Stable Marriage with Multi-Modal Preferences*

Jiehua Chen[†] Ben-Gurion University of the Negev Beer-Sheva, Israel

Rolf Niedermeier TU Berlin Berlin, Germany

Piotr Skowron[‡]
TU Berlin
Berlin, Germany

Abstract

We introduce a generalized version of the famous STABLE MARRIAGE problem, now based on multimodal preference lists. The central twist herein is to allow each agent to rank its potentially matching counterparts based on more than one "evaluation mode" (e.g., more than one criterion); thus, each agent is equipped with multiple preference lists, each ranking the counterparts in a possibly different way. We introduce and study three natural concepts of stability, investigate their mutual relations and focus on computational complexity aspects with respect to computing stable matchings in these new scenarios. Mostly encountering computational hardness (NP-hardness), we can also spot few islands of tractability and make a surprising connection to the GRAPH ISOMORPHISM problem.

Keywords: Stable matching, concepts of stability, multi-layer (graph) models, NP-hardness, parameterized complexity analysis, exact algorithms.

1 Introduction

Information about the same "phenomenon" can come from different, possibly "contradicting", sources. For instance, when evaluating candidates for an open position, data concerning experience and so far achieved successes of the candidates may give different candidate rankings than data concerning their formal qualifications and degrees. In other words, one has to deal with a multi-modal data scenario. Clearly, in maximally objective and rationality-driven decision making, it makes sense to take into account several information resources in order to achieve best possible results. In this work we systematically apply this point of view to the STABLE MARRIAGE problem [25]; a key observation here is that several natural and well-motivated "multi-modal variants" of STABLE MARRIAGE need to be studied. We investigate the complexity of computing matchings that are stable according to the considered definitions.

In the classic (conservative) STABLE MARRIAGE problem [25], we are given two disjoint sets U and W of n agents each, where each of the agents has a *strict preference list* that ranks *every* member of the other set. The goal is to find a bijection (which we call a *matching*) between U and W without any *blocking pair* which can endanger the stability of the matching. A pair of agents is *blocking* a matching if they are not matched to each other but rank each other higher than their respective partners in the matching.

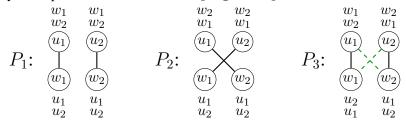
^{*}Work started when all authors were with TU Berlin.

[†]Supported by the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement number 631163.11, and by the Israel Science Foundation (grant number 551145/14).

[‡]Supported by a postdoctoral fellowship of the Alexander von Humboldt Foundation, Bonn, Germany.

Gale and Shapley [25] introduced the STABLE MARRIAGE problem in the fields of Economics and Computer Science in the 1960s. One of their central results was that every STABLE MARRIAGE instance with 2n agents admits a stable matching, which can be found by their algorithm in $O(n^2)$ time. Since then STABLE MARRIAGE has been intensively studied in Economics, Computer Science, and Social and Political Science [1, 27, 29, 30, 35, 37, 39]. Practical applications of STABLE MARRIAGE (and its variants) include partnership issues in various real-world scenarios, matching graduating medical students (so-called residents) with hospitals, students with schools, and organ donors with patients [27, 35, 39], and the design of content delivery systems [36] and other distributed markets [45].

The original model of STABLE MARRIAGE assumes, roughly speaking, that there is a (subjective) criterion and that each agent has a single preference list depending on this criterion. In typically complex real-world scenarios, however, there are usually multiple aspects one takes into account when making a decision. For instance, if we consider the classical partnership scenario, then there could be different criteria such as working hours, family background, physical appearance, health, hobbies, etc. In other words, we face a much more complex multi-modal scenario. Accordingly, the agents may have multiple preference lists, each defined by a different criterion; we call each of these criteria a *layer*. For an illustration, let us consider the following stable marriage example with two sets of two agents each, denoted as u_1 , u_2 , w_1 , and w_2 , and three layers of preferences, denoted as P_1 , P_2 , and P_3 .



In the above diagram the preferences are depicted right above (respectively, right below) the corresponding agents; preferences are represented through vertical lists where more preferred agents are put above the less preferred ones. For example, in the first layer, all agents from the same set have the same preference list, *i.e.* both u_1 and u_2 rank w_1 higher than w_2 while both w_1 and w_2 rank w_1 higher than w_2 . Similarly, in the second layer, both u_1 and u_2 rank w_2 higher than w_1 while both w_1 and w_2 rank w_1 higher than w_2 . In the last layer, the preference lists of two agents from the same set are reverse to each other. For instance, u_1 ranks w_1 higher than w_2 , which is opposite to u_2 . In terms of the classic stable marriage problem, we will have three independent instances, one for each layer. The corresponding stable matching(s) for each instance are depicted through the edges between the agents. For instance, the first layer admits exactly one stable matching, which matches u_1 with w_1 , and u_2 with w_2 . Yet, if we want to take all these layers jointly into account, then we need to extend the traditional concept of stability.

With multiple preference lists for each agent, there are many natural ways to extend the original stability concept. We propose three naturally emerging concepts of stability. Assume each agent has ℓ (possibly different) preferences lists. All three concepts are defined for a certain threshold α with $1 \le \alpha \le \ell$, which quantifies "the strength" of stability. In the following, we briefly describe our three concepts and defer the formal definitions to Section 2.

- The first one, called α -layer global stability, extends the original stability concept in a straightforward way. It assumes that the matched pairs agree on a set S of α layers where no unmatched pair is blocking the matching in any layer from S.

In our introductory example, the matching $M_1 = \{\{u_1, w_1\}, \{u_2, w_2\}\}$ is stable in the first and the last layer, and thus it is a 2-layer globally stable matching.

- The second one, called α -layer pair stability, changes the "blocking ability" of the unmatched pairs. It forbids an unmatched pair to block more than $\ell \alpha$ layers. In other words, each pair of matched agents needs to be stable in some α layers, but the choice of these layers can be different for different pairs.
 - Considering again our running example, we can verify that the 2-layer globally stable matching M_1 is also 2-layer pair stable as each unmatched pair is blocking at most one layer. Indeed, we will see that α -layer pair stability is strictly weaker than α -layer global stability (Proposition 3.1 and Example 3.1).
- The last one, called α -layer individual stability, focuses on the "willingness" of an agent to stay with its partner. It requires that for each unmatched pair, at least one of the agents prefers to stay with its partner in at least α layers.

In our introductory example, the matching M_1 is also 2-layer individually stable. Thus, it is tempting to assume that α -layer individual stability also generalizes α -layer global stability. This is, however, not true as the following matching $M_2 = \{\{u_1, w_2\}, \{u_2, w_1\}\}$ is 2-layer globally stable but not 2-layer individually stable. Neither does the latter implies the former. We refer to Example 3.2 for more explanations.

1.1 Related work

While we are not aware of research on an arbitrary number ℓ of layers, there is some work on $\ell=2$ layers. Weems [48] considered the case where each agent has two preference lists that are the reverse of each other. He provided a polynomial-time algorithm to find a *bistable* matching, *i.e.* a matching that is stable in both layers. Thus, while his concept falls into our α -layer global stability concept for $\alpha=\ell=2$, it is a special case since the preference lists in the two layers are the reverse of each other. In fact, for $\alpha=\ell=2$, we show that the complexity of determining α -layer global stability is NP-hard.

Aggregating the preference lists of multiple layers into one (by comparing each pair of agents) and then searching for a "stable" matching for the agents with aggregated preferences is a plausible approach to multi-modal stable marriages. As already noted by Farczadi et al. [23], the aggregated preferences may be intransitive or even cyclic. Addressing this situation, they consider a generalized variant of STABLE MARRIAGE, where each agent u of one side, say U, has a strict preference list \succ_u (as in the original STABLE MARRIAGE) while each agent w of the other side, say W, may order each possible pair of partners separately, expressed by a subset $B_w \subseteq U \times U$ of ordered pairs. They defined a matching M to be *stable* if no unmatched pair $\{u,w\}$ satisfies " $w \succ_u M(u)$ and $(u,M(u)) \in B_w$ ". It turns out that our concept of individual stability and their concept for a more generalized case where both sides of the agents may have intransitive preferences are related, and we can use one of their results as a subroutine. In a way, our analysis provides a more fine-grained view, since we consider a richer model and thus are able to discuss how certain assumptions on elements of this model (e.g., the number of layers, the threshold value α , etc.) affect the computational complexity of the problem.

Aziz et al. [2] considered a variant of STABLE MARRIAGE, where each agent has a probability for each ordered pair of potential partners. Assigning a probability of 1 to either (x,y) or (y,x) for each x and y, their variant is closely related to the one of Farczadi et al. [23] and is shown to be NP-hard.

We refer to several expositions [35, 27, 31, 39, 33, 7] for a broader overview on STABLE MARRIAGE and related problems.

1.2 Our contributions

We introduce three main concepts of stability for STABLE MARRIAGE with multi-modal preferences. In Section 2 we formally define these concepts, *global stability*, *pair stability*, and *individual stability*, and provide motivating and illustrating examples. In Section 3, we study the relations between the three concepts

Table 1: The computational complexity of finding matchings stable according to the three consideerd definitions— α -layer global stability, α -layer pair stability, and α -layer individual stability—for instances with 2n agents and ℓ layers. All results hold for each value of α specified in the first column. Results marked with * hold even if we assume that each agent of one side has the same preference list in all layers. The NP-hardness results hold even for a fixed number of layers.

Parameters	global stability	pair stability	individual stability
Arbitrary			
$1 = \alpha$	$O(n^2)$ [25]	$O(n^2)$ [25]	$O(n^2)$ [25]
$2 \le \alpha = \ell$	NP-h [T. 4.2+P. 4.4]	NP-h [C. 4.3+P. 4.4]	$O(\ell \cdot n^2)$ [T. 4.1]
$\lfloor \ell/2 \rfloor < \alpha < \ell$	NP-h [P. 5.1]	NP-h [P. 5.4+P. 5.5]	?
$2 \le \alpha \le \lfloor \ell/2 \rfloor$	NP-h [P. 5.1]	NP-h* [C. 5.3]	NP-h* [T. 5.2]
Single-layered			
	NP-h for unbounded α [T. 6.2]	NP-h* when $2 \le \alpha \le \lfloor \ell/2 \rfloor$ [C. 5.3]	NP-h* when $2 \le \alpha \le \lfloor \ell/2 \rfloor$ [T. 5.2]
	W[1]-h & in XP for α [T. 6.2]	$O(\ell \cdot n^2)$ when $\alpha > \lfloor \ell/2 \rfloor$ [P. 6.3]	$O(\ell \cdot n^2)$ when $\alpha > \lfloor \ell/2 \rfloor$ [P. 6.3]
Uniform			
$\alpha \ge \ell/2 + 1$	$O(\ell \cdot n)$ [P. 6.7]	?	$n^{O(\log{(n)})} + O(\ell \cdot n^2)$ [C. 6.6]
$\alpha = \ell/2$	$O(\ell \cdot n)$ [P. 6.7]	?	GRAPH ISOMhard [T. 6.5]

and show that pair stability is the least restrictive form while global and individual stability are in general incomparable (also see Figure 1 for a much refined picture). In Section 4, we consider the special case of all-layers stability ($\alpha = \ell$) for the three concepts. On the one hand, we provide a polynomial-time algorithm for checking individual stability for arbitrary large number of preference lists. On the other hand, through an involved construction, we show NP-hardness for the other two stability concepts, even if there are only two layers. The hardness results demonstrate a complexity dichotomy for both global and pair stability since for single-layer preference lists, all three concepts of stability are the same and polynomialtime computable. In Section 5 we investigate the case of finding stable matchings with respect to less than all layers and only find NP-hardness results. In Section 6, we identify two special scenarios with strong but natural restrictions on the preference lists. For the fist scenario we assume that one side of the agents has single-layered preferences, i.e. on one side the preference list of each agent remains the same in all layers. We find that under such restrictions two out of three studied concepts are equivalent, and can be computed in polynomial time; for global stability we obtain W[1]-hardness (and also NP-hardness) and XP membership for the threshold parameter α . In the second scenario, we assume that the preferences of all agents on each side are uniform in each layer, i.e. when for each fixed layer and side all agents have the same preference list, and when considering individual stability we find surprising tight connections to the complexity of the GRAPH ISOMORPHISM problem. Table 1 gives a broad overview on our complexity results.

2 Definitions and Notations

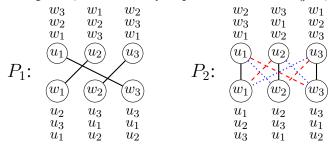
For each natural number t by [t] we denote the set $\{1, 2, \dots, t\}$.

Let $U = \{u_1, \dots, u_n\}$ and $W = \{w_1, \dots, w_n\}$ be two n-element disjoint sets of agents. There are ℓ layers of preferences, where ℓ is a non-negative integer. For each $i \in [\ell]$ and each $u \in U$, let $\succ_u^{(i)}$ be a linear order on W that represents the ranking of agent u over all agents from W in layer i. Analogously, for each $i \in [\ell]$ and each $w \in W$, the symbol $\succ_w^{(i)}$ represents a linear order on U that encodes preferences of w in layer i. We refer to such linear orders as preference lists. A preference profile P_i of layer $i \in [\ell]$ is a collection of preference lists of all the agents in layer i, $\{\succ_a^{(i)} | a \in U \cup W\}$.

Let $U \star W = \{\{u, w\} \mid u \in U \land w \in W\}$. A *matching* $M \subseteq U \star W$ is a set of pairwisely disjoint pairs, *i.e.* for each two pairs $p, p' \in M$ it holds that $p \cap p' = \emptyset$. If $\{u, w\} \in M$, then we also use M(u) to

refer to w and M(w) to refer to u, and we say that u and w are their respective partners under M; otherwise we say that $\{u, w\}$ is an *unmatched pair*. Example 2.1 below shows an example matching and introduces a graphical notation that we will use throughout the paper.

Example 2.1. Consider two sets of agents, $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, w_3\}$, and two layers of preference profiles, P_1 and P_2 . Let us recall that in the following diagram the preferences are represented through vertical lists where more preferred agents are put above the less preferred ones. For instance, in the diagram the preference list of agent u_3 in the first layer (profile P_1) is $w_2 >_{u_3}^{(1)} w_3 >_{u_3}^{(1)} w_1$.



In our diagrams we will depict stable matchings in each layer through edges between matched nodes. If a layer has more than one stable matching, then we will use different types of lines (solid, dashed, dotted) and different colors to distinguish between them. For instance, in the above example profile P_1 has one stable matching $M_1 = \{\{u_1, w_3\}, \{u_2, w_1\}, \{u_3, w_2\}\}$, and P_2 has three stable matchings: (1) $M_2 = \{\{u_1, w_1\}, \{u_2, w_2\}, \{u_3, w_3\}\}, (2)$ $M_3 = \{\{u_1, w_2\}, \{u_2, w_3\}, \{u_3, w_1\}\}, (3)$ M_1 .

Let us now introduce two notions that we will use when defining various concepts of stability.

Definition 2.1 (Dominating pairs and blocking pairs). Let M be a matching over $U \cup W$. Consider an unmatched pair $\{u,w\} \in (U \star W) \setminus M$ and a layer $i \in [\ell]$. We say that $\{u,w\}$ dominates $\{u,v\}$ in layer i if $w \succ_u^{(i)} v$. We say that $\{u,w\}$ is blocking matching M in layer i if it holds that

- (1) u is unmatched in M or $\{u, w\}$ dominates $\{u, M(u)\}$ in layer i, and
- (2) w is unmatched in M or $\{u, w\}$ dominates $\{w, M(w)\}$ in layer i.

For a single layer i, a matching M is *stable in layer* i if no unmatched pair is blocking M in layer i. Let us illustrate the concept of dominating and blocking pairs through Example 2.1. Consider the matching $M_3 = \{\{u_1, w_2\}, \{u_2, w_3\}, \{u_3, w_1\}\}$ and profile P_1 of layer 1. Here, pair $\{u_1, w_3\}$ dominates both $\{u_1, w_2\}$ (since u_1 prefers w_3 to w_2) and $\{u_2, w_3\}$ (since w_3 prefers u_1 to u_2). Thus, $\{u_1, w_3\}$ is a blocking pair and so it witnesses that M is not stable in profile P_1 .

We are interested in matchings which are stable in multiple layers, *i.e.* we aim at generalizing the classic STABLE MARRIAGE problem [25, 35, 27, 39] which is defined for a single layer to the case of multiple layers. The idea behind each of the concepts defined below is similar: in order to call a matching stable for multiple layers we require that it must be stable in at least a certain, given number of layers α (α is a number indicating the "strength" of the stability). However, for different concepts we require a different level of agreement with respect to which layers are required for stability. Informally speaking, on the one end of the spectrum we have a variant of stability where we require a global agreement of the agents regarding the set of α layers for which the matching must be stable. On the other end of the spectrum we have a variant where we assume that the agents act independently: an agent a would deviate if it would find another agent, say a0, such that a1 prefers a2 to its matched partner in some a2 layers, and a3 prefers a4 to its matched partner in another, possibly different, set of a4 layers. In the intermediate case, we require that a deviating pair must agree on the subset of layers which form the reason for deviation. We formally define the three concepts below.

2.1 α -layer global stability

Informally speaking, a matching M is α -layer globally stable if there exist α layers in each of which M is stable.

Definition 2.2 (global stability). A matching M is α -layer globally stable if there exists a set $S \subseteq [\ell]$ of α layers, such that for each layer $i \in S$ and for each unmatched pair $\{u,w\} \in U \star W \setminus M$ at least one of the two following conditions holds:

- (1) pair $\{u, M(u)\}$ dominates $\{u, w\}$ in layer i, or
- (2) pair $\{w, M(w)\}$ dominates $\{w, u\}$ in layer i.

The following example describes a scenario where the above concept of multi-layer stability appears to be useful.

Example 2.2. Assume that the preferences of the agents depend on external circumstances which are not known a priori. Assume that each layer represents a different possible state of the universe. If we want to find a matching that is stable in as many states of the universe as possible, then we need to find an α -layer globally stable matching for the highest possible value of α .

Already for α -layer global stability we see substantial differences compared to the original concept of stability for a single layer. It is guaranteed that such a matching always exists for $\alpha=1$; indeed this would be a matching that is stable in an arbitrary layer. However, one can observe that as soon as $\alpha>1$ an α -layer globally stable matching might not exist (see Example 3.1).

2.2 α -layer pair stability

While α -layer global stability requires that the agents globally agree on a certain subset of α layers for which the matching should be stable, pair stability forbids each unmatched pair to block more than a certain number of layers. The formal definition, using the domination concept, is as follows:

Definition 2.3 (pair stability). A matching M is α -layer pair stable if for each unmatched pair $\{u,w\} \in (U \star W) \setminus M$, there is a set $S \subseteq [\ell]$ of α layers such that for each layer $i \in S$ at least one of the following conditions holds:

- (1) pair $\{u, M(u)\}$ dominates $\{u, w\}$ in layer i, or
- (2) pair $\{w, M(w)\}$ dominates $\{w, u\}$ in layer i.

Definition 2.3 can be equivalently formulated using a generalization of the concept of blocking pairs. Let $\beta \in [\ell]$ be an integer bound. We say that a pair $\{u,w\} \in (U \star W) \setminus M$ is β -blocking M if there exists a subset $S \subseteq \{1,2,\ldots,\ell\}$ of β layers such that for each $i \in S$, pair $\{u,w\}$ is blocking M in layer i.

Proposition 2.1. A matching M is α -layer pair stable if and only if no unmatched pair p is $(\ell - \alpha + 1)$ -blocking M.

Proof. To prove the statement, we show that a matching M is *not* α -layer pair stable if and only if there is an unmatched pair p that is $(\ell - \alpha + 1)$ -blocking M. For the "if" direction, assume that $\{u, w\}$ is an unmatched pair and $R \subseteq [\ell]$ is a subset of $\ell - \alpha + 1$ layers such that $\{u, w\}$ is blocking every layer in R. Now consider an arbitrary subset $S \subseteq [\ell]$ of size α . By the cardinalities of R and S, it is clear that $S \cap R \neq \emptyset$. Let $i \in S \cap R$ be such a layer. Then, by assumption, we have that $\{u, w\}$ is blocking M in layer i. This means that none of the conditions stated in Definition 2.3 holds. Thus, $\{u, w\}$ is an unmatched pair witnessing that M is not α -layer pair stable.

For the "only if" direction, assume that M is not α -layer pair stable and let $\{u,w\}$ be an unmatched pair that witnesses the non- α -layer pair stability of M. We claim that $\{u,w\}$ is $(\ell-\alpha+1)$ -blocking M. Towards a contradiction, suppose that $\{u,w\}$ is not $(\ell-\alpha+1)$ -blocking M. Then, there must be a subset $S\subseteq [\ell]$ of at least α layers where the pair $\{u,w\}$ is not blocking M in each layer in S. Equivalently, we can say that

for each layer $i \in S$, $\{u, M(u)\}$ dominates $\{u, w\}$ in layer i or $\{w, M(w)\}$ dominates $\{u, w\}$ in layer i—a contradiction to $\{u, w\}$ being a witness.

The following example motivates α -layer pair stability.

Example 2.3. Consider the case when the preferences of the agents depend on a context, yet a context is pair-specific. For instance, in matchmaking a woman may have different preferences over men depending on which country they will decide to live in. Thus, a pair of a man and a woman is blocking if they agree on certain conditions, and if they will find each other more attractive than their current partners according to the agreed conditions.

In Section 3, we show that α -layer global stability implies α -layer pair stability (Proposition 3.1). This, among other things, implies that for $\alpha = 1$ an α -layer pair stable matching always exists. However, as soon as $\alpha \geq 2$ the existence is no longer guaranteed (see Example 3.1).

2.3 α -layer individual stability

We move to the third and last concept of stability.

Definition 2.4 (individual stability). A matching M is α -layer individually stable if for each unmatched pair $\{u,w\}\in (U\star W)\setminus M$ there is a set $S\subseteq [\ell]$ of α layers such that at least one of the following conditions holds:

- (1) pair $\{u, M(u)\}$ dominates $\{u, w\}$ in each layer of S, or
- (2) pair $\{w, M(w)\}$ dominates $\{w, u\}$ in each layer of S.

The following example illustrates a potential application in the domain of partnership agencies.

Example 2.4. Assume that each layer describes a single criterion for preferences. The preferences of each agent may differ depending on the criterion. For instance, the two sets of agents can represent, respectively, men and women, as in the traditional stable marriage problem. Different criteria may correspond, for instance, to the intelligence, sense of humor, physical appearance etc. Assume that an agent a will have no incentive to break his or her relationship with b, and to have an affair with c if he or she prefers b to c according to at least α criteria. In order to match men with women so that they form stable relationships, one needs to find an α -layer individually stable matching.

Definition 2.4 can be equivalently formulated using a generalization of the concept of dominating pairs. Let $\beta \in [\ell]$ be an integer bound. We say that a pair $\{u,w\}$ is β -dominating $\{u,w'\}$ if there is a subset $R \subseteq [\ell]$ of β layers such that for each $i \in R$ the pair $\{u,w\}$ dominates $\{u,w'\}$ in layer i.

Proposition 2.2. A matching M is α -layer individually stable if and only if no unmatched pair $\{u, w\}$ exists that is both $(\ell - \alpha + 1)$ -dominating $\{u, M(u)\}$ and $(\ell - \alpha + 1)$ -dominating $\{w, M(w)\}$.

Proof. To prove the statement, we show that a matching M is not α -layer individually stable if and only if there is an unmatched pair p that is $(\ell - \alpha + 1)$ -dominating $\{u, M(u)\}$ and $(\ell - \alpha + 1)$ -dominating $\{w, M(w)\}$. For the "if" direction, assume that $\{u, w\}$ is an unmatched pair and $R_1, R_2 \subseteq [\ell]$ are two (possibly different) subsets of $\ell - \alpha + 1$ layers each, such that $\{u, w\}$ is dominating $\{u, M(u)\}$ in each layer $i \in R_1$ and is dominating $\{w, M(w)\}$ in each layer $j \in R_2$. Now consider an arbitrary subset $S \subseteq [\ell]$ of size α . By the cardinalities of R_1, R_2 , and S, it is clear that $S \cap R_1 \neq \emptyset$ and $S \cap R_2 \neq \emptyset$. Let $i \in S \cap R_1$ and $j \in S \cap R_2$ be two layers in the intersections. Then, by assumption, we have that $\{u, w\}$ is dominating $\{u, M(u)\}$ in layer i and $\{u, w\}$ is dominating $\{w, M(w)\}$ in layer j. This means that none of the conditions stated in Definition 2.4 holds. Thus, $\{u, w\}$ is an unmatched pair witnessing that M is not α -layer individually stable.

For the "only if" direction, assume that M is not α -layer individually stable and let $\{u,w\}$ be an unmatched pair that witnesses the non- α -layer individual stability of M. We claim that $\{u,w\}$ is $(\ell-\alpha+1)$ -dominating $\{u,M(u)\}$ and is $(\ell-\alpha+1)$ -dominating $\{w,M(w)\}$. Towards a contradiction, first suppose

that $\{u,w\}$ is not $(\ell-\alpha+1)$ -dominating $\{u,M(u)\}$, meaning that there are at most $\ell-\alpha$ layers where $\{u,w\}$ dominates $\{u,M(u)\}$. This implies that there is a subset $S\subseteq [\ell]$ of α layers such that for each $i\in S$, the pair $\{u,M(u)\}$ is dominating $\{u,w\}$, a contradiction to $\{u,w\}$ being a witness of the non- α -layer individual stability of M. Analogously, if $\{u,w\}$ was not $(\ell-\alpha+1)$ -dominating $\{w,M(w)\}$, then we could obtain the same contradiction.

For $\alpha=1$ an α -layer individually stable matching always exists (it will follow from Propositions 3.1 and 3.3); however, this is no longer the case when $\alpha \geq 2$ (see Proposition 3.2 and Example 3.1).

Observe that according to α -layer individual stability the preferences of the agents can be represented as *sets* of linear orders: it does not matter which preference order comes from which layer. This is not the case for the other two concepts.

2.4 Central computational problems

In this paper, we study the algorithmic complexity of finding matchings that are stable according to the above definitions. To this end, we first investigate how the three concepts relate to each other. Next, we formally define the search problem of finding an α -layer globally stable matching.

GLOBALLY STABLE MARRIAGE

Input: Two disjoint sets of n agents each, U and W, ℓ preference profiles, and an integer bound $\alpha \in [\ell]$.

Output: Return an α -layer globally stable matching if one exists, or claim there is no such.

The other two problems, PAIR STABLE MARRIAGE and INDIVIDUALLY STABLE MARRIAGE, are defined analogously.

3 Relations Between the Multi-Layer Concepts of Stability

Below we establish relations among the three concepts. We start by showing that α -layer pair stability is a weaker notion than α -layer global stability and α -layer individual stability.

Proposition 3.1. An α -layer globally stable matching is α -layer pair stable.

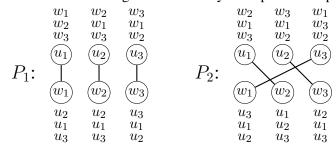
Proof. Let M be an α -layer globally stable matching and let $S \subseteq \{1,\ldots,\ell\}$ be such that $|S| = \alpha$ and that for each $i \in S$, matching M is stable in layer i. For the sake of contradiction let us assume that M is not α -layer pair stable. By Proposition 2.1, let $\{u,w\}$ be an $(\ell-\alpha+1)$ -blocking pair for M. Let $S' \subseteq \{1,\ldots,\ell\}$ be such that $|S'| = \ell-\alpha+1$ and that for each $i \in S'$, pair $\{u,w\}$ blocks M in layer i. Since $|S'| + |S| = \ell + 1$, we get that $S \cap S' \neq \emptyset$. Let $i \in S \cap S'$. This gives a contradiction since on the one hand M is stable in layer i, and on the other hand $\{u,w\}$ blocks M in layer i.

Proposition 3.2. An α -layer individually stable matching is α -layer pair stable.

Proof. Consider an α -layer individually stable matching M. Towards a contradiction, suppose that M is not α -layer pair stable. By Proposition 2.1, this means that there exists an unmatched pair $\{u,w\}$ and a subset $S'\subseteq [\ell]$ of $\ell-\alpha+1$ layers such that $\{u,w\}$ is blocking M in each layer from S'. This means that $\{u,w\}$ is both $(\ell-\alpha+1)$ -dominating $\{u,M(u)\}$ and $(\ell-\alpha+1)$ -dominating $\{w,M(w)\}$. Then, by Proposition 2.2, M is not α -layer individually stable, a contradiction.

Example 3.1, below, shows a matching which is α -layer pair stable, but which is not α -layer globally stable. This example, together with Proposition 3.1, also shows that α -layer global stability is strictly stronger than α -layer pair stability.

Example 3.1. Consider an instance with six agents and two layers of preference profiles.



Observe that matching $M = \{\{u_1, w_1\}, \{u_2, w_3\}, \{u_3, w_2\}\}$ is 1-layer individually stable and, thus 1-layer pair stable. However, M is blocked by pair $\{u_2, w_1\}$ in the first layer and by $\{u_1, w_2\}$ in the second. Thus, M is not 1-layer globally stable. Indeed the only 1-layer globally stable matchings are indicated by the solid lines, which are also 1-layer individually stable (and thus 1-layer pair stable).

As soon as $\alpha \geq 2$, α -layer pair stability is not guaranteed to exist, even if $\ell > \alpha$. To see this we augment the instance with one more layer whose preference lists are identical to the first layer given in Example 2.1. One can verify that for each of all six possible matchings, there is always an unmatched pair that is blocking at least two layers.

For $\alpha = 1$, we observe that 1-layer pair stability is equivalent to 1-layer individual stability.

Proposition 3.3. A matching is 1-layer pair stable if and only if it is 1-layer individually stable.

Proof. By Proposition 3.2, we know that 1-layer individual stability implies 1-layer pair stability. It remains to show the other direction. Let M be a 1-layer pair stable matching. Suppose, for the sake of contradiction, that M is not 1-layer individually stable. By Proposition 2.2, this means that there is an unmatched pair $\{u,w\}$ that is both $\ell-1+1=\ell$ -dominating $\{u,M(u)\}$ and $\ell-1+1=\ell$ -dominating $\{w,M(w)\}$. This implies that the pair $\{u,w\}$ is indeed ℓ -blocking M, which by Proposition 2.1, is a contradiction to M being 1-layer pair stable.

Example 3.2 shows that for $\alpha > 1$, individual stability and pair stability are not equivalent, neither is individual stability equivalent to global stability.

Example 3.2. Consider the example given in Section 1. Recall that the first layer admits exactly one stable matching, namely $M_1 = \{\{u_1, w_1\}, \{u_2, w_2\}\}$ (depicted by solid lines). The second layer also admits exactly one (different) stable matching, namely $M_2 = \{\{u_1, w_2\}, \{u_2, w_1\}\}$ (also depicted by solid lines). The third layer has two stable matchings, M_1 and M_2 (depicted by solid lines and dashed lines, resp.).

Thus, both M_1 and M_2 are 2-layer globally stable (and 2-layer pair stable). However, M_1 is 2-layer individually stable while M_2 is not. To see why M_2 is not 2-layer individually stable, we can verify that the unmatched pair $p = \{u_1, w_1\}$ dominates $\{u_1, w_2\}$ in the first and the third layer and it dominates $\{u_2, w_1\}$ in the first two layers. By Proposition 2.2, M_2 is not 2-layer individually stable since $\ell - \alpha + 1 = 2$.

If we restrict the example to the last two layers only, then matching M_2 is also evidence that an ℓ -layer globally stable (which, by Proposition 3.4, implies ℓ -layer pair stability) is not ℓ -layer individually stable. \diamond

For $\alpha = \ell$ global stability and pair stability are equivalent.

Proposition 3.4. For $\alpha = \ell$, a matching is α -layer globally stable if and only if it is α -layer pair stable.

Proof. By Proposition 3.1, ℓ -layer global stability implies ℓ -layer pair stability. Now, assume that a matching M is ℓ -layer pair stable. For the sake of contradiction, suppose that M is not ℓ -layer globally stable. This means that there exists a pair, say $\{u,w\}$, and a layer, say i, such that $\{u,w\}$ is blocking in layer i. Thus, $\{u,w\}$ is 1-blocking M, and so M cannot satisfy ℓ -layer pair stability. This gives a contradiction. \square

It is somehow counter-intuitive that even an ℓ -layer globally stable matching (*i.e.* a matching that is stable in each layer) may not be ℓ -layer individually stable (see Example 3.2). By Proposition 3.5 we can thus infer that ℓ -layer global stability is strictly weaker than ℓ -layer individual stability.

Proposition 3.5. For $\alpha = \ell$, an α -layer individually stable matching is α -layer globally stable.

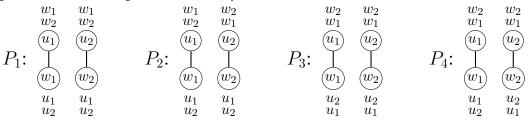
Proof. Proposition 3.2 and Proposition 3.4 imply the statement since $\alpha = \ell$.

By Example 3.2, ℓ -layer global stability does not imply ℓ -layer individual stability. However, it implies $\lceil \ell/2 \rceil$ -layer individual stability.

Proposition 3.6. Every ℓ -layer globally stable matching is $\lceil \ell/2 \rceil$ -layer individually stable. There are instances where ℓ -layer globally stable matchings are not $(\lceil \ell/2 \rceil + 1)$ -layer individually stable.

Proof. For the first statement, let M be an ℓ -layer globally stable matching. Suppose, for the sake of contradiction, that M is not $\lceil \ell/2 \rceil$ -layer individually stable. Let $\beta = \ell - \lceil \ell/2 \rceil + 1$, which is $\lfloor \ell/2 \rfloor + 1$. By Proposition 2.2, let $\{u,w\}$ be an unmatched pair that is both β -dominating $\{u,M(u)\}$ and β -dominating $\{w,M(w)\}$. Since $2 \cdot \beta > \ell$, there is at least one layer i where $\{u,w\}$ is dominating both $\{u,M(u)\}$ and $\{w,M(w)\}$, meaning that $\{u,w\}$ is blocking layer i, a contradiction to M being ℓ -layer globally stable.

To see why ℓ -layer global stability may not imply $(\lceil \ell/2 \rceil + 1)$ -layer individual stability, consider the following instance with four agents and $\ell = 4$ layers.



 $M = \{\{u_1, w_1\}, \{u_2, w_2\}\}$ is the only 4-layer globally stable matching. However, it is not 3-layer individually stable as the unmatched pair $\{u_1, w_2\}$ dominates $\{u_1, w_1\}$ in layers 3 and 4 and dominates $\{u_2, w_2\}$ in layers 1 and 2. By Proposition 2.2, M is not 3-layer individually stable since $\ell - \alpha + 1 = 2$.

The relations among the different concepts of multi-layer stability are depicted in Figure 1.

A 1-layer globally stable matching always exists. Together with Propositions 3.1 and 3.2, we obtain the following.

Proposition 3.7. A preference profile with ℓ layers always admits a matching, which is 1-layer globally stable, 1-layer pair stable, and 1-layer individually stable.

4 All-Layers Stability ($\alpha = \ell$)

In this section, we discuss the special case when $\alpha = \ell$. It turns out that deciding whether a given instance admits an ℓ -layer individually stable matching can be solved in polynomial time. For the other two concepts of stability, however, the problem becomes NP-hard even when $\ell = 2$.

4.1 Algorithm for ℓ -layer individual stability

The algorithm for deciding ℓ -layer individual stability is based on the following simple lemma.

Lemma 4.1. Let $u \in U$ and $w \in W$ be two agents such that w is the first ranked agent of u in some layer $i \in [\ell]$, and let $u' \in U \setminus \{u\}$ be another agent such that w prefers u over u' in some layer $j \in [\ell]$. Then, no ℓ -layer individually stable matching contains $\{u', w\}$.

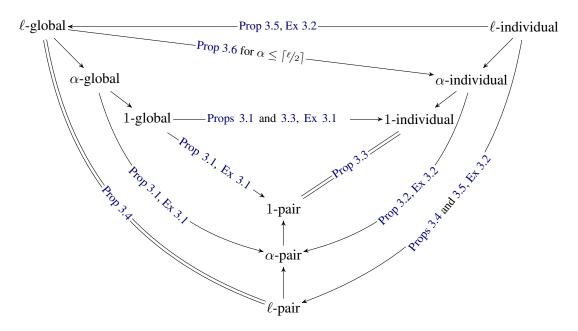


Figure 1: Relations among the different multi-layer concepts of stability for different values of α . Herein, an arc is to be read like an implication: one property implies the other.

Proof. Let u, w, u' be the three agents and let i, j be the two (possibly equal) layers as described by the assumption. Suppose towards a contradiction that there is an ℓ -layer individually stable matching with $\{u', w\} \in M$. This implies that $\{u, w\}$ is an unmatched pair under M. However, w prefers u over u' = M(w) in layer j and u prefers w over M(u) in layer i—a contradiction to M being ℓ -layer individually stable. \square

Lemma 4.1 leads to Algorithm 1 which looks quite similar to the so-called extended Gale-Shapley algorithm by Irving [28]. The crucial difference is that we loop into different layers and we cannot delete a pair p of agents that does not belong to any stable matching, as it may still serve to block certain matchings. Instead of deleting such pair, we will mark it. Herein, $marking\ a\ pair\ \{u,w\}$ means marking the agent u (resp. w) in the preference list of w (resp. u) in every layer.

The correctness of Algorithm 1 follows from Lemmas 4.2 to 4.4.

Lemma 4.2. If a pair $\{u', w\}$ is marked during the execution of Algorithm 1, then no ℓ -layer individually stable matching contains this pair.

Proof. Each pair is marked within two "foreach" loops in Line 2 and Line 3, respectively (we will refer to them as the "outer" loop and the "inner" loop). Let us fix an arbitrary $u \in U$ and $i \in [\ell]$ and consider the pairs which was marked when the outer and the inner loops were run for u and i, respectively. We show the statement via induction on the sequence of pairs which were marked when u and i was considered for the two loops. For the induction to begin, let $\{u', w\}$ with $u' \in U$ and $w \in W$ be the first pair that is marked during the execution. This implies that agent u ranks w in the first position in layer i and that i0 prefers i1 to i2 in some (possibly different) layer. By Lemma 4.1, no ℓ -layer individually stable matching contains $\{u', w\}$.

For the induction assumption, let $\{u',w\}$ be the m^{th} pair (for given u and i) that is marked during the execution and no ℓ -layer individually stable matching contains a pair that is marked prior to $\{u',w\}$. Suppose for the sake of contradiction that there is an ℓ -layer individually stable matching M which contains the marked pair $\{u',w\}$. The fact that $\{u',w\}$ has been marked implies that

Algorithm 1: Algorithm for finding an ℓ -layer individually stable matching.

Input: A set of agents $U \cup W$ and ℓ layers of preferences.

```
1 repeat
 2
        foreach agent u \in U do
             foreach layer i = 1, 2, \dots, \ell do
 3
                   w \leftarrow the first ranked agent in u's preference list in layer i
 4
                   r \leftarrow 1
                   repeat
 6
                        foreach u' with w: u \succ_w^{(j)} u' for some layer i do
                         \mathbf{mark} \{u', w\}
 8
                        w \leftarrow the r^{\text{th}} ranked agent in u's preference list in layer i
10
                   until \{u, w\} is not marked
11
```

- 12 **until** (some agent's preference list consists of only marked agents) **or** (no new pair was marked in the last iteration)
- 13 if some agent's preference list consists of only marked agents then no ℓ -layer individually stable matching exists
- 14 else return $M = \{\{u, w\} \mid w \leftarrow \text{ the first unmarked agent in } u$'s preference list in any layer} as an ℓ -layer individually stable matching
 - 1. u ranks w in the p^{th} position in layer i for some p, and
 - 2. w prefers u over u' in some layer j.

However, by the description of the algorithm in layer i (Lines 4–11) for each agent w' that u prefers to w in layer i, i.e. $w' \succ_u^{(i)} w$, we have that $\{u,w'\}$ is marked (see the "until" condition in Line 11). The induction assumption implies that M does not contain any $\{u,w'\}$ with $w' \succ_u^{(i)} w$. Thus, it follows that u prefers w to M(u) in layer i, i.e. $w \succ_u^{(i)} M(u)$. This is a contradiction to M being ℓ -layer individually stable on the unmatched pair $\{u,w\}$ since there is a layer $j \in [\ell]$ such that $u \succ_w^{(j)} u' = M(w)$.

The following lemma ensures that in Line 14 if w is matched to an agent u, then it is the most preferred unmarked agent of u in *all* layers.

Lemma 4.3. If no agent's preference list consists of only marked agents and there is an agent u and two different layers $i, j \in [\ell]$, $i \neq j$ such that the first unmarked agent in the preference list of u in layer i differs from the one in layer j, then Algorithm 1 will mark at least one more pair.

Proof. Suppose towards a contradiction that no new pair is marked, but there is an agent $u \in U$ such that the first agent unmarked by u is different in different layers, say w and w' in layers i and j, with $w \neq w'$ and $i \neq j$. Since no new pair will be marked, u is the last unmarked agent in the preference lists of w and w' in all layers (see Line 8 of Algorithm 1). Since |U| = |W| there is a different agent $u' \in U \setminus \{u\}$ such that for each agent $w \in W$ we have that u' is *not* the last unmarked agent in the preference list of w in any layer. Since no agent's preference list consists of only marked agents, the preference list of u' (in some layer) contains an agent which is unmarked. Denote this agent as w'. Again, since no new pair will be marked, in the preference list of w'', agent u' is the last unmarked agent—a contradiction.

When Algorithm 1 terminates and no agent contains a preference list that consists of only marked agents, then we can construct an ℓ -layer individually stable matching by assigning to each agent u its first unmarked agent in any preference list (note that Lemma 4.3 ensures on termination that for each agent it holds that in all layers is first unmarked agent is the same).

Lemma 4.4. If upon termination no agent's preference list consists of only marked agents, then the matching M computed by Algorithm 1 is ℓ -layer individually stable.

Proof. Towards a contradiction suppose that M returned by Algorithm 1 is not ℓ -layer individually stable. That is, there is an unmatched pair $\{u,w\} \in (U\star W)\setminus M$ with $u\in U$ and $w\in W$ and two layers $i,j\in [\ell]$ such that $u\colon w\succ_u^{(i)}M(u)$ and $w\colon u\succ_w^{(j)}M(w)$. Observe that agent u is matched with the first agent, denoted as x, in the preference list of u such that the pair $\{u,x\}$ is not marked, and so we infer that $\{u,w\}$ is marked. Thus, the innermost loop of the algorithm has been run for the pair $\{u,w\}$ (see Line 11). By Line 8 of Algorithm 1 for all agents u' where $w\colon u\succ_w^{(j)}u'$ for some layer $j'\in [\ell]$, the pair $\{u',w\}$ is marked. This includes the pair $\{M(w),w\}$ since $w\colon u\succ_w^{(j)}M(w)$ —a contradiction to Lemma 4.2.

Finally, we obtain that Algorithm 1 computes an ℓ -layer individually stable matching if one exists.

Theorem 4.1. For $\alpha = \ell$, Algorithm 1 solves Individually Stable Marriage in $O(\ell \cdot n^2)$ time.

Proof. Let $I = (U, W, P_1, P_2, \dots, P_\ell)$ with $2 \cdot n$ agents be the input of Algorithm 1. By Lemma 4.2, no ℓ -layer individually stable matching contains a marked pair. If there is an agent whose preference list consists of only marked agents, then we can immediately conclude that the given instance is a no-instance.

Otherwise, Lemma 4.4 proves that the algorithm returns an ℓ -layer individually stable matching.

It remains to show that the algorithm terminates and has running time $O(\ell \cdot n^2)$. Since there are in total $O(n^2)$ pairs, the algorithm will eventually terminate, either because some agent's preference list consists of only marked agents or because no new new pair will be marked.

By using a list that points to the first unmarked agent of each agent u in each layer, and by using a table that stores pairs which are already marked and reconsidered by Line 8, the algorithm needs to "touch" each pair at most twice, once when it is not yet marked and a second time when it is already marked.

4.2 NP-hardness for ℓ -layer global stability and ℓ -layer pair stability

In contrast to ℓ -layer individual stability, in this section we show that deciding GLOBALLY STABLE MARRIAGE is NP-hard as soon as $\ell=2$. We establish this by reducing the NP-complete 3-SAT problem [26] to the decision variant of GLOBALLY STABLE MARRIAGE. The idea behind this reduction is to introduce for each variable four agents that admit exactly two possible globally stable matchings, one for each truth value. Then, we construct a satisfaction gadget for each clause by introducing six agents. These agents will have three possible globally stable matchings. We use a layer for each literal contained in the clause to enforce that setting the literal to false will exclude exactly one of the three globally stable matchings. Therefore, unless one of the literals in the clause is set to true, no globally stable matching remains.

Using the above idea, we can already show hardness of deciding ℓ -layer global stability for $\ell=3$. With some tweaks and using a restricted variant of 3-SAT [26] (see Lemma 4.5), we can strengthen our hardness result to hold even for $\ell=2$. From here on, we call a clause *monotone* if the contained literals are either all positive or all negative.

Lemma 4.5. 3-SAT is NP-hard even if each clause has either two or three literals, and no size-three clause is monotone while all size-two clauses are monotone

Proof. We start with a 3-SAT instance and do the following. For each variable x_i introduce a helper variable z_i , and make sure that the helper variable z_i is set to false if and only if the original variable x_i is set to true. To achieve this, we add to the instance two new clauses $(x_i \vee z_i)$ and $(\overline{x_i} \vee \overline{z_i})$. Finally, for each original clause (note that it has size three) that contains only positive literals (resp. only negative literals), say $x_i \vee x_j \vee x_k$ (resp. $\overline{x_i} \vee \overline{x_j} \vee \overline{x_k}$), we replace an arbitrary literal, say x_i (resp. $\overline{x_i}$), with $\overline{z_i}$ (resp. z_i). Observe that in the new instance, each original clause has size three and contains at least one negative and at least one positive literal, and that the newly introduced clauses have size two and are monotone. It is straightforward to see that the original instance is a yes-instance if and only if the new instance is a yes-instance.

Now, we are ready to present one of our main results.

Theorem 4.2. GLOBALLY STABLE MARRIAGE is NP-hard even if $\alpha = \ell = 2$.

Proof. We provide a polynomial-time reduction from an NP-complete restricted variant of 3-SAT as given by Lemma 4.5 to the decision version of GLOBALLY STABLE MARRIAGE. Further, without loss of generality we assume that no clause contains two literals of the form x and \overline{x} as it will be satisfied anyway and can be ignored from the input instance.

Let (X,\mathcal{C}) be an instance of the aforementioned 3-SAT variant with $X=\{x_1,x_2,\ldots,x_n\}$ being the set of variables and $\mathcal{C}=\{C_1,C_2,\ldots,C_m\}$ being the set of clauses of size at most three each. To unify the expression, for each size-three clause $C_j=\ell_j^1\vee\ell_j^2\vee\ell_j^3$ we order the literals so that the first literal is positive and the second one is negative. For each size-two clause $C_j=\ell_j^1\vee\ell_j^2$ (note that it is monotone), we order the literals arbitrarily, and we call it a *positive* clause if it contains only positive literals, otherwise we call it a *negative* clause.

For each variable $x_i \in X$, we create four *variable agents* $x_i, \overline{x}_i, y_i, \overline{y}_i$ (we will make it clear when using x_i and \overline{x}_i whether we are referring to the literals or the variable agents). We will construct the preference lists of the variable agents so that each globally stable matching contains either $M_i^{\text{true}} := \{\{x_i, y_i\}, \{\overline{x}_i, \overline{y}_i\}\}$ or $M_i^{\text{false}} := \{\{x_i, \overline{y}_i\}, \{\overline{x}_i, y_i\}\}$. Briefly put, using M_i^{true} and M_i^{false} will correspond to setting the variable x_i to true or false, respectively.

For each clause $C_j \in \mathcal{C}$, we create six *clause agents* $a_j, b_j, c_j, d_j, e_j, f_j$. We will construct preference lists for these clause agents so that for each size-three-clause, there are exactly three different ways in which these agents are matched in a globally stable matching, and for each size-two-clause, there are exactly two such ways. We use two layers to enforce that setting a different literal contained in the clause C_j to false excludes exactly one of these ways.

In total, we have 4n+6m agents and we divide them into two groups U and W with $U=\{x_i,\overline{x}_i\mid i\in[n]\}\}\cup\{a_i,b_i,c_j\mid j\in[m]\}$ and $W=\{y_i,\overline{y}_i\mid i\in[n]\}\cup\{d_j,e_j,f_j\mid j\in[m]\}.$

Preference lists of the variable agents. The preference lists of the variable agents restricted to the variable agents have the same pattern. We use the symbol " \cdots " to denote some arbitrary order of the remaining agents (that is, agents which were not yet explicitly mentioned in the preference list).

It remains to specify the meaning of symbols A_i, D_i, A'_i , and D'_i .

- A_i denotes a list (in an arbitrary order) of all clause agents a_i that satisfy either of the following conditions:
 - (a) a_j corresponds to a *size-three-clause* C_j such that the *second* literal of clause C_j (which is a negative literal) is \overline{x}_i , or
 - (b) it corresponds to a *negative size-two-clause* C_j such that the *first* literal of clause C_j is \overline{x}_i .
- D_i denotes a list (in an arbitrary order) of all clause agents d_j such that the first literal of clause C_j is x_i (note that in this case C_j has either three literals or exactly two positive literals).
- A'_i denotes a list (in an arbitrary order) of all clause agents a_j such that the last literal of C_j is x_i (note that in this case C_j has either three literals or exactly two positive literals).
- D_i' denotes a list (in an arbitrary order) of all clause agents d_j such that the last literal of C_j is \overline{x}_i (note that in this case C_j has either three literals or exactly two negative literals).

To illustrate the above notation, suppose that variable x_i appears in four size-three-clauses, call them C_1, C_2, C_3 , and C_5 , and in two size-two-clauses: C_4 and C_6 . The positive literal x_i is the first literal in clauses C_1, C_3 , and C_4 . The negative literal \overline{x}_i is the second literal in C_2 , and the last literal in C_5 and C_6 . In this case, $A_i = a_2, D_i$ could be $D_i = d_1 \succ d_3 \succ d_4$, A_i' is empty, and D_i' could be $D_i' = d_5 \succ d_6$.

Preference lists of the clause agents. The preference lists for the clause agents in the first layer are "fixed" when restricted to the clause agents; they only differ in the positions of variable agents. For a clause C_j and an integer $t \in \{1,2,3\}$ let $C_j^{(t)}$ denote the t-th literal in C_j (there will be no $C_j^{(3)}$ if C_j has two literals). For a literal ℓ_i which is x_i or $\overline{x_i}$, by $X(\ell_i)$, $Y(\ell_i)$, $\overline{X}(\ell_i)$, and $\overline{Y}(\ell_i)$, we denote the variable agents x_i , y_i , $\overline{x_i}$, and $\overline{y_i}$, respectively, all corresponding to variable x_i . For instance, for a clause $C_j = x_2 \vee \overline{x_4} \vee x_5$, we have that $\overline{Y}(C_j^{(1)}) = \overline{y_2}$, and $\overline{Y}(C_j^{(2)}) = \overline{y_4}$.

Layer $(1), \forall j \in [m]$:

$$|C_j| = 3: \qquad \qquad a_j \colon d_j \succ e_j \succ Y(C_j^{(2)}) \succ f_j \succ \cdots, \qquad d_j \colon b_j \succ c_j \succ X(C_j^{(1)}) \succ a_j \succ \cdots, \\ b_j \colon e_j \succ f_j \succ d_j \succ \cdots, \qquad e_j \colon c_j \succ a_j \succ b_j \succ \cdots, \\ c_j \colon f_j \succ d_j \succ e_j \succ \cdots, \qquad f_j \colon a_j \succ b_j \succ c_j \succ \cdots, \\ |C_j| = 2 \text{ and } C_j \text{ is } \textit{positive} \colon \quad a_j \colon d_j \succ e_j \succ \cdots, \\ b_j \colon e_j \succ d_j \succ \cdots, \qquad e_j \colon a_j \succ b_j \succ \cdots, \\ c_j \colon f_j \succ \cdots, \qquad f_j \colon c_j \succ \cdots, \\ |C_j| = 2 \text{ and } C_j \text{ is } \textit{negative} \colon \quad a_j \colon d_j \succ Y(C_j^{(1)}) \succ e_j \succ \cdots, \\ b_j \colon e_j \succ d_j \succ \cdots, \qquad e_j \colon a_j \succ b_j \succ \cdots, \\ c_j \colon f_j \succ \cdots, \qquad e_j \colon a_j \succ b_j \succ \cdots, \\ c_j \colon f_j \succ \cdots, \qquad f_j \colon c_j \succ \cdots.$$

The preference lists for the second layer depends on the "positiveness" of the last literal. There are two variants:

Layer (2), $\forall j \in [m]$ with $|C_j| = 3$:

$$\begin{aligned} \text{Variant 1 } (C_j^{(3)} \text{ is } \textit{positive}) \colon & a_j \colon f_j \succ d_j \succ \overline{Y}(C_j^{(3)}) \succ e_j \succ \cdots, \\ & b_j \colon d_j \succ e_j \succ f_j \succ \cdots, \\ & c_j \colon e_j \succ f_j \succ \cdots, \\ & c_j \colon e_j \succ f_j \succ d_j \succ \cdots, \end{aligned} \qquad \begin{aligned} & e_j \colon a_j \succ b_j \succ c_j \succ \cdots, \\ & f_j \colon b_j \succ c_j \succ a_j \succ \cdots, \\ & d_j \colon a_j \succ b_j \succ \overline{X}(C_j^{(3)}) \succ c_j \succ \cdots, \\ & b_j \colon f_j \succ d_j \succ e_j \succ \cdots, \\ & c_j \colon d_j \succ e_j \succ f_j \succ \cdots, \end{aligned} \qquad \begin{aligned} & e_j \colon b_j \succ c_j \succ a_j \succ \cdots, \\ & e_j \colon b_j \succ c_j \succ a_j \succ \cdots, \\ & f_j \colon c_j \succ a_j \succ b_j \succ \cdots, \end{aligned} \end{aligned}$$

Layer (2), $\forall j \in [m]$ with $|C_j| = 2$:

Variant 1 (
$$C_j$$
 is positive): $a_j \colon d_j \succ \overline{Y}(C_j^{(2)}) \succ e_j \succ \cdots$, $d_j \colon b_j \succ a_j \succ \cdots$, $e_j \colon a_j \succ b_j \cdots$, $c_j \colon f_j \succ \cdots$, $f_j \colon c_j \succ \cdots$, Variant 2 (C_j is negative) $a_j \colon d_j \succ e_j \succ \cdots$, $d_j \colon b_j \succ \overline{X}(C_j^{(2)}) \succ a_j \succ \cdots$,

$$b_j : e_j \succ d_j \cdots,$$
 $e_j : a_j \succ b_j \cdots,$ $c_j : f_j \succ \cdots,$ $f_j : c_j \succ \cdots.$

This completes the construction which can be done in polynomial time.

Before we show the correctness of our construction, we first discuss some properties that each 2-layer globally stable matching M must satisfy.

Claim 1. Let M be a 2-layer globally stable matching for our two-layer preference profiles. For each variable $x_i \in X$, it holds that either $M_i^{true} \subseteq M$ or $M_i^{false} \subseteq M$.

Proof. To see this, we distinguish between two cases, depending on whether the partner of \overline{x}_i , $M(\overline{x}_i)$, is \overline{y}_i or not. If $M(\overline{x}_i) \neq \overline{y}_i$, then by the stability of M for the first layer, it follows that \overline{y}_i prefers its partner $M(\overline{y}_i)$ to \overline{x}_i in the first layer as otherwise \overline{x}_i and \overline{y}_i are forming a blocking pair for the first layer. Since x_i is the only agent that \overline{y}_i prefers to \overline{x}_i in this layer, we have that $M(\overline{y}_i) = x_i$. Then, it must hold that $M(\overline{x}_i) = y_i$ as otherwise \overline{x}_i and y_i would block M in the first layer. Thus, $M_i^{\text{false}} \subseteq M$.

Similarly, if $M(\overline{x}_i) = \overline{y}_i$, then by the stability of M and by construction of the preference lists of x_i and y_i in the second layer we must have that $M(x_i) = y_i$. This leads to $M_i^{\text{true}} \subseteq M$. (of Claim 1) \diamond

We obtain a similar result for the clause agents. For each clause $C_j \in \mathcal{C}$ with $|C_j| = 3$, let $N_j^1 =$ $\{\{a_j,d_j\},\{b_j,e_j\},\{c_j,f_j\}\},N_i^2=\{\{a_j,f_j\},\{b_j,d_j\},\{c_j,e_j\}\},N_i^3=\{\{a_j,e_j\},\{b_j,f_j\},\{c_j,d_j\}\}.$

Claim 2. Let $C_i \in \mathcal{C}$ be a size-three-clause, and let x_i be a variable that appears (as either a positive or a negative literal) in C_j . For a 2-layer globally stable matching M the following conditions hold:

- (i) If x_i is the first literal in C_j and if $M_i^{false} \subseteq M$, then either $N_i^2 \subseteq M$ or $N_i^3 \subseteq M$.
- (ii) If \overline{x}_i is the second literal in C_j and if $M_i^{true} \subseteq M$, then either $N_i^1 \subseteq M$ or $N_i^3 \subseteq M$.
- (iii) If x_i is the third literal in C_j and if $M_i^{\text{false}} \subseteq M$, then either $N_j^1 \subseteq M$ or $N_j^2 \subseteq M$. (iv) If \overline{x}_i is the third literal in C_j and if $M_i^{\text{true}} \subseteq M$, then either $N_j^1 \subseteq M$ or $N_j^2 \subseteq M$.

Proof. We consider the four cases separately:

- (i) Assume that x_i is the first literal in C_j and $M_i^{\text{false}} \subseteq M$. This implies that $\{x_i, \overline{y}_i\} \in M$. Consider the preference list of x_i in the first layer, and observe that d_i appears in D_i . Since x_i prefers d_i to its partner \overline{y}_i in the first layer, it follows that d_i must obtain a partner that it prefers to x_i in the first layer. By the preference list of d_j in the first layer, we have that $M(d_j) \in \{b_j, c_j\}$. If $M(d_j) = b_j$, then by the preference list of b_i in the first layer it follows that both e_i and f_i must obtain partners that they find better than b_i in the first layer. This means that $M(f_i) = a_i$ and $M(e_i) \in \{a_i, c_i\}$, implying that $M(e_i) = c_i$. Analogously, if $M(d_i) = c_i$, then $\{a_i, e_i\} \in M$ as otherwise they will block the first layer since the most preferred agents of both a_i and e_i are already assigned to someone else, and a_j and e_j are each other's second most preferred agents. Then, b_j must obtain a partner that it prefers to d_j . Since f_j is the only agent left that b_j prefers to d_j , we get that $M(b_j) = f_j$, and so $N_i^3 \subseteq M$.
- (ii) Assume that \overline{x}_i is the second literal in C_j and $M_i^{\text{true}} \subseteq M$ (thus, in particular, $\{x_i, y_i\} \in M$). Since a_i appears in A_i in the preference list of y_i in the first layer, we infer that y_i prefers a_i to its partner x_i in the first layer. Thus, it follows that a_i must obtain a partner that it prefers to y_i in the first layer, i.e. that $M(a_i) \in \{d_i, e_i\}$. If $M(a_i) = d_i$, then by considering the preference list of d_i in the first layer we infer that both b_i and c_i obtain partners that they find better than d_i in the first layer. This means that $M(c_j) = f_j$, and $M(b_j) \in \{e_j, f_j\}$. Since f_j is taken by c_j , we get that $M(b_j) = e_j$. Together, we have that $N_i^1 \subseteq M$.

Analogously, if $M(a_i) = e_i$, then $\{b_i, f_i\} \in M$ as otherwise they would block the first layer since the most preferred agents of both b_i and f_i are already assigned to someone else, and b_i and f_i are each other's second most preferred agent. Moreover, c_j must obtain a partner that it prefers to e_j . Since d_j is the only agent left that c_j prefers to e_j , we obtain that $M(c_j) = d_j$. This leads to $N_j^3 \subseteq M$.

- (iii) Assume that x_i is the third literal in C_j and $M_i^{\mathrm{false}} \subseteq M$. Observe that in this case a_j appears in A_i' in the preference list of \overline{y}_i in the second layer, and since $\{x_i,\overline{y}_i\}\in M$, that \overline{y}_i prefers a_j to its partner x_i in the second layer. It follows that a_j must obtain a partner that it prefers to \overline{y}_j in the second layer. By investigating the preference list of a_j in the second layer (note that we are in Variant 1), we have that $M(a_j)\in\{f_j,d_j\}$. If $M(a_j)=f_j$, then by looking at the preference list of f_j in the second layer we infer that both b_j and c_j must obtain partners that they find better than f_j in the second layer. Thus $M(c_j)=e_j$, and consequently, $M(b_j)=d_j$. Summarizing, in this case we have that $N_j^2\subseteq M$. Analogously, if $M(a_j)=d_j$, then $\{b_j,e_j\}\in M$ as otherwise they would block the second layer. Moreover, f_j must obtain a partner that it prefers to a_j . Since c_j is the only agent left that f_j prefers to a_j , we obtain that $M(c_j)=f_j$. This leads to $N_j^1\subseteq M$.
- (iv) Finally, assume that \overline{x}_i is the third literal in C_j and $M_i^{\text{true}} \subseteq M$. In this case, we have that d_j appears in D_i' in the preference list of \overline{x}_i in the second layer, and that $\{\overline{x}_i, \overline{y}_i\} \in M$. This means that \overline{x}_i prefers d_j to its partner \overline{y}_i in the second layer, and so it must be the case that d_j obtains a partner that it prefers to \overline{x}_j in the second layer. As a result, we have that $M(d_j) \in \{a_j, b_j\}$. If $M(d_j) = a_j$, then the preference list of a_j indicates that both e_j and f_j must obtain partners that they find better than a_j in the second layer. Thus, $M(f_j) = c_j$, and $M(e_j) \in \{b_j, c_j\}$. Consequently, $M(e_j) = b_j$, and we get that $N_j^1 \subseteq M$. Finally, if $M(d_j) = b_j$, then $\{c_j, e_j\} \in M$ as otherwise they would block the second layer. Further, a_j must obtain a partner that it prefers to d_j , thus $M(a_j) = f_j$. Consequently, $N_j^2 \subseteq M$.

(of Claim 2) ◊

For each clause $C_j \in \mathcal{C}$ with $|C_j| = 2$, let $N_j^1 = \{\{a_j, d_j\}, \{b_j, e_j\}\}, N_j^2 = \{\{a_j, e_j\}, \{b_j, d_j\}\}.$

Claim 3. Let $C_j \in \mathcal{C}$ be a size-two-clause, and let x_i be a variable that appears (as either a positive or a negative literal) in C_j . Assume that M is a 2-layer globally stable matching. The following holds:

- (i) If x_i is the first literal in C_j and if $M_i^{\text{false}} \subseteq M$, then $N_j^2 \subseteq M$.
- (ii) If $\overline{x_i}$ is the last literal in C_j and if $M_i^{true} \subseteq M$, then $N_j^2 \subseteq M$.
- (iii) If x_i is the last literal in C_j and if $M_i^{false} \subseteq M$, then $N_j^1 \subseteq M$.
- (iv) If \overline{x}_i is the first literal in C_j and if $M_i^{true} \subseteq M$, then $N_i^{1} \subseteq M$.

Proof. We show the first two statements together and the last two statements together. Assume that one of the conditions in the first two statements holds, that is,

- 1. x_i is the first literal in C_i and $M_i^{\text{false}} \subseteq M$, or
- 2. \overline{x}_i is the last literal in C_j and $M_i^{\text{true}} \subseteq M$.

This implies that

- 1. either $\{x_i, \overline{y}_i\} \in M$, and in the first layer the list D_i contains d_i , or
- 2. $\{\overline{x}_i, \overline{y}_i\} \in M$, and in the second layer the list D'_i contains d_i and we are in Variant 2.

Since x_i prefers all agents from D_i to \overline{y}_i in the first layer and \overline{x}_i prefers all agents from D_i' to \overline{y}_i in the second layer, we must have that d_j obtains a partner that it prefers to x_i in the first layer or to \overline{x}_i in Variant 2 of the second layer. In either case, b_j is the only agent that fulfills the requirement, implying that $\{b_j, d_j\} \in M$. By looking at the preference lists of b_j and e_j in the first layer, we derive that $\{a_j, e_j\} \in M$. Thus, $N_i^2 \subseteq M$.

Analogously, assume that one of the conditions in the last two statements holds, that is,

- 1. x_i is the last literal in C_j and $M_i^{\text{false}} \subseteq M$, or
- 2. \overline{x}_i is the first literal in C_i and $M_i^{\text{true}} \subseteq M$.

This implies that

- 1. $\{x_i, \overline{y}_i\} \in M$, and in the second layer we have Variant 1 such that the list A_i' contains a_i , or
- 2. $\{x_i, y_i\} \in M$, and in the first layer the list A_i contains a_j .

Since \overline{y}_i prefers all agents from A'_i to x_i in the second layer and y_i prefers all agents from A_i to x_i in the first layer, we must have that a_j obtains a partner that it prefers to \overline{y}_i in the second layer (Variant 1)

or to y_i in the first layer. In either case, d_j is the only agent that fulfills the requirement, implying that $\{a_j,d_j\}\in M$. By the preference lists of d_j and b_j in the first layer, we further derive that $\{b_j,e_j\}\in M$. Thus, $N_j^1\subseteq M$. (of Claim 3) \diamond

Now, we are ready to show that (X, \mathcal{C}) admits a satisfying truth assignment if and only if there exists a 2-layer globally stable matching for the so-constructed instance.

- (\Rightarrow) For the "only if" direction, assume that $\sigma \colon X \to \{T, F\}$ is a satisfying truth assignment for (X, \mathcal{C}) . We claim that the matching M constructed as follows is all-layer globally stable.
- (1) For each variable $x_i \in X$ with $\sigma(x_i) = T$, let $M_i^{\text{true}} \subseteq M$; otherwise let $M_i^{\text{false}} \subseteq M$.
- (2) For each size-three-clause C_j , identify a literal ℓ_j such that $\sigma(\ell_j)$ makes C_j satisfied. If ℓ_j is C_j 's first literal, then let $N_j^1 \subseteq M$. If ℓ_j is the second literal, then let $N_j^2 \subseteq M$. Otherwise let $N_j^3 \subseteq M$.
- (3) For each size-two-clause C_j , let $\{c_j, f_j\} \in M$. Identify a literal ℓ_j such that $\sigma(\ell_j)$ makes C_j satisfied. If ℓ_j is the first literal in C_j , then let $N_j^1 \subseteq M$; otherwise let $N_j^2 \subseteq M$.

Towards a contradiction suppose that M is not 2-layer globally stable, and let $p = \{u, w\}$ be a possible blocking pair with $u \in U$ and $w \in W$. First, we observe that p involves neither two variable agents that correspond to different variables nor two clause agents that correspond to different clauses. Second, p does not involve two variable agents that belong to the same variable since for each layer and for each two variable agents that correspond to the same variable, exactly one of both is already matched to its most preferred agent. Third, p also does not involve two clause agents that belong to the same size-two-clause as either of such clause agents is already matched with its most preferred agent.

Next, consider the case that u and w are two clause agents that belong to the same size-three-clause, say C_j . If $\{u,w\}$ is blocking M in the first layer, then we know that $N_j^3 \subseteq M$ as otherwise either u or w already obtains its most preferred agent. But then $\{u,w\}$ cannot be blocking M in the first layer as for each agent w' that is preferred to M(u) by u in the first layer, we have that w' prefers M(w') to u in the first layer. Similarly, if $\{u,w\}$ is blocking M in the second layer with Variant 1 (resp. Variant 2), then we know that $N_j^1 \subseteq M$ (resp. $N_j^2 \subseteq M$) as otherwise either u or w already obtains its most preferred agent. Since for each agent w' such that u prefers w' to M(u) in the second layer we have that w' prefers M(w') to u, it follows that $\{u,w\}$ cannot be blocking the second layer.

Now, suppose that p involves a variable agent and a clause agent. By the construction of the preference lists, we can assume that the clause agent involved in the blocking pair p is either an a_j or a d_j for some $j \in [m]$. We distinguish between two cases, depending on the size of C_j .

Case 1: $|C_j| = 3$. If $a_j \in p$ and p is blocking the first layer, then by the preference list of a_j in the first layer, we have that $N_j^2 \subseteq M$. By the construction of the matching M, we have that the truth assignment of the second literal, say $\overline{x_i}$, in C_j makes C_j satisfied; note that by our convention, the second literal is always negative. This implies that $M_i^{\text{false}} \subseteq M$. Since p involves a_j and is blocking the first layer, it follows that the agent y_i that corresponds to variable x_i prefers a_i to $M(y_i)$ in the first layer. This means that $M_i^{\text{true}} \subseteq M$, a contradiction.

Analogously, if $a_j \in p$ and p is blocking the second layer, then by the preference list of a_j in the second layer, we have that the preference list of a_j comes from Variant 1 and $N_j^3 \subseteq M$. By the construction of the matching M, we have that the truth assignment of the third literal which is positive in C_j (recall that Variant 1 was used) makes C_j satisfied. Let this literal be x_i . Then, it must hold that $M_i^{\text{true}} \subseteq M$. Since p involves a_j and is blocking the second layer (due to Variant 1), it follows that the agent \overline{y}_i that corresponds to variable x_i prefers a_j to $M(\overline{y}_i)$ in the second layer. This means that $M_i^{\text{false}} \subseteq M$, which results in a contradiction.

If $d_j \in p$ and p is blocking the first layer, then by the preference list of d_j in the first layer, we have that $N_j^1 \subseteq M$. By the construction of the matching M, we have that the truth assignment of the first literal, say x_i , in C_j makes C_j satisfied; note that by convention, the first literal is always positive. This implies that $M_i^{\text{true}} \subseteq M$. Since p involves d_j and is blocking the first layer, it follows that the agent x_i that corresponds

to variable x_i prefers d_j to $M(x_i)$ in the first layer. By the preference list of x_i in the first layer, it follows that $M_i^{\text{false}} \subseteq M$, which also yields a contradiction.

Finally, if $d_j \in p$ and p is blocking the second layer, then by the preference list of d_j in the second layer, we have that the preference list of d_j comes from Variant 2 and $N_j^3 \subseteq M$. By the construction of the matching M, we have that the truth assignment of the third literal which is negative in C_j makes C_j satisfied. Denote this literal by \overline{x}_i . Then, it follows that $M_i^{\text{false}} \subseteq M$. Since p involves d_j and is blocking the second layer (due to Variant 2), it follows that the agent \overline{x}_i that corresponds to variable x_i in C_j prefers d_j to $M(\overline{x}_i)$ in the second layer. By the preference list of x_i in the first layer, this means that $M_i^{\text{true}} \subseteq M$, which is a contradiction.

Case 2: $|C_j| = 2$. If $a_j \in p$, then by the preference lists of a_j in any of the two layers, we have that $N_j^2 \subseteq M$. Hence, the second literal in C_j makes it satisfied. We distinguish between two cases. If the second literal in C_j is positive, say x_i , then $M_i^{\text{true}} \subseteq M$ and the other involved agent in p must be either y_i or \overline{y}_i . Since \overline{y}_i prefers its partner \overline{x}_i to a_j in both layers, it follows that y_i is the other involved agent. However, by the preference list of y_i in the first layer, A_i does not contain a_j , meaning that y_i also prefers its partner x_i to a_j in both layers, which is a contradiction to p being a blocking pair.

If the second literal in C_j is negative, say \overline{x}_i , then $M_i^{\text{false}} \subseteq M$ and our reasoning is very similar. First we infer that the other involved agent in p must be either y_i or \overline{y}_i . Since y_i prefers its partner \overline{x}_i to a_j in both layers, it follows that the other agent in p is \overline{y}_i . However, by the preference list of \overline{y}_i in the second layer, A_i' does not contain a_j , which means that \overline{y}_i prefers its partner to a_j in both layers. Thus, in this case we also get a contradiction.

If $d_j \in p$, then our reasoning is very similar. First, by looking at the preference lists of d_j in any of the two layers, we infer that $N_j^1 \subseteq M$. By the construction of the matching M, we get that the first literal in C_j makes it satisfied. We consider two cases. If this literal is positive, say x_i , then $M_i^{\text{true}} \subseteq M$ and the other involved agent in p must be either x_i or \overline{x}_i . Agent x_i prefers its partner y_i to d_j in both layers and so it cannot be involved in the blocking pair. Thus, \overline{x}_i is the other involved agent. However, by the preference list of \overline{x}_i in the first layer, D_i' does not contain d_j , meaning that \overline{x}_i also prefers its partner \overline{y}_i to d_j in both layers, a contradiction to p being a blocking pair.

If the first literal in C_j is negative, say \overline{x}_i , then $M_i^{\text{false}} \subseteq M$ and the other involved agent in p must be either x_i or \overline{x}_i . Since \overline{x}_i prefers its partner y_i to d_j in both layers, it follows that x_i is the other involved agent. However, by the preference list of x_i in the last layer, D_i does not contain d_j , meaning that x_i prefers its partner \overline{y}_i to d_j in both layers, which is again a contradiction.

(\Leftarrow) For the "if" direction, let M be a 2-layer globally stable matching. We construct a truth assignment σ as follows. For each variable agent x_i , if $M_i^{\text{true}} \subseteq M$, then let $\sigma(x_i) = T$; otherwise by Claim 1 we have that $M_i^{\text{false}} \subseteq M$, and let $\sigma(x_i) = F$. Suppose, towards a contradiction, that σ is not a satisfying assignment and let C_j be a clause where none of the literals is evaluated to true. We distinguish between two cases.

Case 1: $|C_j| = 3$. Let x_r, \overline{x}_s , and ℓ_t be the first, second, and the third literal in C_j . Since none of these literals is evaluated to true, it follows that $M_r^{\mathsf{false}}, M_s^{\mathsf{true}} \subseteq M$. By statements (i) and (ii) in Claim 2, we must have that $N_j^3 \subseteq M$. However, by the statements (iii) and (iv) in Claim 2, applied for ℓ_t , we must have that either $N_j^1 \subseteq M$ or $N_j^2 \subseteq M$, a contradiction.

Case 2: $|C_j| = 2$. Let the first and the second literals in C_j correspond to variables x_i and x_k , respectively. If C_j is positive, then since C_j is not satisfied, we have that M_i^{false} , $M_k^{\text{false}} \subseteq M$. By Claim 3, we have that both N_i^1 and N_i^2 must belong to M, which leads to a contradiction.

Analogously, if C_j is negative, then since C_j is not satisfied, we have that M_i^{true} , $M_k^{\text{true}} \subseteq M$. By Claim 3, we have that both N_i^1 and N_i^2 must belong to M, a contradiction.

Altogether, we showed that the constructed matching is globally stable for two layers. This concludes the proof of Theorem 4.2.

Since ℓ -layer global stability equals ℓ -layer pair stability (Proposition 3.4), by Theorem 4.2 we obtain the following corollary for the pair stability.

Corollary 4.3. PAIR STABLE MARRIAGE is NP-hard even if $\alpha = \ell = 2$.

By adding to the profile constructed in the proof of Theorem 4.2 an arbitrary number of layers with preferences that are stable anyway, we can deduce hardness for arbitrary $\alpha = \ell \geq 2$.

Proposition 4.4. For each $\alpha=\ell\geq 2$, both Globally Stable Marriage and Pair Stable Marriage are NP-hard.

Proof. We add to the profile constructed in the proof of Theorem 4.2 $\ell-2$ layers with preferences of the following form:

It is straightforward that a matching is ℓ -layer globally stable if and only if it is 2-layer globally stable for the first two layers.

5 Multi-Layer Stable Marriage with $\alpha < \ell$

In this section we show that for each of the three concepts that we introduced in Section 2 the problem of computing a multi-layer stable matching is computationally hard as soon as $2 \le \alpha < \ell$.

5.1 NP-hardness for α -layer global stability

To find a matching M that is α -layer globally stable, even if $\alpha < \ell$, the main difficulty is not just to determine α layers where M should be stable. In fact, we sometimes need to find a matching that is stable in some specific layers. This requirement allows us to adapt the construction in the proof of Theorem 4.2 to show hardness for deciding α -layer global stability for the case when $2 < \alpha < \ell$.

Proposition 5.1. For each fixed number α with $\alpha \geq 2$, GLOBALLY STABLE MARRIAGE is NP-hard.

Proof. To prove the NP-hardness, we adapt the reduction in the proof of Theorem 4.2 which shows that deciding α -layer global stability for $\alpha=\ell=2$ is NP-hard. Let $\mathcal P$ be the constructed two-layer instance in the proof of Theorem 4.2. Besides the original agents from $\mathcal P$, we introduce two sets U and W of dummy agents with $|U|=|W|=2\cdot(\ell-\alpha+1)$, where $U=\{u_1,u_2,\ldots,u_{\ell-\alpha+1}\}$ and $W=\{w_1,w_2,\ldots,w_{\ell-\alpha+1}\}$. The idea of introducing such dummy agents is to make sure that each α -layer globally stable matching must include all pairs $\{u_j,w_j\}$, $j\in[\ell-\alpha+1]$. However, this is the case only when such matching is stable in the two layers constructed in the NP-hardness proof of Theorem 4.2; we denote these two layers as layers (1) and (2). In the following, we use " \cdots " to denote an arbitrary order of the unmentioned agents.

Preferences of the original agents. The preferences of the original agents in the first two layers are the same as in \mathcal{P} in the proof of Theorem 4.2. For each other layer, the preferences of the original agents are as follows.

Layers (3)–
$$(\ell)$$
, $\forall i \in [n]$: $x_i : y_i \succ \overline{y}_i \succ \cdots$, $y_i : \overline{x}_i \succ x_i \succ \cdots$,

Preferences of the dummy agents. The preferences of the dummy agents are as follows; let $\hat{\ell} = \ell - \alpha + 1$:

$$\text{Layers } (1)\text{--}(\alpha), \ \, \forall j \in [\hat{\ell}]\colon \quad u_j\colon w_j \succ \cdots, \qquad \qquad w_j\colon u_j \succ \cdots, \\ \text{Layer } (i+\alpha), 1 \leq i \leq \hat{\ell}-1, \ \, \forall j \in [\hat{\ell}]\colon \quad u_j\colon w_{(j \bmod \hat{\ell})+i} \succ \cdots, \qquad w_j\colon u_{(j-1 \bmod \hat{\ell})+i} \succ \cdots.$$

Observe that each dummy agent obtains a different partner in different layers with indices higher than α . More precisely, for each layer $(i+\alpha)$ with $1 \le \alpha \le \hat{\ell} - 1$, the only stable matching in this layer must include $M_i = \{\{u_j, w_{(j \bmod \hat{\ell})+1}\} \mid 1 \le j \le \hat{\ell}\}\}$ since u_j and $w_{(j \bmod \hat{\ell})+1}$ are each other's most preferred agent. Moreover, by the same reasoning, each layer with index at most α admits exactly the same stable matching regarding the dummy agents which is different from any layer with index higher than α , namely $M_0 = \{\{u_j, w_j\} \mid 1 \le j \le \hat{\ell}\}$. For each two distinct values $i, j \in \{0, 1, \dots, \hat{\ell} - 1\}$, however, we have that $M_i \cap M_j = \emptyset$. This means that each α -layer globally stable matching must include M_0 and must be stable in the first α layers, including the first two layers.

Now, it is straightforward to see that a matching M is 2-layer globally stable for \mathcal{P} if and only if $M \cup M_0$ is α -layer globally stable for our new instance.

We remark that our proof for Proposition 5.1 also implies hardness for $\alpha = \ell$ for arbitrary $\ell \geq 2$.

5.2 NP-hardness for α -layer individual stability

For α -layer individual stability, we also obtain a hardness result by reducing from the NP-hard PERFECT SMTI problem, the problem of finding a perfect *SMTI-stable* matching with (possibly) incomplete preference lists and ties [32, 38] which is defined as follows. A preference list is *incomplete* if not all agents from one side are considered acceptable to an agent from the other side. A preference list has a *tie* if there are two agents in the list which are considered to be equally good. As a result, a preference list of an agent u from one side can be considered as a weak (*i.e.* transitive and complete) order \succeq_u on a subset of the agents on the other side. We use \succ_u and \sim_u to denote the asymmetric and symmetric part of the preference list, respectively. Equivalently, two agents x and y are said to be tied by u if $x \succeq_u y$ and $y \succeq_u x$, denoted as $x \sim_u y$. Formally, we say that a matching M for a PERFECT SMTI with two disjoints sets U and U of agents is *SMTI-stable* if there are no SMTI-blocking pairs for U. A pair U is *SMTI-blocking* U if all of the following three conditions are satisfied: (i) U and U appear in the preference lists of each other, (ii) $U \succ_u M(u)$ or U is not matched to any agent from U.

The reduction is based on the following ideas: In a PERFECT SMTI instance I the preference list of an agent z may have ties. To encode ties, we first "linearize" the preference list of z in I to obtain two linear preference lists such that the resulting lists restricted to the tied agents are reverse to each other. Then, we let half of the ℓ layers have one of the preference lists and let the remaining half to have the other list. Since $\alpha < \ell/2$, it is always possible to find half layers which fulfill our α -layer individual stability constraint.

In I, two agents, say x and y, may not be acceptable to each other. To encode this, we introduce to the source instance ℓ pairs of dummy agents with ℓ layers of preferences that preclude x and y from being matched together. However, to make sure that an agent x is not matched to any dummy agent, we have to require that $\ell \geq 4$.

Theorem 5.2. For each fixed number ℓ of layers with $\ell \geq 4$ and for each fixed value α with $2 \leq \alpha \leq \lfloor \ell/2 \rfloor$, INDIVIDUALLY STABLE MARRIAGE is NP-hard, even if on one side, the preference list of each agent is the same in all layers.

Proof. Assume that $\ell \geq 4$ and that $2 \leq \alpha \leq \lfloor \ell/2 \rfloor$. We give a polynomial-time reduction from the NP-hard PERFECT SMTI problem [32, 38].

Let I be an instance of PERFECT SMTI. In I we are given two disjoint sets of agents, U and W, with |U| = |W| = n; each agent $u \in U$ is endowed with a weak order \succeq_u on a subset of W. By the SMTI-stability, an agent u prefers not to be assigned to any agent rather than to be assigned to an agent outside of its preference list.

We assume that ties occur in the preferences of the agents from the side U only, that there is at most one tie per list, and each tie is of length two as this variant remains NP-hard [38, Theorem 2.2].

From I we construct an instance I' of the problem of finding an α -layer individually stable matching in the following way. We copy the sets of agents U and W; further, we introduce a set of $2\ell \cdot n$ dummy agents $P \cup R$ with $|P| = |R| = \ell \cdot n$. We denote the elements in these sets as: $P = \{p_{i,j} \mid 1 \le i \le n \land 1 \le j \le \ell\}$ and $R = \{r_{i,j} \mid 1 \le i \le n \land 1 \le j \le \ell\}$. One side of the bipartite "acceptability graph" will consist of the agents from $U \cup P$, and the other side of the agents from $W \cup R$. We construct the preference orders of the agents as follows:

Agents from U. Consider an agent $u \in U$, and let L_u denote its preference list in I. In I' we construct the preference list of u from L_u as follows. We iterate over L_u starting from the top position. If agent u prefers x over y, then we assume that u also prefers x over y in all layers in I'. If x and y are tied in L_u , then we assume that u prefers x over y in the first $\lceil \ell/2 \rceil$ layers and y over x in the remaining $\lfloor \ell/2 \rfloor$ layers (or the other way around). The remaining parts of the preference orders of u are constructed as follows: first, let us assume that $u = u_i \in U$. Right after all agents from L_u , agent u puts in all layers $r_{i,1}, r_{i,2}, \ldots, r_{i,\ell}$, respectively in the following orders: (1) $r_{i,1} \succ r_{i,2} \succ \ldots \succ r_{i,\ell}$, (2) $r_{i,2} \succ r_{i,3} \succ \ldots \succ r_{i,\ell} \succ r_{i,1}$, and so on, until (ℓ) $r_{i,\ell} \succ r_{i,1} \succ r_{i,2} \succ \ldots \succ r_{i,\ell-1}$. Next, u puts all the remaining agents in an arbitrary order. For example, if $\ell = 5$ and L_u is equal to $w_1 \succ w_4 \succ w_3 \sim w_5$, then the preference lists of u in the five layers will be as follows, where " \cdots " denotes some arbitrary but fixed order of the remaining agents.

```
Layer (1): agent u_i: w_1 \succ w_4 \succ w_3 \succ w_5 \succ r_{i,1} \succ r_{i,2} \succ r_{i,3} \succ r_{i,4} \succ r_{i,5} \succ \cdots,

Layer (2): agent u_i: w_1 \succ w_4 \succ w_3 \succ w_5 \succ r_{i,2} \succ r_{i,3} \succ r_{i,4} \succ r_{i,5} \succ r_{i,1} \succ \cdots,

Layer (3): agent u_i: w_1 \succ w_4 \succ w_3 \succ w_5 \succ r_{i,3} \succ r_{i,4} \succ r_{i,5} \succ r_{i,1} \succ r_{i,2} \succ \cdots,

Layer (4): agent u_i: w_1 \succ w_4 \succ w_5 \succ w_3 \succ r_{i,4} \succ r_{r,5} \succ r_{i,1} \succ r_{i,2} \succ r_{i,3} \succ \cdots,

Layer (5): agent u_i: w_1 \succ w_4 \succ w_5 \succ w_3 \succ r_{r,5} \succ r_{i,1} \succ r_{i,2} \succ r_{i,3} \succ r_{i,4} \succ \cdots.
```

Observe that in the first three layers w_3 is preferred to w_5 and in the remaining layers w_5 is preferred to w_3 .

Agents from W. For each agent w_i from W, we recall that the preferences of w_i in I do not have ties (that was one of the assumptions in the problem we reduce from). Let L_{w_i} denote the preference list of w_i . The preferences of w_i in I' are the same in all layers, where the second " \cdots " denotes some arbitrary but fixed order of the remaining agents.

```
Layers (1)-(\ell): agent w_i: L_{w_i} \succ p_{i,1} \succ p_{i,2} \succ \cdots \succ p_{i,\ell} \succ \cdots.
```

Agents from $P \cup R$. Consider $p_{i,j} \in P$: this agent ranks w_i first, next $r_{i,j}$, and next all the remaining agents in some arbitrary order:

```
Layers (1)-(\ell): agent p_{i,j}: w_i \succ r_{i,j} \succ \cdots.
```

The preferences of an agent from $r_{i,j} \in R$ are constructed analogously:

```
Layers (1)-(\ell): agent r_{i,j}: u_i \succ p_{i,j} \succ \cdots.
```

This completes the description of the construction of instance I'. Obviously, the preferences of each agent from $W \cup R$ are the same in all layers. Now, we will show that there exists a perfect SMTI-stable matching in I if and only if there exists an α -layer individually stable matching in I'.

For the "only if" direction, assume that the instance I has a perfect SMTI-stable matching M. We claim that $M' = M \cup \left\{ \{r_{i,j}, p_{i,j}\} \mid 1 \leq i \leq n \land 1 \leq j \leq \ell \right\}$ is α -layer individually stable for I'. First, M' is a perfect matching for I' as M is a perfect matching for I. Second, observe that no two agents from $R \cup P$ are blocking M' as for each agent a in $R \cup P$ it prefers its partner M'(a) to every other agent in $R \cup P$ in all layers. Third, no agent from $U \cup W$ can form a blocking pair with an agent from $R \cup P$ for the matching M' as each agent a from $U \cup W$ prefers its partner $M'(a) = M(a) \in L_a$ to every other agent in $R \cup P$ in all layers. Now, consider two arbitrary agents u and u with $u \in U$ and $u \in W$ such that $u \in U$ is unmatched. We show that we can always find a set $u \in U$ and $u \in U$ and $u \in U$ in each layer from $u \in U$ of $u \in U$ in each layer from $u \in U$ in each layer from $u \in U$ and $u \in U$ and $u \in U$ in each layer from $u \in U$ in each layer from $u \in U$ in each layer from $u \in U$ and $u \in U$ and $u \in U$ in each layer from $u \in U$ in each layer from $u \in U$ in each layer from $u \in U$ and $u \in U$ in each layer from $u \in U$

Case 1: $u \succ_w M'(w) = M(w)$. By the stability of M we have that $M'(u) = M(u) \succeq_u w$ in instance I. For the case that M'(u) and w tied by u we can identify $\lfloor \ell/2 \rfloor$ layers, either the first $\lfloor \ell/2 \rfloor$ layers or the last $\lfloor \ell/2 \rfloor$ layers where u prefers M'(u) to w. For the case that $M'(u) \succ_u w$ we have that u prefers M'(u) to w in all layers.

Case 2: $M'(w) \succ_w u$. In this case we know that w prefers M'(w) to u in all layers.

For the "if" direction, assume that M' is an α -layer individually stable matching for I'. First, observe that no agent $u_i \in U$ can be matched with an agent that it ranks lower than any agent from $\{r_{i,j} \mid 1 \le j \le \ell\}$ in any layer. Indeed, for such matching each pair $\{u_i, r_{i,j}\}$, with $j \in [\ell]$, would be blocking M' in all layers. Similarly, u_i cannot be matched with $r_{i,j}$ for $j \in [\ell]$ as the pair $\{u_i, r_{i,(j+\ell-2 \mod \ell)+1}\}$ would be blocking M' in exactly $\ell-1$ layers, namely those layers other than the j^{th} layer, which are more than $\ell-\alpha$ layers. For instance, if u_i was matched with $r_{i,1}$, then $\{u_i, r_{i,\ell}\}$ would be blocking the $2^{\text{nd}}, 3^{\text{rd}}, \dots, \ell^{\text{th}}$ layers; if u_i was matched with $r_{i,2}$, then $(u_i, r_{i,1})$ would be blocking the $1^{\text{st}}, 3^{\text{rd}}, 4^{\text{th}}, \dots, \ell^{\text{th}}$ layers, etc. Thus, each agent ufrom U must be matched with someone from L_u , where L_u is the preference list of u in I. Consequently, no agent in W is matched with an agent in P as otherwise, by the pigeonhole principle, some agent in U must be matched with an agent in R which is not possible by our reasoning above. Also, by a similar reasoning we deduce that each agent w from W must be matched with someone from L_w , where L_w is the preference list of w in I. Now, we show that $M = \{\{u, w\} \in M' \mid u, w \in U \cup W\}$ is a perfect SMTI-stable matching for I. Since M' is a perfect matching in I', by the reasoning above, it follows that M is a perfect matching in I. Suppose, for the sake of contradiction, that there is an unmatched pair $\{x,y\} \notin M$ that is SMTI-blocking M. This means that x and y are acceptable to each other, and that $y \succ_x M(x) = M'(x)$ and $x \succ_y M(y) = M'(y)$ in I. By the preference lists of x and y in I' and by the definition of M, it follows that in each layer x prefers y to M'(x) and y prefers x to M'(y)—a contradiction to M' being α -layer individually stable since $\alpha > 2$.

5.3 NP-hardness of α -layer pair stability

We note that in the instance constructed in the proof of Theorem 5.2 every agent from the side $W \cup R$ has the same preference list in all layers. Later on, in Proposition 6.1 we will show that in such a case the concepts

of α -layer individual stability and α -layer pair stability are equivalent, Thus, we obtain the same hardness result for pair stability when $\alpha \leq \lfloor \ell/2 \rfloor$.

Corollary 5.3. For each fixed number ℓ of layers with $\ell \geq 4$ and for each fixed value α with $2 \leq \alpha \leq \lfloor \ell/2 \rfloor$, PAIR STABLE MARRIAGE is NP-hard even if on one side the preference list of each agent is the same in all layers.

For $\alpha > \lceil \ell/2 \rceil$, we use an idea similar to the one used for showing Proposition 5.1.

Proposition 5.4. For each fixed number α with $\lceil \ell/2 \rceil + 1 \leq \alpha \leq \ell$, PAIR STABLE MARRIAGE is NP-hard.

Proof. Assume that $\alpha \geq \lceil \ell/2 \rceil + 1$. To prove the NP-hardness, we slightly adapt the reduction in the proof of Theorem 4.2 which shows that deciding α -layer global stability is NP-hard for $\alpha = \ell = 2$. Let \mathcal{P} be the two-layer instance constructed in the proof of Theorem 4.2. The idea is to copy $\ell/2$ times profile \mathcal{P} and make sure that an α -layer pair stable matching must be stable in the two layers of the original profile.

For the preference lists in the ℓ layers, we do the following; let $k = \lfloor \ell/2 \rfloor$.

- 1. We make k copies of the profile \mathcal{P} .
- 2. If ℓ is odd, then we add an ℓ th layer with the following preference lists:

```
 \begin{split} \text{Layer } (\ell), &\forall i \in [n] \colon & x_i \colon y_i \succ \overline{y}_i \succ \cdots, & y_i \colon \overline{x}_i \succ x_i \succ \cdots, \\ & \overline{x}_i \colon \overline{y}_i \succ y_i \succ \cdots, & \overline{y}_i \colon x_i \succ \overline{x}_i \succ \cdots. \\ & \forall j \in [m] \colon & a_j \colon d_j \succ e_j \succ f_j \succ \cdots, & d_j \colon b_j \succ c_j \succ a_j \succ \cdots, \\ & b_j \colon e_j \succ f_j \succ d_j \succ \cdots, & e_j \colon c_j \succ a_j \succ b_j \succ \cdots, \\ & c_j \colon f_j \succ d_j \succ e_j \succ \cdots, & f_j \colon a_j \succ b_j \succ c_j \succ \cdots. \end{split}
```

This completes the construction, which can clearly be done in polynomial time. We claim that profile \mathcal{P} with two layers has a 2-layer globally stable matching if and only if the new profile with ℓ layers has an α -layer pair stable matching. For the "only if" direction, it is straightforward to see that each 2-layer globally stable matching for \mathcal{P} is ℓ -layer globally stable for the new profile. By Proposition 3.1, M is also ℓ -layer pair stable for the new instance.

For the "if" direction, assume that M is an α -layer pair stable matching for the new instance with ℓ layers. We claim that M is 2-layer globally stable for instance \mathcal{P} . First, for each unmatched pair $\{u,w\}\notin M$, let $S^{\mathrm{unblock}}(\{u,v\})$ be the set that consists of all layers that are *not* blocked by $\{u,w\}$: $S^{\mathrm{unblock}}(\{u,v\}) := \{i \in [\ell] \mid M(u) \succ_u^{(i)} w \text{ or } M(w) \succ_w^{(i)} u\}$. Since M is α -layer pair stable it follows that $|S^{\mathrm{unblock}}(\{u,v\})| \geq \alpha \geq \lceil \ell/2 \rceil + 1$; the last inequality holds by our assumption on the value of α . We claim that $S^{\mathrm{unblock}}(\{u,v\})$ contains at least k+1 layers from the first $2 \cdot k$ layers. If ℓ is odd, then $\lceil \ell/2 \rceil + 1 = k+2$, and the claim follows; otherwise $\ell = 2k$ and $\lceil \ell/2 \rceil + 1 = k+1$, implying our claim. Since the first 2k layers are k copies of the two layers from \mathcal{P} , we can assume without loss of generality that $S^{\mathrm{unblock}}(\{u,v\})$ contains at least the first k+1 layers. Now, it is obvious that $\bigcap_{\{u,v\}\notin M} S^{\mathrm{unblock}}(\{u,v\})$ contains at least the first two layers. This implies that no unmatched pair is blocking the first two layers, and hence that M is 2-layer globally stable for the instance \mathcal{P} .

Corollary 5.3 and Proposition 5.4 cover the whole range of the potential values of α except for $\alpha = \lfloor \ell/2 \rfloor + 1$ when ℓ is odd. As we will see in the next section Theorem 5.2 cannot be strengthened to cover the value $\alpha = \lfloor \ell/2 \rfloor + 1$ (see Proposition 6.3), and so, also Corollary 5.3 cannot be directly strengthened. However, we can tweak the construction from Theorem 5.2, breaking the restriction that on one side the preference list of each agent is the same in all layers, and obtain hardness for α -layer pair stability for $\alpha = \lfloor \ell/2 \rfloor + 1$.

Proposition 5.5. For each fixed odd number ℓ of layers with $\ell \geq 5$ and for the case when $\alpha = \lfloor \ell/2 \rfloor + 1$, PAIR STABLE MARRIAGE is NP-hard.

Proof. We provide a reduction from an instance I of the NP-hard PERFECT SMTI problem which is very similar to the reduction given in the proof of Theorem 5.2. Consequently, instead of describing the new reduction from scratch, we will only explain how it differs from the one from the proof of Theorem 5.2. For each agent $w_i \in W$ in the proof of Theorem 5.2, the preference list of w_i was the same in all layers. Now, in the last $\ell-1$ layers this list is constructed in exactly the same way as in the proof of Theorem 5.2. However, in the first layer we set the first part of the list to be reversed in comparison to the remaining $(\ell-1)$ layers; let L_{w_i} be the reverse of the strict preference list of w in the input instance of PERFECT SMTI and let the second " \cdots " denote some arbitrary but fixed order of the remaining agents:

Layer (1): agent
$$w_i$$
: $\overleftarrow{L_{w_i}} \succ p_{i,1} \succ p_{i,2} \succ \cdots \succ p_{i,\ell} \succ \cdots$.

The preference lists of all other agents are constructed in exactly the same way as in the proof of Theorem 5.2. We prove that I admits a perfect SMTI-stable matching if and only if the constructed instance I' with ℓ layers admits an α -layer pair stable matching with $\alpha = |\ell/2| + 1$.

For the "only if" direction, assume that M is a perfect SMTI-stable matching for I and consider matching $M' = M \cup \big\{ \{r_{i,j}, p_{i,j}\} \mid 1 \leq i \leq n \land 1 \leq j \leq \ell \big\}$. We will show that M' is α -layer pair stable for I'. First of all, just as in the proof of Theorem 5.2 we deduce that no agent from $P \cup R$ and no agent from $U \cup W$ will form a blocking pair. Neither will any two agents that are not acceptable to each other in I form a blocking pair. Now, consider two arbitrary agents u and u with $u \in U$ and $u \in W$ that are acceptable to each other in I. We need to find a subset of $\lfloor \ell/2 \rfloor + 1$ layers, where in each layer in the subset u prefers M'(u) to u or u prefers M'(u) to u. We distinguish between two cases concerning the preference list of u in u; note that it does not have ties:

Case 1: $u \succ_w M'(w) = M(w)$ in I. This implies that w prefers M'(w) to u in the first layer in I'. Thus, it suffices if we can find $\lfloor \ell/2 \rfloor$ layers in the last $\ell-1$ layers, where u prefers M'(u) to w. By the stability of M we have that $M'(u) = M(u) \succeq_u w$ in instance I. For the case that $M'(u) \succ_u w$ in I we have that u prefers M'(u) to w in all layers. For the case that M'(u) and w are tied by u we can identify $\lfloor \ell/2 \rfloor$ layers, either the layers from 2 to $\lfloor \ell/2 \rfloor + 1$ or the layers from $\lfloor \ell/2 \rfloor + 2$ to ℓ , where u prefers M'(u) to w. Additionally, in the first layer w prefers M(w) to u, which gives in total the subset of $\lfloor \ell/2 \rfloor + 1$ layers.

Case 2: $M'(w) \succ_w u$. In this case we know that w prefers M'(w) to u in the last $\ell - 1$ layers.

For the "if" direction, assume that M' is an α -layer pair stable matching for I'. By the same reasoning as in the proof of Theorem 5.2 we deduce that each agent $u \in U$ must be matched with someone that it prefers to each agent from R in each layer, and that $w \in W$ must be matched with an agent that it prefers to each agent from P in all layers. Thus, u and M'(u) (resp. w and M'(w)) must be acceptable to each other in I, meaning that $M = \{\{u, w\} \in M' \mid u \in U, w \in W\}$ is a perfect matching for I. We show that M is SMTI-stable. For the sake of contradiction, suppose that an unmatched pair $\{u, w\}$ is SMTI-blocking M. This means that w prefers u to M(w) = M'(w) in the last $\ell - 1$ layers in I', and that u prefers w to M(u) = M'(u) in all ℓ layers, a contradiction to M' being α -layer pair stable, since $\alpha \geq 2$. \square

6 Two Special Cases of Preferences

In this section we consider two well-motivated special cases of our general multi-layer framework for stable matchings. We will discuss how the corresponding simplifying assumptions affect the computational complexity of finding multi-layer stable matchings. Interestingly, even under seemingly strong assumptions some variants of our problem remain computationally hard.

6.1 Single-layered preferences on one side

Consider the special case where the preferences of the agents from U can be expressed through a single layer. Formally, we model this by assuming that for each agent from U, its preference list is the same in all layers. In this case we say that the agents from U have single-layered preferences.

Single-layered preferences on one side can arise in many real-life scenarios. For instance, consider the standard example of matching residents with hospitals and, similarly as in Example 2.4, assume that each layer corresponds to a certain criterion. It is reasonable to assume that the hospitals evaluate their potential employees with respect to the level of their qualifications only (thus, having a single layer of preferences), while the residents take into account a number of factors such as how far is a given hospital from the place they live, the level of compensation, the reputation of the hospital, etc.

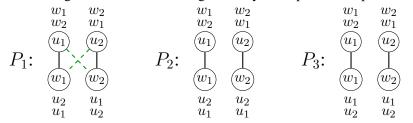
First, we observe that in such a case two out of our three solution concepts from Section 2 are equivalent.

Proposition 6.1. If each agent from U has single-layered preferences, then α -layer pair stability and α -layer individual stability are equivalent for each α .

Proof. The fact that α -layer individual stability implies α -layer pair stability follows from Proposition 3.2. Let us prove the other direction. Let M be an α -layer pair stable matching. Towards a contradiction, suppose that M is not α -layer individually stable. By Proposition 2.2, let $p = \{u, w\}$ be an unmatched pair of agents such that there is a subset S_1 of $\ell - \alpha + 1$ layers where p is dominating $\{u, M(u)\}$, and that there is a subset S_2 of $(\ell - \alpha + 1)$ layers where p is dominating $\{w, M(w)\}$. By our assumption of single-layered preferences for each agent in U, we have that p is dominating $\{u, M(u)\}$ in every layer. Thus, the pair p is blocking M in each layer in S_2 —a contradiction to Proposition 2.1.

For profiles with single-layered preferences of the agents on one side, α -layer global stability is strictly stronger than the other two concepts:

Example 6.1. Consider four agents with the following three layers of preference profiles:



Let $M = \{\{u_1, w_2\}, \{u_2, w_1\}\}$. We show that M is 2-layer pair stable but not 2-layer globally stable. Indeed, M is stable only in the first layer, so it cannot be 2-layer globally stable. To see why M is 2-layer pair stable, consider the unmatched pairs $\{u_1, w_1\}$ and $\{u_2, w_2\}$. The first one only blocks the third layer, and the latter one only blocks the second layer.

6.1.1 Global stability

By Propositions 3.4 and 6.1, we can find out whether an instance with single-layered preferences on one side admits an α -layer globally stable matching in time $O(\ell^{\alpha} \cdot \alpha \cdot n^2)$ by guessing a subset of α layers and using Algorithm 1. However, the following result shows that fixed-parameter tractability (FPT) for the parameter α , *i.e.*, the existence of an algorithm running in $f(\alpha) \cdot (\ell \cdot n)^{O(1)}$ time for some computable function f, is unlikely (for details on parameterized complexity we refer to the books of Cygan et al. [17], Downey and Fellows [19], Flum and Grohe [24], and Niedermeier [42]).

Theorem 6.2. Even if the preferences of the agents from U are single-layered Globally Stable Marriage is NP-hard and is W[1]-hard for the threshold parameter α . It can be solved in $O(\ell^{\alpha} \cdot \alpha \cdot n^2)$ time.

Proof. For the running time statement, let I be an instance of GLOBALLY STABLE MARRIAGE with ℓ layers. We guess a subset $S \subseteq [\ell]$ of α layers and check whether a matching M exists that is stable in all layers of S. Now consider the instance I' restricted to the α layers in S. By Proposition 3.4, M is α -layer globally stable in I' if and only if M is α -layer pair stable in I'. Since each agent on one side has the same preference list in all layers, by Proposition 6.1, M is α -layer pair stable in I' if and only if M is α -layer individually stable in I'. By Theorem 4.1, we can use Algorithm 1 to decide whether M is α -layer individually stable in I' in $O(\alpha \cdot n^2)$ time. The total running time is thus $O(\ell^{\alpha} \cdot \alpha \cdot n^2)$.

For the last statement, we provide a parameterized reduction¹ from the W[1]-complete INDEPENDENT SET problem parameterized by the size k of the solution. We will see that the reduction is also a polynomial-time reduction, showing the first statement since INDEPENDENT SET is also NP-hard.

In the INDEPENDENT SET problem we are given an undirected graph G = (V, E) with vertex set V and edge set E, and a non-negative integer k, and we ask whether G admits a k-vertex independent set, i.e. a vertex subset $V' \subseteq V$ with k pairwisely non-adjacent vertices.

Given an INDEPENDENT SET instance (G=(V,E),k), we construct an instance with |V| layers with the set of agents $U \uplus W$ as follows. Assume that $V=\{v_1,v_2,\ldots,v_n\}$. For each vertex $v_i \in V$ we construct six agents, three for each side, denoted by $u_i,\overline{u}_i,w_i,\overline{w}_i,a_i$ and b_i . Let $U=\{u_i,\overline{u}_i,a_i\mid 1\leq i\leq n\}$ and $W=\{w_i,\overline{w}_i,b_i\mid 1\leq i\leq n\}$. We create a layer for each vertex so that if a matching is stable for a layer, then an independent set solution has to include the corresponding vertex and exclude any of its adjacent vertices. Again, the notation " \cdots " denotes an arbitrary order of the unmentioned agents.

Agents from U. The preference list of each agent in U is the same for all layers.

```
\forall i \in [n]: agent u_i \colon \overline{w}_i \succ w_i \succ \cdots, agent \overline{u}_i \colon w_i \succ \overline{w}_i \succ \cdots, agent a_i \colon w_i \succ b_i \succ \cdots.
```

Agents from W. For each layer i, $1 \le i \le n$, the preference list of agents w_i and \overline{w}_i and of agents w_j and \overline{w}_j for $v_j \ne v_i$ are constructed in such a way that they encode the adjacency structure of graph G.

```
\begin{aligned} \operatorname{agent} \, w_i : u_i \succ a_i \succ \cdots \,, \\ \operatorname{agent} \, \overline{w}_i : \overline{u}_i \succ \cdots \,, \\ \forall j \text{ with } \{v_i, v_j\} \in E \colon \operatorname{agent} \, w_j : \overline{u}_j \succ \cdots \,, \\ \operatorname{agent} \, \overline{w}_j : u_j \succ \cdots \,, \\ \forall k \text{ with } \{v_i, v_k\} \notin E \colon \operatorname{agent} \, w_k \colon u_k \succ \overline{u}_k \succ \cdots \,, \\ \operatorname{agent} \, \overline{w}_k \colon \overline{u}_k \succ u_k \succ \cdots \,. \end{aligned}
```

The preference list of each b_j with $j \in [n]$ remains the same in all layers and is as follows.

agent
$$b_i : a_i \succ \cdots$$
.

Observe that for each layer $i \in [n]$, w_i is the most preferred agent of a_i . Thus, a matching M is stable in layer i only if $M(w_i) \in \{u_i, a_i\}$. From this we infer that for M to be stable in layer i it must also hold that $M(\overline{u}_i) = \overline{w}_i$. This implies that $M(w_i) = u_i$ as otherwise $\{u_i, w_i\}$ would be blocking layer i. Further, if M is stable in layer i, then for each vertex v_j that is adjacent to v_i , it must hold that $M(w_j) = \overline{u}_j$ and $M(\overline{w}_j) = \overline{u}_j$

¹A parameterized reduction from a parameterized problem Π_1 to a parameterized problem Π_2 is an algorithm mapping an instance (x,k) to an instance (x',k') in $f(k) \cdot |x|^{O(1)}$ time such that $k' \leq g(k)$, and that $(x,k) \in \Pi_1$ if and only if $(x',k') \in \Pi_2$, where f and g are some computable functions.

 u_j . In the following, for each $i \in [n]$, let $M_i^{\text{vc}} = \{\{u_i, w_i\}, \{\overline{u}_i, \overline{w}_i\}\}$ and $M_i^{\text{ind}} = \{\{u_i, \overline{w}_i\}, \{\overline{u}_i, w_i\}\}$. Then, we have the following observation for the case that a matching M is stable in layer i:

$$M_i^{\text{vc}} \subseteq M$$
 and for each vertex v_j incident with $\{v_i, v_j\} \in E$ it holds that $M_j^{\text{ind}} \subseteq M$. (1)

Intuitively, the matching M_i^{ind} means that vertex v_j does not belong to an independent set while the matching M_i^{vc} means that vertex v_i belongs to an independent set.

To complete the construction we let $\alpha = k$. Clearly, the construction can be done in polynomial time. Now, we claim that the given graph G has an independent set of size k if and only if the constructed instance has an α -layer globally stable matching.

For the "only if" part, assume that $V'\subseteq V$ is a size-k independent set for G. We show that the matching $M=\bigcup_{v_i\in V'}M_i^{\mathrm{vc}}\cup\bigcup_{v_j\in V\setminus V'}M_j^{\mathrm{ind}}\cup\{\{a_i,b_i\}\mid i\in[n]\}$ is stable in each layer i with $v_i\in V'$. Suppose, for the sake of contradiction, that there is a layer i with $v_i\in V'$ and there is an unmatched pair $p=\{x,y\}$ with $x\in U$ and $y\in W$ that is blocking layer i, i.e. the following holds.

In layer i, agent x prefers y to M(x) and agent y prefers x to M(y).

First, $y \notin \{b_j \mid j \in [n]\}$ since $M(b_j) = a_j$ is the most preferred agent of b_j . Second, $y \notin \{w_j, \overline{w}_j \mid \{v_i, v_j\} \in E\}$ as both w_j and \overline{w}_j already obtain their most preferred agents, namely \overline{u}_j and u_j in layer i. Third, $y \notin \{w_j, \overline{w}_j \mid v_j \in V'\}$, as in this way w_j and \overline{w}_j obtain u_j and \overline{u}_j as their partners, and since V' is an independent set, we have that $\{v_i, v_j\} \notin E$, and so u_j and \overline{u}_j are the most preferred partners of w_j and \overline{w}_j .

Now, the only possible choice for y would be agent w_j or \overline{w}_j such that $\{v_i,v_j\}\notin E$ and $v_j\notin V'$. By our construction of M we have that $M(w_j)=\overline{u_j}$ and $M(\overline{w}_j)=u_j$. First, consider the case when $y=w_j$. By the preference list of w_j in layer i, if $\{x,y\}$ is a blocking pair, then $x=u_j$. However, u_j already obtains its most preferred agent, namely, \overline{w}_j ; a contradiction. Analogously, the case when $y=\overline{w}_j$ also results in a contradiction. Summarizing, matching M is stable in each layer i with $v_i\in V'$.

For the "if" part, assume that our constructed instance admits an α -layer globally stable matching M. We show that the vertex subset $V' := \{v_i \in V \mid M \text{ is stable in layer } i\}$ is an independent set of size k. Clearly, $|V'| = \alpha = k$. Suppose, for the sake of contradiction, that V' contains two adjacent vertices v_i and v_j with $\{v_i, v_j\} \in E$. Since M is stable in layer i and $\{v_i, v_j\} \in E$, by our observation (1) we deduce that $M_i^{\text{vc}} \subseteq M$ and $M_i^{\text{ind}} \subseteq M$. However, this is a contradiction to M being stable in layer j.

6.1.2 Individual stability and pair stability

Observe that, by Theorem 5.2, we know that finding an α -layer individually stable matching with preferences single-layered on one side is NP-hard if $\ell \geq 4$ and $\alpha \leq \lfloor \ell/2 \rfloor$.

Next, we establish a relation between the individual (and thus pair) stability for preferences single-layered on one side, and the stability concept in the traditional single-layer setting, but with general (incomplete and possibly intransitive) preferences as studied by Farczadi et al. [23]. This relation allows us to construct a polynomial-time algorithm for finding an α -layer individually stable (and hence α -layer pair stable) matching with single-layer preferences on one side when $\alpha \ge \lfloor \ell/2 \rfloor + 1$.

Proposition 6.3. If the preferences of the agents on one side are single-layered and $\alpha \ge \lfloor \ell/2 \rfloor + 1$, then PAIR STABLE MARRIAGE and INDIVIDUALLY STABLE MARRIAGE can be solved in $O(\ell \cdot n^2)$ time.

Proof. We will reduce our problem to the STABLE MARRIAGE WITH GENERAL PREFERENCES (SMG) problem [23]. Let I be an instance of our problem, let $U \cup W$ denote the set of agents in I, and let ℓ be the number of layers. We construct an instance I' of SMG as follows. In I' we have the same set of agents as in I. For each agent $u \in U$ we copy the preferences from I to I' (such an agent has the same preference

list in all layers). The preferences of the agents from W can be arbitrary binary relations on the set V. Thus, we use an ordered pair (u,u') of two agents to denote that u is regarded as good as u'. Formally, for each agent $w \in W$ we define the general preferences of w, denoted as R_w , as follows: For each two agents $u,u' \in U$ we let $(u,u') \in R_w$ if and only if w prefers u to u' in at least α layers. First, by our assumption on the value of α , we observe that the preferences of the agents from W are asymmetric as, i.e., for each pair of agents $u,u' \in U$ we have that $|\{(u,u'),(u',u)\} \cap R_w| \leq 1$.

Now, we will prove that a matching M is α -layer pair stable in I if and only if it is stable in I' according to the definition of stability by Farczadi et al. [23]. Indeed, according to Farczadi et al. [23] a pair $\{u,w\}$ is $\mathit{SMG-blocking}$ if and only if the following two conditions hold: (1) $w \succ_u M(u)$ and (2) $(M(w),u) \notin R_w$. The first condition is equivalent to saying that u prefers w to M(u) in each layer in I. The second condition holds if and only if w prefers M(w) to u in less than α layers. This is equivalent to saying that w prefers w to w to w in at least w and so the two conditions are equivalent to saying that w is w in w in w in w in w in w.

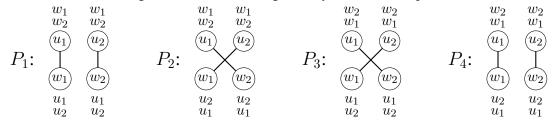
For asymmetric preferences SMG can be solved in $O(n^2)$ time [23, Theorem 2], where the number of agents is 2n. The reduction to an SMG instance takes $O(\ell \cdot n^2)$ time. The number of agents in I equals the number of agents in I'. Thus, the problem (deciding α -layer pair stability) can be solved in $O(\ell \cdot n^2)$ time. By Proposition 6.1, we obtain the same result for α -layer individual stability.

6.2 Uniform preferences in each layer

In this section we consider the case when for each layer the preferences of all agents from U (resp. all agents from W) are the same—we call such preferences uniform in each layer. This special case is motivated with the following observation pertaining to Example 2.4: preferences uniform in a layer can arise if the criterion corresponding to the layer is not subjective. For instance, if a layer corresponds to the preferences regarding the wealth of potential partners, it is natural to assume that everyone prefers to be matched with a wealthier partner (and so the preferences of all agents for this criterion are the same); similarly it is natural to assume that the preferences of all hospitals are the same: candidates with higher grades will be preferred by each hospital.

First, we observe that for uniform preferences no two among the three concepts are equivalent.

Example 6.2. Consider four agents with the following four layers of uniform preferences:



Let $M_1 = \{\{u_1, w_1\}, \{u_2, w_2\}\}$ and $M_2 = \{\{u_1, w_2\}, \{u_2, w_1\}\}$. We observe that M_1 is stable only in layers (1) and (4) whereas M_2 is stable only in layers (2) and (3). Thus, neither is 3-layer globally stable. One can check that neither is 3-layer individually stable. However, both M_1 and M_2 are 3-layer pair stable. To see why this is the case we observe that for M_1 , both unmatched pairs $\{u_1, w_2\}$ and $\{u_2, w_1\}$ are each blocking only one layer, namely layers (2) and (3), respectively. Moreover, if we restrict the instance to only the first three layers, then we have the following results:

- 1. M_1 is not even 2-layer globally stable but it is 2-layer individually stable as for each unmatched pair p with respect to M_1 , at least one agent in p obtains a partner which is its most preferred agent in at least two layers.
- 2. M_2 is 2-layer globally stable (it is stable in layers 2 and 3) but it is not 2-layer individually stable. To see why it is not 2-layer individually stable, we consider the unmatched pair $\{u_1, w_1\}$. There are

3-2+1=2 layers where u_1 prefers w_1 to its partner $M_2(u_1)=w_2$ and there are two layers where w_1 prefers u_1 to its partner $M_2(w_1)$.

 \Diamond

6.2.1 Individual stability

For preferences that are uniform in each layer, we find that there is a close relation between INDIVIDUALLY STABLE MARRIAGE and GRAPH ISOMORPHISM, the problem of finding an isomorphism between graphs or deciding that there exists none. Herein, an instance of GRAPH ISOMORPHISM consists of two undirected graphs G and H with the same number of vertices and the same number of edges. We want to decide whether there is an edge-preserving bijection $f\colon V(G)\to V(H)$ between the vertices, i.e. for each two vertices $u,v\in V(G)$ it holds that $\{u,v\}\in E(G)$ if and only if $\{f(u),f(v)\}\in E(H)$. We call such bijection an isomorphism between G and G.

We explore this relation through the following construction. For an instance I of Individually Stable Marriage we construct two directed graphs G_I and H_I as follows. The agents from U and W will correspond to the vertices in G_I and H_I , respectively. For each two vertices $u,u'\in U$ we add to G_I an arc (u,u') if the agents from W prefer u to u' in at least $(\ell-\alpha+1)$ layers. Analogously, for each two agents w and w', we add to H_I an arc (w,w') if the agents from U prefer w to w' in at least $(\ell-\alpha+1)$ layers. Let $E(G_I)$ and $E(H_I)$ denote the arc sets of G_I and H_I , respectively.

We first explain how the so constructed graphs can be used to find an α -layer individually stable matching in the initial instance I, or to claim there is no such a matching.

Proposition 6.4. A matching M for instance I is α -layer individually stable if and only if the following two properties hold.

- 1. For each two vertices $u, u' \in U$, it holds that: $(u, u') \in E(G_I)$ implies $(M(u'), M(u)) \notin E(H_I)$.
- 2. For each two vertices $w, w' \in W$, it holds that: $(w, w') \in E(H_I)$ implies $(M(w'), M(w)) \notin E(G_I)$.

Proof. For the "only if" direction, assume that M is α -layer individually stable. Towards a contradiction, suppose that one of both properties stated above does not hold. That is, there exist two vertices $u, u' \in U$ such that $(u, u') \in E(G_I)$ and $(M(u'), M(u)) \in E(H_I)$ or there exist two vertices $w, w' \in W$ such that $(w, w') \in E(H_I)$ and $(M(w'), M(w)) \in E(G_I)$. For the first case, it follows that there is a subset S of at least $\ell - \alpha + 1$ layers such that all agents (including M(u')) in W prefer u to u' in each layer of S, and there is a (possibly different) subset R of at least $\ell - \alpha + 1$ layers such that all agents (including u) in u0 prefer u1 to u2 in each layer of u3. This means that the pair u4 is u5 is u6 in each layer of u7. This means that the pair u8 is u9 is u9. Analogously, the second case also leads to a contradiction to Proposition 2.2 regarding the unmatched pair u8 is u9.

For the "if" direction, assume that both properties hold. Towards a contradiction and by Proposition 2.2, suppose that there is an unmatched pair, call it $\{u,w\}$ with $u\in U$ and $w\in W$, which is $(\ell-\alpha+1)$ -dominating $\{u,M(u)\}$ and is $(\ell-\alpha+1)$ -dominating $\{w,M(w)\}$. By our construction of G_I and H_I , it follows that $(w,M(u))\in E(H_I)$ and $(u,M(w))\in E(G_I)$ —a contradiction to the second property. \square

Using a construction by McGarvey [40], given an arbitrary directed graph, we can indeed construct multi-layer preferences that induce this graph.

Lemma 6.1. For each two directed graphs G_I and H_I with m arcs each there exists an instance with $\ell = 2m$ layers of preferences that induces G_I and H_I via McGarvey's construction, where m denotes the number of arcs in G_I (and thus in H_I).

Proof. Our profile uses the vertex sets of G_I and H_I as the two disjoint subsets of agents, denoted as U and W, respectively. For each arc in the graphs G_I (resp. H_I) we construct two layers of preference lists

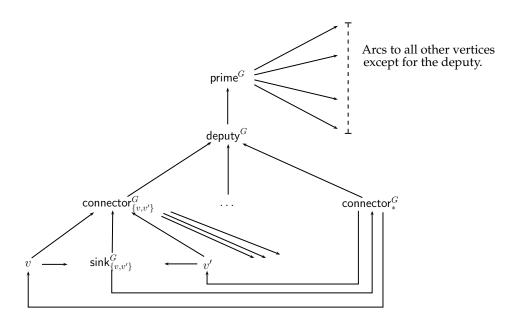


Figure 2: Construction of the graphs G_I and H_I in the proof of Theorem 6.5. The diagram shows the case $\{v, v'\} \in G$.

for the agents from W (resp. U) to "encode" it. We describe how to construct the preference lists for an arc in G_I , say (u, u'). The construction for the arcs in H_I works analogously. To encode the arc (u, u'), we construct two layers where the preference lists for the agents in W are as follows; denote by \overrightarrow{X} some arbitrary but fixed order of the agents other than u and u' and let \overleftarrow{X} be the reverse of \overrightarrow{X} .

One layer:
$$u \succ u' \succ \overrightarrow{X}$$
.
The other layer: $\overleftarrow{X} \succ u \succ u'$.

Finally, we set $\alpha = \ell/2 = m$. Note that for each two agents u and u' in G_I all agents from W prefer u to u' in either exactly m-2 layers, or exactly m layers, or exactly m+2 layers. It is straightforward to see that our constructed instance induces the two input graphs G_I and H_I .

Using Proposition 6.4 and Lemma 6.1 as tools, we can show that for the preferences of the agents being uniform in all layers, the problem of finding an α -layer individually stable matching is at least as hard as the graph isomorphism problem.

Theorem 6.5. Graph Isomorphism is polynomial-time reducible to Individually Stable Marriage, where the preferences of the agents are uniform in all ℓ layers, ℓ is even, and $\alpha = \ell/2$.

Proof. Let G, H be two undirected graphs that form an instance of GRAPH ISOMORPHISM. Without loss of generality, assume that $n \geq 4$ (number of vertices) and $m \geq 3$ (number of edges). From G and H we construct an instance I of the problem of deciding whether there exists an α -layer individually stable matching for preferences being uniform in all ℓ layers, where $\alpha = \ell/2$. By Lemma 6.1 we can describe this instance by providing the corresponding directed graphs G_I and H_I . We show how to construct G_I from G. The graph G_I is constructed from G_I analogously.

Let V and E denote the sets of vertices and edges in G. We copy all vertices from G to G_I (we will refer to these vertices as *non-special*) and we additionally introduce five types of "special" vertices, which we call

prime^G, deputy^G, $\binom{n}{2} + 1$ connectors, m sinks, and $\binom{n}{2} - m$ sources. We specify the connectors, the sinks, and the sources formally: With each pair of vertices $p = \{v, v'\} \in \binom{V}{2}$ we associate one connector, denoted as connector $_{n}^{G}$. One remaining connector is distinguished and denoted as connector $_{n}^{G}$. Further, for each edge $e \in E$ we have one sink, denoted as sink_e^G , and for each non-edge $e = \{v, v'\} \notin E$ we have a source, denoted as $\operatorname{source}_e^G$. The arcs in G_I are constructed as follows. Let $\operatorname{SOURCE}^G = \{\operatorname{source}_e \mid e \in \binom{V}{2} \setminus E\}$, $\mathsf{SINK}^G = \{\mathsf{sink}_e \mid e \in E\}, \text{ and } \mathsf{CONNECTOR}^G = \{\mathsf{connector}_e^G \mid e \in \binom{V}{2}\} \}$ denote the set of all sources, the set of all sinks, and the set of non-special connectors, respectively:

- For each pair $e = \{v, v'\} \subseteq V$ of vertices, we do the following:
 - 1. If $\{v, v'\} \in E$, then we add to G_I the following five arcs: (v, sink_e^G) , (v', sink_e^G) , $(v, \mathsf{connector}_e^G)$, $(v', \mathsf{connector}_e^G)$, and $(\mathsf{sink}_e^G, \mathsf{connector}_e^G)$. Otherwise, we add to G_I the following five arcs: (source_e^G, v) , $(\mathsf{source}_e^G, v')$, $(v, \mathsf{connector}_e^G)$, $(v', \mathsf{connector}_e^G)$, and $(\mathsf{source}_e^G, \mathsf{connector}_e^G)$.

 - 2. For each other non-special vertex $u \in V \setminus \{v, v'\}$, we add to G_I the arc (connector e^G , u). 3. For each other source or sink $x \in \mathsf{SOURCE}^G \cup \mathsf{SINK}^G \setminus \{\mathsf{sink}_e^G, \mathsf{source}_e^G\}$, we add to G_I the arc (connector $_e^G, x$).
- For each source and each sink $x \in \mathsf{SOURCE}^G \cup \mathsf{SINK}^G$ we add to G_I an arc $(x, \mathsf{connector}_*^G)$. For each non-special vertex $v \in V$, we additionally add to G the arc (connector $_*^G, v$).
- For each vertex $x \in V \cup \mathsf{SOURCE}^G \cup \mathsf{SINK}^G \cup \mathsf{CONNECTOR}^G \cup \{\mathsf{connector}^G_*\}$ except for the deputy, we add to G_I an arc (prime G , x).
- For each vertex $x \in \mathsf{SOURCE}^G \cup \mathsf{SINK}^G \cup \{\mathsf{prime}^G\}$ we add to G_I the arc (deputy $^G, x$). For each connector $y \in \mathsf{CONNECTOR}^G \cup \{\mathsf{connector}_*^G\}$ we add to G_I the arc (y, deputy^G) .

This completes the construction of G_I (crucial elements of this construction are illustrated in Figure 2). Observe a useful property: Each vertex in G_I except prime has at least two incoming arcs; prime has exactly one incoming arc, namely from deputy G. Indeed, since G_I has at least three edges (and so at least three connectors), deputy G has at least three incoming arcs from the connectors. Each connector connector $G_{\{v,v'\}}$ has incoming arcs from v and v'; The special connector connector G has arcs from all sources and sinks (there are at least six of them). For each sink, source, and each non-special vertex there are at least two incoming arcs, one from the prime and the other from the deputy.

The arcs in H_I are constructed analogously. We claim that there is an α -layer individually stable matching in the instance (G_I, H_I) with $\alpha = \ell/2$ if and only if there is an isomorphism between G and H.

- (\Leftarrow) For the "if" direction, assume that there is an isomorphism $f:V(G)\to V(H)$ between G to H. We show that the following matching M is m-layer individually stable. For each vertex $v \in V(G)$ let M(v) = $f(v) \in V(H). \text{ Further, for each source or sink } s^G_{\{v,v'\}} \in \mathsf{SOURCE}^G \cup \mathsf{SINK}^G \text{ we let } M\left(s^G_{\{v,v'\}}\right) = s^H_{\{f(v),f(v')\}}. \text{ Why should such a matching exist? If } s^G_{\{v,v'\}} \text{ is a sink, then } \{v,v'\} \in E(G); \text{ since } f \text{ is a sink, then } \{v,v'\} \in E(G)$ an isomorphism, we have that $\{f(v), f(v')\} \in E(H)$ and so the sink $s_{\{f(v), f(v')\}}^H$ exists. Analogously, if $s_{\{v,v'\}}^G$ is a source, then $s_{\{v,v'\}}^H$ is also a source and exists in H. Similarly, we match the corresponding connectors: for each pair of vertices $\{v,v'\}$ we let $M\left(\mathsf{connector}_{\{v,v'\}}^G\right) = \mathsf{connector}_{\{f(v),f(v'))\}}^H$, and we $\mathrm{let}\ M(\mathsf{connector}_*^G) = \mathsf{connector}_*^H. \ \mathrm{Finally,}\ \mathrm{we}\ \mathrm{let}\ M(\mathsf{deputy}^G) = \mathsf{deputy}^H \ \mathrm{and}\ M(\mathsf{prime}^G) = \mathsf{prime}^H.$ Since f is an isomorphism between G and H, by the construction of G_I and H_I , an arc (x,y) in G_I (resp. H_I) implies no arc (M(x), M(y)) in H_I (resp. G_I). By Proposition 6.4, M is m-layer individually stable.
- (\Rightarrow) For the "only if" direction, assume that M is an m-layer individually stable matching for (G_I, H_I) . We start by showing $M(\mathsf{prime}^G) = \mathsf{prime}^H$. For the sake of contradiction, suppose that this is not the case, and let $y = M(\mathsf{prime}^G) \in V(H_I)$, with $y \neq \mathsf{prime}^H$. Since $y \neq \mathsf{prime}^H$, there are at least two vertices $w_1, w_2 \in V(H_I)$ with $(w_1, y), (w_2, y) \in E(H_I)$. Thus, one of them is not matched to deputy^G, say $M(w_1) \neq \text{deputy}^G$. Since $(w_1, y) \in E(H_I)$, by Proposition 6.4, it must hold that $(\text{prime}^G, M(w_1)) =$

 $(M(y), M(w_1)) \notin E(G_I)$. This is a contradiction with $M(w_1) \neq \mathsf{deputy}^G$ since prime^G has outgoing arcs to all vertices but deputy^G .

Second, we show that $M(\operatorname{deputy}^G) = \operatorname{deputy}^H$. The reason for this is that using Proposition 6.4 for the arc $(\operatorname{deputy}^G, \operatorname{prime}^G)$ implies that $(M(\operatorname{prime}^G), M(\operatorname{deputy}^G)) = (\operatorname{prime}^H, M(\operatorname{deputy}^G))$ cannot exists in H_I . By construction, prime^H has an arc to all vertices except deputy^H . Thus, $M(\operatorname{deputy}^G) = \operatorname{deputy}^H$. By an analogous reasoning focusing on the deputy vertex, we deduce that each connector in G_I must be matched with some connector in H_I . Further, for each source or $\operatorname{sink} s^G \in \operatorname{SOURCE}^G \cup \operatorname{SINK}^G$ since $(s^G, \operatorname{connector}^G_*) \in E(G_I)$, by Proposition 6.4, it follows that $(M(\operatorname{connector}^G_*), M(s^G))$ does not exist in H_I ; thus there are at least 6 forbidden arcs from $M(\operatorname{connector}^G_*)$ to some vertices which are sinks, sources, or non-special vertices. This means that $\operatorname{connector}^G_*$ can only be matched to $\operatorname{connector}^H_*$ because each other connector in H_I has all but three outgoing arcs to such vertices. Thus, we deduce that each source/sink must be matched with a source/sink only, and consequently, that non-special vertices in G_I are matched with non-special vertices in H_I . We show that the bijection $f: V(G_I) \to V(H_I)$ derived from M by setting f(v) = M(v) for each $v \in V$ is an isomorphism between G and H.

Consider an arbitrary pair $e = \{v,v'\} \subseteq V(G)$ of non-special vertices and its corresponding connector connector e^G . We know that $M(\operatorname{connector}_e^G) \in \operatorname{CONNECTOR}^H$, say $M(\operatorname{connector}_e^G) = \operatorname{connector}_{\{w,w'\}}^H$. We claim that $\{w,w'\} = \{M(v),M(v')\}$. Towards a contradiction, suppose that $M(v) \notin \{w,w'\}$. Then, $\{v,\operatorname{connector}_e^G\} \in E(G_I)$ and $(\operatorname{connector}_{\{w,w'\}}^G,M(v)) \in E(H_I)$, a contradiction to Proposition 6.4. Finally, we show that $e \in E(G)$ if and only if $\{f(v),f(v')\} = \{M(v),M(v')\} = \{w,w'\} \in E(H)$. If $e \in E(G)$, then sink_e^G exists and $(\operatorname{sink}_e^G,\operatorname{connector}_e^G) \in E(G_I)$. By Proposition 6.4, it follows that $(\operatorname{connector}_{\{w,w'\}}^H,M(\operatorname{sink}_e^G)) = (M(\operatorname{connector}_e^G),M(\operatorname{sink}_e^G)) \notin E(H_I)$. Since $M(\operatorname{sink}_e^G) \in \operatorname{SOURCE}^H \cup \operatorname{SINK}^H$, we have that $M(\operatorname{sink}_e^G) = s_{\{w,w'\}}^H$, where s is a source or a sink. Further, since $(v,\operatorname{sink}_e^G) \in E(G_I)$ we infer that $(s_{\{w,w'\}}^H,w) \notin E(G_I)$, and so s is a sink, and w and w are connected in H. Similarly, if $e \notin E(G)$, then we deduce that $\operatorname{source}_e^G$ and $\operatorname{source}_{\{w,w'\}}^H$ exist, and thus $\{w,w'\} \notin E(H)$. \square

Theorem 6.5 implies that developing a polynomial-time algorithm for our problem is currently out of scope, since the question of whether GRAPH ISOMORPHISM is solvable in polynomial time is still open. Besides Theorem 6.5 there are other interesting implications of Proposition 6.4 and Lemma 6.1. For $\alpha \geq \ell/2 + 1$ our problem can be reduced to the TOURNAMENT ISOMORPHISM problem, which, given two tournament graphs, asks whether there is an arc-preserving bijection between the vertices of the two tournaments [3, 46, 47]. TOURNAMENT ISOMORPHISM has been studied extensively in the literature (for a more detailed discussion see, e.g., [3, 47, 46]), but to the best of our knowledge it is still open whether it is solvable in polynomial time [3]. The best known algorithm solving TOURNAMENT ISOMORPHISM runs in $n^{O(\log n)}$ time, where n denotes the number of vertices.

Corollary 6.6. If the preferences of the agents are uniform in all ℓ layers and $\alpha \ge \ell/2 + 1$, then Individually Stable Marriage can be solved in $n^{O(\log n)} + O(\ell \cdot n^2)$ time, where n denotes the number of agents.

Proof. Assume that $\alpha \geq \ell/2 + 1$ and construct the two directed graphs as we did for Proposition 6.4. Since $\ell - \alpha + 1 \leq \ell/2$, for each pair of vertices x and y in G_I (resp. H_I) at least one arc from (x,y) and (y,x) exists. Note that such graphs are not necessarily tournaments, where for each two vertices x and y exactly one of (x,y) and (y,x) exists. But we can deal with the case when one of the graphs is not a tournament. If both (x,y) and (y,x) exist in G_I or in H_I , then by Proposition 6.4 no α -layer individually stable matchings exists. Thus, the only non-trivial case is when the graphs G_I and H_I are tournaments. Moreover, in such a case, the condition from Proposition 6.4 can be reformulated as $(x,y) \in G_I$ if and only if $(M(x), M(y)) \in H_I$ and this is the condition that M is a tournament isomorphism between G_I and H_I . Consequently, for $\alpha \geq \ell/2 + 1$ the problem of finding an α -layer individually stable matching can be reduced to TOURNAMENT ISOMORPHISM. Note that the number of vertices in the constructed graphs

equals the number of agents in our problem. By the result of Babai and Luks [3], we obtain an algorithm for our problem with the desired running time. \Box

6.2.2 Global stability

There exists a fairly straightforward polynomial-time algorithm for finding α -layer globally stable matchings.

Proposition 6.7. If the preferences of the agents are uniform in all ℓ layers, then GLOBALLY STABLE MARRIAGE can be solved in $O(\ell \cdot n)$ time, where n denotes the number of agents.

Proof. It is apparent that for uniform preferences of the agents, each layer admits a unique stable matching: for each $i \in [n]$ the i-th most preferred agent from U is matched with the i-th most preferred agent from W. Further, such a matching can be computed in O(n) time. Thus, our algorithm proceeds as follows. For each layer we compute a unique stable matching, and we pick the matching which is stable in the largest number of layers. Our algorithm returns this matching if it is stable in at least α layers; otherwise, the algorithm outputs that there exists no α -layer globally stable matchings for the given instance.

7 Open Problems and Conclusions

We have considered a new multi-layer model of preferences in the context of the STABLE MARRIAGE problem. We identified three natural concepts of stability and discussed their relations with each other. Our results show that the algorithmic problem of finding stable matchings according to each of the three concepts is, in general, computationally hard. On the positive side, we also managed to identify a number of natural special cases which are tractable (Table 1 summarizes our results). Interestingly, while in the world of multi-layer stable matchings the case of two layers already leads to most computational hardness results, in the world of maximum-cardinality matching in two-layer graphs one obtains polynomial-time solvability, while in the case of three layers one encounters NP-hardness [14].

Our work provides a rich structure for analyzing computational properties of the problems we considered, and we view our work as only initiating this line of research. Indeed, it directly leads to the following open questions:

- (1) How hard is it to find an α -layer individually stable matching for $\lceil \ell/2 \rceil < \alpha < \ell$?
- (2) When the preferences of the agents are uniform in each layer and $\alpha \geq \ell/2 + 1$, we have shown that the decision variant of INDIVIDUALLY STABLE MARRIAGE is solvable in quasi-polynomial time $n^{O(\log n)} + O(\ell \cdot n^2)$) which implies that the problem is in the complexity class LOGSNP [43]. It would be interesting to know whether it is also complete for LOGSNP.² However, LOGSNP-hardness for our problem would also imply LOGSNP-hardness for GRAPH ISOMORPHISM (see Theorem 6.5).
- (3) When the preferences of the agents are uniform in each layer, how hard is it to search for an α -layer pair stable matching for arbitrary $\alpha>1$ or an α -layer individually stable matching when $\alpha<\ell/2$, or in general when the number of layers is constant.

We also believe that a number of other parameters and special cases can be motivated naturally in the context of our model, in particular parameters quantifying the degree to which the preferences of the agents differ. Analogous parameterizations have been studied in computational social choice, for instance for the NP-hard KEMENY SCORE problem [6, 5].

²LOGSNP-hardness has been encountered and discussed for natural problems in Computational Social Choice [10, 11].

Continuing our research on special cases of input preferences (Section 6), it might be interesting to study stable matching with multi-layer *structured* preferences, such as *single-peaked* [8], *single-crossing* [44, 41], and *1-Euclidean* [16, 34, 15] preferences. We note that it can be detected in polynomial time whether a preference profile has any of these structure [4, 18, 22, 18, 20, 9, 34] and we refer the reader to Bredereck et al. [12] and Elkind et al. [21] for an overview of the literature on single-peakedness and single-crossingness. We also note that Bartholdi III and Trick [4] worked on stable roommates for narcissistic and single-peaked preferences, while Bredereck et al. [13] extended this line by also studying other structured preferences and including preferences with ties and incompleteness.

Finally, our multi-modal view on the bipartite variant (STABLE MARRIAGE) can be generalized to the non-bipartite variant (STABLE ROOMMATES) and the case with incomplete preferences with ties. It would be interesting to see whether our computational tractability results can transfer to these cases.

References

- [1] D. J. Abraham, P. Biró, and D. Manlove. "Almost stable" matchings in the roommates problem. In *Proceedings of the Third International Workshop on Approximation and Online Algorithms (WAOA '05)*, pages 1–14, 2005. 2
- [2] H. Aziz, P. Biró, T. Fleiner, S. Gaspers, R. de Haan, N. Mattei, and B. Rastegari. Stable matching with uncertain pairwise preferences. In *Proceedings of the 16th International Conference on Autonomous Agents and Multiagent Systems (AAMAS '17)*, pages 344–352. AAAI Press, 2017. 3
- [3] L. Babai and E. Luks. Canonical labeling of graphs. In *Proceedings of the 15th Annual ACM Symposium on Theory of Computing (STOC '83)*, pages 171–183. ACM Press, 1983. 33, 34
- [4] J. J. Bartholdi III and M. Trick. Stable matching with preferences derived from a psychological model. *Operations Research Letters*, 5(4):165–169, 1986. 35
- [5] N. Betzler, J. Guo, C. Komusiewicz, and R. Niedermeier. Average parameterization and partial kernelization for computing medians. *Journal of Computer and System Sciences*, 77(4):774–789, 2011.
- [6] N. Betzler, R. Bredereck, and R. Niedermeier. Theoretical and empirical evaluation of data reduction for exact kemeny rank aggregation. *Autonomous Agents and Multi-Agent Systems*, 28(5):721–748, 2014. 34
- [7] P. Biró. Applications of matching under preferences. In *Trends in Computational Social Choice*, pages 345–373. AI Access, 2017. 3
- [8] D. Black. The Theory of Committees and Elections. Cambridge University Press, 1958. 35
- [9] R. Bredereck, J. Chen, and G. J. Woeginger. A characterization of the single-crossing domain. *Social Choice and Welfare*, 41(4):989–998, 2013. 35
- [10] R. Bredereck, J. Chen, P. Faliszewski, J. Guo, R. Niedermeier, and G. J. Woeginger. Parameterized algorithmics for computational social choice: Nine research challenges. *Tsinghua Science and Technology*, 19(4):358–373, 2014. 34
- [11] R. Bredereck, J. Chen, S. Hartung, S. Kratsch, R. Niedermeier, O. Suchý, and G. J. Woeginger. A multivariate complexity analysis of lobbying in multiple referenda. *Journal of Artificial Intellgence Research*, 50:409–446, 2014. 34

- [12] R. Bredereck, J. Chen, and G. J. Woeginger. Are there any nicely structured preference profiles nearby? *Math. Soc. Sci.*, 79:61–73, 2016. 35
- [13] R. Bredereck, J. Chen, U. P. Finnendahl, and R. Niedermeier. Stable roommate with narcissistic, single-peaked, and single-crossing preferences. In *Proceedings of the 5th International Conference on Algorithmic Decision Theory (ADT '17)*, pages 315–330. Springer, 2017. 35
- [14] R. Bredereck, C. Komusiewicz, S. Kratsch, H. Molter, R. Niedermeier, and M. Sorge. Assessing the computational complexity of multi-layer subgraph detection. In *Proceedings of the 10th International Conference on Algorithms and Complexity (CIAC '17)*, pages 128–139. Springer, 2017. 34
- [15] J. Chen, K. Pruhs, and G. J. Woeginger. The one-dimensional Euclidean domain: Finitely many obstructions are not enough. *Social Choice and Welfare*, 48(2):409–432, 2017. 35
- [16] C. H. Coombs. A Theory of Data. John Wiley and Sons, 1964. 35
- [17] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015. 26
- [18] J. Doignon and J. Falmagne. A polynomial time algorithm for unidimensional unfolding representations. *Journal of Algorithms*, 16(2):218–233, 1994. 35
- [19] R. G. Downey and M. R. Fellows. Fundamentals of Parameterized Complexity. Springer, 2013. 26
- [20] E. Elkind, P. Faliszewski, and A. Slinko. Clone structures in voters' preferences. In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC '12)*, pages 496–513. ACM Press, 2012. 35
- [21] E. Elkind, M. Lackner, and D. Peters. Structured preferences. In *Trends in Computational Social Choice*, pages 187–207. AI Access, 2017. 35
- [22] B. Escoffier, J. Lang, and M. Öztürk. Single-peaked consistency and its complexity. In *Proceedings of the 18th European Conference on Artificial Intelligence (ECAI '08)*, pages 366–370. IOS Press, 2008. 35
- [23] L. Farczadi, K. Georgiou, and J. Könemann. Stable marriage with general preferences. *Theory of Computing Systems*, 59(4):683–699, 2016. 3, 28, 29
- [24] J. Flum and M. Grohe. Parameterized Complexity Theory. Springer, 2006. 26
- [25] D. Gale and L. S. Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 120(5):386–391, 1962. 1, 2, 4, 5
- [26] M. R. Garey and D. S. Johnson. *Computers and Intractability—A Guide to the Theory of NP-Completeness*. W. H. Freeman and Company, 1979. 13
- [27] D. Gusfield and R. W. Irving. *The Stable Marriage Problem–Structure and Algorithms*. Foundations of Computing Series. MIT Press, 1989. 2, 3, 5
- [28] R. W. Irving. Stable marriage and indifference. *Discrete Applied Mathematics*, 48(3):261–272, 1994.
- [29] R. W. Irving. Optimal stable marriage. In M. Kao, editor, *Encyclopedia of Algorithms*, pages 1470–1473. Springer, 2016. 2

- [30] R. W. Irving. Stable marriage. In M. Kao, editor, *Encyclopedia of Algorithms*, pages 2060–2064. Springer, 2016. 2
- [31] K. Iwama and S. Miyazaki. A survey of the stable marriage problem and its variants. In *International Conference on Infomatics Education and Research for Knowledge-Circulating Society*, pages 131–136, 2008. 3
- [32] K. Iwama, S. Miyazaki, Y. Morita, and D. Manlove. Stable marriage with incomplete lists and ties. In *Proceedings of the 26th International Colloquium on Automata, Languages, and Programming (ICALP '99)*, pages 443–452, 1999. 21, 22
- [33] B. Klaus, D. F. Manlove, and F. Rossi. Matching under preferences. In *Handbook of Computational Social Choice*. Cambridge University Press, 2016. 3
- [34] V. Knoblauch. Recognizing one-dimensional Euclidean preference profiles. *Journal of Mathematical Economics*, 46(1):1–5, 2010. 35
- [35] D. Knuth. Mariages Stables. Les Presses de L'Université de Montréal, 1976. 2, 3, 5
- [36] B. Maggs and R. Sitaraman. Algorithmic nuggets in content delivery. SIGCOMM Computer Communication Review, 45(3):52–66, 2015. 2
- [37] D. Manlove. Hospitals/residents problem. In M. Kao, editor, *Encyclopedia of Algorithms*. Springer, 2016. 2
- [38] D. Manlove, R. W. Irving, K. Iwama, S. Miyazaki, and Y. Morita. Hard variants of stable marriage. *Theoretical Computer Science*, 276(1-2):261–279, 2002. 21, 22
- [39] D. F. Manlove. Algorithmics of Matching Under Preferences, volume 2 of Series on Theoretical Computer Science. WorldScientific, 2013. 2, 3, 5
- [40] D. C. McGarvey. A theorem on the construction of voting paradoxes. *Econometrica*, 21(4):608–610, 1953. 30
- [41] J. A. Mirrlees. An exploration in the theory of optimal income taxation. *Review of Economic Studies*, 38:175–208, 1971. 35
- [42] R. Niedermeier. Invitation to Fixed-Parameter Algorithms. Oxford University Press, 2006. 26
- [43] C. H. Papadimitriou and M. Yannakakis. On limited nondeterminism and the complexity of the V-C dimension. *Journal of Computer and System Sciences*, 53(2):161–170, 1996. 34
- [44] K. W. Roberts. Voting over income tax schedules. *Journal of Public Economics*, 8(3):329–340, 1977. 35
- [45] A. E. Roth and M. A. O. Sotomayor. *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Cambridge University Press, 1992. Part of Econometric Society Monographs. 2
- [46] P. Schweitzer. A polynomial-time randomized reduction from tournament isomorphism to tournament asymmetry. Technical report, arXiv:1704.08529, 2017. 33
- [47] F. Wagner. Hardness results for tournament isomorphism and automorphism. In *Proceedings of the 32nd International Symposium on Mathematical Foundations of Computer Science (MFCS '07)*, pages 572–583. Springer, 2007. 33
- [48] B. P. Weems. Bistable versions of the marriages and roommates problems. *Journal of Computer and System Sciences*, 59(3):504–520, 1999. 3