# Deterministic Edge Connectivity in Near-Linear Time* 

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October 30, 2018


#### Abstract

We present a deterministic algorithm that computes the edge-connectivity of a graph in near-linear time. This is for a simple undirected unweighted graph $G$ with $n$ vertices and $m$ edges. This is the first $o(m n)$ time deterministic algorithm for the problem. Our algorithm is easily extended to find a concrete minimum edge-cut. In fact, we can construct the classic cactus representation of all minimum cuts in near-linear time.

The previous fastest deterministic algorithm by Gabow from STOC' 91 took $\widetilde{O}\left(m+\lambda^{2} n\right)$, where $\lambda$ is the edge connectivity, but $\lambda$ can be as big as $n-1$. Karger presented a randomized near-linear time Monte Carlo algorithm for the minimum cut problem at STOC'96, but the returned cut is only minimum with high probability.

Our main technical contribution is a near-linear time algorithm that contracts vertex sets of a simple input graph $G$ with minimum degree $\delta$, producing a multigraph $\bar{G}$ with $\widetilde{O}(m / \delta)$ edges which preserves all minimum cuts of $G$ with at least two vertices on each side.

In our deterministic near-linear time algorithm, we will decompose the problem via low-conductance cuts found using PageRank a la Brin and Page (1998), as analyzed by Andersson, Chung, and Lang at FOCS'06. Normally such algorithms for low-conductance cuts are randomized Monte Carlo algorithms, because they rely on guessing a good start vertex. However, in our case, we have so much structure that no guessing is needed.


## 1 Introduction

In this paper we consider classic undirected graphs (i.e., no orientation for edges) where the edges are a set of unordered pairs of vertices. We refer to them as a simple graphs to distinguish them from multigraphs (or pseudographs) allowing parallel edges. For both cases, the edge-connectivity is the smallest number of edges whose removal disconnects the graph. This is a classic global reliability measure for the connectivity of a graph. The set of edges removed are the cut edges of a (global) minimum cut, or for short, a min-cut, and the two components we get when removing them are the sides of the cut. In this paper, we are assuming that the graph is connected, which is trivially checked in linear time.

[^0]Our main result is a deterministic near-linear time algorithm to find the edge connectivity and a global minimum cut of a simple graph. It is based on a new understanding of the cuts in simple graphs that does not hold for multigraphs.

### 1.1 Previous work

We will now discuss previous work on global min-cut algorithms. For the bounds we have $n$ vertices, $m$ edges, and (unknown) edge-connectivity $\lambda$. In the discussion, we consider both simple graphs and multigraphs, but our own results are only for simple graphs. The discussion also considers weighted graphs, where edges have weights. Then edge-connectivity is no longer relevant, but the size of a cut is the total weight of the cut edges. For weighted graphs, parallel edges can be merged adding up the weights, so weighted graphs may be assumed simple.

In 1961, Gomory and Hu [11] showed that the global minimum cut problem can be solved by computing $n-1$ independent minimum $s$ - $t$ cuts, that is, cuts with $s$ and $t$ on different sides. They let $s$ be an arbitrary vertex, and try with $t$ being any of other vertices. The point is that to find a minimum cut, they just have to guess a vertex $t$ on the side that $s$ does not belongs to. The $s$ - $t$ cuts are understood via Menger's classic theorem [25]. We can thus use any $s$ - $t$ cut algorithm. On a multigraph, if we use the classic augmenting path algorithm of Ford and Fulkerson [7], we should work in parallel on all $t$, doing the same number of augmenting paths to each $t$. After $\lambda$ augmentation rounds, for some $t$, we find an $s$ - $t$ cut of size $\lambda$ which is also a global min-cut. The total time is $O(\lambda n m)$. We could also apply the $O\left(\mathrm{~m}^{3 / 2}\right)$ time $s$ - $t$ min-cut algorithm of Even and Tarjan [6], and solve the global min-cut problem for multigraphs in $O\left(n m^{3 / 2}\right)$ time. This improves Ford-Fulkerson when $\lambda=\omega\left(\mathrm{m}^{1 / 2}\right)$.

The first algorithm to compute a global minimum cut faster than $n$ independent $s$ - $t$ cuts is the $O\left(\lambda n^{2}\right)$ time ${ }^{1}$ algorithm of Podderyugin [28] for simple graphs from 1973. For many years, this algorithm did not receive attention until it was rediscovered by Karzanov and Timofeev [21] and by Matula [23], independently.

In the 1990s, the above bounds for simple graphs were generalized to multigraphs and weighted graphs. In 1990, Nagamochi and Ibaraki [26] gave an $O\left(m+\min \left\{\lambda n^{2}, p n+n^{2} \log n\right\}\right)$ time global min-cut algorithm for multigraphs where $p \leq m$ is the number of pairs of vertices between which the graphs has an edge. For weighted graphs, they got a general bound of $O\left(n m+n^{2} \log n\right)$. Hao and Orlin [13] obtained an $O\left(n m \log \left(n^{2} / m\right)\right)$ time algorithm for the directed weighted case. Stoer and Wagner [31] and Frank [8], independently, presented a very simple algorithm finding a global min-cut of an undirected weighted graph within the same $O\left(n m+n^{2} \log n\right)$ time bound as in [26].

The current best deterministic algorithm for simple graphs is from 1991 due to Gabow [9] who, using [27] for preprocessing, gets down to $O\left(m+\lambda^{2} n \log (n / \lambda)\right)$ time. Gabow [9, pp. 268-269] also discuss multigraphs with a slightly worse bound of $O\left(m+\lambda^{2} n \log n\right)$. A linear time $(2+\varepsilon)$-approximation of the edge-connectivity was presented by Matula [24].

All the above-mentioned algorithms have been deterministic. The study of randomized algorithms for the global minimum cut problem was initiated by Karger [17]. We distinguish between Las Vegas algorithms with guaranteed correct answers but expected running times, and Monte Carlo algorithms with bounded running time but some small probability of incorrect answers. Las Vegas algorithms can be converted to Monte Carlo algorithms (making an error if the algorithm doesn't terminate within a certain time bound) so Las Vegas is generally prefered. If we have a way to certify correct answers, then we can also convert Monte Carlo to Las Vegas by rerunning if the certifyer fails. Karger [17] presented Las Vegas algorithms

[^1]yeilding a $(1+o(1))$-approximation to the global minimum cut in $O(m)$ expected time and an exact global minimum cut in $O(m \sqrt{\lambda})$ expected time. Karger and Stein [20] showed that random edge contraction works well for the global minimum cut problem, leading to a Monte Carlo algorithm running in $O\left(n^{2} \log ^{3} n\right)$ time. Finally, Karger [18] gave a randomized $O\left(m \log ^{3} n\right)$ time Monte Carlo algorithm for the global minimum cut problem. Karger [18] points out that we are lacking a correspondingly efficient way to certify the minimality of the returned cut.

For more detailed history for the global minimum cut problem, we refer the reader to the book by Schrijver [29]. We note that a deterministic near-linear time min-cut algorithm is known for planar graphs [4].

### 1.2 Main results

In this paper, we present a deterministic near linear time algorithm for computing the edge connectivity and a global minimum cut for a simple graph. By near-linear time, we mean $\widetilde{O}(m)$ time where $\widetilde{O}$ hides log factors. This is the first $o(m n)$ time deterministic algorithm for the problem. The previous best $\widetilde{O}\left(m+\lambda^{2} n\right)$ time bound of Gabow [9] is good if $\lambda$ is small, but we may have $\lambda=\Omega(n)$.

We note that in this paper, we are not trying to minimize the number of log factors hidden in the $\widetilde{O}$ notation, nor do we count them explicitely, e.g., we will freely use reductions like $\widetilde{O}(f(n)) \widetilde{O}(g(n))=$ $\widetilde{O}(f(n) g(n))$, At the end, we will loosely estimate the number to 12 , thus estimating the running time of our algorithm to $O\left(m \log ^{12} n\right)$. The purpose of this estimate is only to encourage other researchers to get down to a more reasonable number of log factors, providing them a concrete bound to improve on. Indeed Henzinger et al. [14] have recently improved important parts of our construction bringing the total running time down to $O\left(m(\log n)^{2}(\log \log n)^{2}\right)$.

In near-linear time we can also compute the cactus representation of all global minimum cuts introduced in [5]. To do so we involve the previous fastest $\widetilde{O}(\lambda m)$ time algorithm by Gabow [10] as a black-box. We note here that Karger and Panigrahi [19] have presented a near-linear time Monte Carlo algorithm for constructing the cactus data structure.

### 1.3 Technical Result

Henceforth, we are only considering unweighted graphs. We are given a simple graph, and we want to find a min-cut in near-linear time. We may assume that the minimum degree $\delta$ is at least polylogarithmic; for otherwise, we can just use Gabow's [9], algorithm to find the min-cut in $\widetilde{O}(\lambda m)=\widetilde{O}(\delta m)=\widetilde{O}(m)$ time. For our purposes, it suffices to assume $\delta \geq \log ^{6} n$.

By a trivial cut, we mean a cut where one side consists of a single vertex. We note that the minimum degree $\delta$ is an upper bound on the edge-connectivity $\lambda$ since it is the smallest size of a trivial cut. Finding $\delta$ is trivial, but we could have $\lambda<\delta$.

What makes simple graphs special is that if you have a non-trivial cut in a simple graph, then each side needs to have at least $\delta$ vertices. This observation and its relation to conductance will be discussed below in Section 1.4

By contracting a vertex set $U \subseteq V$, we mean identifying the vertices in $U$ while removing the edges between them. We may not check that $U$ is connected, so this may not correspond to edge contractions. The identity of edges not removed are preserved, and a cut is preserved if all the cut edges are preserved.

The basic idea in our min-cut algoritm is do contractions in the graph while preserving all non-trivial min-cuts, continuing until the graph is so small that Gabow's algorithm can finish in $\widetilde{O}(m)$ time where $m$ is
the number of edges in the original simple input graph. The contractions may introduce parallel edges, so internally, the algorithm works with a multigraph.

Our main technical contribution is to prove the following theorem:
Theorem 1 Given a simple input graph $G$ with $n$ vertices, $m$ edges, and minimum degree $\delta$, in near-linear time, we can contract vertex sets producing a multigraph $\bar{G}$ which has only $\bar{m}=\widetilde{O}(m / \delta)$ edges, yet which preserves all non-trivial min-cuts of $G$.

From Theorem 1 , we easily get our near-linear min-cut algorithm:
Corollary 2 We can find a minimum cut of a simple graph $G$ in near-linear time.
Proof Let $\delta$ be the minimum degree of $G$. We apply the Theorem 1 to $G$ producing the graph $\bar{G}$. We now run Gabow's min-cut algorithm [9] on $\bar{G}$, asking it to fail if the edge-connectivity is above $\delta$. This takes $\widetilde{O}(\delta \bar{m})=\widetilde{O}(m)$ time, and now we compare the output with the minimum degree $\delta$.

Likewise, in near-linear time, we can obtain the cactus representation of all global minimum cuts from [5] by applying the cactus algorithm of Gabow [10] to $\bar{G}$. Having produced the cactus $\bar{C}$ of $\bar{G}$, we just need to add min-degree vertices as extra needles so as to get the cactus of the input graph $G$. A description of this including the definition of the min-cut cactus is given in Section 8 .

We note that Theorem 1 cannot hold if the input graph is a multigraph. To see this, consider a cycle of length $n \geq 4$, but where every edge is replaced by $k=(\log n)^{\omega(1)}$ parallel edges. Now every edge is involved in a non-trivial min-cut where each side consists of at least two consecutive vertices, and therefore no edges can be contracted. This shows that the contractions of Theorem 1 are very specific to simple graphs. Also, they can only preserve non-trivial min-cuts, for if we, for example, take a complete graph, then every edge is in a trivial min-cut. However, a complete graph can be contracted to a single vertex since it has no non-trivial min-cut.

Further results While the reduction in Theorem 1 of the number of edges looks like a typical sparsification, it is not, for edges are contracted, not deleted, and the resulting $\bar{G}$ will have much fewer vertices than G. In fact, combining with techniques of Nagamochi and Ibaraki [27], we can make sure that the number of vertices remaining after the contractions is $\widetilde{O}(n / \delta)$. Since the number of min-cuts in any multigraph is at most quadratic, a nice consequence is that that the number of min-cuts in a simple graph with $n$ vertices and minimum degree $\delta$ is at most $n+\widetilde{O}\left((n / \delta)^{2}\right)$. We are not aware of anyone else that has observed that a large minimum degree in a simple graph implies few minimum cuts, though it does apppear that this fact could also be derived from the cactus representation [5].

Our contraction technique can be strengthend to also preserve approximately min-cuts. More precisely, with min-degree $\delta$ and edge connectivity $\lambda$, we will strengthen Theorem 1 to preserve all non-trivial cuts of size at most $\lambda+(1-\varepsilon) \delta$, where $\varepsilon$ is an arbitrarily small positive constant. Since $\lambda \leq \delta$, this implies that we preserve all $(2-\varepsilon)$-approximate min-cuts. Formally, we will prove:

Theorem 3 Let $\varepsilon \in(0,1]$ be a constant. Given a simple input graph $G$ with $n$ vertices, $m$ edges, minimum degree $\delta$, and (unknown) edge connectivity $\lambda$, in $\widetilde{O}(m)$ time, we can contract vertex sets producing a multigraph $\bar{G}$ which has only $\widetilde{O}(n)$ edges and $\widetilde{O}(n / \delta)$ vertices, yet which preserves all non-trivial cuts of size below $\lambda+(1-\varepsilon) \delta$.

We know from [15] that for a multigraph with $n$ vertices and edge connectivity $\lambda$, the number cuts of size $3 \lambda / 2$ is $O\left(n^{2}\right)$. Applying this to the contracted graph from Theorem 3 with $\varepsilon=1 / 2$, we get

Corollary 4 In simple graph with with $n$ vertices, minimum degree $\delta$, and edge connectivity $\lambda$, there are at most $n+\widetilde{O}\left((n / \delta)^{2}\right)$ cuts of size at most $3 \lambda / 2$.

The ability to preserve approximate min-cuts with Theorem 3 has been used for the amortization in a recent efficient algorithm [12] to maintain a min-cut of a simple graph incrementally, paying only polylogarithmic time per edge insertion.

### 1.4 Minimum cuts and low conductance

Our approach to finding a minimum cut involves cuts of low conductance, defined below. Generally we define a cut by specifying one side $U \subset V$. Then the other side $T=V \backslash U$ is implicit. No side is allowed to be empty. Algorithmically, it will typically be the smaller side that we specify explicitly. The edges leaving $U$ are the cut edges, and the set of cut edges is denoted $\partial U=\partial T$. The size of the cut is the number of cut edges $|\partial U|$. We do not require that any side of a cut remain connected if the cut edges are removed.

We are also interested in the sum of the degrees of vertices in $U$ called the volume of $U$ defined as

$$
\operatorname{vol}(U)=\sum_{v \in U} d(v)
$$

Edges with both end-points in $U$ are called internal to $U$, and they are counted twice in the volume of $U$.
Now the conductance of $U$ is defined by

$$
\Phi(U)=\frac{|\partial U|}{\min \{\operatorname{vol}(U), \operatorname{vol}(T)\}}=\Phi(T)
$$

Observation 5 Let $S$ be the smaller side of a min-cut of our simple graph $G$. Then either the cut is trivial with $S$ consisting of a single vertex, or $S$ has volume at least $\delta^{2}$ and the conductance is $\Phi(S) \leq 1 / \delta$.

Proof The graph has minimum degree $\delta$ so the min-cut has at most $\delta$ edges. Since $G$ is simple, a vertex $v \in S$ has at least $\delta-(|S|-1)$ edges leaving $S$. The total number of edges leaving $S$ is thus at least $|S|(\delta+1-|S|)$, and for this to be at most $\delta$, we need $|S|=1$ or $|S| \geq \delta$. In the latter case, we have $\operatorname{vol}(S) \geq \delta^{2}$, so $\Phi(S) \leq 1 / \delta$.

### 1.5 Certify-or-cut

In our algorithm, we are going to assume that the simple input graph $G$ has minimum degree

$$
\delta \geq \lg ^{6} n
$$

By Observation 5, this means that any non-trivial min-cut has very low conductance. With this in mind, we are going to devise a near-linear time deterministic "certify-or-cut" algorithm that will either

1. Certify that there are no non-trivial min-cuts. In particular, this witnesses that any min-degree vertex forms the side of a global min-cut, or
2. Find a low-conductance cut.

We note that each of the above tasks alone is beyond our current understanding of deterministic algorithms. For the first certification task, recall the issue mentioned by Karger [18] that we have no efficient deterministic way of certifying that a proposed minimum cut is indeed minimum. Our task is no easier, for if it was, to certify that a cut of size $k \leq \delta$ is minimum, we could attach a complete graph on $k$ vertices, where $k-1$ of the vertices are new. Each new vertex defines a trivial cut of size $k-1$, and the edge connectivity of the original graph is $k$ if and only if there is no non-trivial minimum cut in the new graph.

For the second task, we want to find a low-conductance cut, e.g., using PageRank [3] as analyzed by Andersson, Chung, and Lang [2]. However, such algorithms for low-conductance cuts are randomized Monte Carlo algorithms, because they rely on guessing a good start vertex. For cut-or-witness, however, we only have to find a low conductance cut if we fail to witness the minimality of the trivial cuts, but then we will have so much structure that no guessing is needed.

Our certify-or-cut algorithm will illustrate some of the basic techniques presented in this paper, including a study of what happens in the endgame of PageRank when most mass has been distributed, yet some vertex is still left out.

### 1.6 The overall algorithm

We will now sketch the basic ideas by using a more elaborate certify-or-cut algorithm for finding a minimum cut, and also point to the issues that arise.

Given a component $C$ of subgraph $H$ of $G$, suppose we can either

1. certify that $C$ is a so-called "cluster" implying that no min-cut of $G$ induces a non-trivial cut of $C$, or
2. find a cut of $C$ of conductance $o(1 / \log m)$.

Then, starting from $H=G$, we will recursively remove the low-conductance cuts, until we have a subgraph $H$ of $G$ where all the components are certified clusters. Inside each cluster $C$, we will identify a so-called "core" $A$ with the property that no non-trivial min-cut of $G$ makes any cut of $A$ (let us observe that $A$ may not be all of $C$ because a non-trivial min-cut of $G$ could induce a trivial cut of $C$ ). Cores can therefore be contracted without affecting any non-trivial min-cut of $G$.

The important observation here is that when removing the low-conductance cuts, most edges survive in $H$. The same observation was used in Spielman and Teng's spectral sparsifiers [30], though they used randomization to find the low-conductance cuts. The reason that only few edges get removed by recursive low-conductance cuts is that we can amortize the edges removed over the edges incident to the smaller side where smaller is measured in terms of volume, that is, number of incident edges. Each edge incident to the smaller side pays $o(1 / \log m)$ (because of the low-conductance cuts), and it can end on the smaller side at most $\lg m$ times, where $\lg =\log _{2}$. The total fraction of edges cut is thus $o(1)$, so most edges remain when we are done removing low-conductance cuts, certifying that each remaining component of $H$ is a cluster. This is important because we want many edges to be contracted when we contract the cores of the clusters in $H$.

We now point out the issues we have to address. The first issue is that as edges get removed, the degrees of the remaining vertices will decrease, and then the minimum degree could fall below $\lg n$, so we can no longer use Observation 5 to conclude that a non-trivial cut has conductance $o(1 / \log m)$. Our fix to this issue will be to not only remove cut edges, but also "trim" the resulting components, removing all vertices that have lost $3 / 5$ of their original edges. As we shall see, this will only increase the number of edges removed by a factor 5 , so most edges will still remain in the final clusters.

The second issue happens when we contract the cluster cores in a graph $\bar{G}$ that preserves all the nontrivial min-cuts of $G$. This may introduce parallel edges, and hence Observation 5 fails completely, e.g., consider a path of length 4 where consecutive vertices are connected by $\delta$ parallel edges. A non-trivial min-cut with two vertices on each side has conductance $1 / 2$. We will, however, argue that if a vertex is dominated by parallel edges, then it is somehow done and can be ignored.

Handling the above two complications will also force us to adopt a more complicated notion of a cluster, but our algorithm will still follow the basic pattern of the above sketch.

The goal is to contract cluster cores until $\bar{G}$ has only $\widetilde{O}(m / \delta)$ edges, yet preserves all non-trivial mincuts from $G$, as desired for Theorem 1. To find a minimum cut of $G$, we finish by applying Gabow's algorithm [9] as described in Corollary 2,

### 1.7 Recent improvement of Henzinger et al.

After this work was announced at STOC' 15 [22], Henzinger et al. [14] have improved the concrete running time from $O\left(m \log ^{12} n\right)$ to $O\left(m(\log n)^{2}(\log \log n)^{2}\right)$, which also improves the $O\left(m \log ^{3} n\right)$ time bound of Karger's Monte Carlo algorithm [18]. The algorithm of Henzinger et al. uses the overall approach developped in this paper. However, instead of using a diffusion based PageRank like us to find low-conductance cuts, they use an interesting local flow based algorithm. We note that diffusion and flow based algorithms have different advantages in different settings, and we hope that our novel use and analysis of the PageRank diffusion will inspire other algorithmic applications.

### 1.8 Notations

As a generic notation, if we have some graph parameter like the edge connectivity $\lambda$, we may use a subscript to specify which graph it is measured on as in $\lambda_{H}$ for the edge connectivity of the graph $H$. We may also put $H$ in paranthesis, e.g., we use $V(H)$ and $E(H)$ to denote the vertices and edges in $H$, and let $n(H)=|V(H)|$ and $m(H)=|E(H)|$ denote the number of vertices and edges in $H$.

To simplify calculations, we will make use of $O-, O$-, and $\widetilde{O}$-notation to hide constants and $\log$ factors when they are not important to our results. Starting with $O$-notation, by definition, $f_{i}(n)=O\left(g_{i}(n)\right)$ means that there constants $n_{i}$ and $c_{i}$ such that $n \geq n_{i}$ implies $f_{i}(n) \leq c_{i} g(n)$. If we have this for $i=1,2$, then $f_{1}(n) f_{2}(n)=O\left(g_{1}(n) g_{2}(n)\right)$ since $n \geq \max \left\{n_{1}, n_{2}\right\}$ implies $f_{1}(n) f_{2}(n)=c_{1} c_{2} g_{1}(n) g_{2}(n)$. Instead of making $f_{1}$ and $f_{2}$ explicit, we can write this rule as $O\left(g_{1}(n)\right) O\left(g_{2}(n)\right)=O\left(g_{1}(n) g_{2}(n)\right)$. Likewise, we have $O\left(g_{1}(n)\right)+O\left(g_{2}(n)\right)=O\left(g_{1}(n)+g_{2}(n)\right)$ and $O(O(g(n)))=O(g(n))$, all illustrating how $O$-notation simplifies calculations. When we say that $n$ is large enough, we mean that it is bigger than any of the $n_{i}$ used in our analysis.

The assumption of a large enough $n$ becomes more important when we combine $O$-notation with $o$ notation. By definition, $f_{i}(n)=o\left(g_{i}(n)\right)$ means that for any constant $c_{i}$ there is a constant $n_{i}$ such that $n \geq n_{i}$ implies $f_{i}(n)<g(n) / c_{i}$. Suppose we have $f(n)=o(g(n))$ and $g(n)=O(h(n))$. Then there exists an $n_{0}$ such that $n \geq n_{0}$ implies $f(n)<h(n)$. More precisely, from $g(n)=O(h(n))$ we get that there are constants $n_{1}$ and $c_{1}$ such that $n \geq n_{1}$ implies $g(n) \leq c_{1} h(n)$. Next, from $f(n)=o(g(n))$, we get that there is a constant $n_{2}$ depending on $c_{1}$ such that $n \geq n_{2}$ implies $f(n)<g(n) / c_{1}$. Thus $n \geq n_{0}=\max \left\{n_{1}, n_{2}\right\}$ implies $f(n)<h(n)$. Our most common use of $n$ being sufficiently large is that $o(1)$ becomes smaller than any concrete constant.

As usual, we have the derived notations $f(n)=\Omega(g(n)) \quad \Longleftrightarrow \quad g(n)=O(f(n)), f(n)=$ $\Theta(g(n)) \Longleftrightarrow f(n)=O(g(n)) \wedge f(n)=\Omega(g(n))$, and $f(n)=\omega(g(n)) \Longleftrightarrow g(n)=o(f(n))$.

Finally, we have the $\widetilde{O}$-notation, where $f_{i}(n)=\widetilde{O}\left(g_{i}(n)\right)$ means that there constants $n_{i}$ and $c_{i}$ such that $n \geq n_{i}$ implies $f_{i}(n) \leq g(n) \lg ^{c_{i}} n$. As for the $O$-notation, we get simplyfying rules like $\widetilde{O}\left(g_{1}(n)\right) \widetilde{O}\left(g_{2}(n)\right)=\widetilde{O}\left(g_{1}(n) g_{2}(n)\right), \widetilde{O}\left(g_{1}(n)\right)+\widetilde{O}\left(g_{2}(n)\right)=\widetilde{O}\left(g_{1}(n)+g_{2}(n)\right)$ and $\widetilde{O}(\widetilde{O}(g(n)))=$ $\widetilde{O}(g(n))$. As for $O$-notation, we are going to use the derived notations $f(n)=\widetilde{\Omega}(g(n)) \Longleftrightarrow g(n)=$ $\widetilde{O}(f(n))$ and $f(n)=\widetilde{\Theta}(g(n)) \Longleftrightarrow f(n)=\widetilde{O}(g(n)) \wedge f(n)=\widetilde{\Omega}(g(n))$. We are only going to use $\widetilde{O} / \widetilde{\Omega} / \widetilde{\Theta}$-notation to hide $\lg n$ factors where $n$ denotes the number of vertices in the simple input graph for which we want to find a minimum cut.

### 1.9 Contents

This paper is structured as follows. First we will show how to implement the certify-or-cut algorithm described above, since it introduces most of the interesting new ideas in a quite clean form. To do so, we will first describe our view of PageRank in Section 2, which includes a new theorem on the endgame. Next we describe the certify-or-cut algorithm in Section 3. After this warm-up, we are ready to present the recursive set-up for our contraction based min-cut algorithm in Section 4. To complete the min-cut algorithm, we present the use of low-conductance cuts in Section 5. In Section6, we show how we can also preserve approximate min-cuts. Finally, in Section 7 we prove the PageRank theorems claimed in Section 2. A cactus construction is given in Section 8.

## 2 Sparse cuts by PageRank

We are going to find sparse cuts using the PageRank algorithm from [2]. We will be running it on a multigraph with $m$ edges. This will be a subroutine of our min-cut algorithm, applied to different minors of the original simple input graph.

The PageRank algorithm is operating with a mass distributions $p \in \mathbb{R}_{\geq 0}^{V}$ assigning non-negative mass to the vertices. Given a subset $U$ of the vertices, $p(U)=\sum_{v \in U} p(v)$ denotes the total mass on the subset. We refer to $p(U) / \operatorname{vol}(U)$ as the density on $U$. For an individual vertex $v$, the density is $p(v) / d(v)=$ $p(v) / \operatorname{vol}(\{v\})$.

We start with some initial mass distribution $p^{\circ} \in \mathbb{R}^{V}$ on the vertices. Most of the time, the total mass is normalized to 1 , corresponding to a probability distribution.

The algorithm has a parameter $\alpha$ called the teleportation constant, and we assume $\alpha \leq 1 / 3$. In our min-cut algorithm, we will have $\alpha=\widetilde{\Omega}(1)$, but this is not assumed for the results in this section.

The algorithm operates by moving mass between two mass distributions: a residual mass $r$ which is initialized as the initial distribution $p^{\circ}$, and a settled mass $p$ which is initially zero on all vertices. Generally we say that the density of mass on a vertex is the mass divided by the degree

The algorithm works by pushing residual mass from vertices. To push the residual mass on $u$, we first settle a fraction $\alpha$ of the residual mass on $u$, and then we spread half the remaining residual mass evenly to the neighbors of $u$. This is described in Algorithm 1. The overall algorithm is flexible in that we can

```
Algorithm 1: \(\operatorname{Push}(\alpha, u)\)
    \(p(u) \leftarrow p(u)+\alpha r(u) ;\)
    for \((u, v) \in E\) do \(r(v) \leftarrow r(v)+(1-\alpha) r(u) /(2 d(u))\);
    \(r(u) \leftarrow(1-\alpha) r(u) / 2\).
```

apply pushes to the vertices in any order we want. To control the amount of work done, [2] introduces a
parameter $\varepsilon$, and they only push from a vertex $u$ if the residual density $r(u) / d(u)$ is at least $\varepsilon$. The resulting PageRank algorithm is described in Algorithm 22 As noted in [2], the time to do a push at $u$ is $d(u)$ and it

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Algorithm 2: \(\operatorname{Page\operatorname {Rank}(\alpha ,\varepsilon ,p^{\circ })~}\)
    \(r \leftarrow p^{\circ} ; \quad p \leftarrow 0^{V} ;\)
    while \(\exists u: r(u) / d(u) \geq \varepsilon\) do \(\operatorname{Push}(\alpha, u)\);
```

settles $\alpha r(u) \geq \alpha d(u) \varepsilon$ of the residual mass. If we thus start with a total residual mass at most 1 , the total amount of work is $O(1 /(\alpha \varepsilon))$. This does assume, however, that $p^{\circ}$ is presented in such a way that we have direct access to vertices density $\varepsilon$ or more. In fact, we typically assume that the vertices with positive mass are listed in order of non-increasing initial density. Then, for any $\varepsilon$, we find those with initial density above $\varepsilon$ as a prefix of this list.

As $\varepsilon$ approaches 0 , the residual mass vanishes, and then, as proved in [2], the settled mass approaches a unique limit denoted $\operatorname{PR}\left(\alpha, p^{\circ}\right)$ that we refer to as the limit mass distribution. The limit mass distribution will play an important role in our analysis, but algorithmically, we will only run the PageRank from Algorithm 2 with $\varepsilon=1 / \log { }^{O(1)} n$. From [2], we get that pushes maintain the following invariant:

$$
\begin{equation*}
\operatorname{PR}\left(\alpha, p^{\circ}\right)=p+\operatorname{PR}(\alpha, r) \tag{1}
\end{equation*}
$$

From [2] we know that $\operatorname{PR}(\alpha, \cdot)$ is a non-negative linear transformation $\mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$, that is, for any teleportation constant $\alpha$, there is an $n \times n$ matrix $M_{\alpha}$ with real non-negative entries such that for any initial distribution vector $p^{\circ}$, we get the limit distribution $\operatorname{PR}\left(\alpha, p^{\circ}\right)=p^{\circ} M_{\alpha}$. For any $\sigma \in \mathbb{R}$, let $\bar{\sigma}$ be the distribution where all vertices have density $\sigma$. From [2] we know that $\bar{\sigma}$ is a fix-point for $\operatorname{PR}(\alpha, \cdot)$, that is, $\operatorname{PR}(\alpha, \bar{\sigma})=\bar{\sigma}$, and we call it a stationary distribution.

Mass can only be moved and settled via pushes. Consider an edge $(u, v) \in E$. Viewing it as directed from $u$ to $v$, we get a positive flow when we push from $u$, pushing $(1-\alpha) r(u) /(2 d(u))$ mass over $(u, v)$ to $v$ while settling $\alpha r(u)$ mass at $u$. Likewise we get a negative flow over $(u, v)$ when we push from $v$. Hence

Fact 6 Recall that $p($.$) is the settled mass. After any sequence of pushes for any (u, v) \in E$, the total net flow of mass over $(u, v)$ is $\frac{1-\alpha}{2 \alpha}(p(u) / d(u)-p(v) / d(v))$.

An important consequence is
Lemma 7 If at some point all residual densities are bounded by $\sigma$, then from this point forward, the net flow over any edge is at most $\sigma /(2 \alpha)$.

Proof The residual distribution $r$ is bounded by the stationary distribution $\bar{\sigma}$ with densities $\sigma$, so $\operatorname{PR}(\alpha, r) \leq \operatorname{PR}(\alpha, \bar{\sigma})=\bar{\sigma}$ where $\leq$ is vector domination. If $p$ is a mass distribution settled from $r$, then $p \leq \operatorname{PR}(\alpha, r) \leq \bar{\sigma}$, so $p(u) / d(u)-p(v) / d(v) \leq \sigma$ for every possible edge $(u, v) \in E$. By Fact 6, the net flow over $(u, v)$ based on $r$ is therefore at most $\sigma / 2 \alpha$.

We are going to find the side $S$ of a low-conductance cut via a so-called "sweep" over the settled mass distributions $p$. To describe the sweep, as general notation, for any comparison operator $\circ \in\{=,<,>, \leq, \geq\}$ and $t \in \mathbb{R}$, define

$$
V_{\circ t}^{p}=\{u \in V \mid p(u) / d(u) \circ t\}
$$

e.g, $V_{\geq t}^{p}=\{u \in V \mid p(u) / d(u) \geq t\}$. Now let $\Phi(p)$ be the smallest conductance we can obtain by picking some threshold $\tau \in[0,1]$, and considering the set of vertices with density at least $\tau$, that is,

$$
\Phi(p)=\min _{\tau \in[0,1]} \Phi\left(V_{\geq \tau}^{p}\right) .
$$

To find $\Phi(p)$, we sweep over the vertices in order of non-increasing settled density. We only have to consider vertices with positive settled mass, including their incident edges, of which there are only $O(1 /(\alpha \varepsilon))$ assuming that the total initial mass is 1 . As described in [2], we can implement the sweep in $O((\log n) /(\alpha \varepsilon))$ time, and we shall further bring the sweep time down to $O(1 /(\alpha \varepsilon))$ in Section 7.1 The important question is, however, when does the sweep give us a cut of low conductance? We will give some sufficient conditions in the next subsection.

### 2.1 Limit concentration and low conductance

We now state conditions under which a PageRank algorithm starting from an initial distribution $p^{0}$ can find a low conductance cut. The conditions are all based on the limit mass distribution $p^{*}=\operatorname{PR}\left(\alpha_{0}, p^{\circ}\right)$. While this limit is unique, there are different ways of running a PageRank, e.g., the choice and use of $\varepsilon$ in Algorithm 2 and which cut we choose to return from the sweep.

As in [2], we first study situations where the limit mass on some set $S$ deviates significantly from the uniform $\operatorname{vol}(S) /(2 m)$, as quantified by

$$
\operatorname{excess}\left(p^{*}, S\right)=p^{*}(S)-\operatorname{vol}(S) /(2 m)
$$

It may seem surprising that we look at this additive excess, rather than the multiplicative difference, but imagine that we have an initial distribution placing the mass 1 on a single vertex $v$ of minimum degree. Then the first push will settle mass $p(v)=\alpha$ on $v$, and in the limit $p^{*}>p(v)=\alpha$. However, on the average $v$ should only have mass $\operatorname{vol}(v) /(2 m) \leq 1 / n$ and in our min-cut algorithm, we will have $\alpha=\widetilde{\Omega}(1)$, so the multiplicative difference on $\{v\}$ is huge.

We think of the additive excess as the mass that gets trapped in $S$ when we push to the limit. As proved in [2], a large excess can only happen if there is a low conductance cut somewhere, and then we can find some low conductance cut efficiently.

The basic result, formalized below in Theorem 8 is that if there exists a set $S$ with limit excess at least $\gamma$, then using a PageRank algorithm, we can find a set $T$ which is the smaller (in volume) side of a cut with conductance

$$
\Phi(T)=O(\sqrt{(\alpha \log m) / \gamma})
$$

in time $O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$. Inside our min-cut algorithm, we will have $\gamma, \alpha=\widetilde{\Omega}(1)$ and then $T$ is found in time $\widetilde{O}(\operatorname{vol}(T))$, so the algorithm is fast if $T$ is small. It is here important that $T$ is the smaller side of the cut. We are not allowed to spend a long time reporting the bigger side instead.

If we are further given an upper bound $s \leq m \gamma / 16$ on the volume of $S$, then the volume of the returned set $T$ has volume bounded by $8 s / \gamma=\widetilde{O}(s)$. In this case, it is further guaranteed that $T$ has excess at least $\gamma /(16 \lg (4 s))$.

When we start running the algorithm, we may not know if the set $S$ exists. If it doesn't exist, the algorithm will either return $T$ as described above, or certify that no such $S$ exists. Inside our min-cut algorithm, there will be cases where certifying the non-existence of $S$ is the prefered outcome. We note that if there is no set $T$ with $\operatorname{vol}(T) \leq 8 s / \gamma$ and excess $\left(p^{*}, T\right) \geq \gamma /(16 \lg (4 s))$, then we must get the certificate that there is no set $S$ with $\operatorname{vol}(S) \leq s$ and $\operatorname{excess}\left(p^{*}, S\right) \geq \gamma$.

The maximal running time of the algorithm is $O(m /(\gamma \alpha))=\widetilde{O}(m)$ without a volume bound, and $O(s /(\gamma \alpha))=\widetilde{O}(s)$ time with the volume bound $s$. This is all formalized in the following theorem which is similar to results proved in [2].

Theorem 8 We are given a multigraph with $m$ edges, an initial mass distribution $p^{\circ}$ of total mass 1 , and a listing of the vertices with positive mass in order of non-increasing density. Let $p^{*}=\operatorname{PR}\left(\alpha, p^{\circ}\right)$. We are also given an excess parameter $\gamma<1$. We have a PageRank algorithm that staring from $p^{\circ}$ will either find $a$ set $T$ with $\operatorname{vol}(T) \leq m$ and conductance

$$
\Phi(T)=O(\sqrt{(\alpha \log m) / \gamma}),
$$

or certify that there is no set $S$ with

$$
\operatorname{excess}\left(p^{*}, S\right) \geq \gamma
$$

The maximal running time is $O(m /(\gamma \alpha))$, but if a set $T$ is returned, then the time is also bounded by $O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$.

If we are further given a volume parameter $s \leq m \gamma / 16$, the algorithm will either find the above $T$ with the additional guarantees that $\operatorname{vol}(T) \leq 8 s / \gamma$ and $\operatorname{excess}\left(p^{*}, T\right) \geq \gamma /(16 \lg (4 s))$, or certify there is no set $S$ with $\operatorname{vol}(S) \leq s$ and $\operatorname{excess}\left(p^{*}, S\right) \geq \gamma$. The maximal running time is $O(s /(\gamma \alpha))$, but if a set $T$ is returned, then the time is also bounded by $O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$.

The proof of Theorem 8 is deferred to Section 7 Without the running time and the part with the volume parameter $s$, Theorem 8 follows from Theorem 4.1 in [2]. However, when it comes to finding low-conductance cuts, [2] is focussed on the case where the initial distribution has all mass on a single "good" vertex. Here we need to find low-conductance cuts starting from initial distributions spreading the mass on many vertices, as supported by our Theorem 8, Another new point in the last part of Theorem 8 is that the set $T$ has $\operatorname{excess}\left(p^{*}, T\right) \geq \gamma /(16 \lg (4 s))$. This will later be critical to the proof of Lemma 27 and 29. The statement of of Theorem 8 is thus tailored for the needs of our min-cut algorithm, but it can be proved using techniques from [1, 2].

The endgame More interesting and novel, we study here the "endgame" of the PageRank algorithm. Suppose there is a single vertex $u$ that even in the limit receives too little mass. Then this is because there is some low conductance cut that prevents mass from reaching $u$. More precisely, suppose there is just a single vertex $u$ with low density

$$
p^{*}(u) / d(u) \leq(1-\gamma) /(2 m) .
$$

Then we will find a low conductance cut like that in Theorem 8 that is, a set $T$ which is the smaller side of a cut with conductance

$$
\Phi(T)=O(\sqrt{(\alpha \log m) / \gamma}) .
$$

We will either find $T$ efficiently in time $O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$ as in Theorem 8 , or in time $O(m /(\gamma \alpha))$ but with the guarantee that $T$ contains all small density vertices $u$ with $p^{*}(u) / d(u) \leq(1-\gamma) /(2 m)$.

If there is no low density vertex $u$, then the algorithm will either return $T$ as described above, or certify that that there is no low density vertex $u$. The maximal running time of the algorithm is $O(m /(\gamma \alpha))$. The full formal details are presented in the theorem below.

Theorem 9 We are given a multigraph with $m$ edges, an initial mass distribution $p^{\circ}$ of total mass 1 , and a listing of the vertices with positive mass in order of non-increasing density. Let $p^{*}=\operatorname{PR}\left(\alpha, p^{\circ}\right)$. We are
also given a parameter $\gamma<1$. We have a PageRank algorithm that staring from $p^{\circ}$ will either find a set $T$ with $\operatorname{vol}(T) \leq m$ and conductance

$$
\Phi(T)=O(\sqrt{(\alpha \log m) / \gamma})
$$

or certify that there is no vertex $u$ with

$$
p^{*}(u) / d(u) \leq(1-\gamma) /(2 m)
$$

The running time of the algorithm is $O(m /(\gamma \alpha))$, and, depending on the input, it will always end in one of the following cases:
(i) The set $T$ is found in time $O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$ and has excess $\left(p^{*}, T\right) \geq \gamma /(64 \lg (8 m))$.
(ii) The set $T$ is guaranteed to contain all small density vertices $u$ with $p^{*}(u) / d(u) \leq(1-\gamma) /(2 m)$. In this case, even if $T$ is small, we have no better time bound than $O(m /(\gamma \alpha))$.
(iii) A certificate that there is no vertex $u$ with $p^{*}(u) / d(u) \leq(1-\gamma) /(2 m)$.

The proof of Theorem 9 is deferred to Section 7 . We note that if we just want a condition for finding a cut with conductance

$$
O(\sqrt{(\alpha \log m) / \gamma})
$$

then Theorem 9 implies Theorem 8 , for if there is a set $S$ with excess $\gamma$ (the condition for Theorem8), then the average density outside $S$ is at most $(1-\gamma) /(2 m)$, so there must exist a vertex $u$ outside $S$ with low density $p^{*}(u) / d(u) \leq(1-\gamma) /(2 m)$ (the condition for Theorem 9). This does not mean that Theorem 9 replaces Theorem 8, for, on a more detailed level, we will make heavy use of the volume parameter $s$ in Theorem 8 ,

Note the big asymmetry in the conditions for the theorems. For Theorem8, we need

$$
p^{*}(S)-\frac{\operatorname{vol}(S)}{2 m} \geq \gamma
$$

The volume of a vertex is its degree, so our new condition for Theorem 9 can be written as

$$
\frac{\operatorname{vol}(u)}{2 m}-p^{*}(u) \geq \gamma \frac{\operatorname{vol}(u)}{2 m}
$$

Thus it takes much less missing mass than excess mass to find a low conductance cut. This asymmetry is necessary, for consider an expander graph where all cuts have conductance $\Omega(1)$. As in the example we gave before Theorem 8 against multiplicative differences, consider an initial distribution $p^{\circ}$ that places all mass on a vertex $u$ of minimum degree $\delta$. Then after the first push, we have settled mass $p(u)=\alpha$, and in the limit $p^{*}(u) \geq p(u)=\alpha$. Assuming $\alpha \geq 2 / n$, we get

$$
p^{*}(u)-\frac{\operatorname{vol}(u)}{2 m} \geq \alpha / 2 \geq \gamma \frac{\operatorname{vol}(u)}{2 m} \text { with } \gamma=n \alpha / 2,
$$

but this does imply any low conductance cut.
The conditions of Theorem 8 and Theorem 9 are used in a complimentary fashion. Suppose we have a low conductance cut around a set $S$. Informally speaking, if our initial distribution concentrates the mass well inside $S$, then a lot of mass will also be trapped in $S$ in the limit, giving us the excess for Theorem 8 Conversely, if we concentrate the initial mass well outside $S$, then only little mass will reach $S$ in the limit, and then there will be some low density vertex $u$ in $S$ for Theorem 9 . We still have the problem of ensuring that we don't start somewhere in the middle, where the initial mass is neither well inside, nor well outside $S$. This will be illustrated in the next section when $S$ is a non-trival min-cut.

### 2.2 PageRank in our applications

In our applications, we are always going to use the same teleportation constant

$$
\alpha_{0}=1 / \lg ^{5} n
$$

Recall that our minimum degree is $\delta \geq \lg ^{6} n$, so $\delta \alpha_{0} \geq \lg n$.
Our initial distribution $p^{\circ}$ will almost always be obtained by distributing all the mass with uniform density on some set $X$ of vertices, that is, $p^{\circ}(v)=1 / \operatorname{vol}(X)$ if $v \in X$; otherwise $p^{\circ}(v)=0$. For short, we call this the uniform density distribution on $X$. We use $p_{X}^{\circ}$ to denote this initial distribution and $p_{X}^{*}=\operatorname{PR}\left(\alpha_{0}, p_{X}^{\circ}\right)$ to denote the corresponding limit distribution. In case we start with all mass on a single vertex $v$, we will write $p_{v}^{\circ}=p_{\{v\}}^{\circ}$ and $p_{v}^{*}=p_{\{v\}}^{*}$. If we need to make explicit which graph $H$ we run PageRank on, we write it as a second subscript as in $p_{X, H}^{\circ}$ and $p_{X, H}^{*}$.

Note that Theorem 8 and 9 both require that the initial mass distribution $p^{\circ}$ is presented with a listing of the vertices with positive mass in order of non-increasing density. For $p_{X}^{\circ}$, we just need to list the vertices in $X$ since they all have the same density, while all other vertices have zero initial mass.

## 3 Certify-or-cut

In this section, using PageRank as described in Theorems 8 and 9 we will implement the "certify-or-cut" algorithm from the introduction, proving
Proposition 10 Given a simple graph with minimum degree $\delta \geq \lg ^{6} n$, in near-linear time, we can either

1. certify that there are no non-trivial min-cuts, or
2. find a cut with conductance $o(1 / \log m)$.

Recall from the introduction that the point of the certify-or-cut is to illustrate our techniques in a simple form on a non-trivial problem. This is also why we will use the same parameters as in the rest of the paper even though the min-degree bound of Proposition 10 could easily be reduced. When we get to our real recursive min-cut algorithm, everything will become far more complicated.

### 3.1 Starting on the small side of a min-cut

Our first important observation is that if we start with a point mass on any vertex $v$ on the small side $S$ of a non-trivial min-cut, and the small side is not too large, e.g., $\operatorname{vol}(S) \leq m / 2$, then in the limit, we get a mass concentration on $S$ so that Theorem 8 applies. This should be contrasted with the results from [2] which say that if $S$ is a side of a low conductance cut, then a large fraction of the vertices can be used as starting points leading to mass concentration. In [2] they have to guess such a good starting vertex, resulting in a randomized algorithm. However, in our min-cut case, any vertex in $S$ will do, which is why we have a chance of a deterministic algorithm.

Note that since $S$ is a min-cut, $v$ can have at most half its edges leaving $S$, for otherwise $S \backslash\{v\}$ would have a smaller cut around it. The result therefore follows from the following more general lemma.

Lemma 11 Consider a vertex $v$ in a non-trivial min-cut side $S$ with $\operatorname{vol}(S) \leq s$. Suppose $S$ contains a fraction $\varepsilon=s /(2 m)+\Omega(1)$ of $v$ 's neighbors (this condition is satisfied for every vertex $v \in S$ if $s \leq m / 2)$. Then, in $\widetilde{O}(s)$ time, we can find a cut with conductance $o(1 / \log m)$. If there is no such min-cut side $S$ containing $v$, in $\widetilde{O}(s)$ time, we will either find the cut with conductance $o(1 / \log m)$, or report an error.

Proof We start PageRank from the initial distribution $p_{v}^{\circ}$ with all mass on $v$. Then we repeatedly push mass from $v$ until its residual mass $r(v)$ is at most $1 / \delta$. The mass from $v$ will be spread evenly to its neighbors, so at the end, we have more than $\varepsilon$ mass staying in $S$. Moreover, the residual mass on any vertex is now bounded by $1 / \delta$. Next we apply the following lemma with $\mu=1 / \delta$ :

Lemma 12 If at some point the residual mass on every vertex is bounded by $\mu$, then from this point forward, at most $\mu /\left(2 \alpha_{0}\right)$ mass can move across any specific min-cut.

Proof Since the minimum degree is $\delta$, the maximal residual density is bounded by $\mu / \delta$. By Lemma 7 from this point forward, the net flow over any edge is at most $\mu /\left(2 \alpha_{0} \delta\right)$. A min-cut has at most $\delta$ edges, so the net flow across any min-cut is therefore at most $\mu /\left(2 \alpha_{0}\right)$.

After pushing the residual mass from its starting point $v$, by Lemma 12, the mass leaving $S$ is at most $1 /\left(2 \alpha_{0} \delta\right)=o(1)$ since $\alpha_{0}=1 /(\lg n)^{5}$ while $\delta \geq \lg ^{6} n$. Thus, in the limit, the mass staying in $S$ is $p_{v}^{*}(S)=\varepsilon-o(1)=s /(2 m)+\Omega(1)$, so $\operatorname{excess}\left(p_{v}^{*}, S\right)=\gamma=\Theta(1)$. By Theorem 8 , we now get a set $T$ with

$$
\Phi(T)=O\left(\sqrt{\left(\alpha_{0} \log m\right) / \gamma}\right)=o(1 / \log m)
$$

in time $O\left(s / \alpha_{0}\right)=\widetilde{O}(s)$. This time bound is immediate from Theorem 8 with a bound $s \leq m \gamma / 16$, but otherwise $s>m \gamma / 16=\Omega(m)$, and then the general time bound is $O\left(m /\left(\gamma \alpha_{0}\right)\right)=\widetilde{O}(s)$, as desired.

Since every vertex $v$ has at least half its neighbors on its side $S$ of a non-trivial min-cut, the conditions of the lemma are satisfied if $1<\operatorname{vol}(S) \leq m / 2$.

### 3.2 Balanced min-cut

We now consider the situation where both sides of some specific min-cut have volume between $m / 2$ and $3 \mathrm{~m} / 2$. We claim that there are less than 16 vertices that we can start from without finding a low-conductance cut. There are at most $2 \delta$ end-points of the cut edges, so there are less than 16 vertices incident to more than $\delta / 8$ cut edges. These are the only bad vertices. Any other vertex $v$ has at least a fraction $\varepsilon=7 / 8$ of its neighbors on its side $S$ of the min-cut. Moreover $\operatorname{vol}(S) \leq s=3 m / 2$, so $\varepsilon=s /(2 m)+1 / 8$. Thus, if we apply Lemma 11 to a vertex $v$, in $\widetilde{O}(s)=\widetilde{O}(m)$ time, we either find a cut of conductance $o(1 / \log m)$, or conclude that $v$ is bad. We run from 16 vertices. If they are all bad, we conclude that there is no mincut where both sides have volume at least $m / 2$. All this takes near-linear time, so to finish the proof of Proposition 10, it suffices to look for non-trivial min-cuts where the small side $S$ has $\operatorname{vol}(S) \leq m / 2$.

### 3.3 Handling any non-trival min-cut using the endgame

We will now assume that we have a bound $s \leq m / 2$ on volume of the small side of any min-cut. If there is a min-cut where one side has volume between $s / 2$ and $s$, then we will find a sparse cut. We are only interested in non-trivial min-cuts. By Observation 5, the smaller side has volume at least $\delta^{2}$, so we will consider $s=m / 2^{i}$ for $i=1, \ldots,\left\lceil\lg \left(m / \delta^{2}\right)\right\rceil$. For a given $s$, consider a min-cut $(S, V \backslash S)$ where $s / 2 \leq \operatorname{vol}(S) \leq s$. We will either find a low-conductance cut, or falsify the existence of $(S, V \backslash S)$.

We pick an arbitrary set $U$ of $4 m /\left(\alpha_{0} s\right)=\widetilde{O}(\mathrm{~m} / \mathrm{s})$ vertices. For each $v \in U$, we apply Lemma 11 either finding a desired low-conductance cut, or determining that $v$ is not in $S$. The check from $v$ takes $\widetilde{O}(s)$ time, so checking all $v \in U$ takes $\widetilde{O}(m)$ time. We now know that $U$ is contained in the big side $V \backslash S$ of our min-cut.

Next, we create the initial distribution $p_{U}^{\circ}$ with uniform density on $U$. None of this mass is in $S$, and the maximal density on any vertex in $G$ is $1 / \operatorname{vol}(U) \leq 1 /(\delta|U|)$. Therefore, by Lemma 12 the netflow over any edge is at most $1 /\left(2 \delta|U| \alpha_{0}\right)$, so the total mass that can move into $S$ through the at most $\delta$ cut edges is at most $1 /\left(2|U| \alpha_{0}\right)=s /(8 m)$, bounding the limit mass $p_{U}^{*}(S)$ on $S$.

Since $\operatorname{vol}(S) \geq s / 2$, the average limit density on $S$ is thus at most $1 /(4 m)$. It follows that some vertex $w \in S$ has limit density $p_{U}^{*}(w) \leq 1 /(4 m)$. This is the endgame considered in Theorem 9 , In $\widetilde{O}\left(m / \alpha_{0}\right)$ time, it finds a cut with conductance $O\left(\sqrt{\alpha_{0} \log m}\right)=o(1 / \log m)$. Otherwise we conclude that $S$ did not exist.

For each of the logarithmic number of values of $s$, we thus spend near-linear time, so our total time bound is near-linear. If no low-conductance cuts are found, we conclude that there is no non-trivial min-cuts. This completes the proof of Proposition 10

### 3.4 Relation to the overall min-cut algorithm

At first sight, it may seem that Proposition 10 is a major step forward in the direction of implementing our min-cut algorithm sketched in Section 1.6 However, in the min-cut algorithm we have a recursion where if there is a non-trivial min-cut, then we will find a low-conductance cuts and split off the smaller side. In order for this to be efficient recursively, the time spent has to be near-linear in the volume of the smaller side that we split off. However, in Proposition 10, we may spend time near-linear in the graph size even if we find a low-conductance cut with a very small side. Indeed we may spend time near-linear in the graph size when we prepare for the endgame in Section 3.3, checking all the $\widetilde{O}(\mathrm{~m} / \mathrm{s})$ vertices in the set $U$. We spend $\widetilde{O}(s)$ time on each vertex $v$, finding either a low-conductance cut where the small side has volume $\widetilde{O}(s)$ (c.f. Theorem [8), or discovering that $v$ is not in $S$. The worst case would be if we only found a low-conductance cut for the last vertex checked. To steer around this issue, we will carefully exploit the guarantee from Theorem 8 that the smaller side $A$ has excess $\left(p^{*}, A\right) \geq \gamma /(16 \lg (4 s))$. The details are found in the proofs of Lemma 27 and 29, but first we have to set up a framework to formalize our min-cut algorithm which has to handle many other tricky issues such as the parallel edges created by contractions.

## 4 The min-cut algorithm: the recursive set-up

In this section, we are going to present the recursive set-up for our min-cut algorithm, doing the min-cut preserving contractions described in Theorem 1:

Given a simple input graph $G$ with $n$ vertices, $m$ edges, and minimum degree $\delta$, in near-linear time, we can contract vertex sets producing a multigraph $\bar{G}$ which has only $\bar{m}=\widetilde{O}(m / \delta)$ edges, yet which preserves all non-trivial min-cuts of $G$.

When first we have the contracted graph $\bar{G}$, we can apply Gabow's min-cut algorithm [9] as described in Corollary 2 and find a min-cut of $G$ in $\widetilde{O}(\delta \bar{m})=\widetilde{O}(m)$ time.

The reader may at this point want to review the sketch of our deterministic min-cut algorithm from Section 1.6. The pseudo-code for the real algorithm is found in Algorithm 3. In the top level repeat-loop, it works with a multigraph $\bar{G}$ obtained from $G$ by contracting vertex sets while preserving all non-trivial mincuts of $G$. The only edges removed from $G$ in the construction of $\bar{G}$ are those with contracted end-points who would otherwise be loops. However, to find out which vertex sets that can be contracted in $\bar{G}$, in each iteration of the repeat-loop, the algorithm works with a subgraph $H$ of $\bar{G}$.

The edge connectivity of $G$ is at most $\delta$, so if the edge connectivity of $\bar{G}$ becomes bigger than $\delta$, then there cannot be any non-trivial min-cuts in $G$, and then we can contract $\bar{G}$ to a single vertex.

We note that if there are more than $\delta$ parallel edges between vertices $u$ and $v$, then we can trivially contract $\{u, v\}$. There are therefore never more than $\delta$ parallel edges between two vertices in $\bar{G}$.

When a vertex set is contracted to a single vertex, we call it a super vertex while the original vertices from $G$ are called regular vertices. If we just say a vertex it can be of either kind. The degrees of the regular vertices (possibly including parallel edges to super vertices) do not decrease, so regular vertices will always have degree at least $\delta$.

```
Algorithm 3: \(\operatorname{Min}-\operatorname{cut}(G)\). Here \(G\) is a simple graph with \(m\) edges and minimum degree \(\delta\)
    if \(\delta \leq \lg ^{5} m\) then
        find min-cut in \(G\) using Gabow's algorithm [9].
    \(\bar{G} \leftarrow G ; \quad / / \bar{G}\) preserves non-trivial min-cuts
    repeat
        \(H \leftarrow \bar{G} ;\)
        Remove passive super vertices from \(H\) and trim \(H\); // section 4.1 and 4.3
        while some component \(C\) of \(H\) is not known to be a cluster do
            Find cut \((A, B)\) of \(C\) with conductance \(\leq \Phi_{0}=1 /(20 \lg m) ; \quad / /\) section 5
            Remove cut edges from \(H\) and trim \(H\)
        Take each cluster component of \(H\) and contract its core to a super vertex in \(\bar{G}\);
        // Preserves all non-trivial cuts from \(G\) by Lemma 15 in Section 4.2
        // Contracts \(\geq 1 / 2\) the edges in \(\bar{G}\) by Lemma 20 in Section 4.4
        While there are vertices \(u\) and \(v\) connected by more than \(\delta\) parallel edges, contract \(\{u, v\}\).
    until \(\geq 1 / 20\) of edges in \(\bar{G}\) are incident to passive super vertices;
    // \(\widetilde{O}(m / \delta)\) edges left in \(\bar{G}\) by Lemma 18 in Section 4.3
```

When we analyze our algorithm, which recursively contracts vertex sets in $\bar{G}$, we will always have $n, m$, and $\delta$ denote the number of vertices and edges, and minimum degree of the simple input graph $G$. This means that our lower bound $\lg ^{6} n$ on the minumum degree $\delta$ is fixed. Likewise, we will never change $\Phi_{0}=1 /(20 \lg m)$ defining low conductance, or our teleportation constant $\alpha_{0}=1 / \lg ^{5} n$.

We assume that $n \geq n_{0}$ for some big enough constant $n_{0}$ so that $o(1)$ is smaller any concrete constant, e.g., that $9 / 5+o(1)<2$ (c.f. discussion in Section 1.8).

In the presentation of the our min-cut algorithm, many (but not all) statements will be made in terms of cuts of $\bar{G}$ of size at most $\delta$. Trivially, this includes all cuts corresponding to non-trivial min-cuts of $G$. When we want to argue that all non-trivial min-cuts of $G$ are preserved in $\bar{G}$, we will further use that every vertex has at least half its vertices on the same side of a non-trivial min-cut in $G$ (c.f. Lemma 15).

In the rest of this section, we will describe the different elements of Algorithm 3, and how they work together to produce the contracted graph $\bar{G}$ from Theorem For now we pretend we have an oracle to find the low-conductance cut in the while-loop, postponing its implementation to Section 5 .

### 4.1 Trimmed clusters

Our min-cut algorithm is centered around finding clusters in $\bar{G}$ defined below.
First, a non-empty set $C \subseteq V$ of vertices is called trimmed if for each $v \in C$, at least $2 / 5$ of the edges from $v$ in $\bar{G}$ don't leave $C$. The set $C$ is called a cluster if it is trimmed and for every cut of size at most $\delta$
in $\bar{G}$, one side contains at most two regular vertices and no super vertices from $C$.
We say a cluster $C$ belongs to a side of a cut of size at most $\delta$ if that side has a super vertex or more than two regular vertices from $C$.

Observation 13 Any cluster $C$ belongs to unique side of any cut of $\bar{G}$ of size at most $\delta$.
Proof From the definition of a cluster, it follows that $C$ belongs to at most one side. However, if $C$ belonged to no side, then each side would have at most 2 regular vertices and no super vertices, implying that $C$ consists of at most 4 regular vertices. However, $C$ is non-empty, so it has at least one regular vertex which has at least $2 \delta / 5$ edges into $C$. If all vertices in $C$ are regular, there are no parallel edges between them, so then we must have at least $2 \delta / 5+1>4$ regular vertices in $C$. Thus we conclude that $C$ belongs to exactly one side of the cut.

The condition of having all but at most two regular vertices from $C$ on the same side of any min-cut may seem a bit ad-hoc, but we have the following lemma stating how more than two makes a big difference.

Lemma 14 Consider a trimmed vertex set $C$ and a cut $(T, U)$ of $\bar{G}$ of size at most $\delta$. If $T \cap C$ has no super vertices and at least 3 regular vertices, then $T \cap C$ has at least $\delta / 3$ regular vertices.

Proof The proof is very similar to that of Observation [5] Consider $T \cap C$ which has no super vertices. Since $C$ is trimmed, the internal degree of regular vertices in $C$ is at least $2 \delta / 5$, so the number of edges crossing from $T \cap C$ to $U \cap C$ is at least $|C \cap T|(2 \delta / 5+1-|C \cap T|)$, but we have at most $\delta$ cut edges. Since $\delta=\omega(1)$, we conclude that $|C \cap T| \leq 2$ or $|C \cap T| \geq 2 \delta / 5-1>\delta / 3$.

### 4.2 Cores and loose vertices

The goal of our algorithm will be to find a family $\mathcal{C}$ of non-overlapping clusters such that the number of edges not internal to clusters is $\bar{m}=\tilde{O}(m / \delta)$. Contracting a "core" of each cluster, as defined below, we will produce a graph $\bar{G}$ with $O(\bar{m})$ edges that preserves all non-trivial cuts of size at most $\delta$. We can then apply Gabow's algorithm [9], and find a minimum cut in $\widetilde{O}(\bar{m} \delta)=\widetilde{O}(m)$ time.

Note that because the clusters in $\mathcal{C}$ are required to be non-overlapping, identifying a subset of vertices in one cluster will not stop any other cluster from being a cluster.

Consider a cluster $C$, and consider any vertex $v \in C$. Recall here that since clusters are trimmed, at most $3 d(v) / 5$ of the edges from $v$ leave $C$ in $\bar{G}$. We say a vertex $v \in C$ is loose if it is regular and at least $d(v) / 2-1$ of its incident edges leave $C$.

If more than $1 / 4$ of the edges incident to $C$ in $\bar{G}$ are internal to $A$ then we define $A$ to be the core of $C$; otherwise the core of $C$ is empty and contracting an empty core has no effect.

The following lemma states that contracting cores preserves non-trivial min-cuts:
Lemma 15 If a non-trivial min-cut of $G$ has survived in $\bar{G}$, then it will also survive when we contract the core of any cluster in $\bar{G}$.

Proof First we note that if a non-trivial min-cut of $G$ survives in $\bar{G}$, then it must also be a min-cut $(T, U)$ of $\bar{G}$. It was a min-cut of $G$, so it has $\lambda \leq \delta$ cut edges. Also, because it was a non-trivial cut in $G$ with at least two vertices on each side, we must have at least two regular vertices or one super vertex both in $T$ and in $U$.

We now consider a cluster $C$ in $\bar{G}$ with a non-empty core. Since $(T, U)$ has at most $\delta$ cut edges, by the definition of a cluster, one side, say $T$, has at most two regular vertices and no super vertices from $C$. We will argue that these vertices in $C \cap T$ must be loose, hence that the vertices identified by the contraction of the core are all in $U$, for then this contraction preserves $(T, U)$.

Let $v$ be one of the vertices from $C \cap T$, and assume for a contradiction that $v$ is not loose. We will prove that we get a smaller cut by moving $v$ to $U$, contradicting that $(T, U)$ was a minimum cut. Since $v$ is regular and both sides have at least one super vertex or two regular vertices, $v$ is not the only vertex in $T$. Therefore we still have a cut after moving $v$ to $U$.

Moving $v$ only affects the cutting of edges incident to $v$. When $v$ is in $T$, we cut all edges from $v$ to $C$, except possibly one to another regular vertex in $C \cap T$. Since $v$ is not loose, it has more than $d(v) / 2+1$ edges from $v$ into $C$, so with $v$ in $T$, we cut more than $d(v) / 2$ edges incident to $v$. Moving $v$ to $U$, we stop cutting these edges, so we cut less than $d(v) / 2$ edges incident to $v$, contradicting that $(T, U)$ was a min-cut.

Next we want to argue that if we have a bound on the number $\bar{m}$ of edges between the clusters, then the number of edges that survive when we contract the cores is $O(\bar{m})$. This is done by the following lemma:

Lemma 16 If a cluster $C$ has $k$ edges leaving it, then there are less than $3 k$ edges incident to $C$ that are not internal to the core. In particular, if the core is empty, we have $\operatorname{vol}(C)<3 k$.

Proof Let $A$ be $C$ without the loose vertices, i.e., $A$ is the core unless the core becomes empty. Let $\ell$ be the number of edges leaving $C$ from loose vertices. Then we have $k-\ell$ edges leaving $C$ from vertices in $A$. Other edges incident to $C$ but not internal to $A$ are all incident to loose vertices.

Consider any loose vertex $v$ in $C$. It has at least $d(v) / 2-1=d(v) /(2+o(1))$ incident edges leaving $C$. Here we used that loose vertices are regular, so $d(v) \geq \delta=\omega(1)$. It follows that the total number of edges incident to loose vertices is at most $(2+o(1)) \ell$. Therefore, the total number of edges not internal to $A$ is at most $(2+o(1)) \ell+(k-\ell) \leq(2+o(1)) k$. This proves the lemma unless the core becomes empty.

The core becomes empty if and only if at most $1 / 4$ of the edges incident to $C$ are internal to $A$, but this implies that the number of edges internal to $A$ is at most $1 / 3$ of the number of edges not internal to $A$. Thus, if $A$ is not the core, there are at most $(2+o(1)) k / 3$ edges internal to $A$, and then we have at most $(2+o(1)) k+(2+o(1)) k / 3<3 k$ edges incident to $C$.

Finally, we claim that each super vertex represents a lot of edges from the orginal graph.
Lemma 17 There are $\Omega\left(\delta^{2}\right)$ edges from $G$ contracted in each super vertex of $\bar{G}$.
Proof Consider the first time a cluster $C$ with a non-empty core $A$ get contracted into a super vertex $v^{*}$. By first we mean that $A$ itself does not already have any super vertices, that is, all vertices in $A$ are regular. By definition of a core, only loose vertices from $C$ are not in $A$, and loose vertices are regular by definition, so we conclude that all vertices in $C$ are regular. But $C$ is also trimmed, so any vertex $v \in C$, has at least $2 / 5$ of its incident edges staying in $C$, and they all go to distinct neighbors since $C$ has no super vertices. Thus $\mid C] \geq 2 \delta / 5$, and hence we have at least $2 \delta^{2} / 5$ edge end-points in $C$, corresponding to at least $\delta^{2} / 5$ distinct edges. By definition of a non-empty core, at least $1 / 4$ of the edges incident to $C$ are internal to $A$,
so we conclude that $A$ has at least $\delta^{2} / 20=\Omega\left(\delta^{2}\right)$ internal edges that all get contracted into $v^{*}$. Now $v^{*}$ may later be contracted with other vertices, but this can only increase the number of edges contracted in $v^{*}$.

### 4.3 Active and passive super vertices

We say that a super vertex is active if it has at least

$$
\delta^{*}=(\lg n) \delta / \alpha_{0}
$$

incident edges in $\bar{G}$; otherwise we call it passive.
Recall that the maximal number of parallel edges between any two vertices is $\delta$; for if it is higher then we can contract the two vertices. The high degree of an active super vertex $v$ therefore implies that at most a fraction $\alpha_{0} /(\lg n)=1 / \lg ^{6} n$ of its outgoing edges go to any single neighbor. This means that when we push residual mass from $v$, then no neighbor receives more than this fraction. Thus we can spread mass from active vertices in a way similar to what was described in Section 3.1 when we had no parallel edges. The point in the low degrees of passive super vertices is the following good bound on the total number of edges incident to passive super vertices.
Lemma 18 The total number of edges leaving passive super vertices is $\widetilde{O}(m / \delta)$.
Proof By Lemma 17 we have $\Omega\left(\delta^{2}\right)$ internal edges contracted into each super vertex $v^{*}$. When $v^{*}$ is passive, the ratio of edges leaving $v^{*}$ to edges contracted in $v^{*}$ is at most $\delta^{*} / \Omega\left(\delta^{2}\right)=O\left((\log n) /\left(\alpha_{0} \delta\right)\right)=$ $\widetilde{O}(1 / \delta)$, and this holds for every passive super vertex.

Our algorithm will terminate successfully if the total number of edges in $\bar{G}$ is less than 20 times the number of edges incident to passive super vertices, for then, by Lemma 18, we have only $\widetilde{O}(m / \delta)$ edges in $\bar{G}$, and then, as described in Section 4.2, we can find a min-cut of $G$ in near-linear time.

### 4.4 Cut, trim, shave, and scrap

An iteration of the repeat-loop in Algorithm 3 generally works by alternation between cutting edges of a subgraph $H$ of $\bar{G}$ and trimming the resulting components of $H$ as described below. We start with $H=\bar{G}$. By cutting we refer to two cases. One is where we cut out a passive super vertex, removing its incident edges. The other is where we remove the edges of a low-conductance cut. By trimming we mean removing any vertex $v$ from $H$ that has lost more than $3 / 5$ of the edges it has in $\bar{G}$. When removing $v$, we also remove all its incident edges from $H$, possibly resulting in more vertices to be trimmed. When no more trimming is possible, each remaining vertex in $H$ satisfies $d_{H}(v) \geq 2 d_{\bar{G}}(v) / 5$ which means that all components of $H$ are trimmed as defined in Section 4.1 .

The while-loop of Algorithm 3 keeps cutting and trimming until we somehow know that all remaining components $C$ in $H$ are clusters in $\bar{G}$. We now shave off loose regular vertices $v$ that have lost at least $d(v) / 2-1$ of their incident edges. Contrary to trimming, the shaving is not recursive. We only shave off vertices that were loose before the shaving started. If we recursively removed vertices $v$ that had lost at least $d(v) / 2-1$ of their incident edges, then we could easily end up losing all vertices in thee graph.

Let $A$ be what is left of a component $C$ after shaving. If less than $1 / 4$ of the edges incident to $C$ are internal to $A$, we scrap $A$ so that nothing remains from $C$. Otherwise $A$ is a core that we contract in $\bar{G}$.

We want to bound the number of edges cut, trimmed, shaved, and scrapped from $H$, for these are the edges that remain in $\bar{G}$ when the cores of the cluster components of $H$ are contracted.

Lemma 19 If the total number of edges cut is $c$, then the total number of edges lost due to trimming, shaving, and scrapping is at most 4 c .

Proof The proof is by amortization. The "lost degree" of a vertex $v \in H$ is the number of incident edges in $\bar{G}$ that are not in $H$. A vertex not in $H$ has no lost degree. We are interested in the total lost degree over all vertices in $H$, and it starts at 0 when $H=\bar{G}$. When we cut an edge, the total lost degree increases by 2 . When we trim a vertex $v$, it has at most $2 d_{\bar{G}}(v) / 5$ incident edges left, so its lost degree is at least $3 d_{\bar{G}}(v) / 5$. The trimming removes $v$ from $H$, so its lost degree is saved. However, each incident edge removed increases the lost degree of the other end-point by 1 . Thus, when we trim $v$, we remove at most $2 d_{\bar{G}}(v) / 5$ edges while reducing the total lost degree by at least $d_{\bar{G}}(v) / 5$. The number of edges removed by trimming is therefore at most twice the decrease in the total lost degree due to triming. The total increase lost degree by cutting is $2 c$ and if the total lost degree is $d$ after all the trimming is done, then the total number of trimmed edges is at most $2(2 c-d)$.

Consider a cluster $C$. At this point, its lost degree $d_{C}$ is exactly the number of edges leaving $C$ in $\bar{G}$, and summing over all clusters $C$ in $H$, we have $\sum_{C} d_{C}=d$. By Lemma 16, there are at most $3 d_{C}$ edges incident to a cluster $C$ in $\bar{G}$ that are not internal to the core of $C$. We have already removed the $d_{C}$ edges leaving $C$, so when we shave $C$ down to its core, scrapping it if too small, then we remove at most $2 d_{C}$ edges. Summing over all clusters, the shaving and scrapping takes out at most $\sum_{C} 2 d_{C}=2 d$ edges.

Summing up, the trimming, shaving, and scrapping of undersized cores, takes out a total of at most $2(2 c-d)+2 d \leq 4 c$ edges.

As mentioned above, we start the round with $H=\bar{G}$. As described at the end of Section 4.3, we are done if more than a fraction $1 / 20$ of the edges are incident to passive super vertices. Otherwise, we cut all edges incident to passive super vertices, and trim the sides.

Next we are repeatedly going to cut and trim using cuts of components of $H$ of conductance at most

$$
\Phi_{0}=1 /(20 \lg m) .
$$

This is what we henceforth regard as a low-conductance cut. Later sections will prove that low-conductance cuts can be found efficiently if a component is not a cluster.

We claim that the total number of edges cut this way is at most a fraction $1 / 20$ of the edges in $\bar{G}$. The point is that the number of edges cut is a fraction $1 /(20 \lg m)$ of the volume of the small side, and the same vertex can end on the smaller side only $\lg m$ times. Here size is measured by volume, that is, number of incident edges.

Including the at most $1 / 20$ of the edges of $\bar{G}$ incident to passive vertices, we thus cut at most a fraction $1 / 10$ of the edges in $\bar{G}$. Hence, by Lemma 19, in total, we lose at most $1 / 2$ of the edges in $\bar{G}$. Summing up,

Lemma 20 Cutting edges around passive vertices and edges of low-conductance cuts, trimming, shaving, and scrapping, leaves at least half the edges of $\bar{G}$ in the resulting cluster cores of $H$. Therefore, when we contract the cluster cores of $H$ in $\bar{G}$ (c.f. Algorithm (3), we reduce the number of edges in $\bar{G}$ by at least a factor 2 .

### 4.5 Graph representation and modification

We will now discuss how to represent the graphs involved in Algorithm 3. Our representations are all fairly standard. They ensure that all the graph manipulations we do in Algorithm 3 can be supported in near-linear total time. The repesenations will also help us when we later search for low-conductance cuts.

We assume that the simple input graph $G=(V, E)$ is represented with a list of neighbors for each vertex. We also assume some linear ordering of $V$.

The contracted graph $\bar{G}$ The graph $\bar{G}$ is obtained from $G$ by contraction of vertex sets. We think of $\bar{G}=(\bar{V}, \bar{E})$ as a multigraph with parallel edges, and the degree $d_{\bar{G}}(v)$ of a vertex in $v$ counts all incident edges, including parallel ones. However, internally, we represent $\bar{G}$ as a simple graph with multiplicities on the edges. To represent the connection to $G$, we store with each vertex in $\bar{V}$, the set of original vertices it represents $V$, and likewise for the edges. Recall here that a vertex $v \in \bar{V}$ is a regular vertex if it represents a single original vertex and that $v$ is a super vertex if it represents several original vertices.

Initially $\bar{G}=G$, and we will always have $\bar{V} \subseteq V$. Each $v \in \bar{V}$ will have its current neighbors, each with a positive multiplicity, stored in a doubly-linked list sorted by the ordering of $V$. The list ordering is maintained by a balanced binary search trees so that we can insert and delete neighbors in $O(\log n)$ time.

We will always do the contractions for two vertices at the time, that is, if we have a set $\left\{v_{1}, \ldots, v_{s}\right\}$ to be contracted, we create a list of pairs $\left\{v_{1}, v_{2}\right\},\left\{v_{2}, v_{3}\right\}, \ldots\left\{v_{s-1}, v_{s}\right\}$ to be contracted.

To make the contractions efficient, for each $v \in \bar{V}$, we maintain the sum of degrees of the original vertices represented by $v$. We note that loops created by the contractions are counted twice even though we remove them from $\bar{G}$. When asked to contract $u$ and $v$, we will use the one with the bigger degree sum, say $u$, as the new representative. As the first step in the contraction, if there is an edge $(u, v)$, we remove it as it would otherwise become a loop. More precisely, we search and remove $v$ from $u$ 's incidence list, and vice versa, all in $O(\log n)$ time thanks to the binary search trees. Next we have to take every edge $(v, w)$ from $v$ and turn it into an edge $(u, w)$ from $u$. This requires replacing $v$ by $u$ in $w$ 's neighbor list, and adding $w$ to $u$ 's neighbor list. Thanks to the ordering, we immediately discover if we already had a $(u, w)$ edge, in which case we merge them into a single edge, adding their multiplicities. This is all done in $O(\log n)$ time per edge incident to $v$. What makes the whole thing efficient is that after the contraction, the degree sum of $u$ is at least twice the degree sum we had for $v$. Since the degree sum cannot exceed $2 m$, we conclude that each edge can be moved at $\operatorname{most}^{\log } \log _{2}(2 m)$ times over all contractions, adding up to a total cost of $O\left(m \log ^{2} n\right)$.

As we do the contractions, we will build a contraction tree, making $u$ a parent of $v$ when $u$ and $v$ are contracted into $u$ as above. The roots are then the current vertices in $\bar{V}$ and their descending nodes are the original vertices they represent. The contraction tree has height at $\operatorname{most}^{\log _{2}(2 m)}$.

Finally, recall that if we get more than $\delta$ parallel edges between two vertices $u$ and $w$, then we want to contract $u$ and $w$. We easily detect this as we merge edges, adding up their multiplicities, and this will result in a list of pairs to be contracted. When get to such a pair $\{u, w\}$, we go to the contraction tree and find the roots $u^{\prime}$ and $w^{\prime}$ of $u$ and $w$. If they have the same root $u^{\prime}=w^{\prime}$, there is nothing to be done. Otherwise, we perform the contraction of $u^{\prime}$ and $w^{\prime}$ as described above. It only takes $O(\log n)$ time to find the roots, so this does not affect our overall time bound of $O\left(m \log ^{2} n\right)$ to support all contractions in $\bar{G}$ done in Algorithm 3

The subgraph $\boldsymbol{H}$ of $\overline{\boldsymbol{G}}$ Inside the repeat-loop of Algorithm 3, we work with a subgraph $H$ of $\bar{G}$. Initially, we just copy $\bar{G}$ into $H$. Let $\bar{n}$ and $\bar{m}$ denote the number of vertices and edges we get from $\bar{G}$.

The subgraph $H$ is decremental in the sense that edges are only removed, not added. We will now describe a dynamic data structure to maintain information on $H$. The data structure only supports the deletion of one edge at the time. If a vertex becomes isolated, it is deleted automatically.

With each vertex $v$, we store its initial degree $d_{\bar{G}}(v)$ from $\bar{G}$ as well as its current degree $d_{H}(v)$ in $H$. For every component $C$ of $H$, we will have a doubly-linked list of its vertices and a balanced binary search tree of height $O(\log \bar{n})$ over this vertex list. For each node in the binary search tree, we maintain the number of descending vertices, the sum of their degrees in $H$, and the sum of their degrees in $\bar{G}$. For the root node of $C$, these numbers represent the number of vertices in $C, \operatorname{vol}_{H}(C)$, and $\operatorname{vol}_{\bar{G}}(C)$. Since no edges leave a component of $H$, we have $m(C)=\operatorname{vol}_{H}(C) / 2$. The root is viewed as representating the component $C$. To check which component a vertex belongs, we simply follow parent pointers to the root in $O(\log n)$ time.

To maintain the components of $H$, we use the dynamic connectivity algorithm from [16] supporting each edge update in $O\left(\log ^{2} \bar{n}\right)$ time. We will only use it decrementally in $\widetilde{O}\left(\bar{m} \log ^{2} \bar{n}\right)$ total time. The algorithm from [16] maintains a spanning forest of $H$. In connection with each edge deletion $(u, v)$, it can tell us if $(u, v)$ was a bridge, and if so, which of $u$ and $v$ that ends in the smaller component.

When we remove an $(u, v)$ edge from $H$, our first action is to decrement $d_{H}(u)$ and $d_{H}(v)$ as well as the degree sums of their ancestors in the binary search trees, all in $O(\log \bar{n})$ time. If $(u, v)$ was a bridge in a component $C$, the algorithm from [16] points us to (a spanning tree of) the smaller new component $A$. We extract these vertices, one by one, from the vertices in $C$ and from the binary search tree over $C$. Next we build a new binary search tree for the vertices in $A$. All this is done in $O(\log n)$ time time per vertex, including all necessary updates to counters in the binary search trees. Every time a vertex is moved, it ends up in a component of half the size, so the total of moving vertices to smaller components is $O\left(\bar{n} \log ^{2} \bar{n}\right)$. Adding up, the total time spent on our decremental data structure for $H$ is $O\left(\bar{m} \log ^{2} \bar{n}\right)$.

Returning to an iteration of the repeat-loop in Algorithm [3, first we copy $\bar{G}$ to $H$, initializing the above decremental data structure. We will generally have a deletion list for vertices to be deleted. When we get to a vertex in the list, we remove all its incident edges, one by one, before removing the vertex.

The first vertices in the deletion list are the passive super vertices, that is, super vertices $v$ with $d_{\bar{G}}(v)<$ $\delta^{*}$. Later we add vertices to be trimmed to the list, that is, any vertex $v$ getting $d_{H}(v)<2 d_{\bar{G}}(v) / 5$. The trimming is not completed until the deletion list is empty.

When we in the while-loop find a low conductance cut, we remove the cut edges one by one. When all cut edges have been removed, we do the trimming as described above.

Recall that we for now ignore the cost of finding the low-conductance cuts in the while-loop. Then the total time spent on reducing $H$ to clusters is $O\left(\bar{m} \log ^{2} \bar{n}\right)$.

Finally, when we are down to clusters, we want to identify the cores to be contracted. This is easily done sequentially in $O(\bar{m})$ total time. More precisely, for each cluster $C$, we first go through all the vertices, identifying the set $L$ of loose vertices, that is, vertices with $d_{H}(v) \leq d_{\bar{G}}(v) / 2+1$. Let $A=V(C) \backslash L$. Next we check if at least $1 / 4$ of the edges incident to $C$ in $\bar{G}$ are internal to $A$. If so we, $A$ is a core to be contracted in $\bar{G}$; otherwise the core is empty with nothing to be contracted from $C$.

Summing up what we have proved in this section, we have
Lemma 21 Ignoring the cost of finding the low-conductance cuts, we can implement Algorithm 3 in $O\left(m \log ^{2} n\right)$ time where $n$ and $m$ are the number of vertices and edges of the simple input graph $G$. The result is the contracted graph $\bar{G}$ described in Theorem $\square$ that is, $\bar{G}$ has only $\widetilde{O}(m / \delta)$ edges, yet it preserves all non-trivial min-cuts in $G$.

Proof Concerning the running time, above we implemented all the contractions in $\bar{G}$ in $O\left(m \log ^{2} n\right)$ total time. Moreover, we implemented each iteration of the repeat-loop in $O\left(\bar{m} \log ^{2} \bar{n}\right)$ time where $\bar{n} \leq n$ and $\bar{m} \leq m$ are the number of vertices and edges in $\bar{G}$ in the beginning of the iteration. By Lemma 20 is reduced by at least a factor 2 by the contractions to $\bar{G}$ at the end of each iteration. Therefore, the total time spent is $O\left(m \log ^{2} n\right)$.

Concerning correctness, Lemma 15 ascerts that the contraction of cluster cores preserve all non-trivial min-cuts of $G$. Moreover, Lemma 18 states that we have only $\widetilde{O}(m / \delta)$ edges leaving the passive super vertices, and the repeat-loop only terminates when these edges constitute at least a fraction $1 / 20$ of all edges, so we conclude that the final contracted graph $\bar{G}$ has only $\widetilde{O}(m / \delta)$ edges.

## 5 Certifying clusters and finding low conductance cuts

We will now, at a high level, describe the process that repeatedly takes a trimmed component $C$ of $H$, cuts the edges of a low-conductance cut and trims the sides, stopping only when all remaining components are known to be clusters. This implements the while-loop in Algorithm 3 .

In the process, $H$ is a subgraph of $\bar{G}$ with no passive super vertices from $\bar{G}$. Since $H$ starts trimmed and since we trim after each cut, we know that all components of $H$ are trimmed when we look for lowconductance cuts.

We need a measure for how close trimmed components of $H$ are at being clusters, but it is useful with a more general mearsure that applies to all subgraphs $B$ of $H$. We say $B$ is $s$-splittable, or has splittability $s$, if every cut $(T, U)$ of $\bar{G}$ with at most $\delta$ cut edges has $\min \left\{\operatorname{vol}_{B}(T \cap B), \operatorname{vol}_{B}(U \cap B)\right\} \leq s$. We want $s$ to be as small as possible, and note that we always have $s \leq m(B)$. A very important part of this definition is that it is inherited by subgraphs, that is, if $A$ is a subgraph of $B$ and $B$ is $s$-splittable, then $A$ is also $s$-splittable. Being $s$-splittable is thus preserved as we cut and trim. Let

$$
s_{0}=64 \delta / \alpha_{0}
$$

Our goal will be to partition $H$ into $s_{0}$-splittable trimmed components, for they are then all clusters:
Lemma 22 If a trimmed component $C$ of $H$ is $s_{0}$-splittable, then $C$ is a cluster.

Proof Suppose that $C$ is not a cluster. Then there is a cut $(T, U)$ of $\bar{G}$ with at most $\delta$ cut edges and such that both $T \cap C$ and $U \cap C$ contain a super vertex or at least 3 regular vertices. Consider $T \cap C$. Suppose $T \cap C$ contains a super vertex $v$. Since $H$ has no passive super vertices, $d_{\bar{G}}(v) \geq \delta^{*}=(\lg n) \delta / \alpha_{0}$. Since $C$ is trimmed, $d_{C}(v) \geq 2 d_{\bar{G}}(v) / 5$. Therefore $\operatorname{vol}_{C}(T) \geq d_{C}(v) \geq 2 \delta^{*} / 5 \gg s_{0}$. Suppose instead that $T \cap C$ contains no super vertices but at least three regular vertices. Then by Lemma 14 , there are at least $\delta / 3$ regular vertices in $C \cap T$. Since $C$ is trimmed, each of them has degree at least $2 \delta / 5$ in $C$, so we conclue that $\operatorname{vol}_{C}(T) \geq 2 \delta^{2} / 5$. Since $\delta \alpha_{0} \geq \lg n$, we again conclude that $\operatorname{vol}_{C}(T) \gg s_{0}$. The same argument holds for $S=U \cap C$, so we conclude that $C$ is not $s_{0}$-splittable.

As we cut and trim components into clusters, for each component $C$ of $H$, we record the smallest $s$ for which we have certified that $C$ is $s$-splittable. By Lemma 22, we certify that $C$ is a cluster if $s \leq s_{0}$. For larger $s$, we will apply the theorem below.

Theorem 23 Let $s \in\left[s_{0}, m(C)\right]$ and $C$ be an $s$-splittable trimmed component of $H$. We have an algorithm that, depending on the input, will do one of the following:
(i) Find a set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C)$ and $\Phi_{C}(A)=o\left(\Phi_{0}\right)$ in time $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$.
(ii) Find a set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C)$ and $\Phi_{C}(A)=o\left(\Phi_{0}\right)$ in time $\widetilde{O}(m(C))$ certifying that the large side $B=V(C) \backslash A$ is $s / 2$-splittable.
(iii) Certifying in $\widetilde{O}(m(C))$ time that $C$ is $s / 2$-splittable.

The proof of Theorem 23 is complicated and deferred to Sections 5.1-5.4.
In Algorithm 4 we use Theorem 23 to efficiently implement the while-loop from Algorithm 3 while carefully recording the certified splittability of each component.

```
Algorithm 4: Partitions \(H\) into certified clusters as in the while-loop of Algorithm 3, Initially \(H\) is a
trimmed subgraph \(H\) of \(\bar{G}\) with no passive super vertices.
    Each component \(C\) of \(H\) is certified \(m(C)\)-splittable;
    while some component \(C\) of \(H\) certified \(s\)-splittable with \(s>s_{0}\) do
        Apply Theorem 23 to \(C\). In case (iii) we certify that \(C\) is \(s / 2\)-splittable;
        In case (i) and (ii) we get the small side \(A\) of a low-conductance cut, remove the cut edges and
        trim the sides;
        // Note that both sides may now have been split into several components
        In case (i), all new components \(D \subseteq C\) are certified \(\min \{s, m(D)\}\)-splittable;
        In case (ii), all new components \(D \subseteq A\) are certified \(\min \{s, m(D)\}\)-splittable
        and all new components \(D \subseteq C \backslash A\) are certified \(\min \{s / 2, m(D)\}\)-splittable
```

Lemma 24 Using Theorem 23 Algorithm 4 implements while-loop of Algorithm 3; starting from a trimmed subgraph $H$ of $\bar{G}$ with no passive super vertices from $\bar{G}$, it repeatedly removes edges from a low-conductance cuts, trimming the sides, until all components of $H$ are certified clusters. The total time spent is near-linear in the initial number of edges in $H$.

Proof Let $n_{H}$ and $m_{H}$ be the initial number of vertices and edges in $H$. It follows directly from the description of Algorithm 4 that the only action it performs on $H$ is to remove the edges of a low-conductance cut and trim the sides. From Section 4.5, we know that the total time spent on modifying the representation of $H$ is $O\left(m_{H} \log ^{2} n_{H}\right)$. What has been added is the certified splittability of components. When it gets down to $s_{0}$-splittable components, we know they are all clusters by Lemma 22, so this a faithful implementation of the while-loop from Algorithm 3.

To amortize the cost of cutting and trimming $H$ into clusters, we say that an edge e pays in each of the following events: (1) when $e$ is removed from $H$, (2) the edge $e$ gets into a component of $H$ of half the volume, or (3) the edge $e$ gets into a component certified to be only half as splittable. The first event happens only once, and the last two events can happen at most $\lg m_{H}$ times each, so an edge can pay at most $2 \lg m_{H}+1$ times. If the payment each time is $\widetilde{O}(1)$, then the total time is $\widetilde{O}\left(m_{H}\right)$, as desired.

Looking at the cost of applying Theorem 23, in case (i), the edges incident to $A$ in $C$ are now either removed or in a component of at most half the size, so they can pay $\widetilde{O}(1)$ each to cover the $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$ time spent in this case. Case (iii) certifies that $C$ is $s / 2$-splittable, so the $\widetilde{O}(m(C))$ time can be paid by the edges in $C$. Finally, case (ii) certifies that $B$ is $s / 2$-splittable. Before removing any edges from $C$, we had $\operatorname{vol}_{C}(B) \geq m(C) \geq \operatorname{vol}_{C}(A)$, hence at least $m(C) / 2$ edges incident to $B$. Each of these edges is either removed as a cut edge or by trimming, or it ends up in a component that is only half as splittable, so they can pay $\widetilde{O}(1)$ each to cover the $\widetilde{O}(m(C))$ time spent. Thus we conclude that the total time spent in Theorem 23 is $\widetilde{O}\left(m_{H}\right)$.

Finally we need to consider the cost of maintaining the certified splittablity of each trimmed component $C$ encountered in the while-loop. Recall from Section4.5that $C$ is represented as the root node of a balanced
binary search tree over all vertices in $C$. This root also knows the number $m(C)$ of edges in $C$. We now further store the splittability of $C$ with the root representing $C$.

For the while-loop we maintain a "to-do" list with all the roots of components with splittability $s>s_{0}$. We are done when this list is empty. For each iteration of the while-loop, we take a component $C$ from the to-do list, copy its splittability $s$, and apply Theorem 23. If we get case (iii), we simply set the splittablity of $C$ to $s / 2$ in constant time, removing $C$ from the to-do list if $s / 2 \leq s_{0}$.

In case (i) and (ii) we get the small side $A$ of a low-conductance cut, remove the cut edges, and trim the sides. This can result in many new components. To track these, recall from Section 4.5 that we via the dynamic connectivity algorithm from [16] will find out which of the edges removed were bridges whose removal split a component. We record all the bridges removed. When the trimming is completed, for each bridge end-point $u$, in $O\left(\log m_{H}\right)$ time, we find its root which represents its new component $D$. In case (i) we just set the splittability of $D$ to $\min \{s, m(D)\}$. In case (ii), we first check if $u$ is in $A$. If so, we set the splittability of $D$ to $\min \{s, m(D)\}$; otherwise we set it to $\min \{s / 2, m(D)\}$. The root of $D$ is added to the to-do list if its splittability is above $s_{0}$. We note that there may several bridge end-points in $D$, leading us to the same root, but we only need to do the above certification once.

The total number of bridges removed in the above process is bounded by $n_{H}-1$ and the total number of trimmed components considered is bounded by $\overline{2} n-1$, so the total time spent on maintaining the certifiability of trimmed components is $O\left(n_{H} \log n_{H}\right)=O\left(m_{H} \log n_{H}\right)$. Adding up, we conclude that the total cost of implementing the while-loop is near-linear in the original number of edges in $H$.

Combining Lemma 21 and 24, we conclude:

## Lemma 25 Theorem 1 follows from Theorem 23

Proof Lemma 24 includes the cost of finding low-conductance cuts ignored in Lemma 21 . Assuming Theorem 23 it says that we can implement the while-loop of Algorithm 3 in near-linear time.

The while-loop is inside an iteration of the repeat-loop in Algorithm 3 where it is applied to a subgraph $H$ of the contracted graph $\bar{G}$. In the first iteration, $\bar{G}$ is identical to the original simple graph with $m$ edges, and by Lemma 20 , the edge number of $\bar{G}$ is halved between iterations. Therefore, in the while-loop, over all iterations of the repeat-loop, the total time spent is near-linear in $m$. From Lemma 21 , we know that everything else is handled in time $O\left(m \log ^{2} n\right)$, so we conclude that Algorithm 3 is implemented in near-linear total time.

Lemma 21 further states that the contracted graph $\bar{G}$ produced by Algorithm 3 has all the desired properties from Theorem 1, which hence follows if we can prove Theorem 23,

The rest of this section is devoted to the proof of Theorem 23 ,

### 5.1 Pushing from a vertex across a small cut-the issue of parallel edges

We are now going to introduce a basic technical lemma that we shall use to find low-conductance cuts. It corresponds to Lemma 12 from Section 3.1, but now we have to handle active super vertices. A new issue is that a vertex might now have many parallel edges to a few neighbors. We cannot handle this situation in general, but in our case, we will argue that it has to be a regular vertex where the parallel edges all go to super vertices, and this special structure will be critical to our solution.

Lemma 26 Consider a trimmed component $B$ of $H$, and let $S$ be one side of a cut of $B$ with $O(\delta)$ cut edges. Start PageRank in $H$ with all mass on a vertex $v$ in $S$ and push to the limit. If $v$ is a super vertex, the mass
leaving $S$ is o(1). If $v$ is a regular vertex with a fraction $\varepsilon$ of its edges leaving $S$, then the mass leaving $S$ is $\varepsilon+o(1)$.

Proof Suppose first that $v$ is a super vertex. Since all super vertices are active, $v$ has at least $\delta^{*}=$ $(\lg n) \delta / \alpha_{0}$ incident edges in $\bar{G}$, and $B$ is trimmed, so $v$ has at least $2 \delta^{*} / 5$ incident edges in $B$. The cut has $O(\delta)$ edges, so the fraction of edges from $v$ leaving $S$ is $O\left(\alpha_{0} / \log n\right)=o(1)$.

We now first push all the initial mass from $v$. The mass is spread evenly over its incident edges, so the mass escaping $S$ is $o(1)$. Moreover, since the maximal number of parallel edges between any pair of vertices is $\delta$, the maximal residual mass ending at any vertex is $\delta /\left(2 \delta^{*} / 5\right)=O\left(\alpha_{0} / \log n\right)$. The minimum degree in $B$ is $2 \delta / 5$, so we end up with a maximum residual density of $O\left(\alpha_{0} /(\delta \log n)\right)$.

By Lemma 7 from this point forward, the net flow over any edge is bounded by $\left.O\left(\alpha_{0} /(\delta \log n)\right) /\left(2 \alpha_{0}\right)\right)=O(1 /(\delta \log n))$, so the net flow over the $O(\delta)$ cut edges is bounded by $O(1 / \log n)=o(1)$. Adding in the $o(1)$ mass leaving $S$ directly at the first push from $v$, we get that the total mass leaving $S$ is $o(1)$.

We now consider the case where $v$ is a regular vertex and where a fraction $\varepsilon$ of its incident edges leave $S$. As above, we first push all the mass from $v$, sending a fraction $\varepsilon$ of the mass out of $S$. We will now study what happens with the remaining residual mass. Recall that the mass pushed from $v$ is distributed evenly along the edges leaving $v$. We now partition the residual mass, recalling from [2] that pushing mass to the limit is a non-negative linear transformation. We can therefore study what happens to different parts separately.

Consider the mass $r$ that the regular vertex $v$ pushed to its regular neighbors. There are no parallel edges between regular vertices, and $v$ has degree at least $2 \delta / 5$, so the residual mass at any regular neighbor is at most $5 /(2 \delta)$ and residual density at most $25 /\left(4 \delta^{2}\right)=O\left(1 / \delta^{2}\right)$. By Lemma 7 starting from $r$, the net flow over any edge is $O\left(1 /\left(\alpha_{0} \delta^{2}\right)\right)$, so the mass leaving $S$ over the $O(\delta)$ cut edges is $O\left(1 /\left(\alpha_{0} \delta\right)\right)=o(1)$.

For each super neighbor $v_{i}$ of the regular vertex $v$, let $r_{i}$ be the residual mass first pushed from $v$ to $v_{i}$. If $v_{i}$ is outside $S$, we already count $r_{i}$ as lost from $S$ in the initial push from $v$, so we can assume that $v_{i}$ is inside $S$. Our analysis above shows that when we push mass starting from a super vertex in $S$, then the mass leaving $S$ is only a fraction $o(1)$, so in this case $o\left(r_{i}\right)$. However, $\sum_{i} r_{i}<1$, so when we add up the limit distributions, we conclude that only $o(1)$ mass leaves $S$ after the initial loss of $\varepsilon$ to the neighbors of $v$ outside $S$.

### 5.2 Starting from a captured vertex

Consider a vertex $v$ in a trimmed component $C$. We say $v$ is captured if there is a set $S \subseteq V(C)$ with $s_{0} \leq \operatorname{vol}_{C}(S) \leq m(C)$ and $\left|\partial_{C}(S)\right| \leq \delta$ such that $S$ contains $v$ and at least $\frac{3}{4}$ of the edges incident to $v$. If $\operatorname{vol}_{C}(S) \leq s$, we further say that $v$ is $s$-captured.

Finding a low-conductance cut is easy if we can somehow guess a captured vertex. More precisely, using Lemma 26 and Theorem 8 , we will prove:

Lemma 27 Given a vertex $v$ in $C$ and a parameter $s \in\left[s_{0}, m(C)\right]$, we have an algorithm that, depending on the input, will do one of the following:
(i) Find a set $A \subseteq V(C)$ with $\Phi_{C}(A)=o\left(\Phi_{0}\right)$ and $\operatorname{vol}_{C}(A) \leq m(C)$ in time $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$. If $s \leq$ $m(C) / 32$, we will further have $\operatorname{vol}_{C}(A) \leq 16 s$ and $\operatorname{excess}_{C}\left(p_{v, C}^{*}, A\right) \geq 1 /(32 \lg (4 s))$. Recall that $p_{v}^{*}$ is the limit distribution when we run PageRank on $C$ starting with all mass on $v$.
(ii) Certify in $\widetilde{O}(s)$ time that $v$ is not $s$-captured.

Proof By Lemma 26, if we start PageRank with all mass on a vertex $v$ that is $s$-captured, and push mass to the limit, we know that $3 / 4-o(1)$ of the mass will stay in $S$. Since $\operatorname{vol}_{C}(S) \leq m(C)$, this corresponds to an excess of at least $3 / 4-o(1)-1 / 2>1 / 5$. Thus, by Theorem 8 with $G=C$ and $\gamma=1 / 5$, we find a cut with small side $A=T$ and conductance $\Phi_{C}(A)=O\left(\sqrt{\left(\alpha_{0} \log m\right) / \gamma}\right)=O\left(1 / \log ^{2} m\right)=o\left(\Phi_{0}\right)$ in time $\widetilde{O}\left(\operatorname{vol}_{C}(T) / \alpha\right)=\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$.

Now, if $s \leq m(C) / 32$ and $\operatorname{vol}_{C}(S) \leq s$, the mass $3 / 4-o(1)$ corresponds to an excess of at least $3 / 4-o(1)-1 / 64>1 / 2$. Thus we can use $\gamma=1 / 2$ in Theorem 8 , noting that $s \leq m(C) / 32=m(C) \gamma / 16$. Then $\operatorname{vol}_{C}(A) \leq 16 s$, and $\operatorname{excess}_{C}\left(p_{v, C}^{*}, A\right) \geq 1 /(32 \lg (4 s))$.

### 5.3 Starting from set of non-captured vertices

Next we consider the case where we somehow manage to guess a large set $X$ of vertices that are not $s$ captured. If $\operatorname{vol}_{C}(X) \geq \min \left\{16 \delta m(C) /\left(\alpha_{0} s\right), m(C) / 4\right\}$, then the lemma below states that we have an algorithm providing all the guarantees required for Theorem 23 ,

Lemma 28 Let $s \in\left[s_{0}, m(C)\right]$ and $C$ be a trimmed component of $H$. Let $X \subseteq V(C)$ be a set of at vertices, none of which are s-captured in $C$, and with total volume $\operatorname{vol}_{C}(X) \geq \min \left\{16 \delta m(C) /\left(\alpha_{0} s\right), m(C) / 4\right\}$. We have an algorithm that, depending on the input, has one of the following outcomes:
(i) Find a set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C)$, $\operatorname{excess}_{C}\left(p_{X, C}^{*}, A\right) \geq 1 /(128 \lg (8 m))$, and $\Phi_{C}(A)=$ $o\left(\Phi_{0}\right)$ in time $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$. Recall that $p_{X, C}^{*}$ is the limit distribution when we run PageRank on $C$ starting with all mass spread on $X$ with uniform density $1 / \operatorname{vol}_{C}(X)$.
(ii) Find a set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C)$ and $\Phi_{C}(A)=o\left(\Phi_{0}\right)$ in time $\widetilde{O}(m(C))$, and certify for the large side $B=V(C) \backslash A$, that every set $S$ in $C$ with $\left|\partial_{C}(S)\right| \leq \delta$ and $\operatorname{vol}_{C}(S) \leq s$ has $\operatorname{vol}_{C}(S \cap B) \leq s / 2$. In particular, if the component $C$ was $s$-splittable, then the large side $B$ is certified s/2-splittable.
(iii) Certify in $\widetilde{O}(m(C))$ time that there is no set $S$ in $C$ with $\left|\partial_{C}(S)\right| \leq \delta$ and $s / 2<\operatorname{vol}_{C}(S) \leq s$. In particular, if the component $C$ was $s$-splittable, then $C$ is now certified $s / 2$-splittable.

Above, it is only case (ii) and (iii) that depend on the assumption that no vertex in $X$ is s-captured, that is, if $X$ does contain some s-captured vertex, then the algorithm will still return in one of three cases where (i) is exactly as stated above while cases (ii) and (iii) are not to be trusted.

Proof We are going to start PageRank from the initial distribution $p_{X, C}^{\circ}$ with all mass spread on $X$ with the uniform density $1 / \operatorname{vol}_{C}(X)$. The limit distribution is denoted $p_{X, C}^{*}$. Since $C$ is trimmed, the minimum degree in $C$ is $2 \delta / 5$.

Exploiting that no vertex $v \in X$ is $s$-captured, we will argue that only little mass can end in a set $S \subseteq V(C)$ with $\left|\partial_{C}(S)\right| \leq \delta$ and $\operatorname{vol}_{C}(S) \leq s$. We assume for now that such a set $S$ exists and that $\operatorname{vol}_{C}(S)>s / 2$, hence that we are not in case (iii).

First we bound the volume of the vertices from $X$ in $S$. Consider a vertex $v \in X \cap S$. Since $v$ is not $s$-captured in $C$, it has at least $1 / 4$ of its edges in $C$ leaving $S$, so we conclude that vol $C(X \cap S) \leq 4\left|\partial_{C}(S)\right|$. It follows that the total initial mass in $S$ is $p_{X, C}^{\circ}(S) \leq 4\left|\partial_{C}(S)\right| / \operatorname{vol}_{C}(X)$.

The maximal initial density on all vertices is $1 / \operatorname{vol}_{C}(X)$, so by Lemma 7, the net flow over any edge is at most $1 /\left(2 \operatorname{vol}_{C}(X) \alpha_{0}\right)$. Hence the total net flow into $S$ at most $\left|\partial_{C}(S)\right| /\left(2 \mathrm{vol}_{C}(X) \alpha_{0}\right)$. The final limit mass on $S$ is thus

$$
\begin{align*}
p_{X, C}^{*}(S) & \leq\left(4+1 /\left(2 \alpha_{0}\right)\right)\left|\partial_{C}(S)\right| / \operatorname{vol}_{C}(X)  \tag{2}\\
& <\left|\partial_{C}(S)\right| /\left(\operatorname{vol}_{C}(X) \alpha_{0}\right)<\delta /\left(\operatorname{vol}_{C}(X) \alpha_{0}\right) .
\end{align*}
$$

The second inequality above uses that $\alpha_{0}=o(1)<1 / 8$. We will now argue that

$$
\begin{equation*}
\delta /\left(\operatorname{vol}_{C}(X) \alpha_{0}\right) \leq s /(16 m(C)) \tag{3}
\end{equation*}
$$

This is trivially true if $\operatorname{vol}_{C}(X) \geq 16 \delta m(C) /\left(\alpha_{0} s\right)$. It is also true if $\operatorname{vol}_{C}(X) \geq m(C) / 4$ because $s \geq$ $s_{0}=64 \delta / \alpha_{0}$. The lemma assumes that $\operatorname{vol}_{C}(X) \geq \min \left\{16 \delta m(C) /\left(\alpha_{0} s\right), m(C) / 4\right\}$, so (3) follows. We conclude that

$$
p_{X, C}^{*}(S) \leq s /(16 m(C)) .
$$

Since $\operatorname{vol}_{C}(S)>s / 2$, this means that vertices $u \in S$ with limit density $p_{X, C}^{*}(u) / d(u) \leq 1 /(4 m(C))$ represent more than half the volume of $S$.

We now apply Theorem 9 with $\gamma=1 / 2$. We get a set $A=T$ with $\operatorname{vol}_{C}(A) \leq m(C)$ and $\Phi_{C}(A)=$ $O\left(\sqrt{\left.\left(\alpha_{0} \log m\right) / \gamma\right)}\right)=O\left(1 /(\log m)^{2}\right)=o\left(\Phi_{0}\right)$. If we end in case (i) of Theorem 9 , the set $A$ is found quickly in time $\widetilde{O}\left(\operatorname{vol}_{C}(A) /\left(\gamma \alpha_{0}\right)\right)=\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$ and then $\operatorname{excess}_{C}\left(p_{X, C}^{*}, A\right) \geq \gamma /(64 \lg (8 m(C)))=$ $1 /(128 \lg (8 m(C)))$ as claimed in case (i) of the lemma.

If we end in case (ii) of Theorem 9 the set $A$ is found in time $\widetilde{O}\left(m(C) /\left(\gamma \alpha_{0}\right)\right)=\widetilde{O}(m(C))$ with the guarantee that $A$ contains all vertices with $p_{X, C}^{*}(u) / d(u) \leq 1 /(4 m(C))$, which implies that $\operatorname{vol}_{C}(A \cap S)>$ $\operatorname{vol}_{C}(S) / 2$. With $B=V(C) \backslash A$, this gives $\operatorname{vol}_{C}(B \cap S) \leq \operatorname{vol}_{C}(S) / 2$, and this holds for any set $S$ with $\left|\partial_{C}(S)\right| \leq \delta$ and $s / 2<\operatorname{vol}_{C}(S) \leq s$, as required for case (ii) of the lemma.

If we end in case (iii) of Theorem 9 , we know that there is no vertex $u$ with $p_{X, C}^{*}(u) / d(u) \leq 1 /(4 m(C))$, but then we conclude, by contradiction, that there was no set $S$ with $\left|\partial_{C}(S)\right| \leq \delta$ and $s / 2<\operatorname{vol}_{C}(S) \leq s$, as required for case (iii) of the lemma.

To finish the proof of case (ii) and (iii), suppose the component $C$ is certified $s$-splittable and consider any cut of $\bar{G}$ with at most $\delta$ cut edges. Let $T$ be the side minimizing $\operatorname{vol}_{C}(C \cap T)$, and set $S=C \cap T$. Since $C$ is $s$-splittable, we know that $\operatorname{vol}_{C}(S) \leq s$. Moreover, $\left|\partial_{C}(S)\right| \leq\left|\partial_{\bar{G}}(T)\right| \leq \delta$.

In case (ii), the algorithm certifies that $\operatorname{vol}_{C}(S \cap B) \leq s / 2$. We also have $\operatorname{vol}_{B}(T \cap B)=\operatorname{vol}_{B}(S \cap B) \leq$ $\operatorname{vol}_{C}(S \cap B)$. Therefore $\operatorname{vol}_{B}(T \cap B) \leq s / 2$, so we conclude that $B$ is $s / 2$-splittable.

In case (iii), the algorithm certifies that we cannot have $s / 2<\operatorname{vol}_{C}(S) \leq s$, $\operatorname{but}^{\operatorname{vol}}{ }_{C}(S) \leq s$, so we conclude that $\operatorname{vol}_{C}(C \cap T)=\operatorname{vol}_{C}(S) \leq s / 2$, implying that $C$ is $s / 2$-splittable.

Above we assumed that no vertex from $X$ was $s$-captured, but even if this is not the case, we only return case (i) if we get it from case (i) of Theorem 9 whose properties do not depend on any assumptions.

### 5.4 Proof of Theorem 23 for highly splittable components

We will now prove Theorem 23 for an $s$-splittable component $C$ where $s=\widetilde{\Omega}(m(C))$, that is,
Let $s \in\left[s_{0}, m(C)\right]$ where $s=\widetilde{\Omega}(m(C))$, and let $C$ be an $s$-splittable trimmed component of $H$. We have an algorithm that, depending on the input, will do one of the following:
(i) Find a set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C)$ and $\Phi_{C}(A)=o\left(\Phi_{0}\right)$ in time $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$.
(ii) Find a set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C)$ and $\Phi_{C}(A)=o\left(\Phi_{0}\right)$ in time $\widetilde{O}(m(C))$ certifying that the large side $B=V(C) \backslash A$ is $s / 2$-splittable.
(iii) Certifying in $\widetilde{O}(m(C))$ time that $C$ is $s / 2$-splittable.

Proof of Theorem 23 when $s=\widetilde{\Omega}(\boldsymbol{m}(\boldsymbol{C}))$ Our algorithm picks an arbitrary set $X \subseteq V(C)$ with $\left\lceil 40 m(C) /\left(s \alpha_{0}\right)\right\rceil=\widetilde{O}(1)$ vertices. Since $C$ is trimmed, the minimum degree in $C$ is at least $2 \delta / 5$, so $\operatorname{vol}_{C}(X) \geq 16 \delta m(C) /\left(s \alpha_{0}\right)$. Now, in parallel alternation, we run Lemma 27 on every vertex $v \in X$, and we run Lemma 28 on the set $X$. We terminate if someone finds a set $A$ with $\Phi_{C}(A)=o\left(\Phi_{0}\right)$ corresponding to case (i) in Lemma 27 or (i) in Lemma 28, calling this early termination; otherwise we continue until all processes have terminated.

In the early termination case, since we run only $\widetilde{O}(1)$ processes in parallel, the total running time is $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$, so we match Theorem 23 (i).

If no process reaches case (i), the total running time is $\widetilde{O}(m(C))$. We get from Lemma 27 (ii) that no vertex $v \in X$ is $s$-captured, which means that we can trust the certifications in case (ii) and (iii) of Lemma 28 , and they match case (ii) and (iii) of Theorem 23.

### 5.5 Proof of Theorem 23 for less splittable components

We will now prove Theorem 23 for less splittable components than those handled above in Section 5.4. Our new proof will work for a trimmed component $C$ that is $s$-splittable for any $s \in\left[s_{0}, m(C) / 32\right]$. This case is far more complicated, and requires several new lemmas.

First let us see what goes right and wrong if we try to do the same as we did with the highly spilttable components. The algorithm would still be correct, but now we have no good bound on the size of the set $X$. This means that the multiplicative slowdown from running $|X|$ process is not bounded.

It is instructive to note that if we apply Lemma27 to all $v \in X$, then the total running time is $\widetilde{O}(m(C))$, for the lemma spends $\widetilde{O}(s)$ time on each of the $|X|=\left\lceil 40 m(C) /\left(s \alpha_{0}\right)\right\rceil$ vertices. However, we cannot afford to spend this much time if for some $v \in X$, we end in case (i) with no certification but a low-conductance cut around a very small side. Our idea to circumvent the problem is to exploit that case (i) implies a lower bound on the excess, both in Lemma 27 and in Lemma 28 , and we want to detect this efficiently in advance. This is the most tricky part of our algorithm, and the motivation for including excess guarantees on the lowconductance set $A$ found in Theorem 8 and Theorem 9 . The following two lemmas address the issue. The reader who wants to fully understand the motivation for these lemmas may want to skip to Theorem 23 and see how they are used in its proof. The first lemma is about identifying a large set of non- $s$-captured vertices in a trimmed component $C$, but not blindly running Lemma 27 from each vertex. If a low conductance cut is found, we will only use time near-linear in the volume of the smaller side.

Lemma 29 For $s \in\left[s_{0}, m(C) / 32\right]$, let $Y$ be a set of at most $m(C) /(1024 s \lg (4 s))$ vertices from a trimmed component $C$. We have an algorithm that, depending on the input, will do one of the following:
(i) Find a set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C)$ and $\Phi_{C}(A)=o\left(\Phi_{0}\right)$ in time $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$.
(ii) Identify a subset $X \subseteq Y, \operatorname{vol}_{C}(X) \geq \operatorname{vol}_{C}(Y) / 2$ in $\widetilde{O}(m(C))$ time, certifying that no vertex in $X$ is s-captured in $C$.

Proof First we consider an optimimistic algorithm that in $\widetilde{O}(m(C))$ time identifies a set $X \subseteq Y$ with no $s$-captured vertices. This is, in itself trivial, since $X=\emptyset$ would do. However, the optimistic algorithm applies Lemma 27 to each $v \in Y$ in $\widetilde{O}(s)$ time. Some vertices will be certified as not $s$-captured in Lemma 27(ii), and they are the ones we place in $X$. The total time we spend is $\widetilde{O}(|Y| s)=\widetilde{O}(m(C))$, so if $X$ ends up with at least half the vertices from $Y$, then we are done as in case (ii).

The bad case for the optimistic algorithm is if we end up with $\operatorname{vol}_{C}(X)<\operatorname{vol}_{C}(Y) / 2$. We will now study the bad case, finding a way to detect it without having to run the optimistic algorithm. Thus, below we pretend we have run the optimistic algorihm ending in the bad case with $\operatorname{vol}_{C}(X)<\operatorname{vol}_{C}(Y) / 2$. For every $v \in Y \backslash X$, when running PageRank from $v$ with Lemma 27, we get Lemma 27(i) with a low conductance cut where the small side $T_{v}$ has $\operatorname{vol}_{C}\left(T_{v}\right) \leq 16 s$ and a limit excess above $1 /(32 \lg (4 s))$. We get the excess guarantee from Lemma 27 (i) because $s \leq m(C) / 32$ and it implies that $p_{v, C}^{*}\left(T_{v}\right)>1 /(32 \lg (4 s))$. Let $S^{\prime}=\bigcup_{v \in Y \backslash X} T_{v}$. Recall that pushing to the limit is a non-negative linear transformation. This means if we run PageRank from any distribution on $Y \backslash X$, then the limit mass on $S^{\prime}$ is above $1 /(32 \lg (4 s))$.

What happens in the real algorithm behind Lemma 29 is that we first run PageRank starting from the distribution $p_{Y, C}^{\circ}$ on $Y$ with uniform density $1 / \operatorname{vol}_{C}(Y)$. Assuming we are in the bad case with $\operatorname{vol}_{C}(X)<$ $\operatorname{vol}_{C}(Y) / 2$, we have $p_{Y, C}^{\circ}(Y \backslash X) \geq 1 / 2$. It follows that when we push to the limit, we end up with mass $p_{Y, C}^{*}\left(S^{\prime}\right)>1 /(64 \lg (4 s))$. We also have $\operatorname{vol}_{C}\left(S^{\prime}\right) \leq 16 s|Y|$. This means that $S^{\prime}$ gets excess at least $\gamma^{\prime}=1 /(64 \lg (4 s))-16 s|Y| /(2 m(C))$. However, we have $|Y| \leq m(C) /(1024 s \lg (4 s))$, and hence $\gamma^{\prime} \geq 1 /(128 \lg (4 s))$. Thus, if we are in the bad case for the optimistic algorithm, then we know that $\operatorname{excess}_{C}\left(p_{Y, C}^{*}, S^{\prime}\right) \geq \gamma^{\prime}$. Applying Theorem 8 to $p_{Y, C}^{\circ}$ with excess parameter $\gamma^{\prime}$, we get one of two outcomes:

- We find a set $A$ with $\operatorname{vol}(A) \leq m(C)$ and $\left.\Phi_{C}(A) \leq O\left(\sqrt{\left(\alpha_{0} \log m\right) / \gamma^{\prime}}\right)=O(1 /(\log m))^{3 / 2}\right)=$ $o\left(\Phi_{0}\right)$ in time $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$, satisfying Lemma $29(\mathrm{i})$.
- In $\widetilde{O}(m(C))$ time, we certify that there is no set $S^{\prime}$ with excess $\gamma^{\prime}$. We now run the above optimistic algorithm with no risk of ending in the bad case, and then we satisfy Lemma 29 (ii).

The next lemma will be used to certify that if we try to run Lemma 28 on any set $X$ with at least half the vertices from a given set $Y$ and we end with the low-conductance cut of Lemma 28 (i), then the smaller side has volume $\widetilde{\Omega}(m(C))$. The new lemma may itself yeild a low-conductance cut, but the difference is that the new lemma does not make any assumptionas about vertices being $s$-captured as required for Lemma 28 ,

Lemma 30 Let $Y$ be any set of vertices from a trimmed component $C$. We have an algorithm that, depending on the input, will do one of the following:
(i) Find a set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C)$ and $\Phi_{C}(A)=o\left(\Phi_{0}\right)$ in time $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$.
(ii) Certify in $\widetilde{O}(m(C))$ time that there is not a subset $X \subseteq Y$ with $\operatorname{vol}_{C}(X) \geq \operatorname{vol}_{C}(Y) / 2$ and a set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C) /(256 \lg (8 m))$ such that excess $_{C}\left(p_{X, C}^{*}, A\right) \geq 1 /(128 \lg (8 m))$. Recall that $p_{X, C}^{*}$ is the limit distribution when we run PageRank on $C$ starting with all mass uniformly spread on $X$.

Proof Let us assume that there is a subset $X \subseteq Y$ with $\operatorname{vol}_{C}(X) \geq \operatorname{vol}_{C}(Y) / 2$ and set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C) /(256 \lg (8 m))$ such that $\operatorname{excess}_{C}\left(p_{X, C}^{*}, A\right) \geq 1 /(128 \lg (8 m))$. Then we have to end in case (i).

We are going to start PageRank with the distribution $p_{Y, C}^{\circ}$ on $Y$ with uniform density $1 / \operatorname{vol}_{C}(Y)$. The limit distribution is $p_{Y, C}^{*}$. We have $p_{X, C}^{\circ}$ and $p_{X, C}^{*}$ denoting the corresponding distributions if we instead started with uniform density $1 / \operatorname{vol}_{C}(X)$ on $X$.

Since $X \subseteq Y$ and $\operatorname{vol}_{C}(X) \geq \operatorname{vol}_{C}(Y) / 2$, we get $p_{Y, C}^{\circ}(v) \geq p_{X, C}^{\circ}(v) / 2$ for every vertex $v \in C$. Since pushing to the limit is a non-negative linear transformation, we conclude that that we also in the limit get $p_{Y, C}^{*}(v) \geq p_{X, C}^{*}(v) / 2$ for every vertex $v \in C$. In particular, we get that

$$
p_{Y, C}^{*}(A) \geq p_{X, C}^{*}(A) / 2>\operatorname{excess}_{C}\left(p_{X, C}^{*}, A\right) / 2 \geq 1 /(256 \lg (8 m)) .
$$

Therefore

$$
\operatorname{excess}_{C}\left(p_{Y, C}^{*}, A\right)=p_{Y, C}^{*}(A)-\operatorname{vol}_{C}(A) /(2 m(C))>1 /(512 \lg (8 m)) .
$$

Thus, starting PageRank from $p_{Y, C}^{\circ}$ using Theorem 8 with $\gamma=1 /(512 \lg (8 m))$, we will get a set $A^{\prime}$ for case (i) with $\operatorname{vol}_{C}\left(A^{\prime}\right) \leq m(C)$ and $\Phi_{C}\left(A^{\prime}\right) \leq O\left(\sqrt{\left(\alpha_{0} \log m\right) / \gamma}\right)=O\left(1 /(\log m)^{3 / 2}\right)=o\left(\Phi_{0}\right)$ in time $\widetilde{O}\left(\operatorname{vol}_{C}\left(A^{\prime}\right)\right)$. We then return $A^{\prime}$ as in case (i).

If no such $A^{\prime}$ is found, we terminate in $\widetilde{O}\left(m(C) /\left(\alpha_{0} \gamma\right)\right)=\widetilde{O}(m(C))$, certifying that our assumptions were false as in case (ii).

We are now ready to complete the proof of Theorem 23,
Let $s \in\left[s_{0}, m(C)\right]$ and $C$ be an $s$-splittable trimmed component of $H$. We have an algorithm that, depending on the input, will do one of the following:
(i) Find a set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C)$ and $\Phi_{C}(A)=o\left(\Phi_{0}\right)$ in time $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$.
(ii) Find a set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C)$ and $\Phi_{C}(A)=o\left(\Phi_{0}\right)$ in time $\widetilde{O}(m(C))$ certifying that the large side $B=V(C) \backslash A$ is s/2-splittable.
(iii) Certifying in $\widetilde{O}(m(C))$ time that $C$ is $s / 2$-splittable.

Proof of Theorem 23 We have already handled the case where $s=\widetilde{\Omega}(m(C))$ in Section 5.4 so we may assume that $s \leq m(C) / 32$, as required for Lemma 29 .

First we assume that $C$ has at least $\left\lceil 80 m(C) /\left(s \alpha_{0}\right)\right\rceil$ vertices, and then we let $Y$ be any set of $\left\lceil 80 \mathrm{~m}(C) /\left(s \alpha_{0}\right)\right\rceil$ from $C$. Since $C$ is trimmed, the minimum degree in $C$ is at least $2 \delta / 5$, so $\operatorname{vol}_{C}(Y) \geq$ $32 \delta m(C) /\left(s \alpha_{0}\right)$. Next we cut $Y$ into $\left\lceil 80 \cdot 1024 \lg (4 s) / \alpha_{0}\right\rceil=\widetilde{O}(1)$ segments $Y_{i}$, each with at most $m(C) /(1024 s \lg (4 s))$ vertices.

Recall from Section 4.5 that we have a balanced binary search tree over the vertex list from $C$, where each tree node knows the number of descendants. Using this tree, we can cut into segments of any desired size in $O(\log n)$ time per segment, so the the total time spent so far is $\widetilde{O}(1)$.

We will then, alternating in parallel, apply Lemma 29 to every $Y_{i}$ while, also in parallel, applying Lemma 30 to $Y$. If any one of these ends in case (i), then this corresponds to case (i) of the theorem. The multiplicative $\widetilde{O}(1)$ slowdown does not affect the time bound. Thus we are done if we get Lemma 29(i) for some $Y_{i}$ or Lemma30(i) for $Y=\bigcup_{i} Y_{i}$.

Assume instead that we get no case (i). Then for each $Y_{i}$, by Lemma 29 (ii), we find a subset $X_{i} \subseteq Y_{i}$ with at least half the volume, that is, $\operatorname{vol}_{C}\left(X_{i}\right) \geq \operatorname{vol}_{C}\left(Y_{i}\right) / 2$, and such that no vertex in $X_{i}$ is $s$-captured. Then no vertex in $X=\bigcup_{i} X_{i}$ is $s$-captured, and $\operatorname{vol}_{C}(X) \geq \operatorname{vol}_{C}(Y) / 2$. Now by Lemma 30 (ii), we know that there is no set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C) /(256 \lg (8 m))$
such that $\operatorname{excess}_{C}\left(p_{X, C}^{*}, A\right) \geq 1 /(128 \lg (8 m))$. We have spent $\widetilde{O}(m(C))$ time so far. We know that $\operatorname{vol}_{C}(X) \geq \operatorname{vol}_{C}(Y) / 2 \geq 16 m(C) /\left(s \alpha_{0}\right)$ and that no vertex from $X$ is $s$-captured.

We now apply Lemma 28 to $X$. If we get case (ii) or (iii) of Lemma 28 , then they directly gives us case (ii) or (iii) of the theorem, so we are done.

Suppose instead we get Lemma 28 (i). Then we find a set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C)$, $\operatorname{excess}_{C}\left(p_{X, C}^{*}, A\right) \geq 1 /(128 \lg (8 m))$, and $\Phi_{C}(A)=o\left(\Phi_{0}\right)$ in time $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$. This would be good enough for case (i) of the theorem, except that the total time we have spent finding $A$ is $\widetilde{O}(m(C))$, and we are only allowed time $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$.

We now combine with the above conclusion from Lemma 30 (ii) which says that there is no set $A \subseteq$ $V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C) /(256 \lg (8 m))$ and $\operatorname{excess}_{C}\left(p_{X, C}^{*}, A\right) \geq 1 /(128 \lg (8 m))$. We conclude that the set $A$ found by Lemma $28\left(\right.$ i) has $\operatorname{vol}_{C}(A)>m(C) /(256 \lg (8 m))=\widetilde{\Omega}(m(C))$, which means that the total time we have spent is $\widetilde{O}(m(C))=\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$, as required for case (i) of the theorem. This completes the proof of Theorem 23 assuming that $C$ has at least $\left\lceil 80 m(C) /\left(s \alpha_{0}\right)\right\rceil$ vertices.

We will now complete the proof of Theorem 23 by handling the case where $C$ has less than $80 m(C) /\left(s \alpha_{0}\right)$ vertices. The proof is very similar to the case where $C$ had more vertices, but this time, we let $Y$ consist of all the vertices from $\underset{\sim}{C}$, that is, $Y=V(C)$.

Next, as above, we partition $Y$ into $\widetilde{O}(1)$ sets $Y_{i}$, each with at most $m(C) /(256 s \lg (4 s))$ vertices. If $Y$ is small, we get fewer sets $Y_{i}$, which is only an advantage.

We apply Lemma 29 in parallel to every $Y_{i}$. If we get Lemma 29(i) for any of $Y_{i}$, then this corresponds to Theorem[23(i), and we are done.

Thus, suppose we for every $Y_{i}$ get Lemma 29 (ii) with a subset $X_{i}$ with no $s$-captured vertices and with $\operatorname{vol}_{C}\left(X_{i}\right) \geq \operatorname{vol}_{C}\left(Y_{i}\right) / 2$. We consider now the set $X=\bigcup_{i} X_{i}$. It has $\operatorname{vol}_{C}(X) \geq \operatorname{vol}_{C}(Y) / 2=m(C)$, and there is no $s$-captured vertex in $X$.

Since $\operatorname{vol}_{C}(X)>m(C) / 4$, we can now apply Lemma 28 to $X$. If we get case (ii) or (iii) of Lemma 28 , then they directly gives us case (ii) or (iii) of the theorem, so we are done.

Suppose instead we get Lemma 28 (i). Then we find a set $A \subseteq V(C)$ with $\operatorname{vol}_{C}(A) \leq m(C)$, $\operatorname{excess}_{C}\left(p_{X, C}^{*}, A\right) \geq 1 /(128 \lg (8 m))$, and $\Phi_{C}(A)=o\left(\Phi_{0}\right)$ in time $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$. This would be good enough for Theorem [23(i), except that the total time we have spent finding $A$ is $\widetilde{O}(m(C))$, and we are only allowed time $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$.

Like in the case when $C$ had more vertices, we want to argue that $\operatorname{vol}_{C}(A)=\widetilde{\Omega}(m(C))$. This time we will not use Lemma 30 (ii), but instead apply a direct argument. Let $p_{X, C}^{\circ}$ be the uniform density distribution on $X$. We have $\operatorname{vol}_{C}(X) \geq m(C)$, so $p_{X}^{\circ}$ is dominated by the stationary distribution $\overline{1 / m(C)}$, and hence so is the limit distribution $p_{X, C}^{*}=\operatorname{PR}_{C}\left(\alpha_{0}, p_{X, C}^{\circ}\right)$. This means that any set $A^{\prime}$ has $\operatorname{excess}_{C}\left(p_{X, C}^{*}, A\right)^{\prime} \leq \operatorname{vol}_{C}\left(A^{\prime}\right)(1 / m(C)-1 /(2 m(C))) \leq \operatorname{vol}_{C}\left(A^{\prime}\right) /(2 m(C))$. But the set $A$ from Lemma 28 (i) had $\operatorname{excess}_{C}\left(p_{X, C}^{*}, A\right) \geq 1 /(128 \lg (8 m))$, so we conclude that $\tilde{\operatorname{vol}}_{C}(A) \geq m(C) /(64 \lg (8 m))=\widetilde{\Omega}(m(C))$. The set $A$ from Lemma 28 (i) is thus found within the $\widetilde{O}\left(\operatorname{vol}_{C}(A)\right)$ time required for Theorem 23 (i). This completes the proof of Theorem 23,

Proof of Theorem 1 We have now completed the proof of Theorem 23, and by Lemma 25, this means that we are also done with the proof of Theorem 1 .

### 5.6 Log-factors

In this paper, we have not worried about the number of log-factors in our near-linear time bound for solving the min-cut problem, nor have we accounted for them. More precisely, as described in Section 1.8, we have freely used simplifications like $\widetilde{O}(\widetilde{O}(f(n)))=\widetilde{O}(f(n))$ and $\widetilde{O}(f(n)) \widetilde{O}(g(n)))=\widetilde{O}(f(n) g(n))$. For a proper accounting, we would have to undo these simplifications. Below we will sketch that 12 log-factors suffices. The purpose is not to prove this, but rather to set up an estimated benchmark that other other researchers can improve on.

Currently, we have $\alpha_{0}=1 /(\log m)^{5}$, but in fact it suffices with $\alpha_{0}=1 /\left(c_{0}(\log m)^{4}\right)$ for some sufficiently large constant $c_{0}$. The place that puts the biggest demand on $\alpha_{0}$ is in the end of the proof of Lemmas 29 and 30 where we need that $\Phi_{C}(A)=O\left(\sqrt{\left(\alpha_{0} \log m\right) / \gamma}\right)=O\left(\sqrt{\alpha_{0} \log ^{2} m}\right) \leq \Phi_{0}=1 /(20 \lg m)$. By definition of the $O$-notation, there exists a large enough constant $c_{0}$ such that $\alpha_{0}=1 /\left(c_{0}(\log m)^{4}\right)$ yields $\Phi_{C}(A) \leq 1 /(20 \lg m)$.

We can also reduce the requirement on $\delta$ to $\delta \geq c_{1} / \alpha_{0}$ and set $\delta^{*}=c_{1} \delta / \alpha_{0}$ for some sufficiently large constant $c_{1}$. The critical place is Lemma 26 which currently says that if we start the PageRank algorithm from a vertex with a fraction $\varepsilon$ of its edges leaving a certain set $S$, then in the limit, the mass leaving $S$ is only $\varepsilon+o(1)$. Lemma 26 was proved for $O(\delta)$ cut edges, but we never need it for more than $2 \delta$ cut edges. With this concrete bound on the cut size, if we parameterize by $c_{1}$ and change the proof of Lemma 26 accordingly, the mass leaving $S$ is at most $\varepsilon+O\left(1 / c_{1}\right)$. When we later apply Lemma 26 to the proof of Lemma 27, what we need is that $3 / 4-O\left(1 / c_{1}\right)-1 / 2>1 / 5$, which is true for some sufficiently large constant $c_{1}$.

The conclusion is that we can run our algorithm with parameters $\alpha_{0}=O\left(\log ^{4} m\right)$ and $\delta^{*}=$ $O\left(\delta \log ^{4} m\right)$. For Lemma [18, this implies that the number of edges leaving passive super vertices is $O\left(m \delta^{*} / \delta^{2}\right)=O\left(m\left(\log ^{4} m\right) / \delta\right)$, which then also bounds the number of edges in $\bar{G}$.

The bottleneck in time originates from Lemma 29 (i), where the set $A$ is really found in time $O\left(\operatorname{vol}_{C}(A)(\log m) /\left(\gamma \alpha_{0}\right)\right)=O\left(\operatorname{vol}_{C}(A) \log ^{6} m\right)$ time. In the proof of Theorem 23, we run $O\left((\log s) / \alpha_{0}\right)=O\left(\log ^{5} m\right)$ such experiments from Lemma 29 in parallel, so the cost is $O\left(\log ^{11} m\right)$ per edge, and the same edge may get charged $2 \lg m$ times as it ends up in components of half the size or half as splittable. Thus a total cost of $O\left(\log ^{12} m\right)$ per edge is needed in order to find the clusters. When we afterwards contract the cores, we halve the number of edges, so it is the cost of the first cluster finding round that dominates. Our total cost for finding $\bar{G}$ is $O\left(m \log ^{12} m\right)$. Since $\bar{G}$ has only $O\left(\left(m \log ^{4} m\right) / \delta\right)$ edges, using Gabow's algorithm, we can now find a minimum cut in $O\left(m \log ^{5} m\right)$ time. Our overall time bound for finding the minimum cut is thus $O\left(m \log ^{12} m\right)=O\left(m \log ^{12} n\right)$. This was recently improved by Henzinger et al. [14] to $O\left(m(\log n)^{2}(\log \log n)^{2}\right)$.

## 6 Approximate cuts with fewer vertices

In this section, we prove Theorem 3. To do this, we will first generalize our min-cut contraction algorithm to preserve approximate min-cuts. Afterwards we will show how to reduce the number of vertices in the contracted graph by a factor $\delta$.

### 6.1 Preserving approximate cuts

We are now going to modify our min-cut contraction algorithm to preserve, not only min-cuts, but also approximate min-cuts:

Theorem 31 Let $\varepsilon \in(0,1]$ be a constant. Given a simple input graph $G$ with $n$ vertices, $m$ edges, and minimum degree $\delta$, and (unknown) edge connectivity $\lambda$, in near-linear time, we can contract vertex sets producing a multigraph $\bar{G}$ which has only $\bar{m}=\widetilde{O}(m / \delta)$ edges, yet which preserves all non-trivial cuts of $G$ of size at most $\lambda+(1-\varepsilon) \delta$.

It turns out that we only need minor modifications of our contraction algorithm for min-cuts to preserve approximate min-cuts and prove Theorem 31. We will first describe the algorithmic changes, later the changes to the analysis.

Algorithmic changes On the high level, we are going to reuse Algorithm 3 for the approximate min-cuts of Theorem 31. However, we need to modify some definitions by changing four thresholds.

The first obvious change is that before we said we could contract two vertices if there were more than $\delta$ parallel edges between them. Now there should be more than $2 \delta$ parallel edges between them.

More interestingly, we have to change the definition of loose vertices from Section 4.2. The original defintion was that a vertex $v$ in a cluster $C$ is loose if it is regular and at least $d(v) / 2-1$ of its edges in $\bar{G}$ leave $C$. Now we say $v$ is loose if at least $\varepsilon d(v) / 2-4$ of its edges in $\bar{G}$ leave $C$. Recall that loose vertices have to be shaved from $C$ to get down to the core, which is scrapped if less than $1 / 4$ of the edges incident to $C$ are internal to the core. With the new definition, we get more loose vertices, hence smaller cores. Essentially this will increase the number of edges removed from cores by a factor $1 / \varepsilon$.

The third change is in Section 4.3 where we said that we terminate the algorithm if more than a fraction $1 / 20$ of the edges in $\bar{G}$ are incident to passive super vertices. Now will we terminate if more than a fraction $\varepsilon / 20$ of the edges are incident to passive super vertices.

The fourth change is in Section 4.4 in the definition of the threshold $\Phi_{0}=1 /(20 \lg m)$ for a low conductance cut. We change it to $\Phi_{0}^{\prime}=\varepsilon /(20 \lg m)$. The last two changes will imply that the fraction of edges cut from passive vertices and from low conductance cuts is reduced by $\varepsilon$ from $1 / 10$ to $\varepsilon / 10$.

Finally, we note an implicit change; namely that new asymptotic calculations may affect the smallest number $n_{0}$ of vertices that our algorithm can handle (c.f. Section 1.8). However, $n_{0}$ remains a constant and Theorem 31] is trivially true for a constant sized graph.

This completes the changes to our contraction algorithm. We will now analyze how the changes help us preserve approximate cuts below.

Modified analysis We want to prove that our modified contraction algorithm preserves cuts of size at most $\lambda+(1-\varepsilon) \delta<2 \delta$. To do that, we will in many cases focus on cuts of size at most $2 \delta$ instead of $\delta$. The first example is in the definition of a cluster from Section 4.1 Originally we said that a trimmed vertex set $C$ was a cluster if for every cut of size at most $\delta$ in $\bar{G}$, one side contains at most two regular vertices and no super vertices from $C$. Now we say a trimmed vertex set $C$ is cluster if for every cut of size at most $2 \delta$ in $\bar{G}$, one side contains at most five regular vertices and no super vertices from $C$. Corrsponding to Lemma 14 , the following lemma says that more than five regular vertices implies many regular vertices:

Lemma 14' Consider a trimmed vertex set $C$ and a cut $(T, U)$ of $\bar{G}$ of size at most $2 \delta$ (before it was $\delta$ ). If $T \cap C$ has no super vertices and at least 6 (before it was 3) regular vertices, then $T \cap C$ has at least $\delta / 3$ regular vertices.

Proof Consider $T \cap C$ which has no super vertices. Since $C$ is trimmed, the internal degree of regular vertices in $C$ is at least $2 \delta / 5$, so the number of edges crossing from $T \cap C$ to $U \cap C$ is at least $\mid C \cap$
$T \mid(2 \delta / 5+1-|C \cap T|)$, but we have at most $2 \delta$ cut edges. Since $\delta=\omega(1)$, we conclude that $|C \cap T| \leq 5$ or $|C \cap T| \geq 2 \delta / 5-4>\delta / 3$.

The most important change to the analysis, however, is to show an analogue to Lemma 15 which said that contracting cores preserved non-trivial min-cuts. With our new definition of clusters and of the loose vertices not in the core, we want to show that contracting cores preserves non-trivial cuts of size up to $\lambda+(1-\varepsilon) \delta$.

Lemma 15' If a non-trivial cut of $G$ of size at most $\lambda+(1-\varepsilon) \delta$ (before it was $\lambda$ ) has survived in $\bar{G}$, then it will also survive when we contract the core of any cluster in $\bar{G}$.

Proof Consider a cut $(T, U)$ of $\bar{G}$ of size at most $\lambda+(1-\varepsilon) \delta$ that was non-trivial in $G$. We must have at least two regular vertices or one super vertex both in $T$ and in $U$.

We now consider a cluster $C$ in $\bar{G}$ with a non-empty core $A$. Since $(T, U)$ has size at most $\lambda+(1-\varepsilon) \delta<$ $2 \delta$, by the new definition of a cluster, one side, say $T$, has at most five regular vertices and no super vertices from $C$. We will argue that these vertices in $C \cap T$ must be loose, hence that the vertices identified by the contraction of $A$ are all in $U$; for then the contraction preserves $(T, U)$.

Let $v$ be one of the regular vertices from $C \cap T$, and assume for a contradiction that $v$ is not loose. By our new definition of loose, this means that $v$ has at most $\varepsilon d(v) / 2-5$ edges leaving $C$ in $\bar{G}$.

We will prove that we get a cut that is more than $(1-\varepsilon) \delta$ edges smaller by moving $v$ to $U$, contradicting that $(T, U)$ had size at most $\lambda+(1-\varepsilon) \delta$.

Moving $v$ only affects the cutting of edges incident to $v$. Recall that $T \cap C$ has at most five regular vertices, including $v$, and no super vertices, so $v$ has at most four edges to other vertices in $T \cap C$. When $v$ is in $T$, we cut all other edges from $v$ to $C$. However, since $v$ is not loose, it has least $(1-\varepsilon / 2) d(v)+5$ edges from $v$ into $C$, so with $v$ in $T$, we cut at least $(1-\varepsilon / 2) d(v)+1$ edges incident to $v$. Moving $v$ to $U$, we cut the at most $\varepsilon d(v) / 2-1$ other edges incident to $v$. Moving $v$ from $T$ to $U$ thus reduces the cut size by at least $(1-\varepsilon) d(v)+2$, yielding the desired contradiction.

We also need an analogue of Lemma 16 bounding the number of edges incident to a cluster that are not internal to the core.

Lemma 16' If a cluster $C$ has $k$ edges leaving it, then there are less than $3 k / \varepsilon$ (before it was $3 k$ ) edges incident to $C$ that are not internal to the core. In particular, if the core is empty, we have $\operatorname{vol}(C)<3 k / \varepsilon$.

Proof Let $A$ be $C$ without the loose vertices, i.e., $A$ is the core unless the core becomes empty. Let $\ell$ be the number of edges leaving $C$ from loose vertices. Then we have $k-\ell$ edges leaving $C$ from vertices in $A$. Other edges incident to $C$ but not internal to $A$ are all incident to loose vertices.

Consider any loose vertex $v$ in $C$. It has at least $\varepsilon d(v) / 2-4=(\varepsilon / 2-o(1)) d(v)$ edges leaving $C$. Here we used that loose vertices are regular, so $d(v) \geq \delta=\omega(1)$. It follows that the total number of edges incident to loose vertices is at most $\ell /(\varepsilon / 2-o(1))=(2 / \varepsilon+o(1)) \ell$. This proves the lemma unless the core becomes empty.

The core becomes empty if and only if at most $1 / 4$ of the edges incident to $C$ are internal to $A$, but this implies that the number of edges internal to $A$ is at most $1 / 3$ of the number of edges not internal to $A$. Thus, if $A$ is not the core, there are at most $(2 / \varepsilon+o(1)) k / 3$ edges internal to $A$, and then we have at most $(2 / \varepsilon+o(1)) k+(2 / \varepsilon+o(1)) k / 3<3 k / \varepsilon$ edges incident to $C$.

There are no changes to Lemma 17 and 18. With our new Lemma 16, we easily get a corresponding change to Lemma 19 :

Lemma 19' If the total number of edges cut is $c$, then the total number of edges lost due to trimming, shaving, and scrapping is at most $4 c / \varepsilon$ (before it was $4 c$ ).

Because of the new bounds, we need to reprove:
Lemma 20 Cutting edges around passive vertices and edges of low-conductance cuts, trimming, shaving, and scrapping, leaves at least half the edges of $\bar{G}$ in the resulting cluster cores of $H$.

Proof Recall that we changed the algorithm to stop when more than a fraction $\varepsilon / 20$ of the edges in $\bar{G}$ were incident to passive vertices, so this limits the fraction that gets cut when we cut out these passive vertices.

Also, now we have defined low conductance cuts to have conductance at most $\Phi_{0}^{\prime}=\varepsilon /(20 \lg m)$, and the fraction of edges cut by recursive low conductance cuts is at most $\lg m$ times bigger, so at most $\varepsilon / 20$. Thus, the total fraction of edges cut is at most $\varepsilon / 10$. Then, by Lemma 19, the total fraction of edges lost due to trimming, shaving, and scrapping is at most $4(\varepsilon / 10) \varepsilon=4 / 10$. All together, the total fraction of edges lost is $\varepsilon / 10+4 / 10 \leq 1 / 2$.

The change of $\Phi_{0}$, multiplying it by $\varepsilon=\Theta(1)$, has no other impact. The reason is that whenever our algorithm produces a low conductance cut, we proved it to be of conductance $o\left(\Phi_{0}\right)<\Phi_{0}^{\prime}$ (c.f. Theorem 23).

When it comes to the graph representations and modifications in Section 4.5, there are only two minor changes. The first is that we now need more than $2 \delta$ parallel edges betweeen vertices before we contract them. The second is the new definition of loose vertices. Before a vertex $v$ was loose if it had $d_{H}(v) \leq$ $d_{\bar{G}}(v) / 2+1$. Now it only needs $d_{H}(v) \leq(1-\varepsilon / 2) d_{\bar{G}}(v)+4$. With the above changes, we can immediately strengthen Lemma 21 to work for approximate min-cuts.

Lemma 21' Ignoring the cost of finding the low-conductance cuts, we can implement Algorithm 3 (with our modified definitions) in $O\left(m \log ^{2} n\right)$ time where $n$ and $m$ are the number of vertices and edges of the simple input graph $G$. The result is the contracted graph $\bar{G}$ described in Theorem 31 that is, $\bar{G}$ has only $\widetilde{O}(m / \delta)$ edges, yet it preserves all non-trivial cuts of $G$ of size at most $\lambda+(1-\varepsilon) \delta$.

Getting to Section [5] like for clusters, we now need to consider cuts of size at most $2 \delta$ instead of just $\delta$, so now we define that a component $C$ of $H$ is s-splittable if every cut $(T, U)$ of $\bar{G}$ of size at most $2 \delta$ has $\min \left\{\operatorname{vol}_{C}(T \cap C), \operatorname{vol}_{C}(U \cap C)\right\}<s$. We do not need to change that $s_{0}=64 \delta / \alpha_{0}$.

Working with cuts of size up to $2 \delta$, we need to reprove Lemma[22] which gives us the main condition for arguing that a trimmed component is a cluster.

Lemma 22 If a trimmed component $C$ of $H$ is $s_{0}$-splittable, then $C$ is a cluster.
Proof Suppose that $C$ is not a cluster. Then there is a cut $(T, U)$ of $\bar{G}$ of size at most $2 \delta$ (before it was $\delta$ ) such that both $T \cap C$ and $U \cap C$ contain a super vertex or at least 6 (before it was 3 ) regular vertices. In the case of at least 6 regular vertices and no super vertices, Lemma 14 tells us that there are at least $\delta / 3$ regular vertices. No other changes are needed to the original proof of Lemma 22 ,

The remaining changes are quite small and local, exploiting that our original analysis has enough slack to accomodate the increase in cut size from $\delta$ to $2 \delta$ as well as the up to $2 \delta$ parallel edges between vertices.

We will see that Theorem 23 holds unchanged like Lemma22, and this means that we need no changes to Lemma 24 or its proof. Therefore, as analogue to Lemma 25] we get

## Lemma 25' Theorem 31 follows from Theorem 23(with our modified definitions).

Moving to Section 5.1, we need no changes to the statement of Lemma 26as it is already handling $O(\delta)$ cut edges. However, inside the proof of Lemma 26 we need a very minor change. We currently say that the maximal fraction of mass that can be pushed from a super vertex to a neighbor is $\delta /\left(2 \delta^{*} / 5\right)=O\left(\alpha_{0} / \log n\right)$. With up to $2 \delta$ parallel edges, this is fraction is increased to $2 \delta /\left(2 \delta^{*} / 5\right)=O\left(\alpha_{0} / \log n\right)$. No other changes are needed to the proof.

With the statement of Lemma 26 unchanged, we do not need to make any changes to Lemma 27 or its proof. The final place were we need a more careful change is

Lemma [28' Lemma 28 holds with the new definition of s-captured for cuts of size up to $2 \delta$ and where we in case (ii) and (iii) double the the cut size bound to $\delta_{C}(S) \leq 2 \delta$.

Proof The proof is essentially the same as that of Lemma 28 , and we will only describe the changes. We double the bound on $\delta_{C}(S)$ to $2 \delta$, and as a result, we may get twice as much limit mass in $S$. More precisely, as in (2) from the proof of Lemma 28, we get

$$
p_{X, C}^{*}(S) \leq\left(4+1 /\left(2 \alpha_{0}\right)\right)\left|\partial_{C}(S)\right| / \operatorname{vol}_{C}(X) .
$$

In the proof of Lemma 28 , we bounded $\left(4+1 /\left(2 \alpha_{0}\right)\right) \leq 1 / \alpha_{0}$, but this time we tighten it to $\left(4+1 /\left(2 \alpha_{0}\right)\right) \leq$ $2 /\left(3 \alpha_{0}\right)$, using $\alpha_{0}=o(1)<1 / 24$, and then we get

$$
p_{X, C}^{*}(S) \leq\left(2 /\left(3 \alpha_{0}\right)\right)\left|\partial_{C}(S)\right| / \operatorname{vol}_{C}(X) \leq 4 \delta /\left(3 \operatorname{vol}_{C}(X) \alpha_{0}\right) .
$$

As in (3) from the proof of Lemma 28, we have

$$
\delta /\left(\operatorname{vol}_{C}(X) \alpha_{0}\right) \leq s /(16 m(C))
$$

so we conclude that

$$
\left.p_{X, C}^{*}(S) \leq s /(12 m(C)) \quad \text { (before it was } s /(16 m(C))\right)
$$

Since $\operatorname{vol}_{C}(S)>s / 2$, this means that vertices $u \in S$ with limit density $p_{X, C}^{*}(u) / d(u) \leq 1 /(3 m(C))$ (before it was $1 /(2 m(C))$ ) represent more than half the volume of $S$. Finally we apply Theorem 9 with $\gamma=1 / 6$ (before it was $1 / 4$ ). The rest of the proof is exactly as the proof of Lemma 28 ,

Since Lemma 27 and Lemma 28 both hold with our new notion of $s$-captured for cuts of size up to $2 \delta$, we do not need to make any changes to the proofs of Lemma 29, Lemma 30, and Theorem 23, Then Theorem 31 follows by Lemma 25 .

This completes the description of how the contraction algorithm and its analysis for Theorem 1 can be modified to prove Theorem 31 We note that the proof is not harder, and we could have proved the stronger Theorem 31 directly to start with. However, the main result of this paper is about min-cuts, and they are cleaner to work with, e.g., exploiting that every vertex has at least half its neighbors on the same side.

### 6.2 Better edge and vertex bounds

We will now show how we can improve the edge and vertex bounds of Theorem 31 to $\widetilde{O}(n)$ and $\widetilde{O}(n / \delta)$, respectively, as required to complete the proof of Theorem 3

Let $k=2 \delta$. First we take out $k$ disjoint forests $F_{1}, \ldots, F_{k}$ from $G$ such that $F_{k}$ is maximal in $G \backslash$ $\bigcup_{j<i} F_{j}$. Nagamochi and Ibaraki [27] have described how to do this in linear time. Then $H=\bigcup_{i} F_{i}$ has less than $m_{H}=n k=O(n \delta)$ edges. Moreover, $H$ preserves all cuts of size $\leq k$ and larger cuts preserve at least $k$ of their edges. Finally, if an edge $(u, v)$ is left in $G \backslash H$, then $u$ and $v$ are at least $k+1$-edge connected in $G$. We are later going to contract the edges from $G \backslash H$, but we cannot destroy simplicity yet.

We now apply Theorem 31 to $H$, obtaining a contracted graph $\bar{H}$ with $\bar{m}_{H}=\widetilde{O}\left(m_{H} / \delta\right)=\widetilde{O}(n)$ edges. We claim that $\bar{H}$ has only $\widetilde{O}(n / \delta)$ vertices. Trivially $\bar{H}$ has at most $2 \bar{m}_{H} / \delta=\widetilde{O}(n / \delta)$ vertices of degree $\geq \delta$, which was the minimum degree in $G$. Lower degree vertices must be super vertices, and by Lemma 17 each super vertex in $\bar{H}$ contracts $\Omega\left(\delta^{2}\right)$ edges from $H$, so we can only have $O\left(m_{H} / \delta^{2}\right)=O(n / \delta)$ super vertices in $\bar{H}$. Thus $\bar{H}$ has $\widetilde{O}(n / \delta)$ vertices and $\widetilde{O}(n)$ edges, and it preserves all non-trivial cuts of $H$ of size at most $\lambda+(1-\varepsilon) \delta$.

Our final step is to contract in $\bar{H}$ the end-points of each edge $(u, v)$ left in $G \backslash H$. More precisely, we find the vertices or super vertices representing $u$ and $v$ in $\bar{H}$, and identify them in a single super vertex. Since $u$ and $v$ were $2 \delta+1$-connected in $G$, these contractions preserve all cuts of size at most $2 \delta$, and the resulting graph $\bar{G}$ is obtained from $G$ by contractions only. The contractions can only decrease the number of edges and vertices, so we conclude that $\bar{G}$ has $\widetilde{O}(n / \delta)$ vertices and $\widetilde{O}(n / \delta)$ edges, and that it preserves all non-trivial cuts of $G$ of size of size at most $\lambda+(1-\varepsilon) \delta$. This completes the proof of Theorem 3,

## 7 Limit concentration and low conductance cuts: the proofs

In this section we will prove Theorem 8 and 9 from Section 2.1. We recommond that the reader reviews Section 2 before continuing. The starting point for both theorems is a multigraph with $m$ edges, an initial mass distribution $p^{\circ}$ of total mass 1 , and a listing of the vertices with positive mass in order of non-increasing density.

First, in Section 7.1, we will show how to implement PageRank followed by a sweep in linear time, improving by a logarithmic factor the bound from [2, §2.2].

In Sections 7.2-7.4 we will study when we from a settled mass distribution get a low-conductance cut in the sweep. Sections 7.2 and 7.3 make symmetric studies of the cases of high versus low settled densities, while Section 7.4 studies the case where a single vertex has $\Omega(1 / m)$ too little mass, as needed for our new endgame result in Theorem 9 The results of Sections 7.2 and 7.3 are essentially equivalent to those in [2], but our proofs are easily modified to prove the new results in Section 7.4. Our proofs are quite similar to those in [1], but cast in terms of the flows from Lemma 7 .

In Section 7.5, we use the results from Sections 7.2 and 7.3 to prove Theorem 8 , which says that if some set has $\Omega(1)$ excess mass in the limit, then we can find a low-conductance cut. The techniques to prove this are not new, but in the previous papers [1, 2] they always assume that the initial distribution is with all mass on a single "good" vertex, while our Theorem 8 holds for arbitrary initial distributions, including our $p_{X}^{\circ}$ where the initial mass is spread with uniform density on an arbitrary subset $X$ of the vertices.

Finally, in Section 7.6 we prove our new endgame Theorem 9 which identifies a low-conductance cut if there is a single vertex whose limit distribution is $\Omega(1 / m)$ too small.

### 7.1 Sweeping for low conductance cuts in linear time

We will first present a simple variant of the PageRank in Algorithm 2 which makes the sweep for a low conductance cut run in linear time. The issue is that in order to do the sweep, we need the vertices with positive settled mass to be sorted in order of non-increasing settled mass density. In [2, §2.2] they used regular sorting to get this order, costing them a logarithmic factor.

First we note that we can make the push more flexible, only pushing part of the residual mass at a vertex as described in Algorithm 5. This more flexible push still satisfies all the basic properties discussed

```
Algorithm 5: Push' \((\alpha, u, q)\)-assumes \(r(u) \geq q\)
    \(p(u) \leftarrow p(u)+\alpha q ;\)
    for \((u, v) \in E\) do \(r(v) \leftarrow r(v)+(1-\alpha) q /(2 d(u))\);
    \(r(u) \leftarrow r(u)-(1+\alpha) q / 2\).
```

in Section 2, e.g., we preserve Invariant (11), Fact 6, and Lemma 7, and as the residual mass goes to 0 , the settled mass converges to the same same unique limit $\operatorname{PR}\left(\alpha, p^{\circ}\right)$ as with PageRank in Algorithm 2 .

Our basic idea is that when we push from a vertex, we will always push $\varepsilon d(u)$ of the residual mass as described in PageRank' in Algorithm 6. This means that the settled density $p(u) / d(u)$ on any vertex is

```
Algorithm 6: PageRank'( }\alpha,\varepsilon,\mp@subsup{p}{}{\circ}
    r\leftarrow\mp@subsup{p}{}{\circ};\quadp\leftarrow\mp@subsup{0}{}{V};
    while }\existsu:r(u)/d(u)\geq\varepsilon do Push'(\alpha,u,\varepsilond(u))
```

always an integer multiple of $\varepsilon$.
Lemma 32 We are given an initial mass distribution $p^{\circ}$ of total mass 1 and a listing of the vertices with positive initial mass in order of non-increasing density. For any $\alpha, \varepsilon \leq 1$, we can implement PageRank' from Algorithm 6 in $O(1 /(\varepsilon \alpha))$ time, producing a settled mass distribion $p$ listing the vertices with positive settled mass in order of non-increasing density. Within this time-bound, we can also find which prefix of list defines the lowest conductance cut.

Proof The initial mass is 1 , so the sum of the degrees of the pushes is bounded by $1 /(\varepsilon \alpha)$, which is within a constant factor of the time bound we want.

We will maintain a doubly-linked push-list with vertices of residual density at least $\varepsilon$. These are the vertices from which PageRank' will have to do a push. Initially the residual mass is the initial mass, so the initial push-list is extracted as a prefix of the list of vertices in order of non-increasing initial/residual density.

We are going to use the general trick that each vertex has a flag telling if it has been affected by the pushes. Moreover, we will have a list with all vertices affected. The maximal length of this list is $1 /(\varepsilon \alpha)$. All vertices have fields for residual density and settled density that we can start using first time they are affected, copying the residual mass from the initial mass and setting the settled mass to zero. At the end, we will have to clean up, going through the list of affected vertices setting their flags to zero.

When we do a push from $u$, we first note that it affects $u$ and all its neighbors, and the neighbors may have to be initialized as described above. Afterwards, we just implement the push as described in Algorithm 5 in constant time per neighbor. Note that when we push from $u$, its residual denisty may drop below $\varepsilon$ and
then it has to be removed from the push-list. Likewise, some neigbors may now get residual density $\varepsilon$ and hence get added to the push-list. This way, the total time spent is $O(1 /(\varepsilon \alpha))$.

We now go through the list of affected vertices, collecting all vertices with positive settled mass. They all have settled density at least $\varepsilon$. Next we go through this positive list extracting all vertices with settled density at least $2 \varepsilon$, and continue this way, for $i=2,3, \ldots$ extracting the vertices with settled density at least $i \varepsilon$, stopping when no more vertices are extracted. This splits the vertices into lists $L_{i}$ with vertices of settled density $i \varepsilon$. The vertices in $L_{i}$ are considered $i+1$ times, but they were pushed $i$ times, and the total number of pushes was bounded by $1 /(\varepsilon \alpha)$, so the total time for this is $O(1 /(\varepsilon \alpha))$. Concatenating the lists in reverse order, we get a list $\left(v_{1}, \ldots, v_{\ell}\right)$ containing the vertices with positive settled mass ordered by non-increasing density, as desired.

We now want to find the lowest conductance cut based on a prefix of the list. The volumes are trivially computed, just adding up degrees for all prefixes. To compute the cut sizes $c_{i}=\left|\partial\left(\left\{v_{1}, \ldots, v_{i}\right\}\right)\right|$, we assume that each vertex has a field that we for an affected vertex assign its index from the list, or set to $\ell+1$ if the vertex is not in the list because it has no settled mass. This means that we for any edge leaving $v_{i}$ can tell if the other end-point is before or after $v_{i}$ in the list. We now sweep the vertices. We start with $c_{0}=0$. When we get to $v_{i}$, we first set $c_{i}=c_{i-1}$. Then we subtract from $c_{i}$ the number of edges from $v_{i}$ to preceeding vertices and add the number of edges to succeeding vertices. Now $c_{i}$ is the desired cut size, which we divide by the volume to get the conductance. At the end we return the lowest conductance cut. The time spent on $v_{i}$ is $O\left(d\left(v_{i}\right)\right)$ but we know that $v_{i}$ was pushed, hence that $\sum_{i} d\left(v_{i}\right) \leq 1 /(\varepsilon \alpha)$. Thus we conclude that the total time spent is $O(1 /(\varepsilon \alpha))$.

### 7.2 High densities

We will now prove that the sweep must find a sparse cut if the settled mass $p$ is concentrated. We will only consider vertices with settled densities above some $t_{0}$ such that

$$
\begin{equation*}
\operatorname{vol}\left(V_{>t_{0}}^{p}\right) \leq m \tag{4}
\end{equation*}
$$

Recall here that $V_{>t_{0}}^{p}$ is the set of vertices with settled density $p(v) / d(v)>t_{0}$. We will focus on cuts defined by sets $V_{\geq t}^{p}$ consisting of vertices with settled density $p(v) / d(v) \geq t$ for some $t>t_{0}$. This implies that vertices with the same settled density will always be on the same side of a cut. It is among the cuts defined by these sets that we will look for low conductance $\Phi\left(V_{\geq t}^{p}\right)=\left|\partial\left(V_{\geq t}^{p}\right)\right| / \operatorname{vol}\left(V_{\geq t}^{p}\right)$.
Lemma 33 Assuming (4), for any $\tau \in\left(t_{0}, 1\right]$,

$$
\begin{equation*}
\min _{t \in\left(t_{0}, \tau\right]} \Phi\left(V_{\geq t}^{p}\right) \leq \sqrt{\frac{18 \alpha}{\left(\tau-t_{0}\right) \operatorname{vol}\left(V_{\geq \tau}^{p}\right)}} . \tag{5}
\end{equation*}
$$

Proof To prove (5), consider any $\phi \leq \min _{t \in\left(t_{0}, \tau\right]} \Phi\left(V_{\geq t}^{p}\right)$. We will prove (5) in the following equivalent form:

$$
\begin{equation*}
\tau-t_{0} \leq \frac{18 \alpha}{\phi^{2} \operatorname{vol}\left(V_{\geq \tau}^{p}\right)} \tag{6}
\end{equation*}
$$

Let $t \in\left(t_{0}, \tau\right]$. By (4), we have vol $\left(V_{\geq t}^{p}\right) \leq m$, so by definition, $\left|\partial\left(V_{\geq t}^{p}\right)\right| \geq \phi \operatorname{vol}\left(V_{\geq t}^{p}\right)$. Consider any edge $(u, v)$ leaving $V_{\geq t}^{p}$ (so $u \in V_{\geq t}^{p}$ but $v \notin V_{\geq t}^{p}$ ). By Fact 6 the net flow over this edge from $u$ to $v$ is
at least $(p(u) / d(u)-p(v) / d(v)) /(3 \alpha)$. Here we used that $\alpha<1 / 3$, so $(1-\alpha) /(2 \alpha)>1 /(3 \alpha)$. Since $p(u) / d(u) \geq t>p(v) / d(v)$ this flow is always positive away from $V_{\geq t}^{p}$.

Let $t^{\prime}$ be the median density $p(v) / d(v)$ of a neighbor of $V_{\geq t}^{p}$, counting $p(v) / d(v)$ with the multiplicity of the number of edges from $V_{\geq t}^{p}$ to $v$. We say that $t^{\prime}$ is derived from $t$ using median expansion. We note that $\left|V_{\geq t^{\prime}}^{p}\right|>\left|V_{\geq t}^{p}\right|$ since there is some vertex with median density $p(v) / d(v)=t^{\prime}$. We also note that we may have $t^{\prime}<t_{0}$ even though $t>t_{0}$.

We have at least $\left|\partial\left(V_{\geq t}^{p}\right)\right| / 2$ edges from $V_{\geq t}^{p}$ to vertices $v$ with $p(v) / d(v) \leq t^{\prime}$, so the net flow out of $V_{\geq t}^{p}$ is at least

$$
\begin{equation*}
\left(\left|\partial\left(V_{\geq t}^{p}\right)\right| / 2\right)\left(t-t^{\prime}\right) / 3 \alpha \geq \phi \operatorname{vol}\left(V_{\geq t}^{p}\right)\left(t-t^{\prime}\right) / 6 \alpha \tag{7}
\end{equation*}
$$

But this can be no more than the total mass, which is 1 , so

$$
\begin{equation*}
\left(t-t^{\prime}\right) \leq \frac{6 \alpha}{\phi \operatorname{vol}\left(V_{\geq t}^{p}\right)} \tag{8}
\end{equation*}
$$

By the median definition of $t^{\prime}$, at least half the edges leaving $V_{\geq t}^{p}$ land in $V_{\geq t^{\prime}}^{p}$, so

$$
\begin{equation*}
\operatorname{vol}\left(V_{\geq t^{\prime}}^{p}\right) \geq \operatorname{vol}\left(V_{\geq t}^{p}\right)+\left|\partial\left(V_{\geq t}^{p}\right)\right| / 2 \geq(1+\phi / 2) \operatorname{vol}\left(V_{\geq t}^{p}\right) . \tag{9}
\end{equation*}
$$

We want to prove for any $t \leq \tau$ that

$$
\begin{equation*}
t-t_{0} \leq \frac{18 \alpha}{\phi^{2} \operatorname{vol}\left(V_{\geq t}^{p}\right)} \tag{10}
\end{equation*}
$$

If $t \leq t_{0}$, the statement of (10) is trivially true, so we can assume $t>t_{0}$. We want to use induction, inductively assuming that (10) holds for the above defined $t^{\prime}<t$. However, we cannot just use induction over the reals. Instead, formally, the induction is over the positive integer $n-\left|V_{\geq t}^{p}\right|$. Since $\left|V_{\geq t^{\prime}}^{p}\right|>\left|V_{\geq t}^{p}\right|$, it is inductively valid to assume that (10) holds for $t^{\prime}$. Combining this with (8) and (9), we get

$$
\begin{aligned}
t-t_{0} & \leq\left(t-t^{\prime}\right)+\left(t^{\prime}-t_{0}\right) \\
& \leq \frac{6 \alpha}{\phi \operatorname{vol}\left(V_{\geq t}^{p}\right)}+\frac{18 \alpha}{\phi^{2} \operatorname{vol}\left(V_{\geq t^{\prime}}^{p}\right)} \\
& \leq \frac{6 \alpha}{\phi \operatorname{vol}\left(V_{\geq t}^{p}\right)}+\frac{18 \alpha}{\phi^{2}(1+\phi / 2) \operatorname{vol}\left(V_{\geq t}^{p}\right)} \\
& \leq \frac{18 \alpha}{\phi^{2} \operatorname{vol}\left(V_{\geq t}^{p}\right)}(\phi / 3+1 /(1+\phi / 2)) \\
& \leq \frac{18 \alpha}{\phi^{2} \operatorname{vol}\left(V_{\geq t}^{p}\right)} .
\end{aligned}
$$

The last inequality uses that $\phi / 3+1 /(1+\phi / 2) \leq 1$ for all $\phi \in[0,1]$. This completes the proof of (10). Now (6) follows with $t=\tau$.

### 7.3 Low densities

We will now make a symmetric study of vertices with settled density below some $t_{0}$ such that

$$
\begin{equation*}
\operatorname{vol}\left(V_{<t_{0}}^{p}\right) \leq m \tag{11}
\end{equation*}
$$

Note that if (4) is false then (11) is true, so for any value of $t_{0}$, the analysis below applies if the analysis from the previous subsection did not apply. We also note that we can pick a median $t_{0}$ satisfing both (4) and (11).

We will consider the conductance $\Phi\left(V_{\leq t}^{p}\right)=\left|\partial\left(V_{\leq t}^{p}\right)\right| / \operatorname{vol}\left(V_{\leq t}^{p}\right)$ of the sets $V_{\leq t}^{p}$ of vertices with settled density $p(v) / d(v) \leq t$ for some $t<t_{0}$. Symmetric to the results from Section7.2, we will prove
Lemma 34 Assuming (II), for any $\tau \in\left[0, t_{0}\right)$,

$$
\begin{equation*}
\min _{t \in\left[\tau, t_{0}\right)} \Phi\left(V_{\leq t}^{p}\right) \leq \sqrt{\frac{18 \alpha}{\left(t_{0}-\tau\right) \operatorname{vol}\left(V_{\leq \tau}^{p}\right)}} . \tag{12}
\end{equation*}
$$

Proof The proof is symmetric to that of Lemma 33, which we assume that the reader is already familar with. For any given $\phi \leq \min _{t \in\left[\tau, t_{0}\right)} \Phi\left(V_{\leq t}^{p}\right)$, we are going to prove (12) in the following equivalent form.

$$
\begin{equation*}
t_{0}-\tau \leq \frac{18 \alpha}{\phi^{2} \operatorname{vol}\left(V_{\leq \tau}^{p}\right)} \tag{13}
\end{equation*}
$$

Consider any $t \in\left[\tau, t_{0}\right)$. By (11), we have $\operatorname{vol}\left(V_{\leq t}^{p}\right) \leq m$, so by definition, $\left|\partial\left(V_{\leq t}^{p}\right)\right| \geq \phi \operatorname{vol}\left(V_{\leq t}^{p}\right)$. Consider any edge $(u, v)$ leaving $V_{\leq t}^{p}$ (so $u \in V_{\leq t}^{p}$ but $v \notin V_{\leq t}^{p}$ ). By Fact 6 the net flow over this edge from $v$ to $u$ is at least $(p(v) / d(v)-p(\bar{u}) / d(u)) /(3 \bar{\alpha})$. Since $p(\bar{u}) / d(u) \leq t<p(v) / d(v)$ this flow is always positive into $V_{\leq t}^{p}$. Let $t^{\prime}$ be the median density $p(v) / d(v)$ of a neighbor of $V_{\leq t}^{p}$, counting $p(v) / d(v)$ with the multiplicity of the number of edges from $V_{\leq t}^{p}$ to $v$. This is the symmetric analogue of the median expansion used in the proof of Lemma 33 ,

We then have at least $\left|\partial\left(V_{\leq t}^{p}\right)\right| / 2$ edges from $V_{\leq t}^{p}$ to vertices $v$ with $p(v) / d(v) \geq t^{\prime}$, so the net flow into $V_{\leq t}^{p}$ is at least

$$
\begin{equation*}
\left(\left|\partial\left(V_{\leq t}^{p}\right)\right| / 2\right)\left(t^{\prime}-t\right) /(3 \alpha) \geq \phi \operatorname{vol}\left(V_{\leq t}^{p}\right)\left(t^{\prime}-t\right) /(6 \alpha) . \tag{14}
\end{equation*}
$$

But this can be no more than the total mass, which is 1 , so symmetric to (8), we get

$$
\begin{equation*}
\left(t^{\prime}-t\right) \leq \frac{6 \alpha}{\phi \operatorname{vol}\left(V_{\leq t}^{p}\right)} \tag{15}
\end{equation*}
$$

Also, since $t^{\prime}$ was the median neighboring density, corresponding to (9), we get

$$
\begin{equation*}
\operatorname{vol}\left(V_{\geq t^{\prime}}^{p}\right) \geq \operatorname{vol}\left(V_{\geq t}^{p}\right)+\left|\partial\left(V_{\geq t}^{p}\right)\right| / 2 \geq(1+\phi / 2) \operatorname{vol}\left(V_{\geq t}^{p}\right) \tag{16}
\end{equation*}
$$

The rest of the argument for Lemma 34 is exactly the same as the argument for Lemma (5).

### 7.4 A single low density

In this section we will show that just a single vertex with low density makes a big difference if we have a good bound on the residual densities $r(v) / d(v)$ for every vertex $v$. More precisely, we will prove
Lemma 35 Assume (IIT), that is, $\operatorname{vol}\left(V_{<t_{0}}^{p}\right) \leq m$. If $r(v) / d(v) \leq \varepsilon \leq 1 /(2 m)$ for all $v \in V$ and there is a vertex $u$ with density $p(u) / d(u) \leq \tau$, then

$$
\begin{equation*}
\min _{t \in\left[\tau, t_{0}\right)} \Phi\left(V_{\leq t}^{p}\right) \leq \sqrt{\frac{12\left(t_{0}+\varepsilon\right) \alpha \lg m}{t_{0}-\tau}} \tag{17}
\end{equation*}
$$

Proof Let $\phi \leq \min _{t \in\left[\tau, t_{0}\right)} \Phi\left(V_{\leq t}^{p}\right)$. We shall reuse a lot of the analysis from Section 7.3 based on some $t<t_{0}$ and the median neighboring density $t^{\prime}$ from the median expansion in the proof of Lemma 34, In Section 7.3, symmetric to the high density case, we said that the total flow into $V_{\leq t}^{p}$ is at most 1 . However, for the current proof we know that the residual density on every vertex is bounded by $\varepsilon$. Then the total mass on $V_{\leq t}^{p}$ is at most $(t+\varepsilon) \operatorname{vol}\left(V_{\leq t}^{p}\right)$. This gives us a different bound on the net flow into $V_{\leq t}^{p}$, which by (14) is at least $\phi \operatorname{vol}\left(V_{\leq t}^{p}\right)\left(t^{\prime}-t\right) /(6 \alpha)$. Thus, as an alternative to (15), we have

$$
\begin{equation*}
\phi \operatorname{vol}\left(V_{\leq t}^{p}\right)\left(t^{\prime}-t\right) /(6 \alpha) \leq(t+\varepsilon) \operatorname{vol}\left(V_{\leq t}^{p}\right) \Longleftrightarrow\left(t^{\prime}-t\right) \leq 6(t+\varepsilon) \alpha / \phi \leq 6\left(t_{0}+\varepsilon\right) \alpha / \phi \tag{18}
\end{equation*}
$$

Starting from $t=\tau$, we consider how many times we can step from $t$ to $t^{\prime}$ using median expansion before reaching or passing $t_{0}$. We know that $u \in V_{\leq \tau}^{p}$ and $d(u) \geq 1$, so we start with $\operatorname{vol}\left(V_{\leq \tau}^{p}\right) \geq 1$.

Now, every time we go from $t$ to $t^{\prime}$, we know from (16) that the volume grows by at least a factor $(1+\phi / 2)$, and by definition, $\operatorname{vol}\left(V_{<t_{0}}^{p}\right) \leq m$, so we can have at most

$$
\log _{(1+\phi / 2)} m<(2 / \phi) \lg m
$$

iterations before we reach or pass $t_{0}$. Therefore

$$
t_{0}-\tau \leq(2 / \phi)(\lg m) 6\left(t_{0}+\varepsilon\right) \alpha / \phi=12\left(t_{0}+\varepsilon\right) \alpha(\lg m) / \phi^{2} .
$$

Thus we have

$$
\phi \leq \sqrt{12\left(t_{0}+\varepsilon\right) \alpha(\lg m) /\left(t_{0}-\tau\right)}
$$

This also holds for $\phi=\min _{t \in\left[\tau, t_{0}\right)} \Phi\left(V_{\leq t}^{p}\right)$, completing the proof of (17).

### 7.5 Exploiting concentration

Our goal in this subsection is to provide an algorithm performing as in Theorem 8, restated below for convenience.

We are given a multigraph with $m$ edges, an initial mass distribution $p^{\circ}$ of total mass 1 , and a listing of the vertices with positive mass in order of non-increasing density. Let $p^{*}=\operatorname{PR}\left(\alpha, p^{\circ}\right)$. We are also given an excess parameter $\gamma<1$. We have a PageRank algorithm that staring from $p^{\circ}$ will either find a set $T$ with $\operatorname{vol}(T) \leq m$ and conductance

$$
\Phi(T)=O(\sqrt{(\alpha \log m) / \gamma})
$$

or certify that there is no set $S$ with

$$
\operatorname{excess}\left(p^{*}, S\right) \geq \gamma
$$

The maximal running time is $O(m /(\gamma \alpha))$, but if a set $T$ is returned, then the time is also bounded by $O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$.

If we are further given a volume parameter $s \leq m \gamma / 16$, the algorithm will either find the above $T$ with the additional guarantees that $\operatorname{vol}(T) \leq 8 s / \gamma$ and excess $\left(p^{*}, T\right) \geq \gamma /(16 \lg (4 s))$, or certify there is no set $S$ with $\operatorname{vol}(S) \leq s$ and $\operatorname{excess}\left(p^{*}, S\right) \geq \gamma$. The maximal running time is $O(s /(\gamma \alpha))$, but if a set $T$ is returned, then the time is also bounded by $O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$.

With a volume bound We will first address the case where we have a volume parameter $s \leq m \gamma / 16$ to bound $\operatorname{vol}(S)$. In this case, we will apply Algorithm 7 Below we analyze this algorithm, but some of the lemmas will be more general so that we can reuse them on other algorithms.

```
Algorithm 7: BoundedNibble \(\left(\alpha, p^{\circ}, \gamma, s\right)\)-assumes \(s \leq m \gamma / 16\)
    \(\varepsilon \leftarrow \gamma / 2 ;\)
    repeat
        \(\varepsilon \leftarrow \varepsilon / 2 ;\)
        \(p \leftarrow \operatorname{PageRank}^{\prime}\left(\alpha, \varepsilon, p^{\circ}\right)\);
        if \(\operatorname{vol}\left(V_{\geq 1 /(2 m)+\varepsilon}^{p}\right) \geq \gamma /(8 \varepsilon \lg (4 s))\) then
            return \(T=V_{\geq t}^{p}\) where \(t \in(1 /(2 m)+\varepsilon / 2,1 /(2 m)+\varepsilon]\) minimizes \(\Phi\left(V_{\geq t}^{p}\right)\).
    until \(\varepsilon \leq \gamma /(8 s)\);
    return "There is no set \(S\) with \(\operatorname{excess}\left(p^{*}, S\right) \geq \gamma\) and \(\operatorname{vol}(S) \leq s\)."
```

Consider an iteration of the loop in Algorithm 7 based on some $\varepsilon>\gamma /(8 s)$. We know from Lemma 32 that it takes $O(1 /(\varepsilon \alpha))$ time to run PageRank' including a sweep for low conductance cuts. Therefore the last iteration will dominate our bound for the total running time. In particular it follows that the total time is at most $O(1 /((\gamma /(8 s)) \alpha))=O(s /(\gamma \alpha))$.

Consider again an arbitrary iteration with some $\varepsilon \geq \gamma /(8 s)$. We claim that

$$
\begin{equation*}
\operatorname{vol}\left(V_{>1 /(2 m)+\varepsilon / 2}^{p}\right) \leq m . \tag{19}
\end{equation*}
$$

To see this, first note by definition that we have settled density $p(u) / d(u)>1 /(2 m)+\varepsilon / 2>\varepsilon / 2$ for all $u \in V_{>1 /(2 m)+\varepsilon / 2}^{p}$. Therefore the settled mass in $V_{>1 /(2 m)+\varepsilon / 2}^{p}$ is bigger than $\operatorname{vol}\left(V_{>1 /(2 m)+\varepsilon / 2}^{p}\right) \varepsilon / 2$, but the mass cannot be bigger than 1 and $s \leq m \gamma / 16$, so $\operatorname{vol}\left(V_{>1 /(2 m)+\varepsilon / 2}^{p}\right)<2 / \varepsilon \leq 8 s / \gamma \leq m$, as claimed in (19).

Suppose the iteration passes the condition of the if-statement and returns a set $T$. This is then the final iteration, so the total time bound is $O(1 /(\varepsilon \alpha))$. The condition of the if-statement together with (19) means that we satisfy the conditions supposed by the lemma below. In return, the lemma states that the set $T$ satisfies all the conditions of Theorem 8 in the case where we have a volume bound parameter $s$.

Lemma 36 Let $p^{*}=\operatorname{PR}\left(\alpha, p^{\circ}\right)$ and $p \leftarrow \operatorname{PageRank}^{\prime}\left(\alpha, \varepsilon, p^{\circ}\right)$. Suppose $\operatorname{vol}\left(V_{\geq 1 /(2 m)+\varepsilon}^{p}\right) \geq \gamma /(8 \varepsilon \lg (4 s))$ and $\operatorname{vol}\left(V_{>1 /(2 m)+\varepsilon / 2}^{p}\right) \leq m$. Let $T=V_{\geq t}^{p}$ where $t \in(1 /(2 m)+\varepsilon / 2,1 /(2 m)+\varepsilon]$ minimizes
$\Phi\left(V_{>t}^{p}\right)$. Then $\operatorname{vol}(T) \leq m, 1 /(\alpha \varepsilon)=O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$, $\operatorname{excess}\left(p^{*}, T\right) \geq \gamma /(16 \lg (4 s)$, and $\Phi(T) \leq \widetilde{O}(\sqrt{(\alpha \log m) / \gamma})$.

Proof First we note that

$$
\gamma /(8 \varepsilon \lg (4 s)) \leq \operatorname{vol}\left(V_{\geq 1 /(2 m)+\varepsilon}^{p}\right) \leq \operatorname{vol}(T) \leq \operatorname{vol}\left(V_{>1 /(2 m)+\varepsilon / 2}^{p}\right) \leq m .
$$

From this we immediately get that $\operatorname{vol}(T) \leq m$ and $1 /(\varepsilon \alpha)=O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$. Moreover, we get

$$
\operatorname{excess}\left(p^{*}, T\right)>\operatorname{vol}(T)(1 /(2 m)+\varepsilon / 2)-\operatorname{vol}(T) / 2 m=(\varepsilon / 2) \operatorname{vol}(T) \geq \gamma /(16 \lg (4 s))
$$

Finally we need to argue about the conductance of $T$. Set $t_{0}^{+}=1 /(2 m)+\varepsilon / 2$ and $\tau^{+}=1 /(2 m)+\varepsilon$. We have $\operatorname{vol}\left(V_{>t_{0}^{+}}^{p}\right) \leq m$ as in (4), so by Lemma33,

$$
\begin{align*}
\min _{t \in\left(t_{0}^{+}, \tau^{+}\right]} \Phi\left(V_{\geq t}^{p}\right) & \leq \sqrt{\frac{18 \alpha}{\left(\tau^{+}-t_{0}^{+}\right) \operatorname{vol}\left(V_{\geq \tau^{+}}^{p}\right)}} \leq \sqrt{\frac{36 \alpha}{\varepsilon \operatorname{vol}\left(V_{\geq 1 /(2 m)+\varepsilon}^{p}\right)}} \\
& \leq \sqrt{\frac{36 \alpha}{\gamma /(8 \lg (4 s))}} \leq O\left(\sqrt{\frac{\alpha \log m}{\gamma}}\right) . \tag{20}
\end{align*}
$$

This completes the proof of Theorem 8 with volume bound $s$ assuming the algorithm terminates satisfying the condition $\operatorname{vol}\left(V_{>1 /(2 m)+\varepsilon}^{p}\right) \geq \gamma /(8 \varepsilon \lg (4 s))$ of the if-statement for some $\varepsilon \geq \gamma /(8 s)$. We need to prove that this happens if there is a set $S$ with $\operatorname{excess}\left(p^{*}, S\right) \geq \gamma$ and $\operatorname{vol}(S) \leq s$, as stated by the following lemma:

Lemma 37 Let $p^{*}=\operatorname{PR}\left(\alpha, p^{\circ}\right)$ and suppose there is a set $S$ with excess $\left(p^{*}, S\right) \geq \gamma$ and $\operatorname{vol}(S) \leq s \leq 2 m$. Then there is an integer $i \leq\lceil\lg (2 s)\rceil$ such that if we set $\varepsilon=\gamma 2^{-i-1}$ and $p \leftarrow \operatorname{PageRank}^{\prime}\left(\alpha, \varepsilon, p^{\circ}\right)$, then $\operatorname{vol}\left(V_{>1 /(2 m)+\varepsilon}^{p}\right) \geq \gamma /(8 \varepsilon \lg (4 s))$.

Proof A vertex $u \in S$ contributes $d(u) \max \left\{0, p^{*}(u) / d(u)-1 /(2 m)\right\}$ to excess $\left(p^{*}, S\right)$, so all together, the vertices $u$ with $p^{*}(u) / d(u) \leq 1 /(2 m)+\gamma /(2 s)$ contribute less than $\gamma / 2$. Let

$$
S_{1}=\left\{u \in S \mid p^{*}(u) / d(u)>1 /(2 m)+\gamma / 2\right\}
$$

and for $i=2, \ldots,\lceil\lg (2 s)\rceil$, define

$$
S_{i}=\left\{u \in S \mid 1 /(2 m)+\gamma 2^{-i}<p^{*}(u) / d(u) \leq 1 /(2 m)+\gamma 2^{1-i}\right\}
$$

Then

$$
\sum_{i=1}^{\lceil\lg (2 s)\rceil}\left(p^{*}\left(S_{i}\right)-\operatorname{vol}\left(S_{i}\right) /(2 m)\right)>\gamma / 2 .
$$

Thus, for some $i=\{1, \ldots,\lceil\lg (2 s)\rceil\}$, we have

$$
p^{*}\left(S_{i}\right)-\operatorname{vol}\left(S_{i}\right) /(2 m)>\gamma /(2 \lg (4 s)) .
$$

This will be the value of $i$ we chose for the lemma. If $i>1$ then

$$
p^{*}\left(S_{i}\right)-\operatorname{vol}\left(S_{i}\right) /(2 m) \leq \gamma 2^{1-i} \operatorname{vol}\left(S_{i}\right) \leq \gamma 2^{1-i} \operatorname{vol}\left(V_{>1 /(2 m)+\gamma 2^{-i}}^{p^{*}}\right)
$$

So

$$
\begin{equation*}
\operatorname{vol}\left(V_{>1 /(2 m)+\gamma 2^{-i}}^{p^{*}}\right)>2^{i-2} / \lg (4 s) \tag{21}
\end{equation*}
$$

This equation is also satisfied if $i=1$, for then $S_{1} \neq \emptyset$, and hence $\operatorname{vol}\left(V_{>1 /(2 m)+\gamma 2^{-1}}^{p^{*}}\right) \geq 1$.
As stated in the lemma, we set $\varepsilon=\gamma 2^{-i-1}$ and $p \leftarrow \operatorname{PageRank}^{\prime}\left(\alpha, \varepsilon, p^{\circ}\right)$. Then our settled distribution $p$ satisfies $p^{*}(u) / d(u)-\varepsilon \leq p(u) / d(u) \leq p^{*}(u) / d(u)$ for all vertices $u$. Therefore

$$
V_{>1 /(2 m)+\varepsilon}^{p} \supseteq V_{>1 /(2 m)+2 \varepsilon}^{p^{*}}=V_{>1 /(2 m)+\gamma 2^{-i}}^{p^{*}}
$$

and by (21),

$$
\operatorname{vol}\left(V_{>1 /(2 m)+\gamma 2^{-i}}^{p^{*}}\right) \geq 2^{i-2} / \lg (4 s)=\gamma /(8 \varepsilon \lg (4 s))
$$

This completes the proof of Theorem 8 when a volume bound $s$ is given.

Without a volume bound With no volume parameter bound $s$, we will run Algorithm 8 below, claiming that it satisfies that the statement of Theorem 8 . Algorithm 8 has a lot of similarities with Algorithm 7

```
Algorithm 8: Nibble \(\left(\alpha, p^{\circ}, \gamma\right)\)
    \(\varepsilon \leftarrow \gamma / 2 ;\)
    repeat
        \(\varepsilon \leftarrow \varepsilon / 2 ;\)
        \(p \leftarrow \operatorname{PageRank}^{\prime}\left(\alpha, \varepsilon, p^{\circ}\right) ;\)
        if \(\operatorname{vol}\left(V_{>1 /(2 m)+\varepsilon / 2}^{p}\right) \leq m\) and \(\operatorname{vol}\left(V_{\geq 1 /(2 m)+\varepsilon}^{p}\right) \geq \gamma /(8 \varepsilon \lg (8 m))\) then
            return \(T=V_{\geq t}^{p}\) where \(t \in(1 /(2 \bar{m})+\varepsilon / 2,1 /(2 m)+\varepsilon]\) minimizes \(\Phi\left(V_{\geq t}^{p}\right)\).
        if \(\operatorname{vol}\left(V_{<1 /(2 m)-\varepsilon}^{p}\right) \leq m\) and \(\operatorname{vol}\left(V_{<1 /(2 m)-2 \varepsilon}^{p}\right) \geq \gamma /(8 \varepsilon \lg (8 m))\) then
            return \(T=V_{\leq t}^{p}\) where \(t \in[1 /(2 m)-2 \varepsilon, 1 /(2 m)-\varepsilon)\) minimizes \(\Phi\left(V_{\leq t}^{p}\right)\).
    until \(\varepsilon \leq \gamma /(16 m)\);
    return "There is no set \(S\) with \(\operatorname{excess}\left(p^{*}, S\right) \geq \gamma\)."
```

applied with the trivial volume parameter bound $s=2 m$, but instead of always returning a set $T$ of high density vertices, it may also return a set of low density vertices. The first condition in each if-statement ensures that the set $T$ returned has $\operatorname{vol}(T) \leq m$. The running time analysis is exactly the same as that for Algorithm 7 with $s=2 m$.

Concerning the first if-statement, we note that the conditions matches exactly the conditions supposed in Lemma 36, which implies that the set returned satisfies all the requirements of Theorem8,

We now need a corresponding lemma for the case where a set is returned from the second if-statement, noting that now we have no no volume bound expect the trival $2 m$.

Lemma 38 Let $p^{*}=\operatorname{PR}\left(\alpha, p^{\circ}\right)$ and $p \leftarrow \operatorname{PageRank}^{\prime}\left(\alpha, \varepsilon, p^{\circ}\right)$. Suppose $\operatorname{vol}\left(V_{\leq 1 /(2 m)-\varepsilon}^{p}\right) \leq m$ and $\operatorname{vol}\left(V_{<1 /(2 m)-2 \varepsilon}^{p}\right) \geq \gamma /(8 \varepsilon \lg (8 m))$. Let $T=V_{\leq t}^{p}$ where $t \in[1 /(2 m)-2 \varepsilon, 1 /(2 m)-\varepsilon)$ minimizes $\Phi\left(V_{\leq t}^{p}\right)$. Then $\operatorname{vol}(T) \leq m, 1 /(\alpha \varepsilon)=O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$, and $\Phi(T) \leq \widetilde{O}\left(\sqrt{\frac{\alpha \log m}{\gamma}}\right)$.

Proof First we note that we have

$$
\gamma /(8 \varepsilon \lg (8 s))<\operatorname{vol}\left(V_{\leq 1 /(2 m)-2 \varepsilon}^{p}\right) \leq \operatorname{vol}(T) \leq \operatorname{vol}\left(V_{<1 /(2 m)-\varepsilon}^{p}\right) \leq m
$$

From this we immediately get that $\operatorname{vol}(T) \leq m$ and $1 /(\varepsilon \alpha)=O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$.
Concerning the conductance, with $t_{0}^{-}=1 /(2 m)-\varepsilon$, we have $\operatorname{vol}\left(V_{<t_{0}^{-}}^{p}\right) \leq m$ as in (11). With $\tau^{-}=1 /(2 m)-2 \varepsilon$, it follows from Lemma 34 that

$$
\begin{aligned}
\min _{t \in\left[\tau^{-}, t_{0}^{-}\right)} \Phi\left(V_{\leq t}^{p}\right) & \leq \sqrt{\frac{18 \alpha}{\left(t_{0}^{-}-\tau^{-}\right) \operatorname{vol}\left(V_{\leq \tau^{-}}^{p}\right)}} \leq \sqrt{\frac{18 \alpha}{\varepsilon \operatorname{vol}\left(V_{\leq 1 /(2 m)-2 \varepsilon}^{p}\right)}} \\
& \leq \sqrt{\frac{18 \alpha(8 \lg (8 m s))}{\gamma}} \leq O\left(\sqrt{\frac{\alpha \log m}{\gamma}}\right) .
\end{aligned}
$$

The above two lemmas imply that if a set $T$ is returned by any of the two if-statements in Algorithm 8 , then $T$ satisfies all the requirements of Theorem 8 in the case without a volume bound. The total running time is $O(1 /(\varepsilon \alpha))=O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$.

To complete the proof of Theorem [8, we need to show that if there is a set $S$ with excess $\left(p^{*}, S\right) \geq \gamma$, then there will be some iteration where we pass the conditions of one of the two if-statements.

Assume first that

$$
\begin{equation*}
\operatorname{vol}\left(V_{\geq 1 /(2 m)}^{p^{*}}\right) \leq m \tag{22}
\end{equation*}
$$

Then we always have

$$
\begin{equation*}
\operatorname{vol}\left(V_{\geq 1 /(2 m)+\varepsilon / 2}^{p}\right) \leq \operatorname{vol}\left(V_{\geq 1 /(2 m)}^{p}\right) \leq \operatorname{vol}\left(V_{\geq 1 /(2 m)}^{p^{*}}\right) \leq m \tag{23}
\end{equation*}
$$

In particular this means that we always satisfy the first condition of the first if-statement. Moreover, by Lemma 37 with $s=2 m$, there will be some iteration satisfying the second condition if there is a set $S$ with excess $\left(p^{*}, S\right) \geq \gamma$. This completes the proof of Theorem 8 when (22) is satisfied. It remains to consider the case where (22) is false, hence where $\operatorname{vol}\left(V_{<1 /(2 m)}^{p^{*}}\right)<m$. Then

$$
\begin{equation*}
\operatorname{vol}\left(V_{<1 /(2 m)-\varepsilon}^{p}\right) \leq \operatorname{vol}\left(V_{<1 /(2 m)}^{p^{*}}\right)<m . \tag{24}
\end{equation*}
$$

The first condition of the second if-statement is thus always satisfied. We need a lemma corresponding to Lemma 37, stating that there will always be an iteration satisfying the last condition of the second ifstatement:

Lemma 39 Let $p^{*}=\operatorname{PR}\left(\alpha, p^{\circ}\right)$ and suppose there is a set $S$ with $\operatorname{excess}\left(p^{*}, S\right) \geq \gamma$. Then there is an integer $i \leq\lceil\lg (2 s)\rceil$ such that if we set $\varepsilon=\gamma 2^{-i-1}$ and $p \leftarrow \operatorname{PageRank}^{\prime}\left(\alpha, \varepsilon, p^{\circ}\right)$, then $\operatorname{vol}\left(V_{<1 /(2 m)-2 \varepsilon}^{p}\right) \geq$ $\gamma /(8 \varepsilon \lg (4 s))$.

Proof Since $p^{*}(V)=1$ and $\operatorname{vol}(V)=2 m$, we have

$$
\begin{aligned}
\operatorname{vol}\left(V_{<1 /(2 m)}^{p^{*}}\right) /(2 m)-p^{*}\left(V_{<1 /(2 m)}^{p^{*}}\right) & =p^{*}\left(V_{\geq 1 /(2 m)}^{p^{*}}\right)-\operatorname{vol}\left(V_{\geq 1 /(2 m)}^{p^{*}}\right) /(2 m) \\
& \geq \operatorname{vol}(S) /(2 m)-p^{*}(S) \geq \gamma .
\end{aligned}
$$

We can now make an analysis of $p^{*}$ for densities below $1 /(2 m)$ which is symmetric to the one we did in the proof of Lemma 39 with densities above $1 /(2 m)$ but based on $V_{\leq 1 /(2 m)}^{p}$ instead of $S$ and with $2 m$ instead of $s$. Corresponding to (21), we find an $i \leq\lceil\lg (4 m)\rceil$ such that

$$
\begin{equation*}
\operatorname{vol}\left(V_{<1 /(2 m)-\gamma 2^{-i}}^{p^{*}}\right) \geq 2^{i-2} / \lg (8 m) \tag{25}
\end{equation*}
$$

Since $p^{*}$ is non-negative, we must have $i \geq \lg (2 \gamma m)$, but we will not exploit this in the analysis. As stated in the lemma, we set $\varepsilon=\gamma 2^{-i-1}$ and $p \leftarrow \operatorname{PageRank}^{\prime}\left(\alpha, \varepsilon, p^{\circ}\right)$. Then our settled distribution $p$ satisfies $p^{*}(u) / d(u)-\varepsilon \leq p(u) / d(u) \leq p^{*}(u) / d(u)$ for all vertices $u$. Since $p^{*}$ dominates $p$, we get

$$
V_{<1 /(2 m)-2 \varepsilon}^{p}=V_{<1 /(2 m)-\gamma 2^{-i}}^{p} \supseteq V_{<1 /(2 m)-\gamma 2^{-i}}^{p^{*}}
$$

so

$$
\begin{equation*}
\operatorname{vol}\left(V_{<1 /(2 m)-2 \varepsilon}^{p}\right) \geq 2^{i-2} / \lg (8 m)=\gamma /(8 \varepsilon \lg (8 m)) \tag{26}
\end{equation*}
$$

Summing up, suppose there is a set $S$ with $\operatorname{excess}\left(p^{*}, S\right) \geq \gamma$. If $\operatorname{vol}\left(V_{\geq 1 /(2 m)}^{p^{*}}\right) \leq m$ as in (22), then (23) and Lemma 37 implies that some iteration passes the conditions of the first if-statement. Otherwise $\operatorname{vol}\left(V_{<1 /(2 m)}^{p^{*}}\right) \leq m$. Then (24) and Lemma 39 implies that some iteration passes the conditions of the second if-statement. In either case we find a set $T$.

Even if the set $S$ does not exist, the quality of $T$ is given by Lemma 36 if it comes from the first if-statement, and by Lemma 38 if it comes from the second if-statement. More precisely, the lemmas state that $\operatorname{vol}(T) \leq m$ and $\Phi(T) \leq \widetilde{O}\left(\sqrt{\frac{\alpha \log m}{\gamma}}\right)$. Moreover, for the finding iteration, they state that $1 /(\varepsilon \alpha)=O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$, which then bounds the total time spent on finding $T$.

If $S$ does not exist we may not find the set $T$. In that case, we report the non-existence of $S$. The total time spent is dominated by the last round, which takes time $O(1 /(\varepsilon \alpha))=O(1 /(\gamma /(16 m)) \alpha))=$ $O(m /(\gamma \alpha))$. This completes the proof of Theorem 8 .

### 7.6 Exploiting single low density

In this subsection we present Algorithm 9 proving that it performs as stated in Theorem 9
We are given a multigraph with $m$ edges, an initial mass distribution $p^{\circ}$ of total mass 1 , and a listing of the vertices with positive mass in order of non-increasing density. Let $p^{*}=\operatorname{PR}\left(\alpha, p^{\circ}\right)$. We are also given a parameter $\gamma<1$. We have a PageRank algorithm that staring from $p^{\circ}$ will either find a set $T$ with $\operatorname{vol}(T) \leq m$ and conductance

$$
\Phi(T)=O(\sqrt{(\alpha \log m) / \gamma})
$$

or certify that there is no vertex $u$ with

$$
p^{*}(u) / d(u) \leq(1-\gamma) /(2 m)
$$

The running time of the algorithm is $O(m /(\gamma \alpha))$, and, depending on the input, it will always end in one of the following cases:
(i) The set $T$ is found in time $O(\operatorname{vol}(T)(\log m) /(\gamma \alpha))$ and has $\operatorname{excess}\left(p^{*}, T\right) \geq$ $\gamma /(64 \lg (8 m))$.
(ii) The set $T$ is guaranteed to contain all small density vertices $u$ with $p^{*}(u) / d(u) \leq(1-$ $\gamma) /(2 m)$.
In this case, even if $T$ is small, we have no better time bound than $O(m /(\gamma \alpha))$.
(iii) A certificate that there is no vertex $u$ with $p^{*}(u) / d(u) \leq(1-\gamma) /(2 m)$.

```
Algorithm 9: SomeSmall \(\left(\alpha, p^{\circ}, \gamma\right)\)
    \(\gamma^{\prime}=\gamma / 4 ;\)
    \(\varepsilon \leftarrow \gamma^{\prime} / 2\);
    repeat
        \(\varepsilon \leftarrow \varepsilon / 2 ;\)
        \(p \leftarrow \operatorname{PageRank}^{\prime}\left(\alpha, \varepsilon, p^{\circ}\right) ;\)
        if \(\operatorname{vol}\left(V_{>1 /(2 m)+\varepsilon / 2}^{p}\right) \leq m\) and \(\operatorname{vol}\left(V_{\geq 1 /(2 m)+\varepsilon}^{p}\right) \geq \gamma^{\prime} /(8 \varepsilon \lg (8 m))\) then
            return Case (i): \(T=V_{\geq t}^{p}\) where \(t \in(1 /(2 m)+\varepsilon / 2,1 /(2 m)+\varepsilon]\) minimizes \(\Phi\left(V_{\geq t}^{p}\right)\).
    until \(\varepsilon<\gamma^{\prime} /(16 m)\);
    \(\varepsilon \leftarrow \gamma /(8 m)\);
    \(p \leftarrow \operatorname{PageRank}^{\prime}\left(\alpha, \varepsilon, p^{\circ}\right)\);
    if \(\exists u: p(u) / d(u) \geq(1-\gamma) /(2 m)\) then
            return Case (ii): \(T=V_{\leq t}^{p}\) where \(t \in[(1-\gamma) /(2 m),(1-0.75 \gamma) /(2 m))\) minimizes \(\Phi\left(V_{\leq t}^{p}\right)\).
    else
            return Case (iii): "There is no vertex \(u\) with \(p^{*}(u) / d(u) \leq(1-\gamma) /(2 m) . "\)
```

Like with the previous algorithms, we note that the running time of Algorithm 9 is $O(1 /(\varepsilon \alpha))$ where $\varepsilon$ has the smallest value encountered. It follows that the maximal running time is $O(m /(\gamma \alpha))$. Also, it follows from Lemma 36 with $\gamma^{\prime}$ instead of $\gamma$ and $s=2 m$ that if the if-statement in the repeat-loop returns a set $T$, then $T$ satisfies all the requirements of Theorem 9 (i).

Assume now that

$$
\begin{equation*}
\operatorname{vol}\left(V_{<(1-\gamma / 2) /(2 m)}^{p^{*}}\right)>m \tag{27}
\end{equation*}
$$

If so, we have negative concentration

$$
\operatorname{vol}\left(V_{<(1-\gamma / 2) /(2 m)}^{p^{*}}\right) /(2 m)-p^{*}\left(V_{<(1-\gamma / 2) /(2 m)}^{p^{*}}\right)>m \gamma /(4 m)=\gamma / 4=\gamma^{\prime}
$$

where $\gamma^{\prime}$ has the value from Algorithm 9. We can only have positive concentration, or excess, on the set $S=V_{\geq 1 /(2 m)}^{p^{*}}$, and negative concentration ourside, so

$$
\begin{aligned}
\operatorname{excess}\left(p^{*}, S\right) & =p^{*}\left(V_{\geq 1 /(2 m)}^{p^{*}}\right)-\operatorname{vol}\left(V_{\geq 1 /(2 m)}^{p^{*}}\right) /(2 m) \\
& =\operatorname{vol}\left(V_{<1 /(2 m)}^{p^{*}}\right) /(2 m)-p^{*}\left(V_{<1 /(2 m)}^{p^{*}}\right) \\
& \geq \operatorname{vol}\left(V_{<(1-\gamma / 2) /(2 m)}^{p^{*}}\right) /(2 m)-p^{*}\left(V_{<(1-\gamma / 2) /(2 m)}^{p^{*}}\right)>\gamma^{\prime}
\end{aligned}
$$

It follows from Lemma 37 with $\gamma^{\prime}$ instead of $\gamma$ that the second condition of the if-statement is satisfied by some iteration of the repeat-loop in Algorithm 9. Moreover, since the total volume is $2 m$, by (27),

$$
m>\operatorname{vol}\left(V_{\geq(1-\gamma / 2) /(2 m)}^{p^{*}}\right) \geq \operatorname{vol}\left(V_{\geq(1-\gamma / 2) /(2 m)}^{p}\right) \geq \operatorname{vol}\left(V_{\geq 1 /(2 m)+\varepsilon / 2}^{p}\right)
$$

This means that the first condition of if-statement is satisfied by all iterations of the repeat-loop. Thus, if (27) is true, then some iteration will return a set $T$, and we saw above that this $T$ satisfies Theorem 9 (i).

We now assume that the repeat-loop fails to find a set $T$, implying that (27) is false. As in Algorithm 9 we set $\varepsilon=\gamma /(8 m)$ and $p \leftarrow \operatorname{PageRank}^{\prime}\left(\alpha, \varepsilon, p^{\circ}\right)$. We also set $\tau=(1-\gamma) /(2 m)$. Since $p^{*}$ dominates $p$, if $p^{*}(u) / d(u) \leq \tau$, then $p(u) / d(u) \leq \tau$. Thus, if there is no $u$ with $p(u) / d(u) \leq \tau$, then we certify that there is no $u$ with $p^{*}(u) / d(u) \leq \tau$ as in Theorem 9 (iii). Thus we may assume that there is a $u$ with $p(u) / d(u) \leq \tau=(1-\gamma) /(2 m)$.

Since (27) is false, with $t_{0}=(1-\gamma / 2) /(2 m)-\varepsilon=(1-0.75 \gamma) /(2 m)$, we get

$$
\begin{equation*}
\operatorname{vol}\left(V_{<t_{0}}^{p}\right) \leq \operatorname{vol}\left(V_{<(1-\gamma / 2) /(2 m)}^{p^{*}}\right) \leq m . \tag{28}
\end{equation*}
$$

Since there is a $u$ with $p(u) / d(u) \leq \tau$, Algorithm 9 returns a set $T=V_{\leq t}^{p}$ where $t \in\left[\tau, t_{0}\right)$ minimizes $\Phi\left(V_{\leq t}^{p}\right)$. By Lemma35, we have

$$
\Phi(T) \leq \sqrt{\frac{12\left(t_{0}+\varepsilon\right) \alpha \lg m}{t_{0}-\tau}}=\sqrt{\frac{12((1-0.5 \gamma) /(2 m)) \alpha \lg m}{\gamma /(8 m)}}=O\left(\sqrt{\frac{\alpha \log m}{\gamma}}\right)
$$

Moreover, the returned set $T=V_{\leq t}^{p}$ has $\operatorname{vol}(T) \leq \operatorname{vol}\left(V_{<t_{0}}^{p}\right) \leq m$ and since $t \geq \tau, T$ includes every $u$ with $p^{*}(u) / d(u) \leq p(u) / d(u) \leq \tau=(1-\gamma) /(2 m)$, as required for Case (ii). This completes the proof of Theorem 9 .

## 8 Cactus

Recall that the set $\partial(U)$ of edges connecting $U$ and $T=V \backslash U$ is called a cut while $U$ and $T$ are the sides of the cut.

We call a loopless and 2-edge-connected graph $G$ a cactus if each edge belongs to exactly one cycle. This is equivalent to saying that all blocks are cycles (allowing two-element cycles). For example, a cactus may be obtained by duplicating each edge of a tree. Note that the minimum cuts of a cactus $C$ are exactly those pairs of edges which belong to the same cycles of $C$.

The following result states that the minimum cuts of an arbitrary graph have the same structure as the minimum cuts of a cactus.

Theorem 40 (Dinits, Karzanov, and Lomonosov, [5]) Let $\lambda$ be an integer and $G=(V, E)$ a loopless graph for which the cardinality of a minimum cut is $\lambda$. There is a cactus $C=(U, F)$ and a mapping $\phi$ from $V$ to $U$ so that the pre-images $\phi^{-1}\left(U_{1}\right)$ and $\phi^{-1}\left(U_{2}\right)$ are the two sides of a minimum cut of $G$ for every 2-element cut of $C$ with sides $U_{1}$ and $U_{2}$. Moreover, every minimum cut of $G$ arises this way.

Gabow's cactus algorithm [10] can construct the cactus and mapping of Theorem 40 in $\widetilde{O}(\lambda m)$ time, but here we will do the construction in near-linear time if the input graph is simple.

Theorem 41 There is a near-linear time algorithm that given a simple graph $G=(V, E)$ constructs a cactus $C=(U, F)$ and a mapping $\phi$ from $V$ to $U$ as described in Theorem 40

Proof Let $\delta$ be the minimum degree in $G$. First we apply our main technical result, Theorem 11 which contracts vertex sets in near-linear time, producing a graph $\bar{G}=(\bar{V}, \bar{E})$ with $\bar{m}=\widetilde{O}(m / \delta)$ edges such that all non-trivial min-cuts of $G$ are preserved in $\bar{G}$.

As in the proof of Corollary [2] we now run Gabow's edge-connectivity algorithm [9] on $\bar{G}$, asking it to fail if the edge-connectivity $\bar{\lambda}$ of $\bar{G}$ is above $\delta$. This takes $\widetilde{O}(\delta \bar{m})=\widetilde{O}(m)$ time, and now we compare $\bar{\lambda}$ with the min-degree $\delta$.

If $\bar{\lambda}>\delta$, then all min-cuts in $G$ are trivial and then the edge connectivity $\lambda$ of $G$ is the minimum degree $\delta$. Let $v_{1}, \ldots, v_{h}$ denote the vertices of degree $\delta$. Let $U=\left\{u_{0}, u_{1}, \ldots, u_{h}\right\}$ be the vertex set of a cactus $C=(U, F)$ in which $u_{0}$ and $u_{i}$ are connected by two parallel edges in $F$ for each $i=1, \ldots, h$. Let $\phi$ be the mapping from $V$ to $U$ defined by $\phi\left(v_{i}\right)=u_{i}$ for $i=1, \ldots, h$ and $\phi(v)=u_{0}$ for $v \in V \backslash\left\{v_{1}, \ldots, v_{h}\right\}$. Then $C$ and $\phi$ form a cactus for $G$ as described in Theorem 40 ,

Suppose instead that $\bar{\lambda} \leq \delta$. Then $\bar{\lambda}$ is also the edge connectivity $\lambda$ of $G$. We then apply the cactus algorithm of Gabow [10] to $\bar{G}$. In $\widetilde{O}(\bar{\lambda} \bar{m})=\widetilde{O}(m)$ time, it produces a cactus $C=(U, F)$ of $\bar{G}$ and a mapping $\bar{\phi}$ from $\bar{V}$ to $U$ as in Theorem 40. Next we turn $\bar{\phi}$ into a mapping $\phi: V \rightarrow U$ from the original vertex set $V$ by reversing the contractions from Theorem 1 , that is, if $v$ got contracted into the super vertex $\bar{v}$, then $\phi(v)=\bar{\phi}(\bar{v})$. Now we have a cactus representing some min-cuts of $G$, including all non-trivial min-cuts of $G$. If $\bar{\lambda}<\delta$, then there are no trivial min-cuts of $G$, and then our cactus $C$ is the final cactus for $G$.

Finally, if $\bar{\lambda}=\delta$, there may be some trivial min-cut of $G$ that are not yet represented. The min-cut around a min-degree vertex $v$ is represented if and only if there is a vertex $u \in U$ such that $\{v\}=\phi^{-1}(u)$ and $u$ has only two incident edge. If this is not the case, let $u=\phi(v)$. To include the min-cut around $v$, we introduce a new vertex $u^{\prime}$ in $U$ and set $\phi(v)=u^{\prime}$. The only neighbor of $u^{\prime}$ is $u$ and we add two parallel edges between them to $F$. This adds the desired trivial min-cut but no other cuts to the cactus representation. We repeat this process for all min-degree vertices whose min-cut is not yet represented. Now $C$ and $\phi$ is a cactus for $G$ as described in Theorem 40, Adding the trivial min-cuts took $O(m)$ time, so the whole construction time is $\widetilde{O}(m)$. This completes the proof of Theorem 41

## 9 Acknowledgments

We would like to thank Hal Gabow and Yuzhou Gu as well as anonymous referees from STOC' 15 and J. ACM for patiently reading earlier versions of this paper, pointing out issues and coming with useful suggestions for the presentation.

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[^0]:    *A preliminary version [22] was presented at the 47th ACM Symposium on Theory of Computing (STOC 2015).
    ${ }^{\dagger}$ Ken-ichi Kawarabayashi's research is partly supported by JST ERATO Kawarabayashi Large Graph Project JPMJER1201 and JSPS KAKENHI JP18H05291.
    ${ }^{\ddagger}$ Mikkel Thorup’s research is supported by his Advanced Grant DFF-0602-02499B from the Danish Council for Independent Research and by his Investigator Grant 16582, Basic Algorithms Research Copenhagen (BARC), from the VILLUM Foundation.

[^1]:    ${ }^{1}$ We know $\lambda n=O(m)$, and this implies $\lambda n^{2}=O(m n)$

