Analysis of a Bounding Box Heuristic for Object Intersection

YUNHONG ZHOU AND SUBHASH SURI

Washington University, St. Louis, Missouri

Abstract. Bounding boxes are commonly used in computer graphics and other fields to improve the performance of algorithms that should process only the intersecting objects. A bounding-box-based heuristic avoids unnecessary intersection processing by eliminating the pairs whose bounding boxes are disjoint. Empirical evidence suggests that the heuristic works well in many practical applications, although its worst-case performance can be bad for certain pathological inputs. What is a pathological input, however, is not well understood, and consequently there is no *guarantee* that the heuristic will always work well in a specific application.

In this paper, we analyze the performance of bounding box heuristic in terms of two natural shape parameters, *aspect ratio* and *scale factor*. These parameters can be used to realistically measure the degree to which the objects are pathologically shaped. We derive *tight worst-case bounds* on the performance for bounding box heuristic. One of the significant contributions of our paper is that we only require that objects be well shaped on *average*. Somewhat surprisingly, the bounds are significantly different from the case when *all* objects are well shaped.

Categories and Subject Descriptors: F.2.2 [**Analysis of Algorithms and Problem Complexity**]: *Nonnumerical Algorithms and Problems*; G.2.1 [**Discrete Mathematics**]: *Combinatorics*; I.3.5 [**Computer Graphics**]: Computational Geometry and Object Modeling

General Terms: Performance, Theory

Additional Key Words and Phrases: Aspect ratio, bounding box, collision detection, scale factor

1. *Introduction*

Many computer graphics algorithms use *bounding boxes* to improve their performance. The bounding box of a geometric object is a simple volume enclosing the object, forming a conservative approximation of the object. The most common form is an *axis-aligned* bounding box, whose extent in each dimension of the space is bounded by the minimum and maximum coordinates of the object in that dimension. But other forms, such as a minimum enclosing sphere or oriented boxes, are also used.

The research for this paper was supported by the National Science Foundation (NSF) grants 96-28190 and 95-01494.

Authors' address: Department of Computer Science, Washington University, St. Louis, MO 63130, e-mail: {yzhou,suri}@cs.wustl.edu.

Permission to make digital/hard copy of part or all of this work for personal or classroom use is granted without fee provided that the copies are not made or distributed for profit or commercial advantage, the copyright notice, the title of the publication, and its date appear, and notice is given that copying is by permission of the Association for Computing Machinery (ACM), Inc. To copy otherwise, to republish, to post on servers, or to redistribute to lists, requires prior specific permission and/or a fee.
© 1999 ACM 0004-5411/99/1100-0833 $05.00

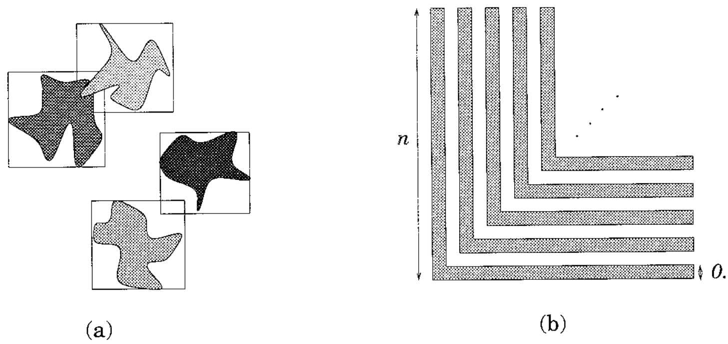

FIG. 1. (a) An example of four objects, where two pairs of bounding boxes and one pair of objects intersect. (b) An example where $\Omega(n^2)$ pairs of bounding boxes intersect, yet no object-pairs intersect.

Bounding boxes are useful in algorithms that should process only the intersecting objects. Due to their simpler shape, checking intersection among bounding boxes is almost always more efficient than intersecting complex objects. Thus, bounding boxes allow an algorithm to quickly perform a "trivial reject" test that prevents more costly processing in unnecessary cases. See Figure 1(a) for an example. For the purposes of this paper, the exact nature of a bounding box heuristic is unimportant. We are only interested in the "filtering" aspect of the heuristic, meaning the extent to which the heuristic is effective in discarding pairs of nonintersecting objects.

The bounding box heuristic is used in rendering algorithms, including the traditional visible-surface determination [Foley et al. 1996], the view-frustum culling [Greene 1994], as well as the recent image-based techniques that reconstruct new images from the reprojected pixels of reference images [McMillan 1997]. The bounding box heuristic is also common in algorithms for modeling, including techniques that define complex shapes as Boolean combinations of simpler shapes [Hoffmann 1989], or that verify the clearance of parts in an assembly [Garcia-Alonso et al. 1995]. Animation algorithms also exploit bounding boxes, especially collision detection algorithms for path planning [Latombe 1991] and the simulation of physically based motion [Cohen et al. 1995; Klosowski et al. 1999; Moore and Wilhelms 1988].

Although there is ample empirical evidence that the bounding box heuristic improves performance in practice, the goal of formally *proving* that bounding boxes maintain high performance in the worst case has remained elusive. Such a proof is important to reassure practitioners that their application will not be the one in which bounding boxes happen to perform poorly. The difficulty in such a proof is illustrated in Figure 1(b), which shows an example where $\Omega(n^2)$ bounding-box pairs intersect even though none of the object-pairs collide. Thus, in this example, the bounding box heuristic only adds unnecessary overhead, and a collision detection algorithm that uses the heuristic is slower than one that naively tests every pair of objects for collision.

Given a set of d-dimensional objects $\mathcal{S}$, let $K_o(\mathcal{S})$ denote the number of colliding object-pairs, and let $K_b(\mathcal{S})$ denote the number of colliding bounding-box pairs. Then, the performance of the bounding box heuristic can be judged by the following ratio:

$$\rho(\mathcal{S}) = \frac{K_b(\mathcal{S})}{n + K_o(\mathcal{S})}.$$

The denominator represents the *best-case* work done by an ideal object intersection algorithm, so the ratio can be seen as the relative performance measure of the heuristic. Ideally, this ratio should be small, although in the pathological case of Figure 1(b), we have $\rho(\mathcal{S}) = \Omega(n)$. The main result of our paper is to prove a tight bound on this ratio in terms of two *shape* parameters, which measure the degree to which objects are pathological. The two parameters are the *aspect ratio* and the *scale factor*.

1.1. ASPECT RATIO AND SCALE FACTOR. The *aspect ratio* measures the *elongatedness* of an object. It is often defined as the ratio between the volumes of the smallest ball enclosing the object and the largest ball contained in the object. We will find it convenient to use the volumes of L_∞-norm balls in the d-space.[1] Given a solid object P in d-space, let $b(P)$ denote the smallest L_∞ ball containing P, and let $c(P)$ denote the largest L_∞ ball contained in P. The aspect ratio of P is defined as

$$\alpha(P) = \frac{\mathrm{vol}(b(P))}{\mathrm{vol}(c(P))},$$

where $\mathrm{vol}(P)$ denotes the d-dimensional volume of P. We will call $b(P)$ the *enclosing box*, and $c(P)$ the *core* of P. Thus, the aspect ratio measures the volume of the enclosing box relative to the core. For a set $\mathcal{S} = \{P_1, P_2, \ldots, P_n\}$ of objects in d-space, its average aspect ratio is defined as:

$$\alpha_{\mathrm{avg}}(\mathcal{S}) = \frac{1}{n} \sum_{i=1}^{n} \alpha(P_i).$$

We also need a bound on the *scale factor*, which measures the *disparity* between the largest and the smallest objects. There are two ways to measure the scale—either as a ratio of the largest to smallest *bounding box*, or as the ratio of largest to smallest *core*. It turns out that the choice affects the results in a nontrivial manner, and so we will consider the two cases separately. For set $\mathcal{S}$ defined before, the *box scale factor* is the ratio between the largest and the smallest bounding boxes in $\mathcal{S}$:

$$\sigma_{\mathrm{box}}(\mathcal{S}) = \max_{i,j} \frac{\mathrm{vol}(b(P_i))}{\mathrm{vol}(b(P_j))}.$$

[1] In two dimensions, for instance, the L_∞ ball of radius r and center o is the axis-aligned square of side length $2r$, with center o. The choice of the norm affects only the dimension-dependent constant factors, and our results apply also to L_2 balls or other commonly used norms with small changes in the constant.

Similarly, the *core scale factor* is defined as the ratio between the largest and the smallest cores:

$$\sigma_{\mathrm{cor}}(\mathcal{S}) = \max_{i,j} \frac{\mathrm{vol}(c(P_i))}{\mathrm{vol}(c(P_j))}.$$

1.2. OUR CONTRIBUTION. The main results of our paper are summarized in the following three theorems.

THEOREM 1.1. *Let $\mathcal{S}$ be a set of n objects in d dimensions, with average aspect ratio α_{avg} and box scale factor σ_{box}, where d is a constant. Then, $\rho(\mathcal{S}) = \Theta(\alpha_{avg}^{2/3}\, \sigma_{box}^{1/3}\, n^{1/3})$.*

THEOREM 1.2. *Let $\mathcal{S}$ be a set of n objects in d dimensions, with average aspect ratio α_{avg} and core scale factor σ_{cor}, where d is a constant. Then, $\rho(\mathcal{S}) = \Theta(\alpha_{avg}^{2/3}\, \sigma_{cor}^{1/2}\, n^{1/3})$.*

THEOREM 1.3. *Let $\mathcal{S}$ be a set of n objects in d dimensions, where each object has aspect ratio at most α and the family has box scale factor σ_{box}, where d is a constant. Then, $\rho(\mathcal{S}) = \Theta(\alpha\, \sigma_{box}^{1/2})$.*

In each case, we show a matching lower bound. To the best of our knowledge, Theorems 1.1 and 1.2 represent the first *average-shape* results in computational geometry. Previously, most shape-dependent work had relied on assuming a worst-case bound for *each* object [Efrat and Sharir 1997; Matoušek et al. 1994]. Such an assumption is frequently hard to justify in practice, since the real-world scenes almost always contain a few unusual and pathologically shaped elements, and yet on average the objects are well shaped.

An unexpected element of our analysis is the sharp distinction between the bounds of Theorems 1.1–1.2, and the bound of Theorem 1.3. In particular, it is surprising that $\rho(\mathcal{S})$ grows with n when the average aspect ratio is bounded, but not if the worst-case aspect ratio is bounded. It was believed that the performance ratio should be *independent* of n even for the average aspect ratio case. In practical terms, however, all the theorems validate the empirical evidence on the bounding box heuristic–since $n^{1/3}$ is a relatively slow-growing function, it seems justifiable that the bound on ρ is small for bounded aspect ratio and scale factor.

The proofs of Theorems 1.1 and 1.2 require several new ideas and combinatorial bounds, because only the average aspect ratio of the input objects is bounded. Consequently, the proofs are significantly more complicated than those in Suri et al. [1998], which deals with the case of *all* objects being well shaped. In particular, we believe that some of the key technical results, such as Lemma 2.1 and Lemma 4.3, will be useful in other applications. Finally, our Theorem 1.3 settles the open problem left in Suri et al. [1998], giving a tight bound for scenes in which all objects are well shaped.

1.3. RELATED WORK. Our analysis of the bounding box heuristic is related to the idea of "realistic input models," which has become a topic of recent interest in computational geometry. The research on realistic input models has been motivated by the observation that ignoring the shapes of geometric objects often leads to overly pessimistic bounds–a few highly artificial and pathological input instances can make a problem or an algorithm appear far worse than practical experience would suggest.

One of the first nontrivial results in this direction is by Matoušek et al. [1994], who showed that the union of n *fat* triangles has complexity $O(n \log \log n)$, as opposed to $\Theta(n^2)$ for arbitrary triangles; a triangle is fat if its minimum angle exceeds δ, for a constant $\delta > 0$. Efrat and Sharir [1997] generalize this result to show that the union of n *convex* objects has complexity $O(n^{1+\epsilon})$ provided that each object is fat and each pair of objects intersects only in a constant number of points. Additional results on fat or uncluttered objects can be found in de Berg [1995], de Berg et al. [1997], and Halperin and Overmars [1994].

Our work is motivated less by the desire to show better combinatorial bounds for well behaved geometric structures, and more by the desire to validate an accepted belief that bounding boxes improve the performance of object intersection algorithms. In an earlier paper, Suri et al. [1998] considered this problem, but their result required *every object to have a bounded aspect ratio*. Briefly, the result in Suri et al. [1998] shows that if there are n object, each with aspect ratio at most α, and if the set has scale factor σ, then $\rho = O(\alpha\, \sigma^{1/2} \log^2 \sigma)$. Requiring a bound on the aspect ratio of *every* object is a strong condition, and rarely met in practice. Most practitioners, however, agree that *on average* the geometric objects are well-behaved, and thus assuming a bound on the average aspect ratio is quite justified.

2. *Preliminaries*

Our proofs all follow a common outline: We decompose the space into square boxes and assign each object to a unique box. We estimate an upper bound on $K_b(\mathcal{S})$ by computing the pairs whose objects are assigned to boxes close enough that they may intersect. We estimate a lower bound on $K_o(\mathcal{S})$ by computing the pairs whose objects are guaranteed to share a point in their cores. We will describe our proof in two dimensions, but all steps extend easily to d dimensions, with only minor changes. We begin with a technical lemma, which will be used frequently.

2.1. A TECHNICAL LEMMA. In our analysis, we will frequently divide the objects into p groups, and within each group, the objects will be further partitioned into q classes. The grouping into p groups is based on the positions of objects. The subgrouping is based on shapes–using exponentially increasing aspect ratios. Since the entire set has a known bound on the average aspect ratio, the total number of objects in class j (over all groups) will be at most $n/2^j$. The following lemma derives an upper bound for an algebraic expression that will arise in our analysis.

LEMMA 2.1. *Let $\{a_{ij}\}$ be a finite sequence of nonnegative numbers, where $1 \leq i \leq p$, and $1 \leq j \leq q$, such that $\Sigma_{i=1}^{p} a_{ij} \leq n/2^j$, for a positive integer n. Let $a_i = \Sigma_j a_{ij}$. Then the following bound holds, for any k, $1 \leq k \leq n$:*

$$\frac{\Sigma_i a_i^2}{n + \Sigma_i \Sigma_j (a_{ij}^2 / k \cdot 2j)} = O(k^{2/3}\, n^{1/3}).$$

PROOF. Let us first consider the case $k = 1$. Define $I_1 = \{j | 2^j \le n^{1/3}\}$ and $I_2 = \{j | 2^j > n^{1/3}\}$. Then, the following bound is easily obtained.

$$a_i^2 = \left(\sum_{j \in I_1} a_{ij} + \sum_{j \in I_2} a_{ij} \right)^2 \le 2 \left(\sum_{j \in I_1} a_{ij} \right)^2 + 2 \left(\sum_{j \in I_2} a_{ij} \right)^2 .$$

We now have the following sequence of inequalities:

$$\begin{aligned} \sum_i \left(\sum_{j \in I_2} a_{ij} \right)^2 &\le \left(\sum_i \sum_{j \in I_2} a_{ij} \right)^2 = \left(\sum_{j \in I_2} \sum_i a_{ij} \right)^2 \\ &\le \left(\sum_{j \in I_2} \frac{n}{2^j} \right)^2 \le \left(2 \frac{n}{n^{1/3}} \right)^2 = 4n^{4/3}. \end{aligned}$$

Next, using Cauchy's Inequality, we get

$$\left(\sum_{j \in I_1} a_{ij} \right)^2 \le \left(\sum_{j \in I_1} \frac{a_{ij}^2}{2^j} \right) \left(\sum_{j \in I_1} 2^j \right) \le \left(\sum_{j \in I_1} \frac{a_{ij}^2}{2^j} \right) (2n^{1/3}).$$

By combining the previous two inequalities, we get

$$\begin{aligned} \frac{\Sigma_i a_i^2}{n + \Sigma_{ij}(a_{ij}^2/2^j)} &\le \frac{2\Sigma_i(\Sigma_{j \in I_2} a_{ij})^2 + 2\Sigma_i(\Sigma_{j \in I_1} a_{ij})^2}{n + \Sigma_{ij}(a_{ij}^2/2^j)} \\ &\le \frac{8n^{4/3}}{n} + \frac{4n^{1/3}\Sigma_i(\Sigma_{j \in I_1}(a_{ij}^2/2^j))}{\Sigma_{ij}(a_{ij}^2/2^j)} \\ &\le 12n^{1/3}. \end{aligned}$$

To prove the lemma for the general case of $k > 1$, we simply scale the variables: Define $a'_{ij} = a_{ij}/k$, $a'_i = a_i/k$ and $n' = n/k$. Then,

$$\begin{aligned} \frac{\Sigma_i a_i^2}{n + \Sigma_{ij}(a_{ij}^2/k \cdot 2j)} &= k \frac{\Sigma_i a_i^2}{kn + \Sigma_{ij}(a_{ij}^2/2^j)} \\ &= k \frac{\Sigma_i (a_i/k)^2}{(n/k) + \Sigma_{ij}((a_{ij}/k)^2/2^j)} \\ &= k \frac{\Sigma_i a_i'^2}{n' + \Sigma_{ij}(a_{ij}'^2/2^j)} \\ &= O\left(k \left(\frac{n}{k} \right)^{1/3} \right) \\ &= O(k^{2/3} n^{1/3}). \end{aligned}$$

This completes the proof. $\square$

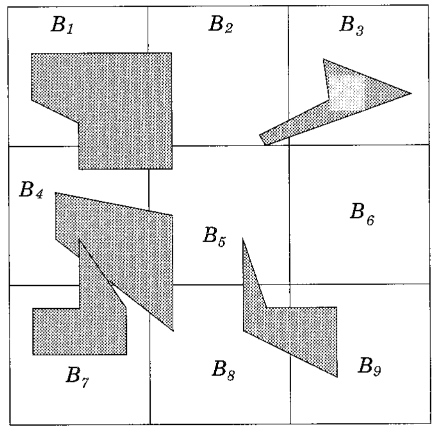

FIG. 2. Tiling of the plane by boxes of size α_{avg}. The light square in the middle of the object in B_3 depicts its square core.

2.2. TILING THE SPACE BY BOXES. We start by considering a special case, where all objects have the same-size bounding box. This case will serve to introduce the basic definitions, constructions, and ideas used in the general proof. Suppose $\mathcal{S}$ is a set of two-dimensional objects, where the bounding box of each object has volume α_{avg}. Recall that a L_∞ box of volume α_{avg} in two dimensions is a square of side length $\sqrt{\alpha_{\text{avg}}}$. We call this a *size* α_{avg} *box*.

Since each object has bounding box size α_{avg} and the average aspect ratio of $\mathcal{S}$ is α_{avg}, we get $(1/n)\ \Sigma_i\ (1/c_i) = 1$, where $c_i = \text{vol}(c(P_i))$. We partition the set $\mathcal{S}$ into $O(\log\ n)$ classes of exponentially decreasing *core size*. Specifically, an object P belongs to class $\mathcal{C}_i$ if $2^{-i} \leq \text{vol}(c(P)) < 2^{-i+1}$, for $i > 0$. The class $\mathcal{C}_0$ contains objects P that satisfy $\text{vol}(c(P)) \geq 1$. Within each class, we can assume that the core size of each object is exactly equal to the lower bound for that class–that is, each object in $\mathcal{C}_i$ has core size 2^{-i}. (This assumption may increase the average aspect ratio by at most a factor of three.)

Consider a tiling of the plane by size α_{avg} boxes that covers the portion of the plane occupied by the bounding boxes of the objects, namely, $\cup b(P_i)$. Let B_1, $B_2, \ldots, B_p$ denote the boxes in this tiling. See Figure 2. We assume that each box is semi-open, so that the boundary shared by two boxes belongs to the one on the left, or above. Thus, each point of the plane belongs to at most one box.

Now imagine superimposing a finer tiling of size $1/n$ boxes on top of the previous tiling. We call this second tiling the *core grid*, which is used only to assign each object to a box B_i. An object P is assigned to the (unique) *lexicographically smallest grid point* contained in the core of P. (Such a point exists because each core is closed and the smallest core has volume at least $1/n$. The assignment of objects to grid points is only for the proof, and we do not need to know the actual core geometry for the bounding box algorithm.)

Let X_{ij} denote the number of objects of class $\mathcal{C}_j$ that are assigned to some grid point contained in the box B_i. Let X_i denote the total number of objects assigned to B_i—that is, $X_i = \Sigma_j X_{ij}$. Clearly, since each object is assigned to a unique box, we have $\Sigma_i X_i = n$. Also, let $n_j = \Sigma_i X_{ij}$ denote the total number of objects of $\mathcal{S}$ that belong to class j. Then, we must have $n_j \leq n/2^j$. (This is because each object in class j has core size 2^{-j} and hence aspect ratio at least $\alpha_{\text{avg}}2^j$. In order

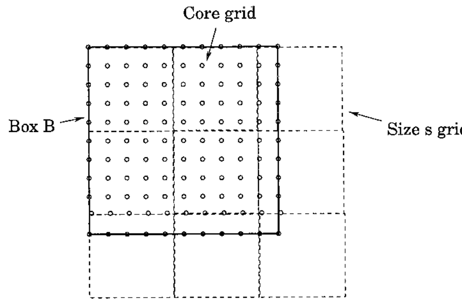

FIG. 3. Proof of Lemma 2.2.

to maintain average aspect ratio bound α_{avg}, there cannot be more than $n/2^j$ such objects.)

We will need the following fact, which follows easily from a result proved in Suri et al. [1998].

LEMMA 2.2. *Let B be a size α_{avg} box in the tiling described above. Suppose that m objects, each with core size at least s, are assigned to B. Then, there are at least $(csm^2/\alpha_{avg}) - (m/2)$ intersections among the objects assigned to B, where c is a constant dependent only on the dimension d.*

PROOF. Consider a *size s grid* superimposed on B, and focus on the portion that covers B. In particular, let $G(B)$ denote the minimal set of size s grid points whose convex hull covers B. If $g = |G(B)|$ denotes the number of grid points in $G(B)$, then it is easy to see that $g = O(\alpha_{\text{avg}}/s)$, where the constant depends on d. (More precisely, one can show the bound $g \leq (\lceil (\alpha_{\text{avg}}/s)^{1/d} \rceil)^d$ in d dimensions.) See Figure 3 for illustration.

Every object assigned to B contains at least one point of $G(B)$. We allocate each of the objects to an arbitrary grid point contained in $G(B)$. If $q \in G(B)$ is a grid point, then let $m(q)$ denote the number of objects assigned to q. Clearly, $\Sigma_{q \in G(B)} m(q) = m$. Now, the number of object pairs whose cores intersect is at least

$$\geq \sum_{q \in G(B)} \binom{m(q)}{2}$$

$$\geq \frac{1}{2} \sum_{q \in G(B)} (m(q)^2 - m(q))$$

$$\leq \frac{1}{2} \left(\sum_{q \in G(B)} m(q)^2 - m \right)$$

$$\geq \frac{1}{2} \left(\frac{(\Sigma_{q \in G(B)} m(q))^2}{g} - m \right)$$

$$\geq \frac{1}{2} \frac{m^2}{c' \alpha_{\text{avg}}/s} - \frac{m}{2}$$

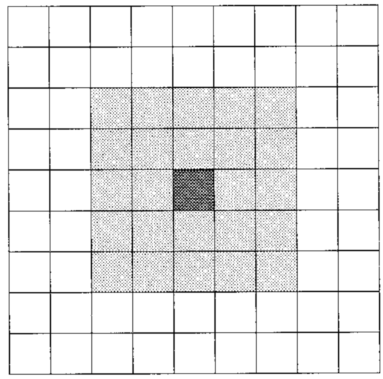

FIG. 4. Neighbors of a box in two dimensional tiling. The lightly shaded 24 boxes are the neighbors of the dark-shaded box in the center.

$$\geq \frac{csm^2}{\alpha_{\text{avg}}} - \frac{m}{2}. \qquad \square$$

3. *Box Scale Factor Theorem*

In this section, we prove Theorem 1.1: if $\mathcal{S}$ is a set of n objects, with average aspect ratio α_{avg} and box scale factor σ_{box}, then $\rho(\mathcal{S}) = \Theta(\alpha_{\text{avg}}^{2/3}\,\sigma_{\text{box}}^{1/3}\,n^{1/3})$. Our proof has three parts. We first consider the case where all objects have the same size bounding boxes. We then consider the case where all bounding boxes have sizes at one of the two extremes: small or large. Finally, the general theorem is established by combining them together.

LEMMA 3.1. *Let $\mathcal{S}$ be a set of n objects with average aspect ratio α_{avg}. Let $\mathcal{S}' \subseteq \mathcal{S}$ be a subset in which each object has the same size bounding box. Then,*

$$\frac{K_b(\mathcal{S}')}{n + K_o(\mathcal{S})} = O(\alpha_{\text{avg}}^{2/3}\,n^{1/3}).$$

PROOF. We will separately estimate bounds on $K_b(\mathcal{S}')$ and $K_o(\mathcal{S})$. Consider the tiling by boxes $B_1, B_2, \ldots, B_p$ described above, but focus on objects in the subset $\mathcal{S}'$. Let $P_i, P_j \in \mathcal{S}'$ be two objects whose bounding boxes intersect, where P_i is assigned to B_i and P_j is assigned to B_j. Since $b(P_i) \cap b(P_j) \neq \emptyset$, the L_∞ norm distance between the boxes B_i and B_j must be less than $2\sqrt{\alpha_{\text{avg}}}$. This means that B_j is among the $c = 25$ (a dimension-dependent constant) boxes that lie within $2\sqrt{\alpha_{\text{avg}}}$ wide corridor around B_i. Call these 25 nearby boxes the *neighbors* of B_i; note that this number includes B_i itself. See Figure 4.

Let X_i^m denote the maximum number of objects of $\mathcal{S}'$ assigned to any neighbor of B_i; that is, $X_i^m = \max\{X_\ell | B_\ell$ is a neighbor of $B_i\}$. Then, we have the following upper bound on $K_b(\mathcal{S}')$:

$$K_b(\mathcal{S}') \leq c \sum_{i=1}^{p} X_i X_i^m \leq c \sum_i (X_i^m)^2 \leq c^2 \sum_i X_i^2,$$

where c is a dimension-dependent constant. (Specifically, in d dimensions, $c = 5^d$.)

Next, we derive a lower bound on $K_o(\mathcal{S})$. Let $K_o^{ij}(\mathcal{S}')$ denote the number of object-pair intersections among the objects of class $\mathcal{C}_j$ assigned to box B_i—recall the classification of objects in logarithmic number of classes depending on the core size. Then, $K_o(\mathcal{S}') = \Sigma_i \Sigma_j K_o^{ij}(\mathcal{S}')$. Since objects in class $\mathcal{C}_j$ have core size at least $1/2^j$, by Lemma 2.2, we have $K_o^{ij}(\mathcal{S}') \geq (c_1 X_{ij}/\alpha_{\text{avg}} 2^j) - (X_{ij}/2)$, for some constant c_1. Thus,

$$
\begin{aligned}
n + K_o(\mathcal{S}) &\geq n + \sum_{ij} K_o^{ij}(\mathcal{S}') \\
&\geq n + \sum_{ij} \left(c_1 \frac{X_{ij}^2}{\alpha_{\text{avg}} 2^j} - \frac{X_{ij}}{2} \right) \\
&\geq \frac{n}{2} + c_1 \sum_{ij} \frac{X_{ij}^2}{\alpha_{\text{avg}} 2^j} \\
&\geq c_2 \left(n + \sum_{ij} \frac{X_{ij}^2}{\alpha_{\text{avg}} 2^j} \right).
\end{aligned}
$$

By putting together the bounds for K_b and K_o, we get

$$
\frac{K_b(\mathcal{S}')}{n + K_o(\mathcal{S})} \leq c \frac{\Sigma_i X_i^2}{n + \Sigma_{ij}(X_{ij}^2/\alpha_{\text{avg}} 2^j)} = O(\alpha_{\text{avg}}^{2/3} n^{1/3}),
$$

where the last inequality follows from Lemma 2.1. This completes the proof of the lemma. $\square$

LEMMA 3.2. *Let $\mathcal{S}$ be a set of n objects with average aspect ratio α_{avg}. Let $\mathcal{S}' \subseteq \mathcal{S}$ be a subset where each object's bounding box has size either α_{avg} or $\alpha_{avg}\, \sigma_{box}$. Then,*

$$
\frac{K_b(\mathcal{S}')}{n + K_o(\mathcal{S})} = O(\alpha_{\text{avg}}^{2/3} \sigma_{\text{box}}^{1/3} n^{1/3}).
$$

PROOF. We label the two classes of objects in $\mathcal{S}'$ *large* and *small*. Each object in the large class has bounding box size $\alpha_{\text{avg}}\, \sigma_{\text{box}}$, while each object in the small class has bounding box size α_{avg}. Let n^ℓ and n^s, respectively, denote the cardinality of large and small sets.

Consider a tiling by size $\alpha_{\text{avg}}\, \sigma_{\text{box}}$ boxes $B_1, B_2, \ldots, B_p$ as before and focus on objects in $\mathcal{S}'$. We partition the objects of the large class into subclasses with core size $\sigma_{\text{box}}\, 2^{-j}$. We let X_{ij} denote the number of large objects in subclass j that are assigned to box B_i. Similarly, we partition the objects of the small class into subclasses with core size 2^{-j}, and use Y_{ij} to denote the number of small objects in subclass j that are assigned to box B_i. Thus, the total number of large objects assigned to B_i is $X_i = \Sigma_j X_{ij}$, while the total number of small objects assigned to B_i is $Y_i = \Sigma_j Y_{ij}$. Note that $\Sigma_i X_{ij} = n_j^\ell \leq n/2^j$ and $\Sigma_i Y_{ij} = n_j^s \leq n/2^j$. (This again follows from the fact that objects in class j have aspect ratio 2^j, and so altogether there cannot be more than $n/2^j$ such objects.)

If we let $K_b^\ell(\mathcal{S}')$ and $K_b^s(\mathcal{S}')$, respectively, denote the number of bounding box intersection pairs among large and small objects, then by Lemma 3.1, we have the following bounds:

$$\frac{K_b^\ell(\mathcal{S}')}{n + K_o(\mathcal{S})} = O(\alpha_{\text{avg}}^{2/3}\, n^{1/3})$$

$$\frac{K_b^s(\mathcal{S}')}{n + K_o(\mathcal{S})} = O(\alpha_{\text{avg}}^{2/3}\, n^{1/3}).$$

Now, let $K_b^{s\ell}(\mathcal{S}')$ denote the number of bounding box intersection pairs where one objects is in large class and the other in small class. Let X_i^m denote the maximum number of large objects of $\mathcal{S}'$ assigned to any neighbor of B_i; that is, $X_i^m = \max\{X_\ell | B_\ell \text{ is a neighbor of } B_i\}$. Y_i^m is defined similarly for small objects. Then, we have

$$\begin{aligned} K_b^{s\ell} &\leq c \sum_i X_i^m Y_i \\ &\leq c \sum_i X_i^m Y_i^m \\ &\leq c \sum_i \left(\sigma_{\text{box}}^{1/3} (X_i^m)^2 + \frac{(Y_i^m)^2}{\sigma_{\text{box}}^{1/3}} \right) \\ &\leq c_1 \sum_i \left(\sigma_{\text{box}}^{1/3}\, X_i^2 + \frac{Y_i^2}{\sigma_{\text{box}}^{1/3}} \right). \end{aligned}$$

The inequality above uses the fact that $xy \leq kx^2 + y^2/k$, for any x, y and $k > 0$. Now, applying Lemma 2.2 twice, we also get

$$n + K_o^\ell(\mathcal{S}') \geq c_2 \left(n + \sum_{ij} \frac{X_{ij}^2}{\alpha_{\text{avg}} 2^j} \right)$$

$$n + K_o^s(\mathcal{S}') \geq c_2 \left(n + \sum_{ij} \frac{Y_{ij}^2}{\alpha_{\text{avg}}\, \sigma_{\text{box}}\, 2^j} \right).$$

By combining these inequalities together with Lemma 2.1, we have

$$\begin{aligned} \frac{K_b^{s\ell}(\mathcal{S}')}{n + K_o(\mathcal{S})} &\leq c_1 \frac{\sigma_{\text{box}}^{1/3} \sum_i X_i^2 + \sum_i Y_i^2 / \sigma_{\text{box}}^{1/3}}{n + K_o} \\ &\leq c_1 \left(\frac{\sigma_{\text{box}}^{1/3} \sum_i X_i^2}{n + K_o^\ell(\mathcal{S})} + \frac{\sum_i Y_i^2}{\sigma_{\text{box}}^{1/3}\, (n + K_o^s(\mathcal{S}))} \right) \end{aligned}$$

$$\leq \frac{c_1}{c_2}\left(\frac{\sigma_{\text{box}}^{1/3}\,\Sigma_i X_i^2}{n + \Sigma_{ij}\dfrac{X_{ij}^2}{\alpha_{\text{avg}}2^j}} + \frac{\Sigma_i Y_i^2}{\sigma_{\text{box}}^{1/3}\left(n + \Sigma_{ij}\dfrac{Y_{ij}^2}{\alpha_{\text{avg}}\sigma_{\text{box}}2^j}\right)}\right)$$

$$\leq c_3(\sigma_{\text{box}}^{1/3}\,\alpha_{\text{avg}}^{2/3}\,n^{1/3} + (\alpha_{\text{avg}}\,\sigma_{\text{box}})^{2/3}n^{1/3}/\sigma_{\text{box}}^{1/3})$$

$$= O(\alpha_{\text{avg}}^{2/3}\,\sigma_{\text{box}}^{1/3}\,n^{1/3}). \qquad \square$$

We are now ready to finish the proof of Theorem 1.1. Without loss of generality, assume that the largest bounding box in the scene has size $\alpha_{\text{avg}}\,\sigma_{\text{box}}$. We partition the objects of $\mathcal{S}$ into $\lceil \log \sigma_{\text{box}} \rceil$ classes, where objects in the jth class have box size in the semi-open range $(\alpha_{\text{avg}}2^{j-1}, \alpha_{\text{avg}}2^j]$. As before, without any loss of generality, we assume that the bounding boxes in class j have size exactly equal to $\alpha_{\text{avg}}2^j$. Now, let $K_b^{ij}(\mathcal{S})$ denote the number of bounding box intersections between class i and class j objects. Similarly, let $K_b^i(\mathcal{S})$ denote the number of box intersections among objects of class i. Then, $K_b(\mathcal{S}) = \Sigma_{i<j} K_b^{ij}(\mathcal{S}) + \Sigma_i K_b^i(\mathcal{S})$. Using the results of Lemmas 3.1 and 3.2, we get

$$\rho(\mathcal{S}) = \frac{K_b(\mathcal{S})}{n + K_o(\mathcal{S})}$$

$$\leq \frac{\Sigma_{i<j} K_b^{ij}(\mathcal{S}) + \Sigma_i K_b^i(\mathcal{S})}{n + K_o(\mathcal{S})}$$

$$\leq \sum_i \frac{K_b^i(\mathcal{S})}{n + K_o^i(\mathcal{S})} + \sum_{i<j} \frac{K_b^{ij}(\mathcal{S})}{n + K_o(\mathcal{S})}$$

$$\leq c_1 \sum_i \alpha_{\text{avg}}^{2/3}\,n^{1/3} + c_2 \sum_{i<j} \alpha_{\text{avg}}^{2/3}(2^{j-i})^{1/3}\,n^{1/3}$$

$$\leq c_1\alpha_{\text{avg}}^{2/3}(\log \sigma_{\text{box}})n^{1/3} + c_3\alpha_{\text{avg}}^{2/3}\,\sigma_{\text{box}}^{1/3}\,n^{1/3}$$

$$= O(\alpha_{\text{avg}}^{2/3}\,\sigma_{\text{box}}^{1/3}\,n^{1/3}). \qquad (*)$$

Ineq. (*) above uses the following fact with $m = \log \sigma_{\text{box}}$:

$$\sum_{0\leq i<j\leq m} (2^{j-i})^{1/3} = \sum_{0\leq k\leq m} (m-k+1)(2^{1/3})^k$$

$$= 2^{m/3} \sum_{0\leq t\leq m} (t+1)(2^{-1/3})^t$$

$$\leq \frac{1}{(1-2^{-1/3})^2}\, 2^{1/3\log\sigma_{\text{box}}}$$

$$= O(\sigma_{\text{box}}^{1/3}).$$

This completes the proof of Theorem 1.1. Next, we show that the bound on ρ is tight by describing a lower bound construction.

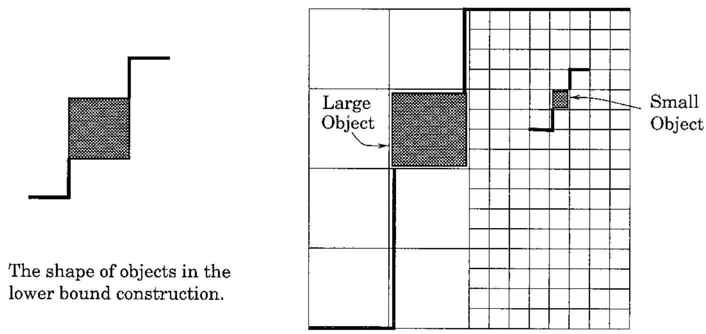

FIG. 5. The lower bound construction showing $\rho(\mathcal{S}) = \Omega(\alpha_{\text{avg}}^{2/3}\, \sigma_{\text{box}}^{1/3}\, n^{1/3})$. The left side of the figure shows the general shape of each object. The right side shows the packing of objects for the lower bound argument.

3.1. A MATCHING LOWER BOUND

LEMMA 3.3. *Given average aspect ratio α_{avg}, box scale factor σ_{box}, and a positive integer n satisfying $n \geq \alpha_{avg}\sigma_{box}$, one can construct a set $\mathcal{S}$ of n objects in d dimensions such that $\rho(\mathcal{S}) = \Omega(\alpha_{avg}^{2/3}\, \sigma_{box}^{1/3}\, n^{1/3})$.*

PROOF. Our construction works in any fixed dimension, but for ease of exposition, we describe it in two dimensions. See Figure 5 for illustration. The construction involves two classes of objects, dubbed *large* and *small*. Each of the objects is essentially a square core, with two wires extending from the opposite corners to the corresponding corners of the bounding box. A large object has bounding box size $\alpha_{\text{avg}}\, \sigma_{\text{box}}$ and core size $\sigma_{\text{box}}\, (\alpha_{\text{avg}}/n)^{1/3}$. A small object has bounding box size α_{avg} and core size $(\alpha_{\text{avg}}\, \sigma_{\text{box}}/n)^{1/3}$. Altogether we have $X = \alpha_{\text{avg}}^{1/3} n^{2/3}$ large objects and $Y = \alpha_{\text{avg}}^{1/3}\, \sigma_{\text{box}}^{1/3}\, n^{2/3}$ small objects. The remaining $n - (X + Y)$ objects have a core of size 1, a bounding box of size α_{avg}, lie far away from other objects, so they are involved in no (object or bounding box) intersections. One can easily check that this family of n objects has the average aspect ratio $\Theta(\alpha_{\text{avg}})$ and the box scale factor σ_{box}.

Now, consider a square box B of size $\alpha_{\text{avg}}\, \sigma_{\text{box}}$. We will use the *left half* of B to place some large objects in it, and the *right half* to put some small objects. Specifically, we can divide the left half of B into

$$\frac{1}{2}\, \frac{\alpha_{\text{avg}}\sigma_{\text{box}}}{\sigma_{\text{box}}(\alpha_{\text{avg}}/n)^{1/3}} = \Theta(\alpha_{\text{avg}}^{2/3}\, n^{1/3})$$

subboxes, each of size equal to the core size of the large objects. We now evenly distribute the X large objects among these subboxes, so that each subbox has about

$$\Theta\left(\frac{X}{\alpha_{\text{avg}}^{2/3}\, n^{1/3}}\right) = \Theta\left(\left(\frac{n}{\alpha_{\text{avg}}}\right)^{1/3}\right)$$

copies of the large-object cores piled on it. The "wire extensions" of the objects are arranged so they run horizontally or vertically avoiding intersections between any two objects whose cores are in disjoint subboxes. Thus, the only object intersections are between objects assigned to the same subbox, meaning that the number of object-pair intersections among the large objects is

$$K_o^\ell = \Theta\left(\left(\left(\frac{n}{\alpha_{\text{avg}}}\right)^{1/3}\right)^2 \alpha_{\text{avg}}^{2/3}\, n^{1/3}\right) = \Theta(n).$$

In a similar manner, we can divide the right half of B into about $\Theta(\alpha_{\text{avg}}^{2/3}\, \sigma_{\text{box}}^{2/3}\, n^{1/3})$ subboxes, each of size equal to the core size of the small objects. We evenly distribute the Y small objects among these subboxes, so that each subbox has about $\Theta((n/\alpha_{\text{avg}}\, \sigma_{\text{box}})^{1/3})$ copies of the small-object cores piled on it. Thus, the number of object-pair intersections among the small objects is at least

$$K_o^s = \Theta\left(\left(\left(\frac{n}{\alpha_{\text{avg}}\sigma_{\text{box}}}\right)^{1/3}\right)^2 \alpha_{\text{avg}}^{2/3}\, \sigma_{\text{box}}^{2/3}\, n^{1/3}\right) = \Theta(n).$$

There are no object intersections among large and small objects, meaning $K_o^{s\ell} = 0$. On the other hand, the number of bounding box intersections is at least $K_b \geq K_b^{s\ell} = XY = \alpha_{\text{avg}}^{2/3}\, \sigma_{\text{box}}^{1/3}\, n^{4/3}$. Thus, we get a lower bound on the ratio ρ

$$\begin{aligned}\rho(\mathcal{S}) &= \frac{K_b}{n + K_o}\\ &\geq \frac{K_b^{s\ell}}{n + K_o^s + K_o^\ell + K_o^{s\ell}}\\ &= \Omega(\alpha_{\text{avg}}^{2/3}\, \sigma_{\text{box}}^{1/3}\, n^{1/3}).\end{aligned}$$

This completes the proof of the lemma. □

4. *Core Scale Factor Theorem*

We now prove Theorem 1.2, which states that if $\mathcal{S}$ is a set of n objects in d dimensions, with average aspect bound α_{avg} and core scale factor σ_{cor}, then $\rho(\mathcal{S}) = \Theta(\alpha_{\text{avg}}^{2/3}\, \sigma_{\text{cor}}^{1/2}\, n^{1/3})$. We begin with a simple lemma, which handles the case of objects with the same bounding box size. This follows easily from a result in Suri et al. [1998] but we include a proof for completeness.

LEMMA 4.1. *Let $\mathcal{S}$ be a set of n objects where each object has core size* 1 *and aspect ratio at most α. Then*, $\rho(\mathcal{S}) = O(\alpha)$.

PROOF. Consider a tiling of the plane by size α boxes, and a unit size core grid superimposed on it. Let $B_1, B_2, \ldots, B_p$ denote the size α boxes in the tiling, and let each object $P \in \mathcal{S}$ be assigned to the lexicographically smallest grid point contained in the core of P. (See Figure 2.) Let $m(q)$ denote the number of objects assigned to a core grid point q, and let X_i denote the total number of objects assigned to the grid points contained in box B_i. That is, $X_i = \Sigma_{q \in B_i} m(q)$. Since the boxes in the tiling are disjoint, we have $\Sigma_i X_i = n$. As before, let X_i^m denote the neighbor[2] of B_i with the maximum number of objects

[2] Recall that a neighbor of B_i is one of the 25 boxes within L_∞ distance $2\sqrt{\alpha}$ of B_i.

assigned to it. Then,

$$K_b(\mathcal{S}) \leq \sum_i c_1 X_i X_i^m \leq \sum_i c_1 (X_i^m)^2 \leq \sum_i c_2 X_i^2,$$

where c_1, c_2 are constants dependent only on the dimension d. Next, by Lemma 2.2, there exists constants c_3, c_4 such that

$$\begin{aligned} n + K_o(\mathcal{S}) &\geq \sum_i (X_i + K_o^i) \\ &\geq \sum_i \left(X_i + \frac{c_3 X_i^2}{\alpha} - \frac{X_i}{2} \right) \\ &= \sum_i \left(\frac{X_i}{2} + \frac{c_3 X_i^2}{\alpha} \right) \\ &\geq c_4 \left(n + \sum_i \frac{X_i^2}{\alpha} \right). \end{aligned}$$

Finally,

$$\rho(\mathcal{S}) = \frac{K_b(\mathcal{S})}{n + K_o(\mathcal{S})} \leq c_5 \frac{\Sigma_i X_i^2}{n + \Sigma_i X_i^2/\alpha} \leq c_5 \alpha = O(\alpha).$$

This completes the proof. □

The following is a simple technical lemma, which will be useful in our proof.

LEMMA 4.2. *Let $\mathcal{S} = \{P_1, P_2, \ldots, P_n\}$ be a set of n objects in the plane, and let $b_j = vol(b(P_j))$. Let $B_1, B_2, \ldots, B_k$ be a partitioning of the plane with size α boxes that covers the portion of the plane occupied by the bounding boxes of P_j, $j = 1, 2, \ldots, n$. Let X_i denote the number of objects whose bounding boxes intersect B_i. Then, $\Sigma_i X_i \leq (c/\alpha) \Sigma_j b_j$, where c is a constant.*

PROOF. Let Y_j denote the number of partitioning boxes that intersect the bounding box of P_j. Clearly, $Y_j \leq cb_j/\alpha$, where c is a constant. Then,

$$\sum_i X_i = \sum_j Y_j \leq \sum_j \frac{cb_j}{\alpha} = \frac{c}{\alpha} \sum_j b_j. \qquad \square$$

We now begin the proof of the main theorem.

LEMMA 4.3. *Let $\mathcal{S}$ be a set of n objects with average aspect ratio α_{avg}. Let $\mathcal{S}' \subseteq \mathcal{S}$ be a subset in which each object has a core of size 1. Then,*

$$\frac{K_b(\mathcal{S}')}{n + K_o(\mathcal{S})} = O(\alpha_{\text{avg}}^{2/3} n^{1/3}).$$

PROOF. We partition the objects of $\mathcal{S}'$ into classes $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_k$, where objects in class $\mathcal{C}_i$ have bounding box size in the range $(\alpha_{\text{avg}} 2^{i-1}, \alpha_{\text{avg}} 2^i]$, for

$1 \leq i \leq \log \alpha_{\text{avg}}$. Every object in $\mathscr{C}_0$ has bounding box size no more than α_{avg}. By scaling, we assume that bounding boxes of objects in class $\mathscr{C}_i$ have size exactly equal to $\alpha_{\text{avg}}\, 2^i$—in the worst-case, this increases the aspect ratio by a factor of 2. Since each object has core size at least one, and the average aspect ratio is α_{avg}, we get

$$\sum_{P_i \in \mathscr{S}'} b_i \leq n\ \alpha_{\text{avg}}, \tag{1}$$

where $b_i = \text{vol}(b(P_i))$ is the volume of P_i's bounding box. Now, since each object in class $\mathscr{C}_i$ has aspect ratio $\alpha_{\text{avg}}\, 2^i$, it follows that the number of objects in class i is $n_i \leq n/2^i$.

Let $i_0 = \log(n/\alpha_{\text{avg}})^{1/3}$, and define $I_1 = \{i | i \leq i_0\}$, $I_2 = \{i | i > i_0\}$. Call the objects in classes $\mathscr{C}_i$, where $i \in I_1$, *small objects*, and those in classes $\mathscr{C}_i$, where $i \in I_2$, *large objects*. Define $n^s = \Sigma_{i \in I_1} n_i$ and $n^\ell = \Sigma_{i \in I_2} n_i$. Finally, let K_b^ℓ denote the number of bounding box intersections among the large objects. Define K_b^s, $K_b^{s\ell}$, similarly. Clearly, $K_b(\mathscr{S}') = K_b^s + K_b^\ell + K_b^{s\ell}$. In the following, we estimate an upper bound for each of them.

We begin with K_b^s. The largest bounding box among the small objects has size $\alpha_{\text{avg}}\, 2^{i_0} = \alpha_{\text{avg}}^{2/3}\, n^{1/3}$. We can enlarge the bounding boxes of all small objects to size $\alpha_{\text{avg}}^{2/3}\, n^{1/3}$. This results in a family of n^s objects in which each object has aspect ratio at most $\alpha_{\text{avg}}^{2/3}\, n^{1/3}$. By Lemma 4.1, therefore, we have

$$\frac{K_b^s}{n + K_o(\mathscr{S})} \leq \frac{K_b^s}{n^s + K_o^s} = O(\alpha_{\text{avg}}^{2/3}\, n^{1/3}).$$

Next, because $n_i \leq n/2^i$ for every i, thus

$$n^\ell \leq \sum_{i > i_0} n_i \leq \sum_{i > i_0} \frac{n}{2^i} \leq \frac{n}{2^{i_0}} \leq \alpha_{\text{avg}}^{1/3}\, n^{2/3}.$$

Thus,

$$\begin{aligned} \frac{K_b^\ell}{n + K_o(\mathscr{S})} &\leq \frac{K_b^\ell}{n} \leq \frac{(n^\ell)^2}{n} \\ &= O((\alpha_{\text{avg}}^{1/3}\, n^{2/3})^2/n) \\ &= O(\alpha_{\text{avg}}^{2/3}\, n^{1/3}). \end{aligned}$$

Finally, we estimate an upper bound on $K_b^{s\ell}/(n + K_o(\mathscr{S}))$. Consider tiling the plane with boxes of size $\alpha_{\text{avg}}\, 2^{i_0} = \alpha_{\text{avg}}^{2/3}\, n^{1/3}$. Let $B_1, B_2, \ldots, B_k$ denote the boxes in this tiling that cover the portion of the plane occupied by the bounding boxes of $\mathscr{S}'$. As before, use the unit size core grid to assign objects of $\mathscr{S}'$ to these boxes. Let X_i denote the number of small objects assigned to B_i, for $i = 1, 2, \ldots, k$. Let Y_i denote the number of large objects that intersect the box B_i. By Lemma 4.2, we know that

$$\sum_i Y_i \leq c_1 \frac{\Sigma_{P_t \in large}\, b_t}{\alpha_{\text{avg}}\, 2^{i_0}} \leq c_1 \frac{\alpha_{\text{avg}}\, n}{\alpha_{\text{avg}}\, 2^{i_0}}$$

$$= c_1 \frac{n}{2^{i_0}} = c_1 \alpha_{\text{avg}}^{1/3} n^{2/3}.$$

(In the first inequality, b_t denotes the bounding box volume of a large object.) Therefore,

$$\begin{aligned} K_b^{s\ell} &\leq c_2 \sum_i X_i Y_i^m \\ &\leq c_2 (\max_i X_i) \left(\sum_j Y_j^m \right) \\ &\leq c_3 (\max_i X_i) \left(\sum_j Y_j \right) \\ &\leq c_4 \; \alpha_{\text{avg}}^{1/3} \, n^{2/3} \left(\max_i X_i \right). \end{aligned}$$

By Lemma 2.2, we also have

$$n + K_o^s \geq c_5 \left(n + \frac{\Sigma_i X_i^2}{\alpha_{\text{avg}} 2^{i_0}} \right) \geq c_5 \left(n + \frac{(\max_i X_i)^2}{\alpha_{\text{avg}}^{2/3} n^{1/3}} \right).$$

Thus,

$$\begin{aligned} \frac{K_b^{s\ell}}{n + K_o(\mathcal{S})} &\leq \frac{c_4 \alpha_{\text{avg}}^{1/3} n^{2/3} (\max_i X_i)}{c_5 (n + (\max_i X_i)^2 / (\alpha_{\text{avg}}^{2/3} \, n^{1/3}))} \\ &\leq c_6 \frac{\alpha_{\text{avg}}^{1/3} n^{2/3} (\max_i X_i)}{n + (\max_i X_i)^2 / (\alpha_{\text{avg}}^{2/3} \, n^{1/3})} \\ &\leq c_6 \frac{\alpha_{\text{avg}}^{1/3} n^{2/3} (\max_i X_i)}{2 \sqrt{n (\max_i X_i)^2 / (\alpha_{\text{avg}}^{2/3} \, n^{1/3})}} \\ &= O(\alpha_{\text{avg}}^{2/3} \, n^{1/3}). \end{aligned}$$

The proof is completed by combining the estimates for K_b^s, K_b^ℓ, $K_b^{s\ell}$:

$$\rho(\mathcal{S}) = \frac{K_b^s + K_b^\ell + K_b^{s\ell}}{n + K_o(\mathcal{S})} = O(\alpha_{\text{avg}}^{2/3} \, n^{1/3}). \qquad \square$$

Next, we generalize to the case where core sizes fall into two classes.

LEMMA 4.4. *Let $\mathcal{S}$ be a set of n objects whose average aspect ratio is α_{avg}, and let $\mathcal{S}' \subseteq \mathcal{S}$ be a subset in which each object's core has size either σ_{cor} or 1. Then,*

$$\frac{K_b(\mathcal{S}')}{n + K_o(\mathcal{S})} = O(\alpha_{avg}^{2/3} \, \sigma_{cor}^{1/2} \, n^{1/3}).$$

PROOF. Fix $\alpha_0 = \alpha_{\text{avg}}^{2/3}\,\sigma_{\text{cor}}\,n^{1/3}$. We tile the plane with boxes of size α_0. We divide the objects into two classes, depending on their core size. In particular, we define the following subsets:

$$\mathcal{S}_1^\ell = \{P \in \mathcal{S}' | c(P) = \sigma_{\text{cor}},\, b(P) \le \alpha_0\}$$

$$\mathcal{S}_2^\ell = \{P \in \mathcal{S}' | c(P) = \sigma_{\text{cor}},\, b(P) > \alpha_0\}$$

$$\mathcal{S}_1^s = \{P \in \mathcal{S}' | c(P) = 1,\, b(P) \le \alpha_0\}$$

$$\mathcal{S}_2^s = \{P \in \mathcal{S}' | c(P) = 1,\, b(P) > \alpha_0\}.$$

We will estimate the number of bounding box pair intersections among these sets. Let n_1^ℓ, n_2^ℓ, n_1^s, n_2^s, respectively, denote the number of objects in these sets. The bound on the average aspect ratio implies that $n_2^\ell \le \alpha_{\text{avg}}\,\sigma_{\text{cor}}\,n/\alpha_0$, and $n_2^s \le \alpha_{\text{avg}}\,n/\alpha_0$. Then, we can bound the total number of bounding box intersections among object pairs of which one belongs to $\mathcal{S}_2^\ell$ and the other to $\mathcal{S}_2^s$ as follows:

$$\begin{aligned}\frac{K_b(\mathcal{S}_2^\ell, \mathcal{S}_2^s)}{n + K_o(\mathcal{S})} &\le \frac{n_2^\ell\, n_2^s}{n} = \frac{(\alpha_{\text{avg}})^2\,\sigma_{\text{cor}}\,n}{\alpha_0^2}\\ &= \alpha_{\text{avg}}^{2/3}\,n^{1/3}/\sigma_{\text{cor}} = O(\alpha_{\text{avg}}^{2/3}\,\sigma_{\text{cor}}^{1/2}\,n^{1/3}).\end{aligned} \tag{2}$$

As before, we use the core grid to assign objects to boxes B_i. Let X_i^ℓ and X_i^s, respectively, denote the number of objects *assigned* to B_i from subsets $\mathcal{S}_1^\ell$ and $\mathcal{S}_1^s$. Similarly, let Y_i^ℓ and Y_i^s, respectively, denote the number of objects in subsets $\mathcal{S}_2^\ell$ and $\mathcal{S}_2^s$ that *intersect* B_i. $X_i^{\ell m}$, X_i^{sm}, $Y_i^{\ell m}$, Y_i^{sm} are defined similarly as before corresponding to X_i^ℓ, X_i^s, Y_i^ℓ, Y_i^s. From Lemma 4.2, we know that

$$\sum_i Y_i^s \le \frac{c}{\alpha_0} \sum_{P_i \in \mathcal{S}_2^s} b_i \le \frac{c}{\alpha_0}\,\alpha_{\text{avg}} n$$

$$\sum_i Y_i^\ell \le \frac{c}{\alpha_0} \sum_{P_i \in \mathcal{S}_2^\ell} b_i \le \frac{c}{\alpha_0}\,\alpha_{\text{avg}}\sigma_{\text{cor}} n.$$

Therefore

$$K_b(\mathcal{S}_1^\ell, \mathcal{S}_2^s) \le c_1\left(\max_i X_i^\ell\right)\left(\sum_j Y_j^s\right) \le c_2\left(\max_i X_i^\ell\right)\frac{(\alpha_{\text{avg}} n)}{\alpha_0}$$

$$K_b(\mathcal{S}_1^s, \mathcal{S}_2^\ell) \le c_1\left(\max_i X_i^s\right)\left(\sum_j Y_j^\ell\right) \le c_2\left(\max_i X_i^s\right)\frac{(\alpha_{\text{avg}}\sigma_{\text{cor}} n)}{\alpha_0}$$

for some constants c_1, c_2. We also have the following three bounds on K_o from Lemma 2.2:

$$n + K_o(\mathcal{S}) \ge c_1\left(n + \sum_i \frac{(X_i^\ell)^2}{\alpha_0/\sigma_{\text{cor}}}\right)$$

$$n + K_o(\mathcal{S}) :\geq c_1\left(n + \sum_i \frac{(X_i^s)^2}{\alpha_0}\right)$$

$$n + K_o(\mathcal{S}) :\geq c_1\left(\sum_i \frac{(X_i^\ell)^2}{\alpha_0/\sigma_{\text{cor}}} + \sum_i \frac{(X_i^s)^2}{\alpha_0}\right)$$

Thus,

$$\begin{aligned}
\frac{K_b(\mathcal{S}_1^\ell, \mathcal{S}_2^s)}{n + K_o(\mathcal{S})} &\leq c_4 \frac{(\max_i X_i^\ell)(\alpha_{\text{avg}} n)/\alpha_0}{n + (\max_i X_i^\ell)^2/(\alpha_0/\sigma_{\text{cor}})} \\
&\leq c_4 \frac{(\max_i X_i^\ell)\alpha_{\text{avg}} n/\alpha_0}{\sqrt{n(\max_i X_i^\ell)^2 \sigma_{\text{cor}}/\alpha_0}} \\
&\leq c_4 \alpha_{\text{avg}} \sqrt{\frac{n}{\sigma_{\text{cor}}\alpha_0}} \\
&= c_4 \frac{\alpha_{\text{avg}}^{2/3} n^{1/3}}{\sigma_{\text{cor}}} \\
&= O(\alpha_{\text{avg}}^{2/3} \sigma_{\text{cor}}^{1/2\, n^{1/3}}).
\end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned}
\frac{K_b(\mathcal{S}_1^s, \mathcal{S}_2^\ell)}{n + K_o(\mathcal{S})} &\leq c_4 \frac{(\max_i X_i^s)(\alpha_{\text{avg}}\sigma_{cor} n)/\alpha_0}{n + (\max_i X_i^s)^2/\alpha_0} \\
&\leq c_4 \frac{(\max_i X_i^s)\alpha_{\text{avg}}\sigma_{cor} n/\alpha_0}{2\sqrt{n(\max_i X_i^s)^2/\alpha_0}} \\
&\leq c_4 \alpha_{\text{avg}} \sigma_{\text{cor}} \sqrt{\frac{n}{\alpha_0}} \\
&= c_4 \alpha_{\text{avg}}^{2/3} \sigma_{\text{cor}}^{1/2}\, n^{1/3} \\
&= O(\alpha_{\text{avg}}^{2/3}\, \sigma_{\text{cor}}^{1/2}\, n^{1/3}).
\end{aligned} \tag{4}$$

Next, it is easily seen that

$$\begin{aligned}
K_b(\mathcal{S}_1^s, \mathcal{S}_1^\ell) &\leq c_5 \sum_i X_i^{sm} X_i^{\ell m} \\
&\leq c_5 \frac{\alpha_0}{\sqrt{\sigma_{\text{cor}}}} \sum_i \left(\frac{(X_i^{\ell m})2}{\alpha_0/\sigma_{\text{cor}}} + \frac{(X_i^{sm})^2}{\alpha_0}\right) \\
&\leq c_6 \frac{\alpha_0}{\sqrt{\sigma_{\text{cor}}}} \sum_i \left(\frac{(X_i^\ell)^2}{\alpha_0/\sigma_{\text{cor}}} + \frac{(X_i^s)^2}{\alpha_0}\right).
\end{aligned}$$

Therefore,

$$\frac{K_b(\mathcal{S}_1^s, \mathcal{S}_1^\ell)}{n + K_o(\mathcal{S})} \leq c_7 \frac{\alpha_0/\sqrt{\sigma_{\mathrm{cor}}}\Sigma_i\left(\dfrac{(X_i^\ell)^2}{\alpha_0/\sigma_{\mathrm{cor}}} + \dfrac{(X_i^s)^2}{\alpha_0}\right)}{\Sigma_i \dfrac{(X_i^\ell)^2}{\alpha_0/\sigma_{\mathrm{cor}}} + \Sigma_i \dfrac{(X_i^s)^2}{\alpha_0}}$$

$$= c_7 \frac{\alpha_0}{\sqrt{\sigma_{\mathrm{cor}}}}$$

$$= c_7 \alpha_{\mathrm{avg}}^{2/3} \sigma_{\mathrm{cor}}^{1/2} n^{1/3}$$

$$= O(\alpha_{\mathrm{avg}}^{2/3} \sigma_{\mathrm{cor}}^{1/2} n^{1/3}). \tag{5}$$

The proof of the lemma is completed by combining the four inequalities (2), (3), (4), (5). □

The preceding lemmas are sufficient to establish Theorem 1.2. We start by partitioning the objects of $\mathcal{S}$ into $\log \sigma_{\mathrm{cor}}$ classes, where the objects in jth class have core size 2^j. (Within each class we shrink the core down to the lower bound.) Let K_b^{ij} denote the number of box intersections between class i and class j objects, and let K_b^i denote the number of box intersections among objects of class i. Then,

$$K_b(\mathcal{S}) = \sum_{i<j} K_b^{ij} + \sum_i K_b^i.$$

Lemma 4.4 gives a bound on the first term, while Lemma 4.3 gives a bound on the second term. (In applying Lemma 4.3 to a collection of objects with the same core size, we scale the scene so that core size become 1.)

$$\rho(\mathcal{S}) \leq \frac{\Sigma_{i<j} K_b^{ij} + \Sigma_i K_b^i}{n + K_o(\mathcal{S})}$$

$$\leq \sum_i \frac{K_b^i}{n + K_o^i} + \sum_{i<j} \frac{K_b^{ij}}{n + K_o}$$

$$\leq c_1 \sum_i \alpha_{\mathrm{avg}}^{2/3} n^{1/3} + c_2 \sum_{i<j} \alpha_{\mathrm{avg}}^{2/3} (2^{j-i})^{1/2} n^{1/3}$$

$$\leq c_1 (\log \sigma_{\mathrm{cor}}) \alpha_{\mathrm{avg}}^{2/3} n^{1/3} + c_2 \alpha_{\mathrm{avg}}^{2/3} \sigma_{\mathrm{cor}}^{1/2} n^{1/3}$$

$$= O(\alpha_{\mathrm{avg}}^{2/3} \sigma_{cor}^{1/2} n^{1/3}).$$

The last inequality again uses the following fact with $m = \log \sigma_{\text{cor}}$:

$$\sum_{o \le i < j \le m} (2^{j-i})^{1/2} = (2^{1/2})^m \sum_{0 \le t \le m} (t+1)(2^{-1/2})^t \le 2^{m/2} \frac{1}{(1-2^{-1/2})^2}$$
$$= O(2^{1/2 \log \sigma_{\text{cor}}}) = O(\sigma_{\text{cor}}^{1/2}).$$

This completes the proof of Theorem 1.2. Please note that a matching lower bound is already given by the construction shown in Section 3.1. In that example, we have n objects, with average aspect ratio α_{avg}. However, the ratio of the large core to the small core is

$$\sigma_{\text{cor}} = \frac{\sigma_{\text{box}}(\alpha_{\text{avg}}/n)^{1/3}}{(\alpha_{\text{avg}}\, \sigma_{\text{box}}/n)^{1/3}} = \sigma_{\text{box}}^{2/3}.$$

If we set $\sigma_{\text{cor}} = (\sigma_{\text{box}})^{2/3}$ and scale the remaining $n - X - Y$ objects so that their core sizes are between the small core size and large core size, then the core scale factor of the family becomes σ_{cor}. Lemma 3.3 already shows that

$$\rho(\mathcal{S}) = \Omega(\alpha_{\text{avg}}^{2/3}\, \sigma_{\text{box}}^{1/3}\, n^{1/3}) = \Omega(\alpha_{\text{avg}}^{2/3}\, \sigma_{\text{cor}}^{1/2}\, n^{1/3}).$$

5. *Bounded Aspect Ratio and Scale Factor Theorem*

In this section, we prove Theorem 1.3, which applies to scenes where every object has aspect ratio at most α and the scene has scale factor σ. Please note that when every object has the same bound on the aspect ratio, the choice of scale factor is immaterial—we get the same result for both the box scale factor and the core scale factor.

LEMMA 5.1. *Let $\mathcal{S}$ be a set of n objects where each object has aspect ratio at most α, and the bounding box of each object has size either α (small) or $\alpha\sigma_{box}$ (large). Then, $\rho(\mathcal{S}) = O(\alpha\sqrt{\sigma_{box}})$.*

PROOF. Let us call an object large if its enclosing box has size $\alpha\sigma_{\text{box}}$, and small otherwise. Clearly, there are only three kinds of intersections: large-large, small-small, and large-small, and we use K_b^ℓ, K_b^s, $K_b^{s\ell}$ to denote the corresponding bounding box intersection numbers. Similarly, we use K_o^ℓ, K_o^s, respectively, to denote the number of core intersections among large and small objects. By Lemma 4.1,

$$\frac{K_b^\ell}{n + K_o} \le \frac{K_b^\ell}{n + K_o^\ell} = O(\alpha) \tag{6}$$

and

$$\frac{K_b^s}{n + K_o} \le \frac{K_b^s}{n + K_o^s} = O(\alpha). \tag{7}$$

Let us tile the plane with boxes of size $\alpha\sigma_{\text{box}}$. Label these boxes $B_1, B_2, \ldots, B_p$. Just like before, we use X_i to denote large objects assigned to box B_i, and X_i^m to denote the maximum of X_j, where B_j runs over B_i's neighbors. Similarly we define Y_i, Y_i^m corresponding to small objects. Then we have the following estimates:

$$
\begin{aligned}
K_b^{s\ell} &\le c_1 \sum_{i=1}^{p} X_i^m Y_i^m \\
&\le c_1 \sqrt{\sigma_{\mathrm{box}}} \sum_{i=1}^{p} ((X_i^m)^2 + (Y_i^m)^2/\sigma_{\mathrm{box}}) \\
&\le c_2 \sqrt{\sigma_{\mathrm{box}}} \sum_{i=1}^{p} (X_i^2 + Y_i^2/\sigma_{\mathrm{box}}) .
\end{aligned}
$$

Let $n^\ell = \Sigma_i X_i$ and $n^s = \Sigma_i Y_i$; thus, $n = n^\ell + n^s$. Also, by Lemma 2.2, $n^\ell + K_o^\ell \ge c_3 \Sigma_i X_i^2/\alpha$ and $n^s + K_o^s \ge c_3 \Sigma_i Y_i^2/\alpha\sigma_{\mathrm{box}}$. Thus,

$$
\begin{aligned}
\frac{K_b^{s\ell}}{N + K_o(\mathcal{S})} &\le \frac{K_b^{s\ell}}{(n^\ell + K_o^\ell) + (n^s + K_o^s)} \\
&\le c_4 \frac{\sqrt{\sigma_{\mathrm{box}}}\, \Sigma_i (X_i^2 + Y_i^2/\sigma_{\mathrm{box}})}{\Sigma_i (X_i^2/\alpha + Y_i^2/(\alpha\sigma_{\mathrm{box}}))} \\
&\le c_4 \alpha \sqrt{\sigma_{\mathrm{box}}} \\
&= O(\alpha \sqrt{\sigma_{\mathrm{box}}}).
\end{aligned} \tag{8}
$$

By combining inequalities (6), (7), (8), we complete the proof of the lemma:

$$
\rho(\mathcal{S}) = \frac{K_b^\ell + K_b^s + K_b^{s\ell}}{n + K_o(\mathcal{S})} = O(\alpha \sqrt{\sigma_{\mathrm{box}}}). \qquad \square
$$

Suppose $\mathcal{S}$ is a set of n objects, with aspect ratio bound α and scale factor σ_{box}. We partition the set $\mathcal{S}$ into $\lceil \log \sigma_{\mathrm{box}} \rceil$ classes, $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_k$, such that P belongs to class $\mathcal{C}_i$ if $\alpha 2^i < \mathrm{vol}(b(P)) \le \alpha 2^{i+1}$. To simplify the analysis, for each class $\mathcal{C}_i$, we enlarge the bounding boxes in it to size $\alpha\, 2^{i+1}$ and shrink their core size to be 2^i. (This only affects the constants in the analysis.) By Lemma 3.1, $\rho(\mathcal{C}_i) = O(\alpha)$, for $i = 0, 1, \ldots, \log \sigma_{\mathrm{box}}$. Let K_b^i denote the number of bounding box intersections within class $\mathcal{C}_i$. Then there exists a constant c_1 such that

$$
\frac{K_b^i}{n + K_o(\mathcal{S})} \le c_1 \alpha, \quad \forall i,\ 0 \le i \le \log \sigma_{\mathrm{box}}.
$$

Let K_b^{ij}, for $0 \le i, j \le \log \sigma_{\mathrm{box}}$, denote the number of object pairs (P, P') whose enclosing boxes intersect and $P \in \mathcal{C}_i$ and $P' \in \mathcal{C}_j$. If we apply Lemma 5.1 to class pair $(\mathcal{C}_i, \mathcal{C}_j)$, then we get

$$
\frac{K_b^{s\ell}}{n + K_o(\mathcal{S})} \le c_2 \alpha (2^{j-i})^{1/2}, \quad \forall i, j\ 0 \le i \le j < \log \sigma_{\mathrm{box}}
$$

Therefore,

$$
\begin{aligned}
\rho(\mathcal{S}) &= \frac{\Sigma_i K_b^i + \Sigma_{i<j} k_b^{ij}}{n + K_o(\mathcal{S})} \\
&\le \sum_{0 \le i \le \log \sigma_{\text{box}}} c_1 \alpha + \sum_{0 \le i < j \le \log \sigma_{\text{box}}} c_2 \alpha \sqrt{2^{j-i}} \\
&= O(\alpha \sqrt{\sigma_{\text{box}}}).
\end{aligned}
$$

The last inequality again uses the fact that

$$
\sum_{0 \le i < j \le \log \sigma_{\text{box}}} (2^{j-i})^{1/2} = O(\sqrt{\sigma_{\text{box}}}).
$$

5.1. A Matching Lower Bound. Finally, we show a lower bound construction that matches the upper bound of Theorem 1.3. Consider a box B of size $\alpha\sigma_{\text{box}}$, we can put $X = n/(1 + \sqrt{\sigma_{\text{box}}})$ large objects with box size $\alpha\sigma_{\text{box}}$ and core size σ_{box} in the left half of it, and $Y = n\sqrt{\sigma_{\text{box}}}/(1 + \sqrt{\sigma_{\text{box}}})$ small objects with box size α and core size 1 in the right half. We evenly distribute the large objects, so that each subbox of size σ_{box} gets an equal number of large objects. This distribution implies that

$$
K_o^{\ell} = \Theta\left(\frac{1}{2}\left(\frac{X}{\alpha/2}\right)^2 \frac{\alpha}{2}\right) = \Theta\left(\frac{n^2}{\alpha\sigma_{\text{box}}}\right).
$$

Similarly, we distribute the small objects into sub-boxes of size 1, so that

$$
K_o^{s} = \Theta\left(\frac{1}{2}\left(\frac{Y}{\alpha\sigma_{\text{box}}/2}\right)^2 \frac{\alpha\sigma_{\text{box}}}{2}\right) = \Theta\left(\frac{n^2}{\alpha\sigma_{\text{box}}}\right).
$$

There are no object intersections among large and small objects, and thus $K_o^{s\ell} = 0$. Finally, every small-large pair of objects contributes an intersecting bounding box pair, and so $K_b \ge XY = \Theta(n^2/\sqrt{\sigma_{\text{box}}})$. Thus

$$
\rho(\mathcal{S}) = \frac{K_b(\mathcal{S})}{n + K_o(\mathcal{S})} = \Omega(\alpha\sqrt{\sigma_{\text{box}}}).
$$

6. *Extension to Higher Dimensions*

Theorems 1.1, 1.2, and 1.3 extends easily to d dimensions, for $d \ge 3$. Since our arguments have been *volume-based*, the structure of proofs remains essentially unchanged– only the implicit constants in the big-Oh notation are affected. In particular, the constant depends on the number of neighboring boxes for a given box B_i in the d-dimensional tiling. While in the plane, a box has at most 5^2 neighboring boxes in the two surrounding layers, this number increases to 5^d in d dimensions.

7. *Discussion*

We have analyzed the performance of a popular bounding box heuristic for collision detection in terms of two shape parameters: aspect ratio and scale factor. Theorems 1.1 and 1.2 make the distinction between *box scale factor* and *core scale factor*. Both measures seem natural, and the choice may be application-dependent. From an analytic point of view, one can always use the one that leads to a tighter estimate. For instance, we can set the scale factor to $\sigma = \min\{\sigma_{\mathrm{box}}, \sigma_{\mathrm{cor}}\}$. Then, the results of Theorems 1.1 and 1.2 can be unified as follows:

Let $\mathcal{S}$ be a set of n objects in d dimensions, with average aspect bound α_{avg} and scale factor $\sigma = min\{\sigma_{box}, \sigma_{cor}\}$, where d is a constant. Then, $\rho(\mathcal{S}) = \Theta(\alpha_{avg}^{2/3} \sigma^{1/2} n^{1/3})$.

Similarly, Theorem 1.3 can be formulated as follows:

Let $\mathcal{S}$ be a set of n objects in d dimensions, with aspect ratio bound α and scale factor $\sigma = min\{\sigma_{box}, \sigma_{cor}\}$, where d is a constant. Then, $\rho(\mathcal{S}) = \Theta(\alpha\, \sigma^{1/2})$.

With this more general definition of scale factor, the average case result becomes a natural extension of the worst-case bound result.

The aspect ratio and scale factor appear to be natural shape parameters for many computer graphics applications, where objects tend to have an intrinsic 3-dimensional solid shape, at least on average. Thus, our analysis applies to such graphics applications as animation and collision detection, where scenes tend to have bounded *average* aspect ratio and bounded scale factor. Our theorems can be used to explain the empirical evidence that bounding boxes tend to improve performance.

There are other applications, however, where aspect ratio or scale factor may not be well-suited shape parameters. For instance, solid modeling applications involve lower-dimensional facets, which would have unbounded aspect ratio by our definition. Similarly, applications in mesh generation involve irregular meshes of widely varying scale. We are currently investigating shape models that might be relevant to these application domains. A key consideration in studying *realistic* shape models is to ensure that they are realistic in the specific application–an inappropriate model can easily *trivialize* the problem. An attractive feature of aspect ratio and scale factor is that they depend on the individual shape of objects and their relative sizes, but not on their *distribution*. By contrast, the geometric models that use *density* and *clutter* as a complexity measure automatically preclude many intersections among objects, and consequently trivialize the collision detection problem.

Finally, another line of future research is to analyze bounding box heuristic for more constrained shapes, such as *convex polyhedra*. We are currently investigating it.

REFERENCES

Cohen, J. D., Lin, M. C., Manocha, D., and Ponamgi, M. K. 1995. I-COLLIDE: An interactive and exact collision detection system for large-scale environments. In *Proceedings of 1995 Symposium on Interactive 3D Graphics* (Monterey, Calif., Apr. 9–12), ACM, New York, pp. 189–196.

de Berg, M. 1995. Linear size binary space partitions for fat objects. In *Proceedings of the 3rd European Symposium on Algorithms*, pp. 252–263.

DE BERG, M., KATZ, M., VAN DER STAPPEN, A. F., AND VLEUGELS, J. 1997. Realistic input models for geometric algorithms. In *Proceedings of the 13th Annual Symposium on Computational Geometry*. (Nice, France, June 4–6). ACM, New York, pp. 294–303.

EFRAT, A., AND SHARIR, M. 1997. On the complexity of the union of fat objects in the plane. In *Proceedings of the 13th Annual ACM Symposium on Computational Geometry*. (Nice, France, June 4–6). ACM, New York, pp. 104–112.

FOLEY, J. D., VAN DAM, A., FEINER, S. K., AND HUGHES, J. F. 1996. *Computer Graphics: Principles and Practice* (2nd Ed.). Addison-Wesley, Reading, Mass.

GARCIA-ALONSO, A., SERRANO, N., AND FLAQUER, J. 1995. Solving the collision detection problem. *IEEE Comput. Graph. Appl. 14* 36–43.

GOTTSCHALK, S., LIN, M. C., AND MANOCHA, D. 1996. OBBTree: A hierarchical structure for rapid interference detection. In *Proceedings of ACM SIGGRAPH (SIGGRAPH '96*) (New Orleans, La., Aug. 4–9). ACM, New York, pp. 171–180.

GREENE, N. 1994. Detecting intersection of a rectangular solid and a convex polyhedron. *Graph. Gems IV*, 83–110.

HALPERIN, D., AND OVERMARS, M. H. 1994. Spheres, molecules, and hidden surface removal. In *Proceedings of the 10th Annual Symposium on Computational Geometry* (Stony Brook, N.Y., June 6–8). ACM, New York, pp. 113–122.

HOFFMANN, C. 1989. *Geometric and Solid Modeling*. Morgan-Kaufmann, San Mateo, Calif.

KLOSOWSKI, J. T., HELD, M., MITCHELL, J. S. B., SOWIZRAL, H., AND ZIKAN, K. 1999. Real-time collision detection for motion simulation within complex environments. Submitted for journal publication.

LATOMBE, J.-C. 1991. *Robot Motion Planning*. Kluwer Academic Publishers, Hingham, Mass.

MATOUŠEK, J., PACH, J., SHARIR, M., SIFRONY, S., AND WELZL, E. 1994. Fat triangles determine linearly many holes. *SIAM J. Computing 23*, 154–169.

MCMILLAN, L. 1997. An Image-Based Approach to Three-Dimensional Computer Graphics. Ph.D. dissertation, Univ. North Carolina at Chapel Hill, Chapel Hill, N.C.

MOORE, M., AND WILHELMS, J. 1988. Collision detection and response for computer animation. In *Proceedings of SIGGRAPH '88* (published as *Comput. Graphics, 22*) (Atlanta, Ga., Aug. 1–5). ACM, New York, pp. 289–298.

PREPARATA, F. P., AND SHAMOS, M. I. 1985. *Computational Geometry: An Introduction*. Springer-Verlag, New York.

SMITH, A., KITAMURA, Y., TAKEMURA, H., AND KISHINO, F. 1995. A simple and efficient method for accurate collision detection among deformable objects in arbitrary motion. In *Proceedings of the IEEE Virtual Reality Annual International Symposium*. IEEE Computer Society Press, Los Alamitos, Calif., pp. 136–145.

SURI, S., HUBBARD, P. M., AND HUGHES, J. F. 1998. Collision detection in aspect and scale bounded polyhedra. In *Proceedings of the 9th Annual ACM Symposium on Discrete Algorithms* (San Francisco, Calif., Jan.) ACM, New York.

RECEIVED OCTOBER 1998; REVISED MARCH 1999; ACCEPTED APRIL 1999