# HOW TO LOSE AT MONTE CARLO: A SIMPLE DYNAMICAL SYSTEM WHOSE TYPICAL STATISTICAL BEHAVIOR IS NON COMPUTABLE 

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#### Abstract

We consider the simplest non-linear discrete dynamical systems, given by the logistic maps $f_{a}(x)=a x(1-x)$ of the interval $[0,1]$. We show that there exist real parameters $a \in(0,4)$ for which almost every orbit of $f_{a}$ has the same statistical distribution in $[0,1]$, but this limiting distribution is not Turing computable. In particular, the Monte Carlo method cannot be applied to study these dynamical systems.


[^0]
## 1. Introduction

For all practical purposes, the world around us is not a deterministic one. Even if a simple physical system can be described deterministically, say by the laws of Newtonian mechanics, the differential equations expressing these laws typically cannot be solved explicitly. This means that predicting the exact evolution of the system is impossible. A classical example is the famous 3 -body Problem, which asks to describe the evolution of a system in which three celestial bodies (the "Earth", the "Sun", and the "Moon") interact with each other via the Newton's force of gravity. Computers are generally not of much help either: of course, a system of ODEs can be solved numerically, but the solution will inevitably come with an error due to round-offs. Commonly, solutions of dynamical systems are very sensitive to such small errors (the phenomenon known as "Chaos"), so the same computation can give wildly different numerical results.

An extreme example of the above difficulties is the art of weather prediction. A realistic weather model will have such a large number of inputs and parameters that simply running a numerical computation will require a massive amount of computing resources; it is, of course, extremely sensitive to errors of computation. A classical case in point is the Lorenz system suggested by meteorologist Edward Lorenz in 1963 [Lor63]. It has only three variables and is barely non-linear (just enough not to have an explicit solution), and nevertheless it possesses a chaotic attractor [Tuc02] - one of the first such examples in history of mathematics- so deterministic weather predictions even in such a simplistic model are practically impossible.

Of course, this difficulty is well known to practitioners, and yet weather predictions are somehow made, and sometimes are even accurate. They are made in the language of statistics (e.g. there is a $40 \%$ chance of rain tomorrow), and are based on what is broadly known as Monte Carlo technique, pioneered by Ulam and von Neumann in 1946 [URvN47, MS49, Met87]. Informally speaking, we can throw random darts to select a large number of initial values; run our simulation for the desired duration for each of them; then statistically average the outcomes. We then expect these averages to reflect the true statistics of our system. To set the stage more formally, let us assume that we have a discrete-time dynamical system

$$
f: D \rightarrow D, \text { where } D \text { is a finite domain in } \mathbb{R}^{n}
$$

that we would like to study. Let $\bar{x}_{1}, \ldots, \bar{x}_{k}$ be $k$ points in $D$ randomly chosen, for some $k \gg 1$ and consider the probability measure

$$
\begin{equation*}
\mu_{k, n}=\frac{1}{k n} \sum_{l=1}^{k} \sum_{m=1}^{n} \delta_{f \circ m\left(\bar{x}_{l}\right)} \tag{1.1}
\end{equation*}
$$

where $\delta_{\bar{x}}$ is the delta-mass at the point $\bar{x} \in \mathbb{R}^{n}$. The mapping $f$ can either be given by mathematical formulas, or stand for a computer program we wrote to simulate our dynamical system. The standard postulate is then that for $k, n \rightarrow \infty$ the probabilities $\mu_{k, n}$ converge to a limiting statistical distribution that we can use to make meaningful long-term statistical predictions of our system.

Let us say that a measure $\mu$ on $D$ is a physical measure of $f$ if its basin $B(\mu) \subset D$ -that is, the set of initial values $\bar{x}$ for which the weak limit of $\frac{1}{n} \sum_{m=1}^{n} \delta_{f^{\circ m}(\bar{x})}$ equals $\mu$ - has positive Lebesgue measure. This means that the limiting statistics of such points will appear in the averages (1.1) with a non-zero probability. If there
is a unique physical measure in our dynamical system, then one random dart in (1.1) will suffice. Of course, there are systems with many physical measures. For instance, Newhouse [New74] showed that a polynomial map $f$ in dimension 2 can have infinitely many attracting basins, on each of which the dynamics will converge to a different stable periodic regime. This in itself, however, is not necessarily an obstacle to the Monte-Carlo method, and indeed, the empirical belief is that it still succeeds in these cases.

Our results are most surprising in view of the above computational statistical paradigm. Namely we consider the simplest examples of non-linear dynamical systems: quadratic maps of the interval $[-1,1]$ of the form

$$
f_{a}(x)=a x(1-x), a \in(0,4]
$$

and find an uncountable set of values of $a$ for which:
(1) there exists a unique physical measure $\mu$ and its basin $B(\mu) \subset[0,1]$ has full Lebesgue measure.
(2) the measure $\mu$ is not computable relative to $a$.

This means that there is no algorithm that correctly computes $\mu$, even if the parameter $a$ is assumed to be provided to the algorithm at no computational cost. Thus, the Monte-Carlo computational approach fails spectacularly for truly simple maps - not because there are no physical measures, or too many of them, but because the "nice" unique limiting statistics cannot be computed, and thus the averages (1.1) will not converge to anything meaningful in practice.

It is worth drawing a parallel with our recent paper [RY19], in which we studied the computational complexity of topological attractors of maps $f_{a}$. Such attractors capture the limiting deterministic behavior of the orbits. They are always computable, and we found that for almost every parameter $a$, the time complexity of computing its attractor is polynomial, although there exist attractors with an arbitrarily high computational complexity. In dynamics, both in theory and in practice, it is generally assumed that long-term statistical properties are simpler to analyze than their deterministic counterparts. From the point of view of computational complexity, this appears to be false.

We note that computability of invariant measures has been studied before [Roj08, GHR10, GR11, BBRY11]. In [GHR10] for instance the authors construct continuous maps of the circle for which computable invariant measures do not exists. In the context of symbolic systems, there have been some recent works studying the computational properties of the limiting statistics, see e.g. [HdMS16], and of thermodynamic invariants (see e.g. in [HM10, BW18]). The computational complexity of individual trajectories in Hamiltonian dynamics has been addressed in e.g. [KTZ18]. Long-term unpredictability is generally associated with dynamical systems containing embedded Turing machines (see e.g. the works [Moo91, MK99, KCG94, BGR12, BRS15]). Dynamical properties of Turing machines viewed as dynamical systems have similarly been considered (cf. [Kur97, Jea14]). Yet we are not aware of any studies of the limiting statistics in this latter context. We also point out that a different notion of statistical intractability in dynamics, based on the complexity of a mathematical description of the set of limiting measures, has been introduced and studied in [Ber17, BB19].

From a practical point of view, some immediate questions arise. Our examples are rare in the one-parameter quadratic family $f_{a}(x)=a x(1-x)$. However, there
are reasons to expect that in more complex multi-parametric, multi-dimensional families they can become common. Can they be generic in a natural setting? As the results of [BB19] suggest, the answer may already be "yes" for quadratic polynomial maps in dimension two. Furthermore, even in the one-dimensional quadratic family $f_{a}$ it is natural to ask what the typical computational complexity of the limiting statistics is - even if it is computable in theory, it may not be in practice.

## 2. Preliminaries

Statistical simulations and computability of probability measures. We give a very brief summary of relevant notions of Computability Theory and Computable Analysis. For a more in-depth introduction, the reader is referred to e.g. [BY08]. As is standard in Computer Science, we formalize the notion of an algorithm as a Turing Machine [Tur36]. We will call a function $f: \mathbb{N} \rightarrow \mathbb{N}$ computable (or recursive), if there exists a Turing Machine $\mathcal{M}$ which, upon input $n$, outputs $f(n)$. Extending algorithmic notions to functions of real numbers was pioneered by Banach and Mazur [BM37, Maz63], and is now known under the name of Computable Analysis. Let us begin by giving the modern definition of the notion of computable real number, which goes back to the seminal paper of Turing [Tur36]. By identifying $\mathbb{Q}$ with $\mathbb{N}$ through some effective enumeration, we can assume algorithms can operate on $\mathbb{Q}$. Then a real number $x \in \mathbb{R}$ is called computable if there is an algorithm $M$ which, upon input $n$, halts and outputs a rational number $q_{n}$ such that $\left|q_{n}-x\right|<2^{-n}$. Algebraic numbers or the familiar constants such as $\pi, e$, or the Feigenbaum constant are computable real numbers. However, the set of all computable real numbers $\mathbb{R}_{C}$ is necessarily countable, as there are only countably many Turing Machines.

We now define computability of functions over $[0,1]$. Recall that for a continuous function $f \in C_{0}([0,1])$, a modulus of continuity consists of a function $\delta: \mathbb{Q} \cap$ $(0, a) \rightarrow \mathbb{Q} \cap(0, a)$ such that $|f(x)-f(y)| \leq \epsilon$ whenever $|x-y| \leq \delta(\epsilon)$. A function $f:[0,1] \rightarrow[0,1]$ is computable if it has a computable modulus of continuity and there is an algorithm which, provided with a rational number which is $\delta(\epsilon)$-close to $x$, outputs a rational number which is $\epsilon$-close to $f(x)$.

Computability of probability measures, say over $[0,1]$ for instance, is defined by requiring the ability to compute the expected value of computable functions.

Definition 2.1. Let $\left(f_{i}\right)$ be any sequence of uniformly computable functions over $[0,1]$. A probability measure $\mu$ over $[0,1]$ is computable if there exist a Turing Machine $M$ which on input $(i, \epsilon)$ (with $\epsilon \in \mathbb{Q}$ ) outputs a rational $M(i, \epsilon)$ satisfying

$$
\left|M(i, \epsilon)-\int f_{i} d \mu\right|<\epsilon
$$

We note that this definition it compatible with the notion of weak convergence (see Section 3.1) of measures in the sense that a measure is computable if and only if it can be algorithmically approximated (in the weak topology) to an arbitrary degree of accuracy by measures supported on finitely many rational points and with rational weights. Moreover, this definition also models well the intuitive notion of statistical sampling in the sense that a measure $\mu$ is computable if and only if there is an algorithm to convert sequences sampled from the uniform distribution into sequences sampled with respect to $\mu$.

In this paper, we will be interested in the computability properties of invariant measures of quadratic maps of the form $a x(1-x)$, with $a \in \mathbb{R}$. As is standard in computing practice, we will assume that the algorithm can read the value of $a$ externally in order to compute $\mu$. More formally, let us denote $\mathcal{D}_{n} \subset \mathbb{R}$ the set of dyadic rational numbers with denominator $2^{-n}$. We say that a function $\phi: \mathbb{N} \rightarrow \mathbb{Q}$ is an oracle for $a \in \mathbb{R}$ if for every $m \in \mathbb{N}$

$$
\phi(m) \in \mathcal{D}_{m} \text { and } d(\phi(m), a)<2^{-(m-1)}
$$

We amend our definitions of computability of a probability measure $\mu$ by allowing oracle Turing Machines $M^{\phi}$ where $\phi$ is any function as above. On each step of the algorithm, $M^{\phi}$ may read the value of $\phi(m)$ for an arbitrary $m \in \mathbb{N}$. This approach, usually referred to as computability relative to $a$, allows us to separate the questions of computability of a parameter $a$ from that of the measure.

## Invariant measures of quadratic polynomials and the statement of the

 main result. As before, we denote$$
f_{a}(x)=a x(1-x)
$$

For $a \in[0,4]$, this quadratic polynomial maps the interval $[0,1]$ to itself. We will view $f_{a}:[0,1] \rightarrow[0,1]$ as a discrete dynamical system, and will denote $f_{a}^{n}$ the $n$-th iterate of $f_{a}$.

A measure $\mu$ is called physical or Sinai-Ruelle-Bowen (SRB) if

$$
\begin{equation*}
\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^{k} x} \rightarrow \mu \tag{2.1}
\end{equation*}
$$

for a set of positive Lebesgue measure. It is known that if a physical measure exists for a quadratic map $f_{a}, a \in[0,4]$, then it is unique and (2.1) is satisfied for Lebesgue almost all $x \in[0,1]$.

Main Theorem. There exists parameters $a \in(0,4)$ for which the quadratic map $f_{a}(x)=a x(1-x)$ has a physical measure $\mu$ which is not computable relative to $a$.

## 3. Proof of the Main Theorem

The proof is based on a delicate construction in one-dimensional dynamics described in [HK90], which will allow us to construct maps $f_{a}$ with physical measures which selectively charge points in a countable set of periodic orbits. To give precise formulations, we will need to introduce some further concepts.
3.1. Setting the stage. It will be convenient to recall that weak convergence of measures on $[0,1]$ is compatible with the notion of Wasserstein-Kantorovich distance, defined by:

$$
W_{1}(\mu, \nu)=\sup _{f \in 1-\operatorname{Lip}([0,1])}\left|\int f d \mu-\int f d \nu\right|
$$

where $1-\operatorname{Lip}([0,1])$ is the space of Lipschitz functions on $[0,1]$, having Lipschitz constant less than one.

For $a \in[0,4]$ and $x \in[0,1]$, we set

$$
\nu_{a}^{n}(x)=\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f_{a}^{k} x}
$$

We will make use of the following folklor fact (see e.g. [dMvS93]):
Proposition 3.1. Suppose, for $a \in[0,4]$ the map $f_{a}$ has an attracting periodic orbit of period $p$ :

$$
x_{0} \stackrel{f_{q}}{\mapsto} x_{1} \stackrel{f_{q}}{\mapsto} \ldots \stackrel{f_{q}}{\mapsto} x_{p-1} \stackrel{f_{q}}{\mapsto} x_{0},\left|\frac{d}{d x} f_{a}^{p}\left(x_{0}\right)\right|<1 .
$$

Let

$$
\mu \equiv \frac{1}{p} \sum_{k=0}^{p-1} \delta_{x_{k}} .
$$

Then $\mu$ is the unique physical measure of $f_{a}$ (so, in particular, the attracting orbit is unique); and

$$
W_{1}\left(\nu_{a}^{n}(x), \mu\right) \rightarrow 0
$$

uniformly on a set of full Lebesgue measure in $[0,1]$.
For $a \in(0,4]$ consider the third iterate $g_{a} \equiv f_{a}^{3}$. We start by noting that there exists a parameter value $c \in(3.85,4)$ such that the following holds:

$$
g_{c}(0.5) \neq g_{c}^{2}(0.5)=g_{c}^{3}(0.5)
$$

If we denote $\beta_{c}=g_{c}^{2}(0.5)$, then $\beta_{c}^{\prime} \equiv g_{c}(0.5)=1-\beta_{c}$, and denoting $I_{c} \equiv\left[\beta_{c}^{\prime}, \beta_{c}\right] \ni$ 0.5 , we have

$$
g_{c}\left(I_{c}\right)=I_{c} ;
$$

both endpoints of $I_{c}$ map to $\beta_{c}$. The restriction $\left.g_{c}\right|_{I_{c}}$ maps both halfs $\left(L_{c}=\left[\beta^{\prime}, 0.5\right]\right.$ and $\left.R_{c}=[0.5, \beta]\right)$ of the interval $I_{c}$ onto the whole $I_{c}$ in a monotone fashion (that is, it folds $I_{c}$ over itself).

For $a \in[c, 4]$, there exists a continuous branch $\beta_{a}$ of the fixed point

$$
g_{a}\left(\beta_{a}\right)=\beta_{a}
$$

and we again set $\beta_{a}^{\prime}=1-\beta_{a}\left(\right.$ so $\left.g_{a}\left(\beta_{a}^{\prime}\right)=\beta_{a}\right)$, and $I_{a} \equiv\left[\beta_{a}^{\prime}, \beta_{a}\right]$. Now, if $a \in(c, 4]$, the image

$$
g_{a}\left(I_{a}\right) \supsetneq I_{c}, \text { with } g_{a}(0,5)<\beta_{a}^{\prime} .
$$

Thus, there is a pair of sub-intervals $L_{a}=\left[\beta_{a}^{\prime}, l_{a}\right], R_{a}=\left[r_{a}, \beta_{a}\right]$ inside $I_{a}$ which are mapped monotonely over $I_{a}$ by $g_{a}$ (the endpoints $l_{a}, r_{a}$ are both mapped to $\beta_{a}^{\prime}$. See Figure 1 for an illustration.

Assigning values 0 to $L_{a}$, and 1 to $R_{a}$ we obtain symbolic dynamics on the set of points

$$
C_{a} \equiv\left\{x \in I_{a} \text { such that } g_{a}^{n}(x) \in I_{a} \text { for all } n \in \mathbb{N}\right\}
$$

If $a=c$, then, of course, $C_{a}=I_{a}$. Otherwise, the following is well-known:
Proposition 3.2. If $a \in(c, 4)$ then $C_{a}$ is a Cantor set, and the symbolic dynamics conjugates $\left.g_{a}\right|_{C_{a}}$ to the full shift on $\{0,1\}$.


Figure 1. Some iterates of $f \equiv f_{a}$ for $a \in(c, 4]$ (we drop the subscript $a$ for simplicity in all notations in the figure).

In particular, every periodic sequence of 0's and 1's corresponds to a unique periodic orbit in $C_{a}$ with this symbolic dynamics. These orbits clearly move continuously with $a$, and can be easily computed given $a$ and the symbolic sequence as the unique fixed points of the corresponding monotone branches of iterates of $g_{a}$.

We enumerate all periodic sequences of 0 ' $s$ and 1 's as follows. A sequence with a smaller period will precede a sequence with a larger period. Within the sequences of the same period, the ordering will be lexicographic, based on the convention $1 \prec 0$. We let

$$
\left\{p_{a, n}^{1}, \ldots, p_{a, n}^{k_{n}}\right\}
$$

be the periodic orbit of $g_{a}$ in $C_{a}$ which corresponds to the $n$-th symbolic sequence in this ordering (note that the first one is $\beta_{a}$ ). We denote

$$
\operatorname{Per}_{a}(n)=\cup_{j=0}^{2} f_{a}^{j}\left(\left\{p_{a, n}^{1}, \ldots, p_{a, n}^{k_{n}}\right\}\right)
$$

which is, clearly, a periodic orbit of $f_{a}$. Let us denote

$$
\lambda_{a}(n)=\frac{1}{\left|\operatorname{Per}_{a}(n)\right|} \sum_{x \in \operatorname{Per}_{a}(n)} \delta_{x}
$$

3.2. Main construction. Our arguments will be based on the results of F. Hofbauer and G. Keller in [HK90]; see also the earlier paper of S. Johnson [Joh87], which uses similar language to ours.

Let us develop some further notation. For $x \in[0,1]$ and $a \in[0,4]$ we let $\Omega_{a}(x)$ denote the set of weak limits of the sequence $\nu_{a}^{n}(x)=\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f_{a}^{k} x}$. Let us denote $\mathcal{P} \subset(c, 4]$ the collection of parameters $a$ such that the following holds:

- $f_{a}$ has a unique physical probability measure $\mu_{a}$;
- denoting $m_{a}(n) \equiv \mu_{a}\left(\operatorname{Per}_{a}(n)\right)$, we have

$$
\begin{equation*}
\sum_{n=1}^{\infty} m_{a}(n)=1 \tag{3.1}
\end{equation*}
$$

Thus, the charge of the physical measure $\mu_{a}$ resides in the periodic orbits in the Cantor set $C_{a}$.

We will formulate the following direct consequence of the main result (Theorem 5) of $[\mathrm{HK} 90]^{1}$ :

Theorem 3.3. There exists an infinite set $\hat{\mathcal{P}} \subset \mathcal{P}$ such that the following holds:
(1) $\Omega_{a}(x)=\Omega_{a}(0.5)=\left\{\mu_{a}\right\}$ for Lebesgue almost every $x$;
(2) for any sequence of non-negative reals $\left\{l_{n}\right\}_{n \in \mathbb{N}}$ with $\sum l_{n}=1$, the subset of $a \in \hat{\mathcal{P}}$ for which $m_{a}(n)=l_{n}$ is dense in $\hat{\mathcal{P}}$.

We will outline the idea of the construction of such maps below, but the complete proof of Theorem 3.3 is quite technical and goes beyond the scope of this paper.

We start with the following "simple" example:
Example 1: The set

$$
\Omega_{a}(0.5)=\lambda_{a}(1)=\frac{1}{3}\left(\delta_{\beta_{a}}+\delta_{f_{a}\left(\beta_{a}\right)}+\delta_{f_{a}^{2}\left(\beta_{a}\right)}\right)
$$

and $\Omega(x)=\Omega(0.5)$ for almost every $x$ (compare with Theorem 1 of [HK90]).
Consider again Figure 1 as an illustration. We note that there exists an interval $J_{0}$ to the right of the fixed point $\beta_{a}$ such that the following holds:

- $f_{a}^{2}\left(J_{0}\right) \supseteq\left[\beta_{a}^{\prime}, \beta_{a}\right]$;
- Denote by $\psi_{a}$ the branch of $g_{a}^{-1}$ which fixes $\beta_{a}$. Then the interval $J_{0}$ is contained in the domain of definition of $\psi_{a}$. Thus, there is an orbit

$$
J_{-n} \equiv \psi_{a}^{n}\left(J_{0}\right) \rightarrow \beta_{a}\left(\text { here } f_{a}^{3 n}\left(J_{-n}\right)=J_{0}\right)
$$

Moving the parameter $a \in(c, 4]$, we can place the image $g_{a}^{2}(0.5)$ at any point of $J_{-n_{1}}$, for an arbitrary $n_{1}$. If the value of $n_{1}$ is large, then the $g_{a}$-orbit of 0.5 will spend a long time in a small neighborhood of $\beta_{a}$, before hitting some $x_{1} \in J_{0}$. Adjusting the position of $a \in(c, 4]$, we can ensure that $f_{a}^{2}\left(x_{1}\right)$ is inside $J_{-n_{2}}$ for an even larger $n_{2}$, so the orbit returns to an even smaller neighborhood of $\beta_{a}$ where it will spend an even longer time. Continuing increasing $n_{k}$ 's as needed so the orbit of 0.5 spends most of its time in ever smaller neighborhoods of $\beta_{a}$, we can ensure that the averages $\nu_{a}^{n}(0.5)=\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f_{a}^{k}(0.5)}$ converge to the delta masses supported on the orbit of $\beta_{a}$.

Proceeding in this way, for an arbitrarily large $l \in \mathbb{N}$ and $x \in\left[\beta_{a}^{\prime}, \beta_{a}\right]$ we can find $a \in(c, 4]$ and $m>2^{-l}$ such that:
(1) the distance

$$
W_{1}\left(\nu_{a}^{m}(0.5)-\lambda_{a}(1)\right)<2^{-l} ;
$$

(2) the iterate $f_{a}^{m}(0.5)$ lies in $J_{0}$;
(3) the next iterate $f_{a}^{m+1}(0.5)=0.5$.

[^1]Property (3) ensures that the critical point 0.5 is periodic with period $m+1$. Since $f_{a}^{\prime}(0.5)=0$, we have $\left(f_{a}^{m+1}\right)^{\prime}(0.5)=0$, so this is a (super) attracting periodic point. Proposition 3.1 implies that the physical measure $\mu_{a}$ for $f_{a}$ is supported on the orbit of 0.5 , and thus

$$
W_{1}\left(\mu_{a}-\lambda_{a}(1)\right)<2 \cdot 2^{-l}
$$

Again, by Proposition 3.1 and considerations of continuity, there exist $n>m$ and $\epsilon>0$ such that for any $a^{\prime}$ with $\left|a^{\prime}-a\right|<\epsilon$, we have

$$
W_{1}\left(\nu_{a^{\prime}}^{n}(x)-\lambda_{a}(1)\right)<4 \cdot 2^{-l}
$$

for any $x$ in a set of length $1-2^{-l}$.
Assuming $\epsilon$ is small enough, we again have $g_{a^{\prime}}(0.5)$ slightly to the right of $\beta_{a^{\prime}}$ and we can repeat the above steps inductively to complete the construction.

As a next step, we construct an asymptotic measure supported on two periodic orbits:
Example 2: the set $\Omega(0.5)=a_{1} \lambda_{1}(1)+a_{2} \lambda_{a}(n)$ for $n>1$ and $a_{1}+a_{2}=1$.
Let $p \in \operatorname{Per}_{a}(n)$ and, as before, denote by $3 k_{n}$ its period. Letting $\phi$ denote the branch of $f_{a}^{-3 k_{n}}$ fixing $p$, we again find a $\phi$-orbit

$$
J_{0}^{\prime} \mapsto J_{-1}^{\prime} \mapsto J_{-2}^{\prime} \mapsto \cdots, \text { with } J_{-k}^{\prime} \rightarrow p
$$

where

$$
f_{a}^{s}\left(J_{0}\right) \ni\left[\beta_{a}^{\prime}, \beta_{a}\right]
$$

for a univalent branch of the iterate $f_{a}^{s}$.
Now we can play the same game as in Example 1, alternating between entering the orbit $J_{-k}$ close to the point $\beta_{a}$, and the orbit $J_{-k}^{\prime}$ close to $p$. In this way, we can achieve the desired limiting asymptotics with any values $a_{1}, a_{2}$.

The above construction can be clearly modified for any countable collection of periodic orbits in $C_{a}$, as required for the proof of Theorem 3.3.

### 3.3. Constructing non computable physical measures.

Definition 3.1. Let us define a very particular subset $\tilde{\mathcal{P}} \subset \hat{\mathcal{P}}$ as follows: $a \in \tilde{\mathcal{P}}$ if

$$
\begin{equation*}
m_{a}(2 n-1)+m_{a}(2 n)=2^{-n} \quad \text { for all } n \in \mathbb{N} \tag{3.2}
\end{equation*}
$$

For convenience of reference, let us formulate a corollary of Theorem 3.3:
Proposition 3.4. Suppose, $a \in \tilde{\mathcal{P}}$. Then, for every $\epsilon>0, l \in \mathbb{N}$ and $s \in\{0,1\}$, there exists $a^{\prime}>a$ such that

- $a^{\prime} \in \tilde{\mathcal{P}}$;
- $\left|a-a^{\prime}\right|<\epsilon$;
- $m_{a}(n)=m_{a^{\prime}}(n)$ for all $n \notin\{2 l, 2 l-1\}$ and
- $m_{a^{\prime}}(2 l-s)=2^{-l}$.

Let $\left(\tau_{i}\right)_{i \in \mathbb{N}}$ be the smallest collection of functions containing the step continuous functions of rational intervals, and which is closed by rational linear combinations and scalar multiplication. Note that this is a countable collection of functions that can be enumerated in an effective way.

We construct a parameter $a$ for which the map $f_{a}=a x(1-x)$ has a unique physical measure $\mu_{a}$ such that for any Turing Machine $M^{\phi}$ with an oracle $\phi$ for $a$, that computes a probability measure, there exists $i$ and $\epsilon>0$ such that

$$
\left|M^{\phi}(i, \epsilon)-\int \tau_{i} d \mu\right|>\epsilon
$$

Our construction can be thought of as a game between a Player and infinitely many opponents, which will correspond to the sequence consisting of machines $M_{n}^{\phi}$ that compute some probability measure. The opponents try to compute $\mu_{a}$ by asking the Player to provide an oracle $\phi$ for $a$, while the Player tries to chose the bits of $a$ in such a way that none of the opponents correctly computes $\mu_{a}$.

We show that the Player always has a winning strategy: it plays against each machine, one by one, asking the machine to compute the expected value of a particular function $\tau_{i}$ to a certain degree of accuracy. The machine then runs for a while, asking the Player to provide more and more bits of $a$, until it eventually halts and outputs a rational number. Then the Player reveals the next bit of $a$ and shows that the machine's answer is incompatible with $\mu_{a}$. The details are as follows.

We will proceed inductively. Let $M_{1}^{\phi}, M_{2}^{\phi}, \ldots$ be some enumeration of all the machines with an oracle for $a$ that compute some probability measure. At step $n$ of the induction, we will have a parameter $a_{n} \in(c, 4)$ and a natural number $l_{n}$ such that:
(1) $a_{n} \in \tilde{\mathcal{P}}$;
(2) there exists $i=i(n) \in \mathbb{N}$ such that either

- $M_{n}^{\phi}\left(\tau_{i}, 2^{-n} / 100\right) \leq 2^{-n} / 2$ whereas $\mu_{a_{n}}\left(\tau_{i}\right) \sim 2^{-n}$; or
- $M_{n}^{\phi}\left(\tau_{i}, 2^{-n} / 100\right)>2^{-n} / 2$ whereas $\mu_{a_{n}}\left(\tau_{i}\right) \sim 0$

In other words, given an oracle for $a_{n}$, the machine $M_{n}^{\phi}$ cannot correctly approximate the value of $\mu_{a_{n}}$ at $\tau_{i}$;
(3) $\left|\mu_{a_{n-1}}\left(\tau_{i(k)}\right)-\mu_{a_{n}}\left(\tau_{i(k)}\right)\right|<2^{-3 n}$ for all $k<n$;
(4) $\left|a_{n}-a_{n-1}\right|<2^{-3 l_{n}}$.

Base of the induction. We start by letting $a$ be any of the parameters in $\tilde{\mathcal{P}}$. We note that $m_{a}(1)+m_{a}(2)=2^{-1}$. It follows that there exists $\tau=\tau_{i(1)}$ such that $\left|\mu_{a}(\tau)-m_{a}(1)\right|<2^{-1} / 200^{2}$, and

$$
\begin{equation*}
\operatorname{Supp} \tau \cap \operatorname{Per}(j)=\emptyset \text { for all } 1<j<10 \tag{3.3}
\end{equation*}
$$

We now let the machine $M_{1}^{\phi}$ compute the expected value of $\tau$ with precision $2^{-1} / 100$, giving it $a$ as the parameter. Let $l_{1}$ be the last time a bit of $a$ is queried by $M_{1}^{\phi}$ during the computation. By Proposition 3.4, for any $s \in\{0,1\}$ there exists $a^{\prime}$ such that

- $\left|a-a^{\prime}\right|<2^{-3 l_{1}}$;
- $a^{\prime} \in \tilde{\mathcal{P}}$;
- $m_{a^{\prime}}(2-s)=2^{-1}$.

Let $q=M_{1}^{\phi}\left(\tau, 2^{-1} / 100\right)$. There are two possibilities:
Case 1. If $q \leq 2^{-1} / 2$, we chose $a^{\prime}$ above so as to have $m_{a^{\prime}}(1)=2^{-1}$;
Case 2. If $q>2^{-1} / 2$, we chose $a^{\prime}$ above so as to have $m_{a^{\prime}}(2)=2^{-1}$ (and therefore $\left.m_{a^{\prime}}(1)=0\right)$;

[^2]We then let $a_{1} \equiv a^{\prime}$. By 3.3, $\left|\mu_{a_{1}}(\tau)-m_{a_{1}}(1)\right|<2^{-10}$. Note that up to the first $l$ bits, $a_{0}$ and $a_{1}$ are indistinguishable and therefore the machine $M_{1}^{\phi}$ will return the same answer for both parameters. It follows that the machine $M_{1}^{\phi}$ cannot correctly approximate $\mu_{a_{1}}$ at $\tau$.

Step of the induction. Assume $a_{n-1} \in \tilde{\mathcal{P}}$ has been defined. Then it holds

$$
m_{a_{n-1}}(2 n-1)+m_{a_{n-1}}(2 n)=2^{-n}
$$

and there exists $\tau=\tau_{i(n)}$ such that $\left|\mu_{a_{n-1}}(\tau)-m_{a_{n-1}}(2 n-1)\right|<2^{-n} / 200$ and

$$
\begin{equation*}
\operatorname{Supp} \tau \cap \operatorname{Per}(j)=\emptyset \text { for all } 2 n-1<j<10 n \tag{3.4}
\end{equation*}
$$

Once again, we let the machine $M_{n}^{\phi}$ compute the expected value of $\tau$ with precision $2^{-n} / 100$, giving it $a_{n-1}$ as the parameter. Let $l_{n}$ be the last time a bit of $a_{n-1}$ is queried by $M_{1}^{\phi}$ during the computation. By Proposition 3.4 again, for any $s \in\{0,1\}$ there exists $a^{\prime}$ such that

- $\left|a_{n-1}-a^{\prime}\right|<2^{-3 l_{n}}$;
- $a^{\prime} \in \tilde{\mathcal{P}}$;
- $m_{a_{n-1}}(t)=m_{a^{\prime}}(t)$ for all $t \notin\{2 n-1,2 n\}$ and
- $m_{a^{\prime}}(2 n-s)=2^{-n}$.

Let $q=M_{n}^{\phi}\left(\tau, 2^{-n} / 100\right)$. There are two possibilities:
Case 1. If $q \leq 2^{-n} / 2$, we chose $a^{\prime}$ above so as to have $m_{a^{\prime}}(2 n-1)=2^{-n}$;
Case 2. If $q>2^{-n} / 2$, we chose $a^{\prime}$ above so as to have $m_{a^{\prime}}(2 n)=2^{-n}$ (and therefore $\left.m_{a^{\prime}}(2 n-1)=0\right) ;$
We then let $a_{n} \equiv a^{\prime}$. Since $\tau$ satisfies property 3.4 , we have that $\mid \mu_{a_{n}}(\tau)-m_{a_{n}}(2 n-$ $1) \mid<2^{-10 n}$. Note that up to the first $l_{n}$ bits, $a_{n-1}$ and $a_{n}$ are indistinguishable, and thus the machine $M_{n}^{\phi}$ will return the same answer for both parameters. It follows that the machine $M_{n}^{\phi}$ cannot correctly approximate $\mu_{a_{n}}$ at $\tau$. Moreover, property 3.4 again and the fact that (by construction) $a_{n}$ satisfies

$$
m_{a_{n-1}}(t)=m_{a_{n}}(t) \text { for all } t \notin\{2 n-1,2 n\}
$$

guarantee that Condition (3) is satisfied as well. We now let $a_{\infty}=\lim _{n} a_{n}$ and claim that $\mu_{a_{\infty}}$ has the required properties. Indeed, Condition (2) ensures that for every $n$ there is a step function $\tau_{i(n)}$ at which machine $M_{n}^{\phi}$ fails to compute correctly the expected value for $\mu_{a_{n}}$, and Condition (3) guarantees that the same holds for $\mu_{a_{\infty}}$.

## 4. Conclusion

Ever since the first numerical studies of chaotic dynamics appeared in the early 1960's (such as the work of Lorenz [Lor63]), it has become commonly accepted among practitioners that computers cannot, in general, be used to make deterministic predictions about future behavior of nonlinear dynamical systems. Instead, the standard practice now is to make statistical predictions. This approach is based on the Monte Carlo method, pioneered by Ulam and von Neumann at the dawn of the computing age. It is universal and powerful - and only requires access to the dynamical system as a black box, which is then subjected to a number of statistical trials. Applications of the Monte Carlo technique are ubiquitous, ranging from weather forecasts to simulating nuclear weapons tests (nuclear weapons design was, of course, the original motivation of its inventors).

Our result raises a disturbing possibility that even for the most simple family of examples of non-linear dynamical systems the Monte Carlo method can fail. Given one of our examples as a black box, no algorithm can find its limiting statistics. How common such examples are in higher-dimensional families of dynamical systems, and whether one is likely to encounter one in practice remain exciting open questions.

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[^0]:    2010 Mathematics Subject Classification. 68Q17 and 37E05.
    Key words and phrases. Non-computability, unimodal maps, physical measures, Monte Carlo simulation.
    C.R was partially supported by the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie grant agreement No 731143 , by project FONDECYT Regular No 1190493 and project Basal PFB-03 CMM-Universidad de Chile. M.Y. was partially supported by NSERC Discovery grant.

[^1]:    ${ }^{1}$ Note that the set of physical measures constructed in Theorem 5 of [HK90] includes convex combinations of $\lambda_{a}(n)$. Compare also with Theorem 1 of [HK90]

[^2]:    ${ }^{2}$ Note that the mass of the higher periodic points that may fall in an open set containing $\operatorname{Per}_{a}(1)$ goes to zero as the diameter of the open set goes to zero.

