j, y_q{}^l$ and $\psi_n(\pi,u) = \psi_n(x_p{}^j,u), \psi_n(y_q{}^l, u + \Sigma(M_c \mid \pi))$.
$m > 1$. In this case, π consists of a subpath π_1 from $x_p{}^j$ to $y_q{}^l$ of length m and a subpath π_2 from $y_q{}^l$ to $z_s{}^k$ of length 1. Then $\psi_n(\pi,u) = \psi_n(\pi_1,u), \psi_n(z_s{}^k, u + \Sigma(M_c \mid \pi))$.

As an example, consider the path $\pi = a_1{}^1, r_1{}^0, t_4{}^0, r_2{}^0, a_2{}^0, t_4{}^0, a_1{}^0$ in the signal graph of Figure 1. Then

$$\psi_3(\pi, 0) = a_1{}^1, r_1{}^2, t_4{}^1, r_2{}^1, a_2{}^1, t_4{}^2, a_1{}^4, \quad \psi_3(\pi, 1) = a_1{}^3, r_1{}^4, t_4{}^2, r_2{}^2, a_2{}^2, t_4{}^0, a_1{}^0,$$

and $\psi_3(\pi, 2) = a_1{}^5, r_1{}^0, t_4{}^0, r_2{}^0, a_2{}^0, t_4{}^1, a_1{}^2$.

[5] If $t_v{}^0$ is an internal transition, then we set $\mathbf{v} = 1$.

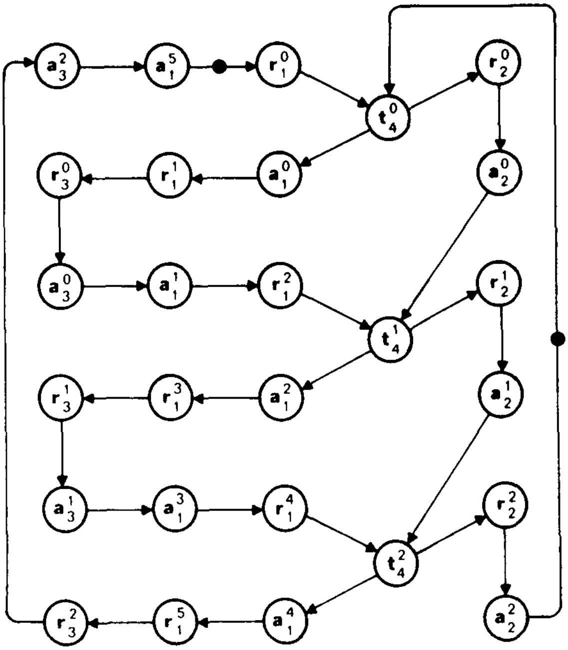

FIG. 4. Expansion of MGCS $C1$

Definition. Let $G_c = (T_c, P_c, M_c)$ be the signal graph of the MGCS C and let n be a positive integer. Then the *n-th expansion* of G_c is the marked graph $G_{nc} = (T_{nc}, P_{nc}, M_{nc})$ where:

$$T_{nc} = \{\psi_n(x_p{}^j,u) \mid x_p{}^j \in T_c \text{ and } 0 \leq u < n\},$$

$$P_{nc} = \{\psi_n(e,u) \mid e \in P_c \text{ and } 0 \leq u < n\}, \text{ and}$$

$$M_{nc} = \{\psi_n(e, n - 1) \mid e \in M_c\}.$$

The expansion of the signal graph in Figure 1 for $n = 3$ is shown in Figure 4.

In order to prove that G_{nc} is a signal graph of an MGCS that is equivalent to C, we must first establish a relationship between the paths of G_c and the paths of G_{nc}.

Definition. θ denotes the function from T_{nc} to T_c defined by $\theta(x_p{}^j) = x_p^{R[j/\mathbf{p}]}$, for all $x_p{}^j$ in T_{nc}.

Given a path $\pi = x_p{}^j, y_q{}^l, \cdots, z_s{}^k$ in G_{nc}, $\theta(\pi)$ denotes the sequence $\theta(x_p{}^j)$, $\theta(y_q{}^l), \cdots, \theta(z_s{}^k)$ of elements of T_c. The functions ψ_n and θ will now be used to relate the paths of G_{nc} and G_c.

LEMMA 4.1. *Let π be a path of length m from $x_p{}^j$ to $y_q{}^l$ in G_c and let u and n be integers such that $0 \leq u < n$. Then $\pi' = \psi_n(\pi,u)$ is a path of length m from $\psi_n(x_p{}^j,u)$ to $\psi_n(y_q{}^l, u + \Sigma(M_c \mid \pi))$ in G_{nc} and $\Sigma(M_{nc} \mid \pi') = Q[u + \Sigma(M_c \mid \pi)/n]$.*

PROOF. From the definition of M_{nc} and ψ_n, it can be easily seen that for any $v \geq 0$, $\psi_n(e, v) \in M_{nc}$ iff $e \in M_c$ and $R[v/n] = n - 1$. Using this observation, the proof of the lemma proceeds by induction on m.

Basis step. Let $m = 1$. Then $0 \leq \Sigma(M_c \mid \pi) \leq 1$ since π is an edge and G_c is safe. $\pi' = \psi_n(\pi,u)$ is a path of length 1 from $\psi_n(x_p{}^j,u)$ to $\psi_n(y_q{}^l, u + \Sigma(M_c \mid \pi))$ by the definition of P_{nc}. Moreover, $0 \leq \Sigma(M_{nc} \mid \pi') \leq 1$. Also, $\Sigma(M_{nc} \mid \pi') = 1$ iff $R[u/n] = u = n - 1$ and $\Sigma(M_c \mid \pi) = 1$. Hence $\Sigma(M_{nc} \mid \pi') = 1$ iff $Q[u + \Sigma(M_c \mid \pi)/n] = 1$, since $u < n$.

Induction step. Assume that the lemma holds for all paths of length m and let π be a path of length $m + 1$. Then π consists of a subpath π_1 of length m from $x_p{}^j$ to $z_s{}^k$ and subpath π_2 of length 1 from z_s^k to y_q^l, for some z_s^k. Hence $\pi_1' = \psi_n(\pi_1, u)$ is a path in G_{nc} of length m from $\psi_n(x_p^j,u)$ to $\psi_n(z_s^k, u + \Sigma(M_c \mid \pi_1))$ with $\Sigma(M_{nc} \mid \pi') = Q[u + \Sigma(M_c \mid \pi_1) \mid n]$. Moreover, $\pi_2' = \psi_n(\pi_2, u + \Sigma(M_c \mid \pi_1))$ is a path from $\psi_n(z_s^k, u + \Sigma(M_c \mid \pi_1))$ to

$$\psi_n(y_q^l, u + \Sigma(M_c \mid \pi_1) + \Sigma(M_c \mid \pi_2)) = \psi_n(y_q^l, u + \Sigma(M_c \mid \pi)).$$

Hence $\pi' = \psi_n(\pi,u)$ is the path formed by composing π_1' and π_2'. If $\Sigma(M_c \mid \pi_2) = 0$ or $R[u + \Sigma(M_c \mid \pi_1)/n] < n - 1$, then $\Sigma(M_{nc} \mid \pi_2') = 0$ and $Q[u + \Sigma(M_c \mid \pi_1)/n] = Q[u + \Sigma(M_c \mid \pi)/n]$. Hence $\Sigma(M_{nc} \mid \pi') = Q[u + \Sigma(M_c \mid \pi)/n]$. If $\Sigma(M_c \mid \pi_2) = 1$ and $R[u + \Sigma(M_c \mid \pi_1)/n] = n - 1$, then $\Sigma(M_{nc} \mid \pi_2') = 1$ so that $\Sigma(M_{nc} \mid \pi') = Q[u + \Sigma(M_c \mid \pi_1)/n] + 1$. But then

$$\begin{aligned} Q[u + \Sigma(M_c \mid \pi)/n] &= Q[u + \Sigma(M_c \mid \pi_1) + \Sigma(M_c \mid \pi_2)/n] \\ &= Q[u + \Sigma(M_c \mid \pi_1) + 1/n] = Q[u + \Sigma(M_c \mid \pi_1)/n] + 1 \\ &= \Sigma(M_{nc} \mid \pi'). \end{aligned}$$ □

LEMMA 4.2. *If π is a path of length m from x_p^j to y_q^l in G_{nc}, then $\pi' = \theta(\pi)$ is a path of length m from $\theta(x_p^j)$ to $\theta(y_q^l)$ in G_c and $\Sigma(M_c \mid \pi') = (\Sigma(M_{nc} \mid \pi))\, n + Q[l/\mathbf{q}] - Q[j/\mathbf{p}]$.*

PROOF. The proof is by induction on m.

Basis step. Let $m = 1$. Then there is an edge $\pi' = (x_p^i, y_q^k)$ in P_c such that $j = i + R[u/n]\mathbf{p}$ and $l = k + R[u + \Sigma(M_c \mid \pi')/n]\mathbf{q}$ for some $0 \leq u < n$. But then $R[j/\mathbf{p}] = i$ and $R[l/\mathbf{q}] = k$ since $i < \mathbf{p}$ and $k < \mathbf{q}$. Also, $\Sigma(M_{nc} \mid \pi) = Q[u + \Sigma(M_c \mid \pi')/n]$. Hence

$$\begin{aligned} u + \Sigma(M_c \mid \pi') = (Q[u + \Sigma(M_c \mid \pi')/n])\, n + R[u + \Sigma(M_c \mid \pi')/n] \\ = (\Sigma(M_{nc} \mid \pi))n + Q[l/q]. \end{aligned}$$

But $Q[j/\mathbf{p}] = R[u/n] = u$ so that $\Sigma(M_c \mid \pi') = (\Sigma(M_{nc} \mid \pi))n + Q[l/\mathbf{q}] - Q[j/\mathbf{p}]$.

Induction step. Assume that the lemma holds for all paths of length m and let π be a path of length $m + 1$. Then there is a path π_1 from x_p^j to z_s^k of length m and a path π_2 from z_s^k to y_q^l of length 1. Hence $\pi_1' = \theta(\pi_1)$ is a path of length m from $\theta(x_p^j)$ to $\theta(z_s^k)$ and $\Sigma(M_c \mid \pi_1') = (\Sigma(M_{nc} \mid \pi_1))n + Q[k/\mathbf{s}] - Q[j/\mathbf{p}]$. Also, $\pi_2' = \theta(\pi_2)$ is a path of length 1 from $\theta(z_s^k)$ to $\theta(y_q^l)$ with $\Sigma(M_c \mid \pi_2') = (\Sigma(M_{nc} \mid \pi_2))n + Q[l/\mathbf{q}] - Q[k/\mathbf{s}]$. Hence $\pi' = \theta(\pi)$ is a path from $\theta(x_p^j)$ to $\theta(y_q^l)$ and

$$\begin{aligned} \Sigma(M_c \mid \pi') &= \Sigma(M_c \mid \pi_1') + \Sigma(M_c \mid \pi_2') \\ &= (\Sigma(M_{nc} \mid \pi_1))n + Q[k/\mathbf{s}] - Q[j/\mathbf{p}] + (\Sigma(M_{nc} \mid \pi_2))n + Q[l/\mathbf{q}] \\ &\quad - Q[k/\mathbf{s}] \\ &= (\Sigma(M_{nc} \mid \pi))n + Q[l/\mathbf{q}] - Q[j/\mathbf{p}]. \end{aligned}$$ □

We now use these two lemmas to show that the marked graph G_{nc} is both live and safe.

THEOREM 4.2. *G_{nc} is a live marked graph.*

PROOF. Let π be a cycle from x_p^j to x_p^j in G_{nc}. Then $\pi' = \theta(\pi)$ is a cycle in G_c. If $\Sigma(M_{nc} \mid \pi) = 0$, then $\Sigma(M_c \mid \pi') = 0 \cdot n + Q[j/\mathbf{p}] - Q[j/\mathbf{p}] = 0$. But this contradicts the liveness of G_c. □

THEOREM 4.3. *G_{nc} is a safe marked graph.*

PROOF. Let $e = (x_p^j, y_q^l)$ be an edge in P_{nc}. Then $e' = \theta(e) = (\theta(x_p^j), \theta(y_q^l))$ is an edge in P_c. Since G_c is safe, there is a synchronizing loop π' of G_c containing e'. Then π' may be viewed as a path from $\theta(x_p^j)$ to $\theta(x_p^j)$ consisting of the edge e' composed with the subpath of π' from $\theta(y_q^l)$ to $\theta(x_p^j)$. Let π_n' denote the path obtained by composing π' with itself n times. Let $\pi = \psi_n(\pi_n', Q[j/\mathbf{p}])$. Using Lemma 4.1, it can be easily shown that π is a cycle of G_{nc} containing the edge e. Also, $\Sigma(M_{nc} \mid \pi) = Q[Q[j/\mathbf{p}] + n/n] = 1$ since $Q[j/\mathbf{p}] < n$. Hence π is a synchronizing loop containing e. □

THEOREM 4.4. *Let G_c be the signal graph of an MGCS C and let G_{nc} be the n-th expansion of G_c. Then there exists an MGCS nC such that: (1) G_{nc} is the signal graph of nC, (2) C and nC have identical link sets, and (3) the multiplicity of link p, relative to nC, is n times the multiplicity of link p, relative to C.*

PROOF. Let L be the link set and α the link assignment for the MGCS C. Define $n\alpha$ as the function from $\{\psi_n(e,u) \mid e \in \operatorname{dom} \alpha \text{ and } 0 \leq u < n\}$ to L such that $n\alpha(\psi_n(e,u)) = \alpha(e)$. Then it can be shown, using Lemmas 4.1 and 4.2, that $n\alpha$ is a valid link assignment for the marked graph G_{nc}. Moreover, G_{nc} is live and safe by Theorems 4.2 and 4.3. It is strongly connected due to the strong-connectedness of G_c and Lemma 4.1. Hence the triple $nC = (G_{nc}, n\alpha, L)$ is an MGCS which can easily be shown to satisfy the three conditions in the statement of the theorem. □

Since G_{nc} is the signal graph for an MGCS nC, the elements of T_{nc} can be classified as either internal vertices or internal signals, relative to the link assignment $n\alpha$. Indeed, if the element x of T_c is an internal vertex (signal) of G_c, then $\psi_n(x, u)$ is an internal vertex (signal) of G_{nc}, for $0 \leq u < n$.

We now show that the MG-control systems C and nC are equivalent. For this purpose, we state the following result, which can easily be derived from Lemmas 4.1 and 4.2.

LEMMA 4.3. *If π is a marker-minimal, signal-free path in G_{nc}, then $\theta(\pi)$ is a marker-minimal, signal-free path in G_c. Conversely, if π is a marker-minimal, signal-free path in G_c, then $\psi_n(\pi, u)$ is both marker-minimal and signal-free in G_{nc}, for $0 \leq u < n$.*

THEOREM 4.5. *The MG-control systems C and nC are equivalent.*

PROOF. Let $\bar{G}_c = (\bar{T}_c, \bar{P}_c, \bar{M}_c)$ and $\bar{G}_{nc} = (\bar{T}_{nc}, \bar{P}_{nc}, \bar{M}_{nc})$ be the behavior graphs of G_c and G_{nc}, respectively. We will prove that C and nC are equivalent by showing that $\bar{P}_c = \bar{P}_{nc}$ [8].

Let e be an edge in $\bar{P}_c$. It was shown in [8] that e must have the form $e = (x_p^{j+u\mathbf{p}}, y_q^{l+u\mathbf{q}})$ for some $u \in Z$, and there is a marker-minimal, signal-free path π from $x_p{}^j$ to $y_{\mathbf{q}}^{R[l/\mathbf{q}]}$ in G_c with $\Sigma(M_c \mid \pi) = Q[l/\mathbf{q}]$. By Lemmas 4.1 and 4.3, there is a marker-minimal, signal-free path π' from $\psi_n(x_p{}^j, R[u/n]) = x_p^{j+R[u/n]\mathbf{p}}$ to

$$\psi_n(y_q^{R[l/\mathbf{q}]}, R[u/n] + Q[l/\mathbf{q}]) = y_q^{R[l/\mathbf{q}]+R[R[u/n]+Q[l/\mathbf{q}]/n]\mathbf{q}}$$

in G_{nc}, with $\Sigma(M_{nc} \mid \pi') = Q[R[u/n] + Q[l/\mathbf{q}]/n]$. But if there is a marker-minimal, signal-free path τ from $x_p{}^i$ to $y_q{}^k$ in G_{nc} with $\Sigma(M_{nc} \mid \tau) = m$, then the ordered pair $(x_p^{i+vn\mathbf{p}}, y_q^{k+(v+m)n\mathbf{q}})$ is in $\bar{P}_{nc}$ [8]. Setting $i = j + R[u/n]\mathbf{p}$ and $v = Q[u/n]$, we have that

$$i + vn\mathbf{p} = j + R[u/n]\mathbf{p} + Q[u/n]n\mathbf{p} = j + u\mathbf{p}.$$

Setting $k = R[l/\mathbf{q}] + R[R[u/n] + Q[l/\mathbf{q}]/n]\mathbf{q}$ and $m = Q[R[u/n] + Q[l/\mathbf{q}]/n]$, we have

$$\begin{aligned} k + (v + m)n\mathbf{q} &= R[l/\mathbf{q}] + R[R[u/n] + Q[L/\mathbf{q}]/n]\mathbf{q} + Q[R[u/n] \\ &\quad + Q[l/\mathbf{q}]/n]n\mathbf{q} + Q[u/n]n\mathbf{q} \\ &= R[l/\mathbf{q}] + (R[u/n] + Q[l/\mathbf{q}])\mathbf{q} + Q[u/n]n\mathbf{q} \\ &= R[l/\mathbf{q}] + Q[l/\mathbf{q}]\mathbf{q} + (R[u/n] + Q[u/n]n)\mathbf{q} \\ &= l + u\mathbf{q}. \end{aligned}$$

Therefore, $e = (x_p^{j+u\mathbf{p}}, y_q^{l+u\mathbf{q}})$ is an element of $\bar{P}_{nc}$ so that $\bar{P}_c \subseteq \bar{P}_{nc}$.

The proof that $\bar{P}_{nc} \subseteq \bar{P}_c$ uses Lemma 4.2 in a way that is similar to the use of Lemma 4.1 above. The details are left to the reader. □

We summarize this section with the following theorem.

THEOREM 4.6. *Let $C1$ and $C2$ be distinct MG-control systems and let p be an output link of $C1$ and q an input link of $C2$. Then the system obtained by connecting links p and q can be represented by an MGCS.*

5. *Networks of Control Systems*

In Sections 2–4 we have completely characterized the effect of connecting a single-output link to a single-input link. We now show how these results can be used to solve the more general problem of analyzing a network of several control systems interconnected by means of several pairs of links.

We first note that the general problem can immediately be reduced to analyzing the effect of connecting several output links to several input links of the same system. To see this, consider an arbitrary network of control systems. The interconnection pattern of the network can be represented by an undirected graph $\Gamma = (V,E)$ where the elements of the vertex set V correspond to the control systems in the network and there is an edge in E connecting two vertices iff their corresponding control systems are connected through a pair of links. Let $T = (V,E')$ be a spanning tree for Γ and consider the subnetwork formed by connecting only those pairs of links that correspond to edges in E'. Using Theorem 4.6, it is easy to see that if each of the component systems is represented by an MGCS, then there is an MGCS C which represents the behavior of this subnetwork. Hence each edge of Γ that is not in T (i.e. edges in $E - E'$) corresponds to the connection of two links of the system. Therefore, the problem of determining whether or not the behavior of a network of control systems can be represented by an MGCS has been reduced to analyzing the effect of making those connections which correspod to edges in $E - E'$.

We represent the effect of connecting several pairs of links of the same ACS by means of the following marked graph.

Definition. Let $\bar{G}_c = (\bar{T}_c, \bar{P}_c, \bar{M}_c)$ be the behavior graph of an MGCS C and let f be a one-to-one function from a nonempty subset X of the set of output links of C into the set of input links of C. Then $\bar{G}_f$ denotes the marked graph $(\bar{T}_f, \bar{P}_f, \bar{M}_f)$ where

(1) $\bar{T}_f = \bar{T}_c$,

(2) $\bar{P}_f = (\bar{P}_c - \{(r_p{}^j, a_p{}^j), (a_p{}^j, r_q^{j+1}) \mid p \in X, f(p) = q \text{ and } j \geq 0\})$
$\cup \{(r_p{}^j, r_p{}^j), (a_p{}^j, a_p{}^j) \mid p \in x, f(p) = q \text{ and } j \geq 0\}$, and

(3) $\bar{M}_f = \bar{M}_c \cap \bar{P}_f$.

Hence $\bar{G}_f$ represents the behavior of the system obtained by connecting link p to link $f(p)$ for all p in X.

We have seen in Section 3 that if the connection of an input and an output link of the same MGCS produces a system that cannot be modeled as an MGCS, then this new system contains deadlock. We now show that this deadlock cannot be removed by connecting additional pairs of links.

PROPOSITION 5.1. *Let X be a nonempty set of output links and f a one-to-one function from X into the set of input links of an MGCS $C = (G,\alpha,L)$. Then for all p in L and $j \geq 0$, either there is a path in $\bar{G}_f$ from $r_p{}^j(a_{f(p)}^j)$ to $a_p{}^j(r_{f(p)}^{j+1})$ or the vertex $a_p{}^j(r_{f(p)}^{j+1})$ is dead in $\bar{G}_f$.*

PROOF. The proof is by induction on $|X|$.

Basis step. If $|X| = 1$, then this proposition follows immediately from Corollary 3.3.

Induction step. Assume that the lemma holds for all sets of output links with n elements and let X be a set of output links such that $|X| = n + 1$. Let $q \in X$ and $X' = X - \{q\}$. Let f be any 1-1 function from X into the set of input links of C and let f' be the restriction of f to X'. By the induction hypothesis, either there is a path π in $\bar{G}_{f'}$ from $r_p{}^j$ to $a_p{}^j$, or an infinite, marker-free path τ in $\bar{G}_{f'}$ directed into $a_p{}^j$, for all p in L.

If π exists in $\bar{G}_{f'}$ and is also a path in $\bar{G}_f$, then there is a path from $r_p{}^j$ to $a_p{}^j$ in $\bar{G}_f$. Assume that π exists in $\bar{G}_{f'}$ but there is no path in $\bar{G}_f$ from $r_p{}^j$ to $a_p{}^j$. Then for some $l \geq 0$, either

(1) $(r_q{}^l, a_q{}^l)$ is an edge on π and there is a path from $a_q{}^l$ to $a_p{}^j$ but none from $r_q{}^l$ to $a_q{}^l$ in $\bar{G}_f$, or

(2) $(a_{f(q)}^l, r_{f(q)}^{l+1})$ is an edge on π and there is a path from $r_{f(q)}^{l+1}$ to $a_p{}^j$ but none from $a_{f(q)}^l$ to $r_{f(q)}^{l+1}$ in $\bar{G}_f$.

Similarly, if τ exists in $\bar{G}_{f'}$ and is also a path in $\bar{G}_f$, then $a_p{}^j$ is dead in $\bar{G}_f$. If τ exists in $\bar{G}_{f'}$ but is not a path in $\bar{G}_f$, then there is an $l \geq 0$ such that either

(3) (r_q^l, a_q^l) is an edge on τ and there is a path from a_q^l to a_p^j but none from r_q^l to a_q^l in $\bar{G}_f$, or
(4) $(a_{f(q)}^l, r_{f(q)}^{l+1})$ is an edge on τ and there is a path from a_q^l to a_p^j but none from $a_{f(q)}^l$ to $r_{f(q)}^{l+1}$ in $\bar{G}_f$.

In all four cases, a proof similar to that of Lemma 5.1 can easily be constructed to show the existence of an infinite, marker-free path in $\bar{G}_f$ directed into a_p^l. Hence a_p^l is dead in $\bar{G}_f$. □

COROLLARY 5.1. *If $\bar{G}_f$ contains a dead vertex, then for every link signal x_p of C, there is an $l \geq 0$ such that x_p^l is a dead vertex in $\bar{G}_f$.*

COROLLARY 5.2. *Let X' be a nonempty subset of X, and let f' be the function f restricted to X'. Then $\bar{G}_f$ contains a dead vertex whenever $\bar{G}_{f'}$ contains a dead vertex.*

From Corollaries 5.1 and 5.2, we may conclude that if connecting one or more pairs of links of an *MGCS* produces a system with a deadlock, then connecting additional pairs of links can never eliminate this deadlock. Hence an arbitrary network of control systems can be analyzed in the following way. First pick a spanning tree $T = (V,E')$ for the interconnection graph $\Gamma = (V,E)$ of the network. Then construct an *MGCS* C for the subnetwork obtained by connecting only the pairs of links that correspond to edges in T. Check each pair of links that corresponds to an edge in $E - E'$ to see if the two links have equal multiplicities. If not, the network contains a deadlock. Otherwise, pick an edge e in $E - E'$ and determine if the pair of links corresponding to e is buffered in C. If not, the network contains a deadlock. If it is buffered, then form a new *MGCS* C' by connecting the pair. Next pick a new edge in $(E - E') - \{e\}$ and check the corresponding pair of links to see if they are buffered in C'. This procedure can obviously be repeated until either it is established that the network contains deadlock or an *MGCS* representation for its behavior is obtained.

6. *Conclusion*

In this paper we have completely characterized the effect of interconnecting links of MG-control systems. It has been shown that there are two possibilities when two links are connected. Either the resulting system is an MGCS or it contains deadlock. When two links of disjoint control systems are connected, the composite system is always an MGCS. If the two links are from the same MGCS, then the new system is an MGCS iff the links are buffered. We now compare these results with previous work on the interconnection of asynchronous systems.

The problem of detecting deadlock in networks of control modules has been investigated by Bruno and Altman [1] and Friedman and Menon [5]. The control modules used by Bruno and Altman were the WYE, SEQUENCE, JUNCTION, ITERATE, and SELECT modules. They characterized the class of networks which are free of deadlock in terms of the interconnection pattern of the network. The WYE, SEQUENCE, and JUNCTION modules can easily be represented as MG-control systems in which the multiplicity of every link is 1 [4]. Hence their characterization for deadlock-free networks composed of these modules can also be derived from Theorems 3.4, 3.5, and 4.1.

Friedman and Menon consider only one type of module. It is similar to modules proposed by Muller [11] in that a communication cycle on a link involves a more complicated "pipelining" operation. While Bruno and Altman assume that all modules start in a quiescent state with all links idle, Friedman and Menon allow each module to be initialized to one of two stable states. Hence the existence of deadlock depends on the initial state of the network as well as its topology. As a result, their characterizations of networks with deadlock cannot be derived from the interconnection theorems of this paper. However, given the initial state of a network of these modules, its behavior can be represented by a marked graph. Changing the initial state corresponds to using a different initial marking of the same graph. In this case, the detection of deadlock is equivalent to determining whether or not the marked graph is live.

Muller [9,10] has also considered the problem of interconnecting asynchronous systems. He was primarily concerned with guaranteeing that the composite system display a certain type of determinacy called semimodularity. This property of systems is roughly equivalent to safeness in marked graphs. However, if we restrict our attention to MG-control systems in which every link has multiplicity 1, then Theorem 4.1 can be easily derived from his basic interconnection theorem. It would appear to be more difficult to formulate the interconnection of links with multiplicity greater than 1 and the interconnection of two links of the same MGCS in terms of this more general theory. The problem of preserving determinacy under the interconnection of systems has also been investigated by Patil [13].

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