Unified method for determining canonical forms of a matrix

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Abstract

Let K be a field and A a matrix with entries in K. It is well known that if $\det(xI - A)$ splits in K, there exists a regular matrix P with entries in K, such that $P^{-1}AP$ is a diagonal matrix of Jordan blocks. Two problems arise that have been well treated in the literature. What should be done if the roots of $xI - A$ do not belong to K? How can one compute P? In this note, I answer these questions in a unified way.

1 Introduction

Let A be a matrix with entries in a field K. One of the classical problems in linear algebra is to find canonical representatives in the similarity class of A. Canonical stands for two things: 1) it is the most simple (perhaps sparse) possible and 2) it is uniquely determined by A, up to permutations of diagonal blocks. Among these representatives, the most classical are the Frobenius[1] form and the Jordan form of A (see [GF], [GE], [GE2], [GI], [GT], [J], [K], [L], [MF], [MK], [OJ], [OP], [S]).

A non-canonical representative, which is still interesting for its simplicity and efficiency in computation, is any cyclic[2] form C of A (see [GF], [K], [MF], [MK], [OJ], [OP]); that is, a block-diagonal $C = \mathrm{diag}[C(f_1), \ldots, C(f_r)]$ of companion matrices, similar to A. There are in the literature several polynomial algorithms (see [D], [MF], [OJ], [OP]) of low degree to obtain a regular matrix Q such that $Q^{-1}AQ$ is a cyclic form C of A. In order to compute the Jordan form J of A, once we have computed Q and C, it suffices to find for each companion matrix a regular matrix T_i such that

$$T_i^{-1}C(f_i)T_i = J_i \tag{1}$$

is the Jordan form of $C(f_i)$. In fact, letting $T = \mathrm{diag}[T_1, \ldots, T_r]$, and $P = QT$, one can show that $P^{-1}AP = \mathrm{diag}[J_1, \ldots, J_r] = J$ is the Jordan form of A.

For the algebraically closed case, several authors have published efficient algorithms to find T_i (see [GI], [GT], [OP]). However, what should be done for the cases that is not algebraically closed?

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[1] Rational for some authors

[2] Quasi-Frobenius, for some authors

A first step is to reduce the size of the companion matrices. It is known that if $f = p_1^{m_1} \cdots p_r^{m_r}$ is the irreducible factorization of f, there exists a regular matrix S such that

$$S^{-1}C(f)S = \operatorname{diag}[C(p_1^{m_1}), \ldots, C(p_r^{m_r})] \ . \tag{2}$$

Once we know an algorithm to find S, it will be straightforward, for any matrix A, to obtain a regular matrix P such that $P^{-1}AP = G$ is a block-diagonal matrix of the companion matrices associated with the elementary divisors of A. This matrix G is known as the irreducible canonical form of A (see [OJ]). In consequence, it would be very interesting to have a unified algorithm to solve both problems (1) and (2).

This is the aim of the first section of these notes; the regular change matrix obtained contains in its columns the coefficients of certain polynomials uniquely determined by f (see theorem (1.5) and definition (1.1)). In the second section, the same method is used to quasi-split by similarity, with no restriction over the field, a companion matrix $C(p^n)$, given $m = \deg(p)$ in

$$G = \left.\begin{pmatrix} C(p) & & & & \\ E & C(p) & & & \\ & E & \ddots & & \\ & & & C(p) & \\ & & & E & C(p) \end{pmatrix}\right] n$$

where E is the zero matrix, except for the element $E_{1m} = 1$; moreover, the matrix Q such that $Q^{-1}C(p^n)Q = G$ is easily described: its columns contain the coefficients of p^k, $k = 0, \ldots, n-1$ suitably moved down (see theorem (2.1)).

Finally in the third section, for the real case, similarity operations are stated which yield, as the canonical representation for $C((x-c)^2 + d^2)^n)$, its real-Jordan form

$$\left.\left(\begin{array}{cc|cc|c|cc}
c & -d & & & & & \\
d & c & & & & & \\ \hline
 & 1 & c & -d & & & \\
 & & d & c & & & \\ \hline
 & & & 1 & \ddots & & \\
 & & & & \ddots & & \\ \hline
 & & & & 1 & c & -d \\
 & & & & & d & c
\end{array}\right)\right] n$$

So, assuming a polynomial factorization in the field of real numbers, theorem (3.2) yields as canonical representation of any real matrix a block-diagonal matrix of Jordan-blocks and real-Jordan blocks.

2 The irreducible and Jordan forms of a companion matrix

Let K be a field. For any monic polynomial f, the companion matrix of f will be denoted $C(f)$.

Definition 2.1 *Given $f \in K[x]$ of degree n, a factor p of f of degree $\geq m$ and $t \in K$ we define a (t,p,m)-pillar of f to be the $n \times m$ matrix whose k-th column contains the coefficients, padded with 0 if necessary, of the polynomial $(x-t)^{k-1}(f/p)$.*

Examples

1. Let $f = (x^2+2)(x^2-2)$. The $(0, x^2+2, 2)$-pillar of f is

$$P_1 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the $(0, x^2-2, 2)$-pillar of f is

$$P_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Setting $P = [P_1, P_2]$,

$$P^{-1}C(f)P = \left(\begin{array}{c|c} C(x^2+2) & 0 \\ \hline 0 & C(x^2-2) \end{array}\right)$$

splitting, as it is well known, the companion matrix $C(f)$ into two smaller companion matrices (see [GF] chapter VII, section 2, theorem 1).

2. Let $f = (x^2-6x+9)(x-2)(x+2)$. The $(3, x^2-6x+9, 2)$-pillar of f is

$$P_1 = \begin{pmatrix} -4 & 12 \\ 0 & -4 \\ 1 & -3 \\ 0 & 1 \end{pmatrix};$$

the $(2, x-2, 1)$ pillar of f is

$$P_2 = \begin{pmatrix} 18 \\ -3 \\ -4 \\ 1 \end{pmatrix},$$

and the $(-2, x+2, 1)$ pillar of f is

$$P_3 = \begin{pmatrix} -18 \\ 21 \\ -8 \\ 1 \end{pmatrix}.$$

Setting $P = [P_1, P_2, P_3]$,

$$P^{-1}C(f)P = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

the well known Jordan form of the companion matrix $C(f)$.

As is well known, swapping the columns of the pillar P_1 gives

$$P^{-1}C(f)P = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

the preferred Jordan form for some authors.

3. Let $f = (x^2 + 2x + 5)^3$. The $(0, (x^2 + 2x + 5)^3, 2)$-pillar of f is

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix};$$

the $(0, (x^2 + 2x + 5)^2, 2)$ pillar of f is

$$P_2 = \begin{pmatrix} 5 & 0 \\ 2 & 5 \\ 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and the $(0, (x^2 + 2x + 5), 2)$ pillar of f is

$$P_3 = \begin{pmatrix} 25 & 0 \\ 20 & 25 \\ 14 & 20 \\ 4 & 14 \\ 1 & 4 \\ 0 & 1 \end{pmatrix}.$$

Setting $P = [P_1, P_2, P_3]$,

$$P^{-1}C(f)P = \left(\begin{array}{cc|cc|cc} 0 & -5 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & -5 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right),$$

which is called the quasi-Jordan form of the companion matrix $C(f)$.

Definition (1.1) gives the following

Procedure **PILLAR**:

Input: (t,p,m,f), where p is a factor of a monic polynomial $f \in K[x]$, $t \in K$, $m \leq deg(p)$

Output: The (t,p,m)-pillar of f

[P1]: $n := \deg(f); Q := \text{matrix}(n,m)$

[P2]: $q := f/p$

For i from 1 to n do $Q[i,1] := \text{coeff}(q, x^{i-1})$

[P3]: For k from 2 to m do

[P3.1]: $q := (x-t)q$

[P3.2]: for i from 1 to n do $Q[i,k] := \text{coeff}(q, x^{i-1})$

Remark 2.2 The above examples show that the processes to split or to obtain the Jordan form or the quasi-Jordan form of any companion matrix are essentially the same. It is the aim of the following results:

Lemma 2.3 *Let $C = C(f)$, a companion matrix. If $\deg g < \deg f$, the first column of $g(C)$ contains the coefficients of g.*

Proof: Let $g = \sum d_s x^s$ and $e_s = (0,\ldots,1,\ldots,0)^t$, then

$$g(C)e_1 = \sum d_s C^s e_1 \overset{s<n}{=} \sum d_s e_{s+1} \; .$$

Lemma 2.4 *The k-th column of the (t,p,m)-pillar Q of f is $(C-tI)^{k-1}Qe_1$.*

Proof: I argue by induction on k, the case $k=1$ being obvious. By the above lemma, the kth column is

$$(C-tI)^{k-1} f/p(C)e_1 = (C-tI)[(C-tI)^{k-2} f/p(C)e_1] \; ,$$

but $(C-tI)^{k-2} f/p(C)e_1$ is the preceding column and by induction is $(C-tI)^{k-2}Qe_1$.

Theorem 2.5 *Let $C = C(f)$ a companion matrix and $f = p_1 \cdots p_r$ a coprime factorization of f. For $j = 1,\ldots,r$ let $m_j = deg(p_j)$, $t_j \in K$ and P_j the (t_j,p_j,m_j)-pillar of f. Let $P = [P_1,\ldots,P_r]$; then P is regular and*

$$P^{-1}CP = \text{diag}[t_1 I + C(q_1),\ldots,t_r I + C(q_r)], \qquad \text{where } q_j = p_j(x+t_j)$$

Proof: Let $p_j = x^{m_j} + \sum_{s=0}^{m_j-1} a_{js}x^s$;

Step 1): $p_j(C)P_j e_1 = (0)$, $\forall j = 1,\ldots,r$:

$$\begin{aligned}(0) &= (0)e_1 = f(C)e_1 = (p_j f/p_j)(C))\, e_1 \\ &= (p_j f/p_j(C))\, e_1 = p_j(C)\,(f/p_j(C))\, e_1 \overset{(1.3)}{=} p_j(C)P_j e_1 \; .\end{aligned}$$

Step 2): If $q_j = x^{m_j} + \sum c_{js}x^s$ then $(C-t_jI)^{m_j}P_j^1 + \sum_{s=0}^{m_j-1} c_{js}(C-t_jI)^s P_j^1 = (0)$:

This is a direct consequence of the hypothesis $q_j(x-t_j)=p_j(x)$ and the above step.

Step 3): $(C-t_jI)^{m_j}P_j^1 = P_j\begin{pmatrix}-c_{j0}\\ \vdots \\ -c_{jm_j-1}\end{pmatrix}, j=1,\ldots,r$

By lemma (1.4),

$$P_j\begin{pmatrix}-c_{j0}\\ \vdots \\ -c_{jm_j-1}\end{pmatrix} = [P_j^1,(C-t_jI)P_j^1,\ldots,(C-t_jI)^{m_j-1}P_j^1]\begin{pmatrix}-c_{j0}\\ \vdots \\ -c_{jm_j-1}\end{pmatrix}$$

$$= \sum_{s=0}^{m_j-1} -c_{js}(C-t_jI)^sP_j^1 \stackrel{2)}{=} (C-t_jI)^{m_j}P_j^1$$

Step 4): $CP_j = P_j(t_jI+C(q_j)), j=1,\ldots,r$

By lemma (1.4),

$$(C-t_jI)P_j = [(C-t_jI)P_j^1,(C-t_jI)^2P_j^1,\ldots,(C-t_jI)^{m_j}P_j^1]$$

and

$$P_jC(q_j) = \left[(C-t_jI)P_j^1,(C-t_jI)^2P_j^1,\ldots,(C-t_jI)^{m_j-1}P_j^1,P_j\begin{pmatrix}-c_{j0}\\ \vdots \\ -c_{jm_j-1}\end{pmatrix}\right]$$

It suffices to apply the above step.

Step 5): $CP = P\,\mathrm{diag}[t_1I+C(q_1),\ldots,t_rI+C(q_r)]$:

$$CP=[CP_1,\ldots,CP_j,\ldots,CP_r]$$

$P\,\mathrm{diag}[t_1I+C(q_1),\ldots,t_rI+C(q_r)]$

$$=[P_1(t_1I+C(q_1)),\ldots,P_j(t_jI+C(q_j)),\ldots,P_r(t_rI+C(q_r))]$$

It suffices to apply 4).

Step 6): P is a regular matrix.

Since $(p_1,\ldots,p_r)$ is a coprime factorization of f, by means of the extended Euclidean algorithm, we may compute polynomials g_j such that

$$1=g_1(f/p_1)+\cdots+g_r(f/p_r)$$

So, $I=g_1(C)(f/p_1)(C)+\cdots+g_r(C)(f/p_r)(C)$ and

$$\mathbf{e}_1 = g_1(C)P_1\mathbf{e}_1+\cdots+g_r(C)P_r\mathbf{e}_1$$

Letting $g_j(x) = h_j(x - t_j)$,

$$g_j(C)P_j\boldsymbol{e}_1 = h_j(C - t_jI)P_j\boldsymbol{e}_1 \stackrel{2)}{=} d_0 + d_1(C - t_jI)P_j\boldsymbol{e}_1 + \cdots + d_{m_j-1}(C - t_jI)^{m_j-1}P_j\boldsymbol{e}_1 .$$

This shows that $\boldsymbol{e}_1$ is a linear combination of the columns of the whole matrix P, and there must be a vector $B^1 \in K^n$ such that $\boldsymbol{e}_1 = PB^1$. Finally,

$$\boldsymbol{e}_{i+1} = C^i\boldsymbol{e}_1 = C^iPB^1 \stackrel{5)}{=} PC^{*i}B^1 = B^i$$

and, letting $B = [B^1, \ldots, B^n]$, $I_n = PB$.

We obtain the following (see also [GF] chapter VII, section 2, theorem 1) known

Corollary 2.6 *Let $C = C(f)$ a companion matrix and $f = p_1 \cdots p_r$ a coprime factorization of f. For $k = 1, \ldots, r$ let $m_k = \deg(p_k)$ and P_k the $(0, p_k, m_k)$-pillar of f. Let $P = [P_1, \ldots, P_r]$; then P is regular and*

$$P^{-1}CP = \mathrm{diag}[C(p_1), \ldots, C(p_r)] .$$

Proof: It suffices to see that, for $t_k = 0$, $q_k = p_k$.

This result yields

Procedure **SPLIT**:

Input: A coprime factorization $p_1 \cdots p_r$ of a monic polynomial $f \in K[x]$.

Output: A regular matrix P over K such that $P^{-1}C(f)P = \mathrm{diag}[C(p_1), \ldots, C(p_r)]$

[SP1]: For k from 1 to r do $P_k :=$ **PILLAR**$(0, p_j, deg(p_k), f)$

[SP2]: $P := [P_1, \ldots, P_r]$

Definition 2.7 *For a complete factorization of f, the above matrix is called the irreducible form of $C(f)$.*

We must note that it is uniquely determined by f, up to permutations of blocks.

Notation 2.8 Given a natural number m and $t \in K$, we denote by the t-Jordan block of order m the matrix

$$J_m(t) = tI_m + C(x^m) .$$

Corollary 2.9 *Let $C = C(f)$ a companion matrix and assume that f splits in K as $f = (x - t_1)^{m_1} \cdots (x - t_r)^{m_r}$. For $k = 1, \ldots, r$ let P_k the $(t_k, (x - t_k)^{m_k}, m_k)$-pillar of f. Let $P = [P_1, \ldots, P_r]$; then P is regular and*

$$P^{-1}CP = \mathrm{diag}[J_{m_1}(t_1), \ldots, J_{m_r}(t_r)] = J,$$

the Jordan form of $C(f)$.

Proof: It suffices to note that $p_k(x+t_k) = x^{m_k}$.

This result yields

> Procedure **JORDAN**:
>
> *Input*: A monic polynomial f splitting as $(x-t_1)^{m_1}\cdots(x-t_r)^{m_r}$ in $K[x]$
>
> *Output*: A regular matrix P over K such that $P^{-1}C(f)P = \mathrm{diag}[J_{m_1}(t_1),\ldots,J_{m_r}(t_r)]$
>
> [J1]: For k from 1 to r do
> $p_k := (x-t_k)^{m_k}$
> $P_k :=$ **PILLAR**(t_k, p_k, m_k, f)
> [J2]: $P := [P_1,\ldots,P_r]$

Remark 2.10 As we have stated, if we swapped the columns of P_k in the above corollary and procedure, we obtain J^t, the preferred Jordan form for some authors.

We must note that our theorem applies to any matrix, by means of its cyclic form (see [MK], [OJ], [OP]), giving not only a unified proof for the existence of its irreducible and Jordan forms, but a unified[3] method to obtain them, with their change matrix.

3 The quasi-Jordan form of a companion matrix

Let $C = C(p^n)$ a companion matrix. The above results do not permit the splitting of C into a block-diagonal matrix of lower size companion matrices. However, the same method of suitable pillars yields the following:

Theorem 3.1 *Let p be a monic polynomial of degree m over K and C the companion matrix $C(p^n)$. For $j = 1,\ldots,n$ let Q_j the $(0, p^{n-j+1}, m)$-pillar of p^n. Let $Q = [Q_1,\ldots,Q_n]$; then Q is regular and*

$$Q^{-1}CQ = \begin{pmatrix} C(p) & & & & \\ E & C(p) & & & \\ & E & \ddots & & \\ & & & C(p) & \\ & & & E & C(p) \end{pmatrix} = G\,,$$

where E is the zero matrix $m\times m$, except for $E_{1m} = 1$.

Proof: The sketch of proof is very similar to the above theorem (1.5).

Step 1): Q is regular; in fact, recalling that, for $k = 1,\ldots,m$ the k-th column of the pillar Q_j contains the coefficients of $x^{k-1}p^{j-1}$; so, it is clear that Q is upper triangular with $Q_{ii} = 1$, for all i.

[3]Note the close analogy between the procedures **SPLIT** and **JORDAN**

Step 2): $Q_{j+1}e_1 = p^j(C)e_1$: It is the lemma (1.3).

Step 3): $Q_{j+1} = [p^j(C)e_1, Cp^j(C)e_1, \ldots, C^{m-1}p^j(C)e_1]$:

By lemma (1.4) $Q_j = [Q_je_1, CQ_je_1, \ldots, C^{m-1}Q_je_1]$. It suffices to apply the above step.

Step 4): Let $p = x^m + \sum_{s=0}^{m-1} a_sx^s$. Then

- (4.1) $CQ_ne_m = Q_n \begin{pmatrix} -a_0 \\ \vdots \\ -a_{m-1} \end{pmatrix}$

- (4.2) $CQ_je_m = Q_j \begin{pmatrix} -a_0 \\ \vdots \\ -a_{m-1} \end{pmatrix} + Q_{j+1}Ee_m, \quad$ for $j < n$

Since $p^j = pp^{j-1} = (x^m + \sum_{s=0}^{m-1} a_sx^s)p^{j-1}$,

$$p^j(C) = (C^m + \sum_{s=0}^{m-1} a_sC^s)p^{j-1}(C) = C^mp^{j-1}(C) + \left(\sum_{s=0}^{m-1} a_sC^sp^{j-1}(C)\right)$$

therefore

$$C^mp^{j-1}(C) = \left(\sum_{s=0}^{m-1} -a_sC^sp^{j-1}(C)\right) + p^j(C)$$

and for its first column:

$$\begin{aligned} C(C^{m-1}p^{j-1}(C)e_1) &= \left(\sum_{s=0}^{m-1} -a_sC^sp^{j-1}(C)e_1\right) + p^j(C)e_1 \\ &\overset{3)}{=} \left(\sum_{s=0}^{m-1} -a_sQ_je_{s+1}\right) + p^j(C)e_1 \\ &= Q_j \begin{pmatrix} -a_0 \\ \vdots \\ -a_{m-1} \end{pmatrix} + p^j(C)e_1 \end{aligned}$$

- For $j = n$, $p^n(C) = (0)$ and, by step 3), $C^{m-1}p^{n-1}(C)e_1 = Q_ne_m$; we obtain (4.1).

- For $j < n$,

  – $C(C^{m-1}p^{j-1}(C)e_1) = CQ_je_m$, by step 3).

  – By step 2), $p^j(C)e_1$ is the first column of Q_{j+1}, i.e., $Q_{j+1}Ee_m$

  which yields (4.2)

Step 5): $CQ_n = Q_nC(p)$ and $CQ_j = Q_jC(p) + Q_{j+1}E, \quad j = 1, \ldots, n-1$, directly from step 4).

Step 6): $CQ = QG$, directly from step 5).

Note that the first pillar always contains the m first vectors of the canonical basis; so the above statement yields the following

Procedure **QUASI_JORDAN_AUX:**

Input: A monic polynomial p^n.

Output: A regular matrix Q over K such that $Q^{-1}C(p^n)Q =$

$$\begin{pmatrix} C(p) & & & & \\ E & C(p) & & & \\ & E & \ddots & & \\ & & & C(p) & \\ & & & E & C(p) \end{pmatrix}, \quad \text{where} \quad E = \begin{pmatrix} 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

[QJA1]: $m := \deg(p)$
[QJA2]: To assign to Q_1 the zero matrix $mn \times m$
[QJA3]: For j from 1 to m do $Q_1[j,j] := 1$
[QJA4]: For j from 2 to n do
$Q_j := \mathbf{PILLAR}(0, p^{n-j+1}, m, p^n)$
[QJA5]: $Q := [Q_1, \ldots, Q_n]$

We must note the simplicity of the matrix Q. On the other hand, some readers will probably prefer the following version of (2.1):

Theorem 2.1 bis *Let p be a monic polynomial of degree m over K and C the companion matrix $C(p^n)$. For $j = 1, \ldots, n$ let Q_j the $(0, p^j, m)$-pillar of p. Let $Q = [Q_1, \ldots, Q_n]$; then Q is regular and*

$$Q^{-1}CQ = \begin{pmatrix} C(p) & E & & & \\ & C(p) & E & & \\ & & \ddots & & \\ & & & C(p) & E \\ & & & & C(p) \end{pmatrix} = G$$

where E is the zero matrix $m \times m$, except for $E_{1m} = 1$.

Proof: It suffices to note that we have swapped the pillars, which yields the *new* canonical form.

Corollary 3.2 *Let $C = C(f)$ a companion matrix and $f = p_1^{m_1} \cdots p_r^{m_r}$ the complete factorization of f over K. Then it is possible to compute a regular matrix P such that*

$$P^{-1}CP = \mathrm{diag}[G_1, \ldots, G_r]\ ,$$

where G_j is the correlative matrix obtained in the theorem (2.1).

Proof: By corollary (1.6), by means of the suitable pillars, we compute Q such that $Q^{-1}CQ = \mathrm{diag}[C(p_1)^{m_1}), \ldots, C(p_r^{m_r})] = D$.

Now applying (2.1) to every one of the above companion matrices, we compute R_j such that $R_j^{-1}C(p_j)^{m_j})R_j = G_j$.

Letting $R = \mathrm{diag}[R_1, \ldots, R_r]$, $R^{-1}DR = \mathrm{diag}[G_1, \ldots, G_r]$. Finally, taking $P = QR$,

$$P^{-1}CP = R^{-1}Q^{-1}CQR = R^{-1}DR = \mathrm{diag}[G_1, \ldots, G_r]\ .$$

Remark 3.3 We must note that we do not need the irreducibility of p in the above results. On the other hand, if all factors p_i are linear in the above corollary, we obtain the Jordan form of $C(f)$. This suggest the following

Definition 3.4 *Given a monic polynomial f, the matrix G obtained in* (2.1) *is called the quasi-Jordan form of $C(f)$.*

We must note that is uniquely determined by f, up to permutation of blocks. We have obtained the following

Procedure **QUASI_JORDAN**:

Input: The complete factorization $p_1^{m_1} \cdots p_r^{m_r}$ of a monic polynomial $f \in K[x]$

Output: A regular matrix P over K such that $P^{-1}C(f)P$ is the quasi-Jordan form of $C(f)$

[QJ1]: $Q :=$ **SPLIT**$(p_1^{m_1} \cdots p_r^{m_r})$
[QJ2]: For j from 1 to r do
    [QJ2.1]: $R_j :=$ **QUASI_JORDAN_AUX**$(p_j^{m_j})$
[QJ3]: $R := \mathrm{diag}[R_1, \ldots, R_r]$
[QJ4]: $P := QR$

Remark 3.5 In the real case several irreducible factors may be linear and others may be of the form $(x-c)^2 + d^2$. The theorem (2.1) yields also an interesting canonical form for a companion matrix. This is the object of the next section.

4 The real-Jordan form of a companion matrix

The quasi-Jordan form of a real matrix will have some Jordan blocks[4] and some blocks that are companion matrices of $(x-c)^2+d^2 = x^2-2cx+(c^2+d^2)$, $d \neq 0$:

$$\begin{pmatrix} 0 & -(c^2+d^2) \\ 1 & 2c \end{pmatrix}.$$

But this companion matrix is similar to the most *geometric*[5]

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix}.$$

Lemma 4.1 *Let $p=(x-c)^2+d^2, 0 \neq d \in K$ and G the $2n \times 2n$ matrix*

$$\begin{pmatrix} C(p) & & & & \\ E & C(p) & & & \\ & E & \ddots & & \\ & & & C(p) & \\ & & & E & C(p) \end{pmatrix} \quad \text{where} \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

obtained in (2.1).

Then, letting $T = \left(\prod_{j=1}^{n} P_{2j-1,2j}(-c)\right)\left(\prod_{j=1}^{n-1} Q_{2j}(1/d^j)Q_{2j+1}(1/d^j)\right) Q_{2n}(1/d^n)$,

$$T^{-1}GT = \left(\begin{array}{cc|cc|c|cc}
c & -d & & & & & \\
d & c & & & & & \\ \hline
 & 1 & c & -d & & & \\
 & & d & c & & & \\ \hline
 & & & 1 & \ddots & & \\ \hline
 & & & & \ddots & & \\
 & & & & 1 & c & -d \\
 & & & & & d & c
\end{array}\right) = (RJ)_n(c,d)$$

Proof: It is straigthforward to see that letting $R = \left(\prod_{j=1}^{n} P_{2j-1,2j}(-c)\right)$

$$R^{-1}GR = \left(\begin{array}{cc|cc|c|cc}
c & -d^2 & & & & & \\
1 & c & & & & & \\ \hline
 & 1 & c & -d^2 & & & \\
 & & 1 & c & & & \\ \hline
 & & & 1 & \ddots & & \\ \hline
 & & & & \ddots & & \\
 & & & & 1 & c & -d^2 \\
 & & & & & 1 & c
\end{array}\right) = H$$

[4] corresponding to linear irreducible factors

[5] a rotation for $c^2+d^2=1$

Now letting $S_k = \left(\prod_{j=1}^{k-1} Q_{2j}(1/d^j)Q_{2j+1}(1/d^j)\right) Q_{2k}(1/d^k)$, an easy inductive argument over $k = 1, \ldots, n-1$ yields

$$S_k^{-1}HS_k = \left(\begin{array}{cc|cc|cc|cc|cc}
c & -d & & & & & & & & \\
d & c & & & & & & & & \\
\hline
 & 1 & c & -d & & & & & & \\
 & & d & c & & & & & & \\
\hline
 & & & 1 & \ddots & & & & & \\
 & & & & & \ddots & & & & \\
\hline
 & & & & & 1/d^k & c & -d^2 & & \\
 & & & & & & 1 & c & & \\
\hline
 & & & & & & & 1 & \ddots &
\end{array}\right) = H_k$$

So, $S_{n-1}HS_{n-1} = H_{n-1}$ and

$$T^{-1}GT = Q_{2n}(d^n)Q_{2n-1}(d^{n-1})H_{n-1}Q_{2n-1}(1/d^{n-1})Q_{2n}(1/d^n) = (RJ)_n(c,d)$$

Theorem 4.2 *Let $C = C(f)$ a companion matrix and assume that f splits over K as*

$$f = \prod_{j=1}^{r}(x - t_j)^{m_j} \prod_{k=1}^{s}[(x - c_k)^2 + d_k^2]^{n_k}, d_k \neq 0$$

Then it is possible to compute a regular matrix P such that

$$P^{-1}CP = \text{diag}[J_{m_1}(t_1), \ldots, J_{m_r}(t_r), (RJ)_{n_1}(c_1, d_1), \ldots, (RJ)_{n_s}(c_s, d_s)] = (RJ)(f)$$

Proof: By corollary (1.6), by means of the correspondant pillars, we may compute Q such that

$$Q^{-1}CQ = \text{diag}[C((x - t_1)^{m_1}), \ldots, C((x - t_r)^{m_r}),$$
$$C([(x - c_1)^2 + d_1^2]^{m_1}), \ldots, C([(x - c_s)^2 + d_s^2]^{m_1})] = D$$

By applying (2.1) to any one of the above companion matrices, we compute R such that

$$R^{-1}DR = \text{diag}[J_{m_1}(t_1), \ldots, J_{m_r}(t_r), G_1, \ldots, G_s] = F$$

where

$$G_k = \begin{pmatrix} C(p_k) & & & & \\ E & C(p_k) & & & \\ & E & \ddots & & \\ & & & C(p_k) & \\ & & & E & C(p_k) \end{pmatrix} \text{ and } E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} p_k = (x - c_k)^2 + d_k^2 .$$

Now, in the above lemma we have computed T_k such that $T_k^{-1}G_kT_k = (RJ)_{n_k}(c_k, d_k)$
Letting $T = \text{diag}[I_{m_1}, \ldots, I_{m_r}, T_1, \ldots, T_s]$,

$$T^{-1}FT = \text{diag}[J_{m_1}(t_1), \ldots, J_{m_r}(t_r), (RJ)_{n_1}(c_1, d_1), \ldots, (RJ)_{n_s}(c_s, d_s)] = (RJ)(f)$$

Finally, taking $P = QRT$,

$$P^{-1}CP = T^{-1}R^{-1}Q^{-1}CQRT = T^{-1}R^{-1}DRT = T^{-1}FT = (RJ)(f)$$

Definition 4.3 *The matrix $(RJ)(f)$ obtained above is called the real Jordan form of $C(f)$.*

We must note that it is uniquely determined by f, up to the permutation of blocks. The procedure to compute P is

Procedure **REAL_JORDAN**:

Input: The complete factorization $\prod_{j=1}^{r}(x-t_j)^{m_j}\prod_{k=1}^{s}[(x-c_k)^2+d_k^2]^{n_k}$ of a monic polynomial $f \in \mathbf{R}[x]$

Output: A real matrix P such that $P^{-1}C(f)P$ is the real-Jordan form of $C(f)$

[RJ1]: $Q := \mathbf{SPLIT}(\prod_{j=1}^{r}(x-t_j)^{m_j}\prod_{k=1}^{s}[(x-c_k)^2+d_k^2]^{n_k})$
[RJ2]: For j from 1 to r do $R_j := \mathbf{QUASI\_JORDAN\_AUX}((x-t_j)^{m_j})$
[RJ3]: For k from 1 to s do $S_k := \mathbf{QUASI\_JORDAN\_AUX}([(x-c_k)^2+d_k^2]^{n_k})$
$$T_k := S_k\left(\prod_{j=1}^{n_k} P_{2j-1,2j}(-c_k)\right)\left(\prod_{j=1}^{n_k} Q_{2j}(1/d_k^j)Q_{2j+1}(1/d_k^j)\right)Q_{2n_k}(1/d_k^{n_k})$$
[RJ4]: $R := \mathrm{diag}[R_1,\ldots,R_r,T_1,\ldots,T_s]$
[RJ5]: $P := QR$

Remark 4.4 We must emphasize the symbolic character of the above procedure, which assumes a real factorization of the polynomial $f \in \mathbf{R}[x]$.

References

[B] E.T. Browne. *On the reduction of a matrix to a canonical form.* American Mathematical Monthly, pp. 437-450. Vol. 47 (1940).

[BU] A. Bujosa et al. *Jordan Normal Form via elementary transformations*, SIAM REV. pp. 947-956. Vol. 40 (1998).

[D] A. Danilevski. *On a numerical solution of Vekua's equation.* Mat. Sb. Vol. 2, pp. 169-171 (Russian) (1937).

[GF] F.R. Gantmacher. *Theory of Matrices.* Vol. I. Chelsea Publishing Company. New York (1977).

[GE] M. Giesbrecht. *Nearly optimal Algorithms for Canonical Matrix Forms.* SIAM J. of Computing, Vol. 24, pp. 948-969 (1995).

[GE2] M. Giesbrecht. *Fast Algorithms for Rational Forms of Integer Matrices.* Proc. ISSAC'94, pp 305-311.

[GI] I. Gil. *Computation of the Jordan Canonical Form of a Square Matrix (Using the Axiom Programming Language).* ISSAC'92-7/92/CA USA.

[GT] T. Gómez-Diaz. *Quelques applications de l'evaluation dynamique.* Doctoral Thesis. University of Limoges (1994).

[J] N. Jacobson. *Lectures in Abstract Algebra.* Vol. II. Linear Algebra. Springer-Verlag (1953).

[K] E. Kaltofen et al. *Parallel Algorithms for Matrix Normal Forms.* Linear Algebra and its Applications, Vol. 136, pp. 189-208 (1990).

[L] H. Lüneburg. *On the Rational Normal Form of Endomorphisms.* Mannheim: BI-Wissenschaftverlag (1981).

[MF] M. Mathieu and D. Ford. *On p-adic Computation of the Rational Form of a Matrix.* J. Symb. Comput. Vol. 10, pp. 453-464 (1990).

[MK] K. Martín *Cálculo simbólico de la forma racional de una matriz.* Doctoral Thesis. University of Cantabria (1994).

[ML] T.M.L. Mulders and A.H.M. Levelt *A package for the computation of several matrix normal forms.* Share Library. MAPLE V. (1993)

[OJ] J. M. de Olazábal. *Procedimientos simbólicos en Álgebra Lineal.* Servicio de Publicaciones de la Universidad de Cantabria. Santander, Spain (1998).

[OP] P. Ozello. *Calcul Exact des formes de Jordan et de Frobenius d'une Matrice.* Doctoral Thesis. University of Grenoble (1987).

[S] A. Stojohann. *An $O(n^3)$ algorithm for the Frobenius Normal Form.* Proc. ISSAC'98, pp. 101-104.