Odd-Even Reduction for Banded Linear Equations

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ABSTRACT The method of odd-even reduction for tridiagonal systems is generalized to banded systems The method is developed so that it can be easily implemented on a vector processor such as the CDC STAR-100 Results are presented which describe when this odd-even reduction can be performed on a pentadiagonal system A computational example is given

KEY WORDS AND PHRASES parallel processing, banded linear systems, CDC-STAR-100, cyclic reduction, odd-even reduction, biharmonic problem

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Introduction

With the installation of the CDC STAR-100's at Lawrence Livermore Laboratory and NASA Langley Research Center, the problem of how one solves a linear system of equations on these machines has received considerable attention. The objective in designing an algorithm for the STAR is to try to use the available vector instructions whenever possible (cf [4]). In addition one would like to use these vector instructions on vectors whose lengths are as large as possible. In the case when the matrix of the linear system is full, defining the vectors to be the columns of the associated matrix problem enables the system to be efficiently solved by Gaussian elimination (via outer products in contrast to inner products [7]); cf. [2]. However, in the case when the matrix of the linear system has a banded structure, Gaussian elimination is no longer practical when the bandwidths are relatively small. This is due to the fact that using the columns of the matrix as vectors gives rise to short vectors.

In [8] it is shown that for a tridiagonal system of equations having no zero principal minors, Gaussian elimination could be implemented in a practical manner on a vector processor by defining the vectors to be the diagonals of the corresponding matrix In the case when the tridiagonal system is positive definite, it is shown in [9] that the method of cyclic odd-even reduction is faster than the recursive doubling algorithm of [8]. A comparison of cyclic reduction with Gaussian elimination is given in [4].

The problem of the general banded system has yet to be effectively resolved. For positive definite systems, Lambiotte in his thesis [3] recommends that the band structure be approached in terms of a block tridiagonal structure. The main problem with this approach is that it involves considerable effort to define the vectors at each stage of the cyclic reduction process.

In this paper the method of odd-even reduction is generalized to banded linear systems. The algorithm is developed so that it can be easily implemented on a vector processor. For

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clarity and ease of presentation, we will describe the odd-even reduction algorithm for matrices whose entries are scalars (real numbers). Actually, the algorithm is easily seen to be generalizable to the case of block matrices whose entries are themselves square submatrices.

It should be noted that a variant of the method described in this paper has been used by Bauer and Reiss [1] to solve the two-dimensional biharmonic problem. Their method, however, only applies to the constant diagonal case and does not generalize easily to the nonconstant case.

1. *Vector Notation*

If R is the set of real numbers, then by

$$v = \begin{bmatrix} v_1 \\ \cdot \\ v_m \end{bmatrix}$$

we denote a vector of *length m* having elements in R. We define $v(i) = v_i$, and we adopt the "offset" notation of [5]:

$$v(k,l) = \begin{bmatrix} v_{k+1} \\ \cdot \\ v_{m-l} \end{bmatrix}.$$

In addition we use the shortened notation

$$v(k;) = v(k;0), \qquad v(,l) = v(0;l)$$

This notation is actually the syntax used in LRLTRAN [6], the vector extended Fortran at the Lawrence Livermore Laboratory. If $c \in R$, then $v = c$ implies $v(i) = c$ for all i (similarly, $v \neq c$ implies $v(i) \neq c$ for all i). If v and w are vectors of length m, then the vector operations $v \pm w$, vw, and v/w are defined by componentwise operations. In this paper if the division operation is used, then it is assumed to be well defined (i.e. all components of w are nonzero).

The above notation is very useful for describing particular data structures for an $n \times n$ matrix $A = (\alpha_{ij})$, $\alpha_{ij} \in R$ We define the vectors a_j, a_{-j}, $j = 0, 1, \dots, n-1$, by

$$\begin{aligned} a_j(i) &= \alpha_{i,i+j}, \quad i = 1, \dots, n-j, \\ a_{-j}(i) &= \alpha_{j+i,i}, \quad i = 1, \dots, n-j \end{aligned} \tag{1.1}$$

More explicitly, for $j > 0$, a_j is the jth superdiagonal of A and a_{-j} is the jth subdiagonal of A. The length of the vectors $a_{\pm j}$ is $n - j$. The matrix A is denoted by $A = (a_j)$, $-(n-1) \leq j \leq (n-1)$, in the case when A has a banded structure, i.e $a_{-j} = 0$ for $j > l$ and $a_j = 0$ for $j > u$, we will use the notation

$$A = (a_j), \quad -l \leq j \leq u.$$

2. *Cyclic Reduction*

We now turn our attention to the method of cyclic reduction for solving a system $Ax = b$ where $A = (a_j)$, $-m \leq j \leq m$, and a_0 is of even length n The main idea behind the method is to reduce the size of the problem by suitable row additions on the matrix A. Specifically, if P is a permutation matrix and PAP^{T} is of the form

$$PAP^{\mathrm{T}} = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]\}n/2, \tag{2.1}$$

then the idea is to construct a matrix Q so that

$$QPAP^{\mathrm{T}} = \left[\begin{array}{c|c} D & U \\ \hline 0 & \tilde{A} \end{array}\right]\}n/2, \tag{2 2}$$

where D is an "easily" invertible matrix (relative to the invertibility of A) and $\tilde{A}$ has the same band structure as that of A.

The solution of the system $Ax = b$ is then equivalent to the solution of the system $[QPAP^T]Px = QPb$. Letting

$$y = Px = \left[\frac{y_1}{y_2}\right]\}n/2 \quad \text{and} \quad \tilde{b} = QPb = \left[\frac{b_1}{b_2}\right]\}n/2,$$

the solution of the system $[QPAP^T] = \tilde{b}$ is given by

$$y_2 = \tilde{A}^{-1}b_2, \tag{2.3}$$

$$y_1 = D^{-1}(b_1 - Uy_2). \tag{2.4}$$

The final solution of $Ax = b$ is then $x = P^Ty$. Note that in (2.3) the system $\tilde{A}y = b_2$ still must be solved. However, since $\tilde{A}$ has the same band structure as A, the process described above can be (in principle) performed on $\tilde{A}$ which is half the size of the original system. This successive application of the above process defines the method of *cyclic reduction.* Note that these successive reduction stages can be terminated when the matrix system remaining to be solved is of sufficiently low order so that it can be solved more efficiently by other means (e.g. Gaussian elimination).

The desired matrix Q will be one such that for the operation described in (2.2) the resultant matrix D will be a diagonal matrix. In effect Q will decouple all of the even variables from the odd variables and we say that Q performs an *odd-even reduction* on A.

3. *Odd-Even Reduction*

The key issues in odd-even reduction are the choice of the permutation matrix P and the construction of the matrix Q. The specific permutation matrix with which we will be concerned will be called an odd-even permutation matrix since its primary function is to isolate the odd and even rows and columns of A from each other. The Q matrix will result from "diagonal elimination," a process whose name will become more meaningful later.

The $n \times n$ odd-even permutation matrix P is defined by requiring that

$$P(1, 2, \dots, n)^T = (1, 3, 5, \dots, n-1, 2, 4, 6, \dots, n)^T.$$

Specifically, P is obtained by permuting the rows of the $n \times n$ identity matrix so that the odd rows appear sequentially in the top half of the matrix and the even rows are sequentially at the bottom. Multiplication of a matrix or a vector by a permutation matrix is easily accomplished on the STAR-100 by vector instructions. It follows that PAP^T is of the form (2.1) where

$$\begin{aligned}
A_1 &= (b_j), \quad -[m/2] \le j \le [m/2],\\
A_2 &= (c_j), \quad -[(m+1)/2] \le j \le [(m-1)/2],\\
A_3 &= (d_j), \quad -[(m-1)/2] \le j \le [(m+1)/2],\\
A_4 &= (e_j), \quad -[m/2] \le j \le [m/2]
\end{aligned}$$

(here $y = [x]$ if y is the greatest integer such that $y \le x$) and

$$\begin{aligned}
b_j(i) &= a_k(2i-1), \quad k = 2j,\\
c_j(i) &= a_k(2i-1), \quad k \doteq 2j+1, \quad \text{when } j \ge 0,\\
c_j(i) &= a_k(2i), \quad k = 2j+1, \quad \text{when } j < 0,\\
d_j(i) &= a_k(2i), \quad k = 2j-1, \quad \text{when } j > 0,\\
d_j(i) &= a_k(2i-1), \quad k = 2j-1, \quad \text{when } j \le 0,\\
e_j(i) &= a_k(2i), \quad k - 2j.
\end{aligned}$$

Hence, if $m = 2k + 1$,

$$\begin{aligned} A_1 &= (b_j), \quad -k \le j \le k, & A_3 &= (d_j), \quad -k \le j \le k+1, \\ A_2 &= (c_j), \quad -(k+1) \le j \le k, & A_4 &= (e_j), \quad -k \le j \le k, \end{aligned} \tag{3.1}$$

and if $m = 2k$,

$$\begin{aligned} A_1 &= (b_j), \quad -k \le j \le k, & A_3 &= (d_j), \quad -(k-1) \le j \le k, \\ A_2 &= (c_j), \quad -k \le j \le k-1, & A_4 &= (e_j), \quad -k \le j \le k \end{aligned} \tag{3.2}$$

DIAGONAL ELIMINATION. The matrix Q will be the product of a sequence of matrices Q_i that have the form

$$Q_i = \left[\begin{array}{c|c} I & 0 \\ \hline \begin{smallmatrix} \ddots & \ddots & 0 \\ 0 & \ddots & \ddots \end{smallmatrix} & I \end{array}\right] \begin{matrix} \}\, n/2 \\ \end{matrix}$$

or

$$Q_i = \left[\begin{array}{c|c} I & \begin{smallmatrix} \ddots & \ddots & 0 \\ 0 & \ddots & \ddots \end{smallmatrix} \\ \hline 0 & I \end{array}\right] \begin{matrix} \}\, n/2 \\ \end{matrix}$$

Our approach will be in two stages. First we will determine how the Q_i's are calculated and then we will show how their product does indeed effect an odd-even reduction of A.

To start, suppose $A = (a_j)$, $-m \le j \le m$, and that P is the odd-even permutation matrix such that PAP^T has the form (2.1) where (recalling the notation in (1.1))

$$\begin{aligned} A_1 &= (b_j), \quad -l_1 \le j \le u_1, & A_3 &= (d_j), \quad -l_3 \le j \le u_3, \\ A_2 &= (c_j), \quad -l_2 \le j \le u_2, & A_4 &= (e_j), \quad -l_4 \le j \le u_4. \end{aligned} \tag{3 3}$$

As will be seen later, only two cases are of importance.

Case 1. m is odd.

$$u_3 = u_1 + 1, \quad l_1 = l_3, \quad u_4 = u_2, \quad l_2 = l_4 + 1.$$

We construct a matrix Q of the form

$$Q = \left[\begin{array}{c|c} I & 0 \\ \hline \begin{smallmatrix} \ddots & \ddots & & 0 \\ x_0 & x_1 & & \\ 0 & & \ddots & \ddots \end{smallmatrix} & I \end{array}\right] \begin{matrix} \}\, n/2 \\ \end{matrix} \tag{3.4}$$

so that

$$QPAP^T = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline \bar{A}_3 & \bar{A}_4 \end{array}\right],$$

where

$$\bar{A}_3 = (\bar{d}_j), \quad -(l_3 - 1) \le j \le u_3 - 1$$

That is, Q will eliminate a superdiagonal and a subdiagonal from the lower left block. We say that the matrix Q in (3 4) is of Type 1.

Now, block multiplication yields

$$\begin{bmatrix} \ddots & \ddots & 0 \\ & x_0 \quad \ddots \; x_1 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix} A_1 + A_3 = \bar{A}_3,$$

so that the desired effect will be obtained if

$$x_1(,u_1)b_{u_1}(1) + d_{u_3} = 0, \qquad x_0(l_1,)b_{-l_1} + d_{-l_3} = 0 \tag{3.5}$$

or

$$x_1(,u_1) = -d_{u_3}/b_{u_1}(1;), \qquad x_0(l_1;) = -d_{-l_3}/b_{-l_1}. \tag{3.6}$$

Note that (3.6) does not define all of the elements of x_1 and x_0. However, the only property that is required of the vectors x_1 and x_0 is that they effect the calculation (3.5). For definiteness we will assume that all of the undefined elements of x_0 and x_1 are zero.

Block multiplication also yields

$$\bar{A}_4 = (\bar{e}_j), \quad -(l_4 + 1) \leq j \leq u_4 + 1, \tag{3.7}$$

so that a superdiagonal and a subdiagonal are added to the lower right block.

Case 2. m is even.

$$u_1 = u_3, \quad l_1 = l_3 + 1, \quad l_2 = l_4, \quad u_4 = u_2 + 1.$$

For this case we construct a matrix Q of the form

$$Q = \left[\begin{array}{c|c} I & \begin{matrix} \ddots & \ddots & 0 \\ \ddots & x_0 & \\ x_{-1} & \ddots & \ddots \\ 0 & & \ddots \end{matrix} \\ \hline 0 & I \end{array}\right] \Big\} n/2 \tag{3.8}$$

so that

$$QPAP^{\mathrm{T}} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ A_3 & A_4 \end{bmatrix},$$

where

$$\bar{A}_1 = (\bar{b}_j), \quad -(l_1 - 1) \leq j \leq u_1 - 1. \tag{3.9}$$

That is, Q will eliminate a superdiagonal and a subdiagonal from the upper left block. We say that the matrix Q in (3.8) is of Type 2. Again, block multiplication yields

$$A_1 + \begin{bmatrix} \ddots & \ddots & 0 \\ x_{-1} & x_0 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix} A_3 = \bar{A}_1,$$

so that the desired effect will be obtained if

$$b_{-l_1} + x_{-1}(l_3;)d_{-l_3}(,1) = 0, \qquad b_{u_1} + x_0(,u_3)d_{u_3} = 0 \tag{3.10}$$

or

$$x_{-1}(l_3;) = -b_{-l_1}/d_{-l_3}(;1), \qquad x_0(;u_3) = -b_{u_1}/d_{u_3}. \tag{3.11}$$

As in case 1, (3.11) does not define all of the elements of x_{-1} and x_0 and again we will assume them to be zero.

For the remaining block, it is easy to see that

$$\bar{A}_2 = (\bar{c}_j), \quad -(l_2 + 1) \leq j \leq u_2 + 1, \tag{3.12}$$

so that a superdiagonal and a subdiagonal are added to the upper right block.

We summarize the essential points of this section

PROPOSITION 3.1. *Let A be permuted to have the form given by* (2.1). *Suppose the conditions of case* 1 *are satisfied* (m *is odd*). *If Q is of Type* 1, *then $QPAP^T$ satisfies the conditions of case* 2. *Conversely, suppose the conditions of case* 2 *are satisfied* (m *is even*). *If Q is of Type* 2, *then $QPAP^T$ satisfies the conditions of case* 1.

We now use the concept of diagonal elimination to describe the method of odd-even reduction. Let $A = (a_j)$, $-m \leq j \leq m$, and P be an odd-even permutation matrix. Then PAP^T is of the form (2.1) where $A_1 - A_4$ are defined in (3.1) or (3.2) depending on whether m is odd or even, respectively. In the case when m is odd, A satisfies the conditions of case 1 and if m is even, A satisifes the conditions of case 2. We thus have demonstrated the following:

PROPOSITION 3.2. *Let $A = (a_j)$, $-m \leq j \leq m$, and PAP^T be as in* (2.1). *Then, the matrix $Q = \prod_{i=1}^{m} Q_i$ will effect an odd-even reduction of PAP^T where*

(1) *if m is odd, then* (*a*) *for odd i, Q_i is of Type* 1, (*b*) *for even i, Q_i is of Type* 2;
(2) *if m is even, then* (*a*) *for odd i, Q_i is of Type* 2, (*b*) *for even i, Q_i is of Type* 1.

Suppose a matrix Q as defined in Proposition 3 2 effects an odd-even reduction of PAP^T. Then

$$QPAP^T = \left[\begin{array}{c|c} \diagdown & U \\ \hline 0 & \tilde{A} \end{array}\right] \tag{3.13}$$

We now state the following proposition concerning the diagonal structure of U and $\tilde{A}$.

PROPOSITION 3.3. *Suppose $A = (a_j)$, $-m \leq j \leq m$, and Q as defined in Proposition* 3.2 *effects the odd-even reduction* (3.13). *Then,*

$$\tilde{A} = (\tilde{a}_j),\ -m \leq j \leq m, \quad \text{and} \quad U = (u_j),\quad -m \leq j \leq m-1$$

PROOF.

Case 1. $m = 2k + 1$. By Proposition 3.2, $Q = \prod_{i=1}^{m} Q_i$ where for odd i, Q_i is of Type 1 and for even i, Q_i is of Type 2. By (3.1),

$$A_2 = (c_j),\quad -(k+1) \leq j \leq k, \quad \text{and} \quad A_4 = (e_j),\quad -k \leq j \leq k.$$

By (3.7) a matrix of Type 1 adds a superdiagonal and a subdiagonal to the lower right block and by (3.12) a matrix of Type 2 adds a superdiagonal and a subdiagonal to the upper right block. Since the odd-even reduction process involves $k + 1$ matrices of Type 1 and k matrices of Type 2,

$$\begin{aligned} \tilde{A} &= (\tilde{a}_j),\quad -(k+k+1) \leq j \leq (k+k+1),\\ &= (\tilde{a}_j),\quad -(2k+1) \leq j \leq (2k+2), \end{aligned}$$

and

$$\begin{aligned} U &= (u_j),\quad -(k+1+k) \leq j \leq k+k,\\ &= (u_j),\quad -(2k+1) \leq j \leq 2k. \end{aligned}$$

Case 2. $m = 2k$. By Proposition 3.2, $Q = \prod_{i=1}^{m} Q_i$ where for odd i, Q_i is of Type 2, and for even i, Q_i is of Type 1. By (3.2),

$$A_2 = (c_j),\quad -k \leq j \leq k-1, \quad \text{and} \quad A_4 = (e_j),\quad -k \leq j \leq k.$$

Since the odd-even reduction process involves k matrices of Type 1 and Type 2, we have by (3 7) and (3 12) that

$$\begin{aligned} \tilde{A} &= (\tilde{a}_j),\quad -(k+k) \leq j \leq (k+k),\\ &= (\tilde{a}_j),\quad -2k \leq j \leq 2k, \end{aligned}$$

and

$$\begin{aligned} U &= (u_j),\quad -(k+k) \leq j \leq (k-1+k),\\ &= (u_j),\quad -(2k) \leq j \leq (2k-1). \end{aligned}$$

4. *Some Sufficient Conditions*

In this section we establish some properties on the original matrix $A = (a_j)$, $-m \leq j \leq m$, so that a single odd-even reduction can be performed.

TRIDIAGONAL SYSTEMS ($m = 1$). For $m = 1$, only the matrix Q_1 is generated. Using the notation established in (1.1), Q_1 is given explicitly by

$$Q_1 = \left[\begin{array}{c|c} I & 0 \\ \hline \begin{matrix} \ddots & \ddots & & 0 \\ & x_0 & x_1 & \\ 0 & & \ddots & \ddots \end{matrix} & I \end{array}\right],$$

where

$$x_1 = -d_1/b_0(1,), \qquad x_0 = -d_0/b_0.$$

The vector b_0 consists of the odd elements from the main diagonal of the original matrix A. Hence the odd-even reduction can be performed on A if the main diagonal elements of A are nonzero. The question of whether odd-even reduction can be continued cyclically and be performed on $\tilde{A}$ is a more difficult one to answer. However, it is well known that odd-even reduction on a scalar tridiagonal system A is equivalent to block Gaussian elimination on PAP^{T} where the blocks are as in (3.1). Hence, all of the well-known results for Gaussian elimination can be applied. That is, if A has certain properties such as positive-definiteness or irreducible diagonal dominance, then cyclic reduction can be carried out.

PENTADIAGONAL SYSTEMS ($m = 2$). For $m = 2$, two matrices Q_1 and Q_2 are generated. For Q_1, (3.8) applies so that

$$Q_1 = \left[\begin{array}{c|c} I & \begin{matrix} \ddots & \ddots & 0 \\ \ddots & x_0 & \ddots \\ 0 & x_{-1} & \ddots \end{matrix} \\ \hline 0 & I \end{array}\right],$$

where (using the notation of (3.2))

$$x_{-1} = -b_{-1}/d_0(;1), \qquad x_0(;1) = -b_1/d_1. \tag{4.1}$$

Then,

$$Q_1(PAP^{\mathrm{T}}) = \left[\begin{array}{c|c} \begin{matrix} \ddots & & 0 \\ & s_0 & \\ 0 & & \ddots \end{matrix} & \times \\ \hline \begin{matrix} \ddots & \ddots & 0 \\ 0 & \ddots & \ddots \end{matrix} & \times \end{array}\right],$$

where

$$s_0 = b_0 + x_0 d_0, \qquad s_0(1;) = s_0(1;) + x_{-1} d_1. \tag{4.2}$$

Q_2 now takes the form (3.4),

$$Q_2 = \left[\begin{array}{c|c} I & 0 \\ \hline \begin{matrix} \ddots & \ddots & & 0 \\ & y_0 & y_1 & \\ 0 & & \ddots & \ddots \end{matrix} & I \end{array}\right].$$

where

$$y_0 = -d_0/s_0, \qquad y_1 = -d_1/s_0(1;). \tag{4.3}$$

It follows from (4.1) and (4.3) that odd-even reduction can be performed on A if the vectors d_0, d_1, and s_0 have no zero elements. By (3.2), d_0 and d_1 originate from the first off-diagonals of A. Thus, d_0 and d_1 will have no zero elements if the first super- and subdiagonals of A have no zero elements. More explicitly, d_0 and d_1 will have no zero elements if every even-numbered variable has a nonzero coupling to its adjacent odd-numbered variables. We now suppose this to be the case and turn to the analysis of s_0.

If $A = (a_{ij})$, $1 \le i, j \le n$ (i.e. the standard notation), then the elements of s_0 are given by

$$s_0(1) = a_{11} - \frac{a_{13}a_{21}}{a_{23}},$$

$$s_0(k) = a_{ii} - \frac{a_{i,i+2}a_{i+1,i}}{a_{i+1,i+2}} - \frac{a_{i,i-2}a_{i-1,i}}{a_{i-1,i-2}}, \quad k = 2, \ldots, n/2 - 1, \; i = 2k - 1,$$

$$s_0(n/2) = a_{n-1,n-1} - \frac{a_{n-2,n-1}a_{n-1,n-3}}{a_{n-2,n-3}}.$$

Consider the following submatrices of A:

$$M_1 = \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix},$$

$$M_k = \begin{bmatrix} a_{i-1,i-2} & a_{i-1,i} & 0 \\ a_{i,i-2} & a_{ii} & a_{i,i+2} \\ 0 & a_{i+1,i} & a_{i+1,i+2} \end{bmatrix}, \quad k = 2, \ldots, n/2 - 1, \; i = 2k - 1,$$

$$M_{n/2} = \begin{bmatrix} a_{n-2,n-3} & a_{n-2,n-1} \\ a_{n-1,n-3} & a_{n-1,n-1} \end{bmatrix}$$

We then get the following theorem:

THEOREM 4.1. *Suppose $A = (a_j)$, $-2 \le j \le 2$, such that $a_{\pm 1} \ne 0$. Then $s_0(i) \ne 0$ if and only if M_i is nonsingular*

PROOF. Note that

$$\begin{aligned} a_{23}s_0(1) &= \text{determinant } (M_1), \\ a_{i+1,i+2}a_{i-1,i-2}s_0(k) &= \text{determinant } (M_k), \\ a_{n-2,n-3}s_0(n/2) &= \text{determinant } (M_{n/2}). \end{aligned}$$

Since $a_{\pm 1} \ne 0$, the above determinants are nonzero if and only if $s_0 \ne 0$. Q.E.D

We get the following corollaries:

COROLLARY 4.1. *If $A = (a_{ij})$ satisfies the following properties:*

(*i*) $|a_{ii}| > |a_{i-1,i}| + |a_{i+1,i}|$,
(*ii*) $|a_{i,i-1}| > |a_{i+1,i-1}|$,
(*iii*) $|a_{i,i+1}| > |a_{i-1,i+1}|$,

then $s_0 \ne 0$.

PROOF The above assumptions force each of the matrices M_i to be strictly column diagonally dominant. Q E D.

COROLLARY 4.2. *If A is symmetric and satisfies the following:*

(*i*) $|a_{ii}| > |a_{i,i-1}| + |a_{i,i+1}|$,
(*ii*) $|a_{i,i+1}| > |a_{i,i+2}|$,
(*iii*) $|a_{i,i-1}| > |a_{i,i-2}|$,

then $s_0 \ne 0$.

In the case when A is symmetric with constant diagonals we get the following:

COROLLARY 4.3. *If $A = (a_j)$, $-2 \le j \le 2$, such that*

$$a_0 = 1, \quad a_{\pm 1} = b \ne 0, \quad a_{\pm 2} = c,$$

then $s_0 \neq 0$ *if and only if* $c \neq \frac{1}{2}$ *and* $c \neq 1$.

PROOF. Note that

$$\text{determinant}\,(M_1) = \text{determinant}\,(M_{n/2}) = b(1-c)$$

and

$$\text{determinant}\,(M_k) = b^2(1-2c), \quad k = 2, \dots, n/2 - 1,$$

so that Theorem 4.1 now applies. Q E.D.

All of the above results determine properties of a pentadiagonal matrix A so that a single odd-even reduction can be performed Specific properties of A which will guarantee that cyclic reduction can be performed still remain to be determined. Also, conditions for odd-even reduction to work on a general banded system are still not known.

COMPUTATIONAL EXAMPLE. Cyclic reduction was used to solve the system $Ax = b$ where A was the biharmonic matrix

$$\begin{bmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & 0 & \\ 1 & -4 & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & 1 \\ & 0 & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{bmatrix}$$

and $b = [1, 1, \dots, 1]^T$. If n is the order of A, then it is known that the condition number of A is approximately $16(n/\pi)^{-4}$. The cyclic odd-even reduction algorithm was coded in single-precision LRLTRAN [6] at the Lawrence Livermore Laboratory and compared with a single-precision Fortran version of Gaussian elimination (Cholesky square-root free variation) on a CDC-STAR-100. The timing analysis (in milliseconds) is given in Table I. A least squares fit of the timing data in Table I indicates that cyclic reduction behaves like

$$0\,0053n + 0\,807 \log_2 n - 1.431$$

and Gaussian elimination behaves like

$$0.302n - 0.147.$$

The particular forms of these equations arise from operation counts performed for cyclic reduction and Gaussian elimination in [3]

To give some feeling for the stability of the new algorithm the "exact" solution $\bar{x}$ was computed (using 96-bit mantissas) for $n = 128$ and $n = 512$ If x_{CR} and x_{GE} represent the computed solutions from cyclic reduction and Gaussian elimination, respectively (using 48-bit mantissas), then for $n = 128$,

$$\|\bar{x} - x_{\text{GE}}\|_\infty / \|\bar{x}\|_\infty = 3 \times 10^{-8}, \qquad \|\bar{x} - x_{\text{CR}}\|_\infty / \|\bar{x}\|_\infty = 3 \times 10^{-12},$$

and for $n = 512$,

$$\|\bar{x} - x_{\text{GE}}\|_\infty / \|\bar{x}\|_\infty = 8 \times 10^{-6}, \qquad \|\bar{x} - x_{\text{CR}}\|_\infty / \|\bar{x}\|_\infty = 1 \times 10^{-11}$$

5. *Conclusions*

The major purpose of this paper has been to introduce a new method of odd-even reduction for banded systems of linear equations. This new algorithm is unique because the basic reduction process is performed using the longer diagonals of the banded matrix rather than the shorter rows or columns. This fact makes the algorithm attractive for possible implementation on vector processors such as the CDC STAR-100 which require long vectors for efficient operation.

TABLE I

n	Cyclic reduction	Gaussian elimination
8	1 059	2 292
16	1 872	4 653
32	2 749	9 485
64	3 793	19 191
128	4 831	38 465
256	6 303	77 168
512	8 694	154 399
1024	12 008	308 929

We have also established some conditions which are sufficient to guarantee that odd-even reduction can be performed. However, there remain several interesting unanswered questions.

Conditions which are sufficient to guarantee that this new odd-even reduction algorithm can be applied in a cyclical manner need to be established. Since the off-diagonal matrix elements are used in the odd-even reduction process in a manner which is analogous to the use of pivot elements in Gaussian elimination, the overall stability of this cyclic reduction process needs to be theoretically analyzed. We hope that this paper will serve to stimulate the investigation of some of these questions.

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(Note Reference [10] is not cited in the text)

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