A Note on Product-Form Solution for Queuing Networks with Poisson Arrivals and General Service-Time Distributions with Finite Means

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Abstract. The steady-state joint probability distribution of queue lengths is obtained for queuing networks with Poisson arrivals in which some of the service-time distribution functions are general (e g , not even differentiable). In particular, an analytical model for queuing networks which is more general than those considered to date is produced by using the concept of generalized function Previous results on the relationships between the properties of queuing discipline, product form, and local balance can be shown to hold in this more general setting.

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1. *Introduction*

During the past few years, the scope of queuing network analysis has been enlarged considerably by various authors [1, 2, 4–11]. The results associated with the single queue [4] can be summarized as follows:

(1) A differential equation describing the state probabilities of the queue for a class of queuing disciplines can be derived.
(2) Any queue which satisfies station balance satisfies local balance and has product-form steady-state probabilities. Furthermore, the steady-state probabilities of a queue which satisifes station balance are functions of the mean service times and are otherwise independent of the service distribution.
(3) Any queue with exponential service times for all classes and product-form steady-state probabilities must satisfy station balance.
(4) In any queue which satisfies station balance, arriving customers must commence service immediately (immediate service discipline).
(5) Product form implies local balance for queues with class-independent disciplines, that is, disciplines which treat all customers alike.

The difference between the above results and the results of this paper are that (i) the differential equation now involves generalized functions and (ii) all the above results have been extended to arbitrary service distributions with finite mean (the

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nondifferentiable case included). In this paper only (1) and (2) for a single queue will be proved, since this illustrates the use of the proof techniques.

The results for an open network of queues [4] can be summarized as follows:

(1) A differential equation describing the state probabilities in the open network can be derived.
(2) If every queue in the network satisfies local balance in isolation with Poisson arrivals, then the entire network satisfies local balance and the steady-state probabilities for the network are in product form.
(3) If all queues in the network have class-independent disciplines, then the network has product-form steady-state probabilities for all customer classes if and only if each queue in the network has product-form steady-state probabilities in isolation.

The difference between the above results and the results obtainable by the methods of this paper are that in (1) the differential equation now involves generalized functions and all the results can be extended to arbitrary service distributions with finite mean (the nondifferentiable case included). The results associated with networks of queues will not be presented here, but the details can be obtained from the authors.

2. *Notation*

From this point on, the word distribution will mean generalized function. Distribution function and probability density function will retain their usual meanings from probability theory.

Any Markov process considered in this paper is such that its time-dependent state probability density function satisfies some partial differential integrodifference equations determined by the system. These Markov processes are assumed to be ergodic in the sense that the time-dependent state probability density functions are uniquely determined (after normalization) by the corresponding partial differential integrodifference equations and initial conditions, and the time-dependent state probability density functions converge to unique steady-state probability density functions independent of what the initial time-dependent state probability functions were. The important point about the stochastic processes considered in this paper is that by the introduction of supplementary variables, one has a parent Markov process which is assumed to be ergodic. This allows one to interpret the steady-state probabilities as proportions of time.

DISTRIBUTIONS. The theory of distributions is a topic of functional analysis which would require too much detail to discuss fully here. In this section we merely seek to develop an intuitive understanding of distributions, referring the reader to [3, 12] for additional information.

The following notation is defined for general use:

$$\begin{aligned} R &= \text{real numbers}, \\ R^n &= \{(x_1, \ldots, x_n): x_i \in R,\ i = 1, \ldots, n\}, \\ \Omega_n &= \{(x_1, \ldots, x_n): x_i \geq 0,\ i = 1, \ldots, n\}, \\ C &= \text{complex numbers}. \end{aligned}$$

Let Ω be any open connected subset of R^n. $C_0^\infty(\Omega)$ is the vector space (over C) of all complex-valued functions defined on Ω which have compact support and are infinitely differentiable. Let $C^n(\Omega)$ be the vector space (over C) of all functions defined on Ω whose nth order derivatives are continuous for $n = 0, 1, \ldots, +\infty$. Let

$C_B^n(\Omega)$ be the vector subspace of $C^n(\Omega)$ of functions whose derivatives, up to and including the nth order, are bounded in the norm

$$f \to \sup_{x\in\Omega} |f(x)|.$$

Let $\phi \in C_0^\infty(\Omega)$. Let $p = (p_1, \ldots, p_n)$ be such that $p_1, \ldots, p_n$ are nonnegative integers. Let $N = \sum_{i=1}^n p_i$. Then, define $(\partial/\partial x)^p(\phi)$ by

$$\left(\frac{\partial}{\partial x}\right)^p (\phi) = \frac{\partial^N \phi}{\partial x_1^{p_1} \cdots \partial x_n^{p_n}}.$$

Give $C_0^\infty(\Omega)$ the canonical LF topology described in [12, Ch. 13-6, Ex. II]. $C_0^\infty(\Omega)$ is generally referred to as the space of *test functions* A linear functional on $C_0^\infty(\Omega)$ which is continuous in the canonical LF topology is called a *distribution.* More precisely, a linear functional $L: C_0^\infty(\Omega) \to C$ is a distribution if $L(\phi_k) \to 0$ as $k \to +\infty$ for any sequence of test functions (ϕ_k) which satisfy:

(a) for each $p = (p_1, \ldots, p_n)$, where p_i is a nonnegative integer for $i = 1, \ldots, n$, the sequence $((\partial/\partial x)^p(\phi_k))$ converges uniformly to 0, and
(b) there is a compact set $K \subset \Omega$ such that the support of ϕ_k is contained in K for all k.

Let $\mathscr{D}'(\Omega)$ be the vector space of the distribution on $C_0^\infty(\Omega)$.

The following examples should give an intuitive understanding of distributions.

Example 2.1. Consider $C_0^\infty(R)$, and define $\delta: C_0^\infty(R) \to C$ by

$$\delta(f) = f(0)$$

for all $f \in C_0^\infty(\Omega)$. δ is a distribution, and it is the Dirac delta "function."

Electrical engineers sometime write

$$\delta(f) = \int_{-\infty}^{+\infty} \delta(x) f(x)\, dx = f(0).$$

The use of the integral is motivated by the fact that every locally integrable function can be viewed as a distribution. This can be seen in the second example. □

Example 2.2. Let f be locally integrable on R. This means that for every compact set $K \subset R$, $\int_K |f(x)|\, dx < \infty$. Define $T_f: C_0^\infty(R) \to C$ by

$$T_f(g) = \int_{-\infty}^{+\infty} f(x) g(x)\, dx$$

for all $g \in C_0^\infty(R)$. T_f is a distribution on $C_0^\infty(R)$ which is uniquely determined by f. In this sense, f is identified with T_f. □

Finite-order distributions (see [12]) can be written as a sum of certain order derivatives (in the distribution sense) of continuous functions. That is, if T is a finite order distribution on $C_0^\infty(R)$, then there exist continuous functions $f_1, \ldots, f_p$ and nonnegative integers $q_1, \ldots q_p$ such that

$$T(\phi) = \int_{-\infty}^{+\infty} \sum_{j=1}^{p} (-1)^{q_j} f_j(x) \phi^{(q_j)}(x)\, dx \tag{2.1}$$

for all $\phi \in C_0^\infty(R)$. If each f_j is continuously differentiable of order q_j, then (2.1) becomes

$$T(\phi) = \int_{-\infty}^{+\infty} \left(\sum_{j=1}^{p} f_j^{(q_j)}(x)\right)\phi(x)\, dx$$

for all $\phi \in C_0^\infty(R)$.

The above motivates the following notation. Let $T \in \mathscr{D}'(\Omega)$. Denote $T(f)$ for $f \in C_0^\infty(\Omega)$ by $\langle T, f\rangle$. That is, $T(f) = \langle T, f\rangle$. This notation is very close to that of inner product of two elements in a Hilbert space.

Now, if $T = T_f\,(=f)$, then $\langle T, \phi\rangle = \langle f, \phi\rangle = \int_\Omega f(x)\phi(x)\,dx$ for all $\phi \in C_0^\infty(\Omega)$.

The theory of distributions has been built primarily to extend some of the basic operations of analysis to functions for which these operations were not well defined in the usual sense. For instance, the derivative of a function that is not differentiable cannot be said to produce another well-defined function, but it does produce a distribution.

The case of differentiation is considered first. Let $f \in C^1(\Omega)$. Then, owing to the compactness of support of the test functions, one has

$$\left\langle \frac{\partial f}{\partial x_i}, \phi \right\rangle = \left\langle -f, \frac{\partial \phi}{\partial x_i} \right\rangle$$

for all $\phi \in C_0^\infty(\Omega)$. With the above in mind, one can define partial derivatives and higher order partial derivatives of any distribution in Ω. Let $T \in \mathscr{D}'(\Omega)$. Let $p = (p_1, \ldots, p_n)$, where $p_1, \ldots, p_n$ are nonnegative integers. Let $|p| = \sum_{i=1}^n p_i$. Define $(\partial/\partial x)^p T$ by

$$\left\langle \left(\frac{\partial}{\partial x}\right)^p T, \phi \right\rangle = (-1)^{|p|} \left\langle T, \left(\frac{\partial}{\partial x}\right)^p \phi \right\rangle$$

for all $\phi \in C_0^\infty(\Omega)$.

Example 2.3. Let $f: R \to R$ be defined by

$$f(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases}$$

Then for all $\phi \in C_0^\infty(R)$,

$$\left\langle \frac{df}{dx}, \phi \right\rangle = -\left\langle f, \frac{d\phi}{dx} \right\rangle = -\int_{-\infty}^{+\infty} f(x)\phi'(x)\, dx$$

$$= -\int_0^{+\infty} \phi'(x)\, dx = \phi(0) = \langle \delta, \phi \rangle.$$

That is,

$$\frac{df}{dx} = \delta, \quad \text{the Dirac delta "function."}$$

(Note that as a function, df/dx does not make sense.) □

One other operation that should be considered is multiplication by any infinitely differentiable function. Let $T \in \mathscr{D}'(\Omega)$ and $\phi \in C^\infty(\Omega)$. Define $\phi T: C_0^\infty(\Omega) \to C$ by

$$\langle \phi T, \Psi \rangle = \langle T, \phi \Psi \rangle$$

for all $\Psi \in C_0^\infty(\Omega)$.

Defined in this way, ϕT is a distribution. This is the natural extension of multiplication by a function in $C^\infty(\Omega)$ to a locally integrable function.

With the above extended operations, one can consider partial differential equations whose solutions might be distributions.

Example 2.4. Consider the equation

$$\frac{dT}{dx} = \delta,$$

where δ is the Dirac delta and T is unknown. We have already seen that one solution is $T_0 = f$, where

$$f(x) = \begin{cases} 0 & x \leq 0, \\ 1 & x > 0. \end{cases} \qquad \square$$

This concept of using partial differential equations in distribution sense will be indispensible in later sections of the paper.

Instead of developing a notation which describes a variety of queuing disciplines and loads, the authors use the same notation as found in Chandy et al. [4]. In order to illustrate the proof techniques, the differential equation that describes the equilibrium state probabilities in the case of a single queue will be derived, as well as the theorem on station balance and product form.

3. *Single Queue*

Let $P(t, X)$ be the time-dependent state probability density function associated with state $X = (S, X_1, \ldots, X_n)$, $S = (k(1), \ldots, k(n))$ of the queue. Since $X_1, \ldots, X_n$ are continuous random variables (residual life) which have right-continuous decreasing (in X_i) probability density functions, $P(t, X)$ will be right continuous in the X_i and $P(t, X) \to 0$ as $X_i \to +\infty$ for any i. However, this time we can only assume that $\partial P(t, X) | \partial X_i$ is a distribution. Let

$$\mathcal{H}_n = \operatorname{Span}\left\{ \phi \in C^{(1)}(\Omega_n) : \phi(X) = \int_0^{X_i} \psi(X)\, dX_i \right.$$
$$\left. \text{for some } \psi \in C_0^\infty(\Omega_n) \text{ and } i = 1, \ldots, n \right\},$$

where Ω_n is as defined in the introduction and Span indicates that $\mathcal{H}_n$ consists of all finite linear (complex) combinations of such functions. Thus $\mathcal{H}_n$ is a vector space containing $C_0^\infty(\Omega_n)$ as a subspace.

Now for each $t \geq 0$, let $L_t : \mathcal{H}_n \to C$ be given by

$$L_t(\phi) = \int_{\Omega_n} \phi(X) P(t, X)\, dX$$

for all $\phi \in \mathcal{H}_n$. It is obvious that L_t is a linear functional on $\mathcal{H}_n$ whose restriction to $C_0^\infty(\Omega_n)$ is a distribution. For the rest of the paper it will be convenient to abuse notation slightly by writing

$$L_t(\phi) = \langle P(t, X), \phi \rangle$$

for all $\phi \in \mathcal{H}_n$ to emphasize the idea that for each $t \geq 0$, $P(t, X)$ is also a distribution. The above abuse of notation will also be present in some of the other linear functionals defined on $\mathcal{H}_n$ later in this paper. It will be useful to note that

$\partial P(t, X)/\partial X_\iota$ is also defined and linear on $\mathscr{H}_n$ and is given by

$$\left\langle \frac{\partial P(t, X)}{\partial X_\iota}, \phi \right\rangle = -\int_{\Omega_n} P(t, X) \frac{\partial \phi(X)}{\partial X_\iota} dX$$

for all $\phi \in \mathscr{H}_n$. This is just the extension of its action as a distribution on $C_0^\infty(\Omega_n)$.

Balance equations will now be obtained by equating $\langle P(t + \Delta t, X), \phi\rangle$ to an expression involving $P(t, X)$, $\partial P(t, X)/dX_\iota$, Δt, and ϕ. In the time interval $(t, t + \Delta t)$ the state $X = (S, X_1, \ldots, X_n)$, $S = (k(1), \ldots, k(n))$ is reached as the result of an arrival (arr), departure (dep), or no change in occupancy (nc). For small values of Δt the effect of multiple events will be of order $o(\Delta t)$ and thus can be ignored. More specifically we have that

$$\begin{aligned}\langle P(t + \Delta t, X), \phi\rangle = \langle P(t + \Delta t, X), \phi\rangle_{\text{arr}} + \langle P(t + \Delta t, X), \phi\rangle_{\text{dep}} \\ + \langle P(t + \Delta t, X), \phi\rangle_{\text{nc}}.\end{aligned} \tag{3.1}$$

for all $\phi \in \mathscr{H}_n$, where

$\langle P(t + \Delta t, X), \phi\rangle_{\text{arr}}$	is the contribution to
$\langle P(t + \Delta t, X), \phi\rangle$	due to arrivals,
$\langle P(t + \Delta t, X), \phi\rangle_{\text{dep}}$	is the contribution to
$\langle P(t + \Delta t, X), \phi\rangle$	due to departures, and
$\langle P(t + \Delta t, X), \phi\rangle_{\text{nc}}$	is the contribution to
$\langle P(t + \Delta t, X), \phi\rangle$	due to no change in occupancy (no arrival and no departure).

Arrivals. Suppose that X results from the arrival of a single customer during the interval t to $t + \Delta t$. If the new customer was inserted at station i at time τ, $t \le \tau < t + \Delta t$, then the state of the system at time t was $X_{\Delta t} - i$, where

$$\begin{aligned}X_{\Delta t} - i = (S - i, X_1 + r(1 \mid S - i)\Delta t, \ldots, X_{\iota-1} + r(i - 1 \mid S - i)\Delta t, \\ X_{\iota+1} + r(i + 1 \mid S - i)\Delta t, \ldots, X_n + r(n \mid S - i)\Delta t),\end{aligned}$$

$r(i \mid S)$ is the service rate for the customer at station i given occupancy S, and $R(S) = \sum_{i=1}^{n} r(i \mid S)$. The new customer is of class $k(i)$, so his initial service requirement $Y_{k(i)}$ has probability measure $T_{k(i)}$. Customers of class $k(i)$ arrive in accordance with a Poisson process with rate $\lambda_{k(i)}$, so the probability of such an arrival is $\lambda_{k(i)}\Delta t + o(\Delta t)$. Station i was chosen with probability $a(i \mid S - i, k(i))$ conditional on the class $k(i)$ and the prior occupancy $S - i$. We thus have

$$\begin{aligned}\langle P(t + \Delta t, X), \phi\rangle_{\text{arr}} = \Delta t \sum_{\iota=1}^{n} \lambda_{k(\iota)} a(i \mid S - i, k(i)) \langle T_{k(\iota)}, \langle P(t, X - i), \phi\rangle_\iota\rangle \\ + o(\Delta t),\end{aligned}$$

where $\langle P(t, X - i), \phi\rangle_\iota = \int_{\Omega_{n_\iota}} \phi(X) P(t, X - i)\, dX^\iota$, Ω_{n_ι} is the image of Ω_n under the map $(x_1, \ldots, x_n) \to (x_1, \ldots, x_{\iota-1}, x_{\iota+1}, \ldots, x_n)$, $dX^\iota = \prod_{j=1, j\ne\iota}^{n} dX_j$, and $\langle T_{k(\iota)}, \Psi(x_\iota)\rangle = \int_0^{+\infty} \Psi(x_\iota) T_{k(\iota)}\, dx_\iota$ for any $\Psi \in C_B^{(1)}(0, +\infty)$. This notation will be used many times in the following discussion. The error incurred by the replacement of $X_{\Delta t} - i$ by $X - i$ is absorbed into $o(\Delta t)$.

No change. The state at time t must be

$$\begin{aligned}X_{\Delta t} &= (S, X_1 + r(1 \mid S)\Delta t, \ldots, X_n + r(n \mid S)\Delta t) \\ &= X + r(S)\Delta t.\end{aligned}$$

It is easily seen that

$$\langle P(t + \Delta t, X), \phi\rangle_{\mathrm{nc}} = \left(1 - \sum_{k=1}^{K} \lambda_k \Delta t - o(\Delta t)\right)\langle P(t, X + r(S)\Delta t), \phi\rangle.$$

Keeping in mind that the boundary of Ω_n is some positive distance from $\mathrm{supp}(\Phi)$, we have

$$\begin{aligned} P(t, X + r(S)\Delta t), \phi) &= \int_{\Omega_n} P(t, X + r(S)\Delta t)\phi(X)\, dX \\ &= \int_{\Omega_n} P(t, X)\phi(X - r(S)\Delta t)\, dX \\ &= \int_{\Omega_n} P(t, X)\left[\phi(X) - \Delta t \sum_{j=1}^{n} r(j|S)\frac{\partial\phi(X)}{\partial X_j}\right] dX + o(\Delta t) \\ &= \langle P(t, X), \phi\rangle + \Delta t \sum_{j=1}^{n} r(j|S)\left\langle\frac{\partial P(t, X)}{\partial X_j}, \phi\right\rangle + o(\Delta t) \end{aligned}$$

for Δt sufficiently small. Combining the above, we obtain

$$\begin{aligned} \langle P(t + \Delta t, X), \phi\rangle_{\mathrm{nc}} = \langle P(t, X), \phi\rangle &- \sum_{k=1}^{K} \lambda_k \Delta t\langle P(t, X), \phi\rangle \\ &+ \sum_{i=1}^{n}\left\langle\frac{\partial P(t, X)}{\partial X_i}, \phi\right\rangle r(i|S)\Delta t + o(\Delta t). \end{aligned}$$

Departures. If state X is to result from the departure of a single customer during the interval t to $t + \Delta t$, the customer must have occupied some station $j = 1, \ldots, n + 1$, been of some class k, and had remaining service time $0 < \tau < r(j|S + (j, k))\Delta t$. Thus

$$\langle P(t + \Delta t, X), \phi\rangle_{\mathrm{dep}} = \sum_{k=1}^{K}\sum_{j=1}^{n+1}\int_0^{r(j|S+(j,k))\Delta t} \langle P(t, X_{\Delta t} + (j, k, \tau)), \phi\rangle\, d\tau.$$

Since $P(t, X_{\Delta t} + (j, k, \tau))$ is right continuous in τ and Δt, we have

$$\lim_{\Delta t\to 0}\left[\frac{1}{\Delta t}\int_0^{r(j|S+(j,k))\Delta t} \langle P(t, X_{\Delta t} + (j, k, \tau)), \phi\rangle\, d\tau\right]$$

$$= \lim_{\Delta t\to 0} r(j|S + (j, k))\frac{1}{r(j|S + (j|k))\Delta t}\int_0^{r(j|S+(j,k))\Delta t} P(t, X_{\Delta t} + (j, k, \tau)), \phi\rangle\, d\tau$$

$$= \langle P(t, X + (j, k, 0+)), \phi\rangle r(j|S + (j, k)).$$

We are now in a position to produce the balance equation. Subtracting $\langle P(t, X), \phi\rangle$ from both sides of (3.1) and dividing by Δt, we compute the limit as Δt tends to 0. We then obtain the following time dependent balance equation:

$$\begin{aligned} \left\langle\frac{\partial P(t, X)}{\partial t}, \phi\right\rangle = &\sum_{i=1}^{n} \lambda_{k(i)} a(i|S - i, k(i))\langle T_{k(i)}, \langle P(t, X - i), \phi\rangle_i\rangle \\ &- \sum_{k=1}^{K} \lambda_k\langle P(t, X), \phi\rangle + \sum_{i=1}^{n}\left\langle\frac{\partial P(t, X)}{\partial X_i}, \phi\right\rangle r(i|S) \\ &+ \sum_{k=1}^{K}\sum_{j=1}^{n+1} (r(j|S + (j, k))\langle P(t, X) + (j, k, 0+), \phi\rangle \end{aligned}$$

for all $\phi \in \mathscr{H}_n$.

Because the system is assumed to be ergodic, it has a steady-state solution $P(X)$ and $\langle P(t, X), \phi\rangle \to \langle P(X), \phi\rangle$ as $t \to +\infty$ for every bounded continuous function ϕ on Ω_n. It is important to realize that the introduction of supplementary variables has made X a Markov process. Thus $P(S) = \int_{\Omega_n} P(X)\,dX$ basically represents the proportion of time that the system is in state S. If S did not have this ergodic parent Markov process, this interpretation of $P(S)$ would not be possible. Continuing with this idea of steady state, it is clear that

$$\begin{aligned}\lim_{t\to+\infty}\left\langle \frac{\partial P(t, X)}{\partial X_i}, \phi\right\rangle &= \lim_{t\to+\infty}\left\langle -P(t, X), \frac{\partial\phi}{\partial X_i}\right\rangle \\ &= -\left\langle P(X), \frac{\partial\phi}{\partial X_i}\right\rangle\end{aligned}$$

for all $\phi \in \mathscr{H}_n$. So there exists a unique linear functional on $\mathscr{H}_n$ which is denoted by $\partial P(X)/\partial X_i$ to which $\partial P(t, X)/\partial X_i$ converges (pointwise), and it is defined by

$$\left\langle \frac{\partial P(X)}{\partial X_i}, \phi\right\rangle = -\int_{\Omega_n} P(X)\frac{\partial\phi(X)}{\partial X_i}\,dX$$

for all $\phi \in \mathscr{H}_n$. Thus we have the following balance equation:

$$\begin{aligned}&\sum_{k=1}^{K}\left[\lambda_k\langle P(X), \phi\rangle - \sum_{j=1}^{n+1} r(j\,|\,S + (j, k))\langle P(X + (j, k, 0+)), \phi\rangle\right] \\ &\quad= \sum_{i=1}^{n}\left[\lambda_{k(i)}a(i\,|\,S - i, k(i))\langle T_{k(i)}, \langle P(X - i), \phi\rangle_i\rangle\right. \\ &\qquad\left. + r(i\,|\,S)\left\langle \frac{\partial P(X)}{\partial X_i}, \phi\right\rangle\right]\end{aligned} \tag{3.2}$$

This equation balances the loss in expectation of ϕ with the gain in expectation of ϕ. $\lambda_k\langle P(X), \phi\rangle$ is the loss in expectation of ϕ due to arrivals;

$$\sum_{j=1}^{n+1} r(j\,|\,S + (j, k))\langle P(X + (j, k, 0+)), \phi\rangle$$

is the gain in expectation of ϕ due to departures of such customers. On the right-hand side,

$$\lambda_{k(i)}a(i\,|\,S - i, k(i))\langle T_{k(i)}, \langle P(X - i), \phi\rangle_i\rangle$$

is the gain in expectation of ϕ due to arrivals of customers at station i, and $r(i\,|\,S)\langle\partial P(X)/\partial X_i, \phi\rangle$ is the loss due to service at station i.

$P(X)$ is said to satisfy *local balance* if

$$\lambda_k\langle P(X), \phi\rangle = \sum_{j=1}^{n+1} r(j\,|\,S + (j, k))\langle P(X + (j, k, 0+)), \phi\rangle \tag{3.3}$$

and

$$\sum_{i=1}^{n}\left[\lambda_{k(i)}a(i\,|\,S - i, k(i))\langle T_{k(i)}, \langle P(X - i, \phi\rangle_i\rangle + r(i\,|\,S)\left\langle \frac{\partial P(X)}{\partial X_i}, \phi\right\rangle\right] = 0$$

for all $\phi \in \mathscr{H}_n$. This balances the rate of loss of expectation of ϕ due to arrivals of customers of class k against the rate of gain of expectation of ϕ due to departures of such customers.

Similarly, $P(X)$ is said to satisfy *station balance* if

$$\lambda_{k(i)}a(i\,|\,S-i,k(i))\langle T_{k(i)}, \langle P(X-i),\phi\rangle_i\rangle = -r(i\,|\,S)\left\langle \frac{\partial P(X)}{\partial X_i},\phi\right\rangle \tag{3.4}$$

for $i = 1, \ldots, n$, all $\phi \in \mathscr{H}_n$, and all admissible states of S. This equation balances the gain in expectation of ϕ due to arrivals of customers to station i to the loss in expectation of ϕ due to departures from that station.

We say that $p(X)$ is in *product form* if

$$P(X) = Cq(S)\prod_{i=1}^{n}\lambda_{k(i)}(1-F_{k(i)}(X_i))$$

for all admissible $X = (S, X_1, \ldots, X_n)$, where C is a constant, $q(\phi) = 1$, and $q(S)$ is algebraically independent of the various X_i and λ_k. This means that $q(S)$ will be invariant under arbitrary changes in the λ_k and X_i. However, C may change owing to the normalization constraint.

Proceeding just as in [4], we get three forms of balance:

Product-form balance:

$$\sum_{k=1}^{K}\lambda_k\left[q(S)-\sum_{j=1}^{n+1}r(j\,|\,S+(j,k))q(S+(j,k))\right]\langle P(X),\phi\rangle$$
$$= -\sum_{i=1}^{n}\left[a(i\,|\,S-i,k(i))q(S-i)-r(i\,|\,S)q(S)\right]\left\langle P(X),\frac{\partial\phi}{\partial X_i}\right\rangle \tag{3.5}$$

for all $\phi \in \mathscr{H}_n$.

Product-form local balance:

$$q(S) = \sum_{j=1}^{n+1} r(j\,|\,S+(j,k))q(S+(j,k)),$$
$$\sum_{i=1}^{n} a(i\,|\,S-i,k(i))q(S-i) = \sum_{i=1}^{n} r(i\,|\,S)q(S), \tag{3.6}$$

for all admissible $S = (k(1), \ldots, k(n))$.

Product-form station balance:

$$a(i\,|\,S-i,k(i))q(S-i) = r(i\,|\,S)q(S) \tag{3.7}$$

for all admissible $S = (k(1), \ldots, k(n))$ and $i = 1, \ldots, n$.

We are now in a position to illustrate the proof techniques needed for extending the results of [4] to the case where the service requirements are no longer assumed to have differentiable distribution functions.

THEOREM 3.1. *A state probability density function satisfies station balance if and only if it is in product form and satisfies product-form station balance.*

PROOF. If one assumes product form and product-form station balance, it is trivial to verify (3.4). So assume that we have station balance. That is,

$$\lambda_{k(i)}a(i\,|\,S-i,k(i))\langle T_{k(i)}, \langle P(X-i),\phi\rangle_i\rangle = -r(i\,|\,S)\left\langle \frac{\partial P(X)}{\partial X_i},\phi\right\rangle$$

for all $\phi \in \mathscr{H}_n$. The proof is completed by induction on n, the number of customers in the queue. For $n = 0$ this is trivial, so let $n > 0$. Let

$$H_i^r(X_1, \ldots, X_r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \lambda_{k(j)}(1 - F_{k(j)}(X_j)).$$

Then on one side we have

$$-r(i \mid S)\left\langle \frac{\partial P(X)}{\partial X_i}, \phi \right\rangle = r(i \mid S)\left\langle P(X), \frac{\partial \phi}{\partial X_i} \right\rangle.$$

On the other side we have

$$\begin{aligned}
&\lambda_{k(i)} a(i \mid S - i, k(i)) \langle T_{k(i)}, \langle P(X - i), \phi \rangle_i \rangle \\
&= \lambda_{k(i)} a(i \mid S - i, k(i)) \left\langle \frac{dF_{k(i)}(X_i)}{dX_i}, \langle P(X - i), \phi \rangle_i \right\rangle \\
&= -\lambda_{k(i)} a(i \mid S - i, k(i)) \left\langle \frac{d(1 - F_{k(i)}(X_i))}{dX_i}, \langle P(X - i), \phi \rangle_i \right\rangle \\
&= \lambda_{k(i)} a(i \mid S - i, k(i)) \left\langle (1 - F_{k(i)}(X_i)), \frac{d\langle P(X - i), \phi \rangle_i}{dX_i} \right\rangle \\
&= \lambda_{k(i)} a(i \mid S - i, k(i)) \left\langle (1 - F_{k(i)}(X_i)), \left\langle P(X - i), \frac{\partial \phi}{\partial X_i} \right\rangle_i \right\rangle \\
&= Cq(S - i) a(i \mid S - i, k(i)) \left\langle H_{n+1}^{n+1}(X_1, \ldots, X_{n+1}), \frac{\partial \phi}{\partial X_i} \right\rangle.
\end{aligned}$$

The last equality was gained by way of induction hypothesis. So for all $\phi \in \mathscr{H}_n$,

$$\begin{aligned}
&r(i \mid S)\left\langle P(X), \frac{\partial \phi}{\partial X_i} \right\rangle \\
&\quad = Cq(S - i) a(i \mid S - i, k(i)) \left\langle H_{n+1}^{n+1}(X_1, \ldots, X_{n+1}), \frac{\partial \phi}{\partial X_i} \right\rangle.
\end{aligned}$$

Since $C_0^\infty(\Omega_n) \subset \{\partial\phi/\partial X_i : \phi \in \mathscr{H}_n\}$, we have

$$r(i \mid S)\langle P(X), \psi \rangle = Cq(S - i) a(i \mid S - i, k(i)) \langle H_{n+1}^{n+1}(X_1, \ldots, X_{n+1}), \psi \rangle$$

for all $\psi \in C_0^\infty(\Omega_n)$. Thus the two distributions are the same. That is,

$$r(i \mid S) P(X) = Cq(S - i) a(i \mid S - i, k(i)) \prod_{j=1}^{n} \lambda_{k(j)}(1 - F_{k(j)}(X_j)).$$

That is,

$$P(X) = C \frac{a(i \mid S - i, k(i))}{r(i \mid S)} q(S - i) \prod_{j=1}^{n} \lambda_{k(j)}(1 - F_{k(j)}(X_j)),$$

which says that $P(X)$ is in product form, and

$$q(S) = \frac{a(i \mid S - i, k(i)) q(S - i)}{r(i \mid S)},$$

which is product-form station balance. □

Again, it is to be emphasized that all of the theorems of [2] and [4] can be extended to the case of general service-time distributions in a similar fashion.

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