# Lower Bounds for QBFs of Bounded Treewidth 

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#### Abstract

The problem of deciding the validity (QSAT) of quantified Boolean formulas (QBF) is a vivid research area in both theory and practice. In the field of parameterized algorithmics, the well-studied graph measure treewidth turned out to be a successful parameter. A well-known result by Chen 10 is that QSAT when parameterized by the treewidth of the primal graph and the quantifier rank of the input formula is fixedparameter tractable. More precisely, the runtime of such an algorithm is polynomial in the formula size and exponential in the treewidth, where the exponential function in the treewidth is a tower, whose height is the quantifier rank. A natural question is whether one can significantly improve these results and decrease the tower while assuming the Exponential Time Hypothesis (ETH). In the last years, there has been a growing interest in the quest of establishing lower bounds under ETH, showing mostly problem-specific lower bounds up to the third level of the polynomial hierarchy. Still, an important question is to settle this as general as possible and to cover the whole polynomial hierarchy. In this work, we show lower bounds based on the ETH for arbitrary QBFs parameterized by treewidth and quantifier rank. More formally, we establish lower bounds for QSAT and treewidth, namely, that under ETH there cannot be an algorithm that solves QSAT of quantifier rank i in runtime significantly better than $i$-fold exponential in the treewidth and polynomial in the input size. In doing so, we provide a reduction technique to compress treewidth that encodes dynamic programming on arbitrary tree decompositions. Further, we describe a general methodology for a more fine-grained analysis of problems parameterized by treewidth that are at higher levels of the polynomial hierarchy. Finally, we illustrate the usefulness of our results by discussing various applications of our results to problems that are located higher on the polynomial hierarchy, in particular, various problems from the literature such as projected model counting problems.


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## 1 Introduction

Treewidth, which was introduced specifically for graph problems by Robertson and Seymour in a series of papers [55-59], is a popular parameter in the community of parameterized complexity [14, 17, 33] and according to Google Scholar mentioned in 20,000 results (queried on April 27, 2020). Treewidth is a combinatorial invariant that renders a large variety of NPcomplete or \#P-complete graph problems tractable [6, 11]. Among these problems are for example deciding whether a graph has a Hamiltonian cycle, whether a graph is 3 -colorable, or determining the number of perfect matchings of a graph [13]. Still, treewidth has also been widely employed for important applications that are defined on more general input structures such as Boolean satisfiability (SAt) [60] and constraint satisfaction (CSP) [15, 34]. Even problems that are located "beyond NP" such as probabilistic inference [51, problems in knowledge representation and reasoning [20, 37, 54] as well as deciding the validity (QSAT) of quantified Boolean formulas (QBF) can be turned tractable using treewidth; for QSAT we also parameterize by the number of alternating quantifier blocks (quantifier rank) [10. However, QSat remains intractable when parameterized by treewidth alone [1], which is established using a particular fragment of path decompositions for QBF. QSAT is also known as the prototypical problem for the polynomial hierarchy in descriptive complexity [38, 40]. Indeed, an encoding in QBF allows the characterization of problems on certain levels of the hierarchy using results by Fagin [26]. This has, for instance, been done for reasoning problems [21-23].

The meta results on treewidth are the well-known Courcelle's theorem [12] and its logspace version [24], which states that whenever one can encode a problem into a formula in monadic second order logic (MSO), then the problem can be decided in time linear in the input size and some function in the treewidth. While Courcelle's theorem provides a full framework for classifying problems concerning the existence of a tractable algorithm, its practical application is limited due to potentially huge constants, and the exponential runtime in the treewidth (upper bound) may result in a tower of exponents that is far from optimal. In contrast, the available upper bounds are more immediate for QSat: Chen 10 showed that one can decide validity for a given QBF in time exponential in the treewidth where the treewidth is on top of a tower Pf iterated exponentials of height that equals the quantifier rank in the formula. Since the quantifier rank required to encode a problem directly matches the level on which the problem is located in the polynomial hierarchy, reductions to QSAt seem natural. Lampis, Mitsou, and Mengel 47] employed this fact and proposed reductions from a collection of reasoning problems in AI to QSAT that yield quite precise (up to a constant factor) upper bounds on the runtime. In consequence, these results highlight QBF encodings as a very handy and precise alternative to Courcelle's theorem. A natural question is whether one can significantly improve existing algorithms or establish limits that, unless very bad things happen in computational complexity theory, an algorithm with a certain runtime cannot exist. Lampis, Mitsou, and Mengel also consider this question using results [46] for QBF of quantifier rank two (2-QSat). While these results for the second level are applicable to numerous important problems, there is also a plethora of interesting problems that are even harder, which underlines the need for further research in this direction.

In this paper, we address lower bounds for the runtime of an algorithm that exploits treewidth in a more general setting. We establish results for QBFs of bounded treewidth and of arbitrary quantifier rank, thereby providing a novel method to generalize the result for 2-QSAT in a non-incremental way.

[^1]A way to establish tight lower bounds in parameterized complexity theory is to assume the exponential time hypothesis (ETH) [41] and construct reductions. ETH is a widely accepted standard hypothesis in the fields of exact and parameterized algorithms. ETH states that there is some real $s>0$ such that we cannot decide satisfiability of a given 3-CNF formula $F$ in time $2^{s \cdot|F|} \cdot\|F\|^{\mathcal{O}(1)}[14$, Ch.14], where $|F|$ refers to the number of variables and $\|F\|$ to the size of $F$, which is number of variables plus number of clauses in $F$. Recently, Lampis and Mitsou [46] established that 2-QSat ( $\exists \forall$-SAT and $\forall \exists$-SAT) cannot be solved by an algorithm that runs in time single exponential in the treewidth of the primal graph (primal treewidth) when assuming ETH. The primal graph of a QBF $Q$ has as vertices the variables of $Q$ and there is an edge between two variables if they occur together in a clause or term, respectively. Pan and Vardi [52] mention in an earlier work that this extends to 3 -QSAT $(\forall \exists \forall$-SAT and $\exists \forall \exists$-SAT), and $\ell$-QSAT, if $\ell$ is an odd number. But it does not extend constructively to the case, where $\ell$ is even. Therefore, a new approach is needed to show the complete picture for QSat. While Marx and Mitsou [50] considered certain graph problems that are located on the third level of the polynomial hierarchy 62], they emphasize that the classical complexity results do not provide sufficient explanation why double- or triple-exponential dependence on treewidth is needed and one requires quite involved proofs for each problem separately. However, they state that intuitively the quantifier rank of the problem definitions are the common underlying reason for being on higher levels of the polynomial hierarchy and for requiring high dependence on treewidth. A natural generalization of the statement to arbitrary QBFs is formally stated in the following hypothesis.

Claim 1. Under ETH, QSAT for a closed formula $Q$ in prenex normal form with $n$ variables, primal treewidth $k$, and quantifier rank $\ell$ cannot be decided in time tow $(\ell, o(k)) \cdot \operatorname{poly}(n)$.

Contributions. In this paper, we prove Claim 1, which strengthens the importance of QBF encodings for problems parameterized by treewidth, and establish a general methodology to obtain treewidth lower bounds for problems of the polynomial hierarchy. Our contributions are as follows:

1. We consider ETH and QSAT and establish the full picture of runtime lower bounds for algorithms parameterized by treewidth in connection to the quantifier rank of the formula.

We present a reduction that significantly compresses treewidth and applies to any instance of QSAT without restricting the quantifier rank while only assuming ETH. Note that this "compression" is constructive and independent of the original instance size, which is different from existing methods, e.g., [46, 50, 52]. In fact, compression only depends on the original parameter treewidth.
2. We provide a novel methodology for a more fine-grained analysis of algorithms parameterized by treewidth. This methodology relies only on the ETH and allows:
(a) for simply using reductions from QBF to exclude runtime results (height of the tower) for treewidth and
(b) for directly concluding lower bounds for projected model counting problems (PQSAT), that is, $\# \Sigma_{\ell}$ SAT and $\# \Pi_{\ell}$ SAT [19], which serve as canonical problems in counting complexity, as well as for various other problems.
Instead of establishing problem-specific reductions from SAT for problems higher in the polynomial hierarchy, e.g., [46, 50], our reduction and methodology are very general. We illustrate its applicability in the showcases in Section 3.2.

Novel Techniques. We constructively encode core ideas of a dynamic programming algorithm on tree decompositions into a QBF that expresses solving an instance of QSAT by means of a QSAT oracle of one level higher in the hierarchy (self-reduction) while achieving a certain compression of treewidth. More precisely, we provide a reduction that reduces any instance $Q$ of QSAT of treewidth $t$ and quantifier rank $\ell$ into an instance $Q^{\prime}$ of QSAT of treewidth $\mathcal{O}(\log t)$ and quantifier rank $\ell+1$, while the size of $Q^{\prime}$ is linearly bounded in the size of $Q$. Notice that the treewidth of $Q^{\prime}$ only depends on the treewidth of $Q$, but is independent of, e.g., the number of variables and quantifier rank of $Q$. Hence, treewidth of $Q^{\prime}$ is compressed compared to the original treewidth of $Q$. Atserias and Oliva [1] cover a related setting: compressing pathwidth ${ }^{2}$ for a fragment of path decompositions of QBFs thereby increasing the quantifier rank by two. However, we require a general, constructive method to compress the width of arbitrary tree decompositions of any QBF, thereby increasing quantifier rank by only one, and improve their result (Corollary 29).

Our reduction is novel in the following sense:

1. We use a given tree decomposition to guide the evaluation of the considered formula, which allows us to decouple the variables sufficiently to decrease treewidth and we thereby achieve exponential compression of the parameter treewidth. By construction of the reduction, the lower bound results carry over to the larger parameter pathwidth ${ }^{22}$ and even 2-local pathwidth, where each variable occurs at most twice. Note that this direction is by construction and does not hold in general. However, particular novelty lies in encoding essentials of dynamic programming, which will be presented in the more general context of treewidth (tree decompositions).
2. In the proof, we use a reduction approach that balances redundancy and structural dependency (captured by treewidth or pathwidth), which allows us to apply this method to QBFs of arbitrary quantifier rank, thereby increasing quantifier rank by only one.
3. Our approach might help to improve solvers utilizing treewidth, as instances of huge treewidth might become solvable in practice (cf., [9, 32]) after applying our reduction. Indeed, our reduction encodes dynamic programming on tree decompositions into a Boolean formula, namely, guessing of finite states for table entries of decomposition nodes, checking whether certain entries sustain, and propagating entries among different nodes. As this technique, although presented for QBFs, does not explicitly encode quantifier dependencies into the Boolean formula, the technique is hopefully of general use.

Connection to kernels. Note that our approach is orthogonal to kernelization as kernelizations tackle bounds of the instance size by the considered parameter, whereas here we target reducing the parameter itself and not the size of the input instance.

## 2 Preliminaries

Basics. For a set $X$, let $2^{X}$ be the power set of $X$. The function tow $(\ell, k)$ is defined as in Footnote 1. The domain $\mathcal{D}$ of a function $f: \mathcal{D} \rightarrow \mathcal{A}$ is given by $\operatorname{dom}(f)$. By $f^{-1}: \mathcal{A} \rightarrow \mathcal{D}$ we denote the inverse function $f^{-1}:=\{f(d) \mapsto d \mid d \in \operatorname{dom}(f)\}$ of a given function $f$, if it exists. To permit operations such as $f \cup g$ for functions $f$ and $g$, the functions may be viewed as relations. We use the symbol "." as placeholder for a value of an argument, which is clear from the context and the actual value is negligible. We let $\mathbb{N}$ contain all positive

[^2]integers and $\mathbb{N}_{0}$ all non-negative integers. Throughout this paper, we refer by $\log (\cdot)$ to the binary logarithm.
Computational Complexity. We assume familiarity with standard notions in computational computational complexity [53, counting complexity classes [19, and parameterized complexity [14, 17, 33]. We recall some basic notions. Let $\Sigma$ and $\Sigma^{\prime}$ be some finite alphabets. We call $I \in \Sigma^{*}$ an instance and $\|I\|$ denotes the size of $I$. Let $L \subseteq \Sigma^{*} \times \mathbb{N}$ and $L^{\prime} \subseteq \Sigma^{\prime *} \times \mathbb{N}$ be two parameterized problems. An fpt-reduction $r$ using $g$ from $L$ to $L^{\prime}$ is a many-to-one reduction from $\Sigma^{*} \times \mathbb{N}$ to $\Sigma^{\prime *} \times \mathbb{N}$ such that for all $I \in \Sigma^{*}$ we have $(I, k) \in L$ if and only if $r(I, k)=\left(I^{\prime}, k^{\prime}\right) \in L^{\prime}$ with $k^{\prime} \leq g(k)$, where $f, g: \mathbb{N} \rightarrow \mathbb{N}$ are fixed computable functions such that $r$ is computable in time $f(k) \cdot \operatorname{poly}(\|I\|)$. We call $r$ also an $f$-bounded fpt-reduction using $g$ for given $f$ and $g$.
Quantified Boolean Formulas (QBFs). We define Boolean formulas and their evaluation in the usual way and literals are variables or their negations. For a Boolean formula $F$, we denote by $\operatorname{var}(F)$ the set of variables of $F$. Logical operators $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ are used in the usual meaning. A term is a conjunction of literals and a clause is a disjunction of literals. $F$ is in conjunctive normal form (CNF) if $F$ is a conjunction of clauses and $F$ is in disjunctive normal form ( $D N F$ ) if $F$ is a disjunction of terms. In both cases, we identify $F$ by its set of clauses or terms, respectively. From now on assume that a Boolean formula is either in CNF or DNF. A formula is in $c-C N F$ or $c-D N F$ if each set in $F$ consists of at most $c$ many literals. Let $\ell \geq 0$ be integer. A quantified Boolean formula $Q$ (in prenex normal form) is of the form $Q_{1} V_{1} \cdot Q_{2} V_{2} \cdots Q_{\ell} V_{\ell} . F$ where $Q_{i} \in\{\forall, \exists\}$ for $1 \leq i \leq \ell$ and $Q_{j} \neq Q_{j+1}$ for $1 \leq j \leq \ell-1$; and where $V_{i}$ are disjoint, non-empty sets of Boolean variables with $\bigcup_{i=1}^{\ell} V_{i} \subseteq \operatorname{var}(F)$; and $F$ is a Boolean formula. We call $\ell$ the quantifier rank of $Q$ and let matrix $(Q):=F$. Further, we denote the set $\operatorname{fvar}(Q)$ of free variables of $Q$ by $\operatorname{fvar}(Q):=\operatorname{var}($ matrix $(Q)) \backslash\left(\bigcup_{i=1}^{\ell} V_{i}\right)$. If $\operatorname{fvar}(Q)=\emptyset$, then $Q$ is referred to as closed, otherwise we say $Q$ is open. Unless stated otherwise, we assume open QBFs. The truth (evaluation) of QBFs is defined in the standard way. An assignment is a mapping $\iota: X \rightarrow\{0,1\}$ defined for a set $X$ of variables. An assignment $\iota^{\prime}$ extends $\iota\left(\operatorname{by} \operatorname{dom}\left(\iota^{\prime}\right) \backslash \operatorname{dom}(\iota)\right)$ if $\operatorname{dom}\left(\iota^{\prime}\right) \supseteq \operatorname{dom}(\iota)$ and $\iota^{\prime}(y)=\iota(y)$ for any $y \in \operatorname{dom}(\iota)$. Given a Boolean formula $F$ and an assignment $\iota$ for $\operatorname{var}(F)$. Then, for $F$ in CNF, $F[\iota]$ is a Boolean formula obtained by removing every $c \in F$ with $x \in c$ and $\neg x \in c$ if $\iota(x)=1$ and $\iota(x)=0$, respectively, and by removing from every remaining clause $c \in F$ literals $x$ and $\neg x$ with $\iota(x)=0$ and $\iota(x)=1$, respectively. Analogously, for $F$ in DNF values 0 and 1 are swapped. For a given $\operatorname{QBF} Q$ and an assignment $\iota, Q[\iota]$ is a QBF obtained from $Q$, where variables $x \in \operatorname{dom}(\iota)$ are removed from preceding quantifiers accordingly, and matrix $(Q[\iota]):=(\operatorname{matrix}(Q))[\iota]$. A Boolean formula $F$ evaluates to true if there exists an assignment $\iota$ for $\operatorname{var}(F)$ such that $F[\iota]=\emptyset$ if $F$ is in CNF or $F[\iota]=\{\emptyset\}$ if $F$ is in DNF. A closed QBF $Q$ evaluates to true (or is valid) if $\ell=0$ and the Boolean formula matrix $(Q)$ evaluates to true. Otherwise, i.e., if $\ell \neq 0$, we distinguish according to $Q_{1}$. If $Q_{1}=\exists$, then $Q$ evaluates to true if and only if there exists an assignment $\iota: V_{1} \rightarrow\{0,1\}$ such that $Q[\iota]$ evaluates to true. If $Q_{1}=\forall$, then $Q[\iota]$ evaluates to true if for any assignment $\iota: V_{1} \rightarrow\{0,1\}$, $Q[\iota]$ evaluates to true. An (open or closed) QBF $Q$ is satisfiable if there is a truth assignment $\iota: \operatorname{fvar}(Q) \rightarrow\{0,1\}$ such that resulting closed QBF $Q[\iota]$ evaluates to true. Otherwise $Q$ is unsatisfiable. Given a closed QBF $Q$, the evaluation problem QSAT of QBFs asks whether $Q$ evaluates to true; $\ell$-QSAT refers to the problem QSAT on QBFs of quantifier rank $\ell$. The problem QSAT is PSPACE-complete and is therefore believed to be computationally harder than Sat [44, 53, 62]. For more details on QBFs we refer to [3, 44].

The projected model counting problem PQSat takes an open QBF $Q$ and asks to output the number of distinct assignments $\iota: \operatorname{fvar}(Q) \rightarrow\{0,1\}$ such that $Q[\iota]$ evaluates to true.

Example 2. Consider the closed $Q B F Q=\exists w, x . \forall y, z . D$, where $D:=d_{1} \vee d_{2} \vee d_{3} \vee d_{4}$,


Figure 1: Primal graph $P_{Q}$ of $Q$ from Example 2 (left) with TDs $\mathcal{T}_{1}, \mathcal{T}_{2}$ of graph $P_{Q}$ (right).
and $d_{1}:=w \wedge x \wedge \neg y, d_{2}:=\neg w \wedge \neg x \wedge y, d_{3}:=w \wedge y \wedge \neg z$, and $d_{4}:=w \wedge y \wedge z$. Observe that $Q[\iota]$ is valid under assignment $\iota=\{w \mapsto 1, x \mapsto 1\}$. In particular, $Q[\iota]$ can be simplified to $\forall y, z .(\neg y) \vee(y \wedge \neg z) \vee(y \wedge z)$, which is valid, since for any assignment $\kappa:\{y, z\} \rightarrow\{0,1\}$ the formula $Q[\iota][\kappa]$ (and therefore $Q$ ) evaluates to true.

Tree Decompositions (TDs). For basic terminology on graphs and digraphs, we refer to standard texts [7, 16]. For an arborescence $T=(N, A, r)$, which is a directed, rooted tree with root $r$ and a node $t \in N$, we let $\operatorname{cld}(t, T)$ be the set of all child nodes $t^{\prime}$, which have an outgoing edge $\left(t, t^{\prime}\right) \in A$ from $t$ to $t^{\prime}$. Let $G=(V, E)$ be a graph. A tree decomposition (TD) of graph $G$ is a pair $\mathcal{T}=(T, \chi)$ where $T=(N, A, r)$ is an arborescence with root $r \in N$, and $\chi$ is a mapping that assigns to each node $t \in N$ a set $\chi(t) \subseteq V$, called a bag, such that the following conditions hold: (i) $V=\bigcup_{t \in N} \chi(t)$ and $E \subseteq \bigcup_{t \in N}\{\{u, v\} \mid u, v \in \chi(t)\}$; and (ii) for each $q, s, t$, such that $s$ lies on the path from $q$ to $t$, we have $\chi(q) \cap \chi(t) \subseteq \chi(s)$. Then, width $(\mathcal{T}):=\max _{t \in N}|\chi(t)|-1$. The treewidth $\operatorname{tw}(G)$ of $G$ is the minimum width $(\mathcal{T})$ over all tree decompositions $\mathcal{T}$ of $G$. For arbitrary but fixed $w \geq 1$, it is feasible in linear time to decide if a graph has treewidth at most $w$ and, if so, to compute a tree decomposition of width $w$ [5]. Further, we call a tree decomposition $\mathcal{T}=(T, \chi)$ a path decomposition (PD) if $T=(N, \cdot, r)$ and $|\operatorname{cld}(t)| \leq 1$ for each node $t \in N$. Analogously, we define pathwidth $\mathrm{pw}(G)$ as the minimum $\operatorname{width}(\mathcal{T})$ over all path decompositions of $G$. Similarly, for $m \geq 2$, let $m$-local pathwidth of $G$ refer to the pathwidth over all path decompositions of $G$, where each vertex in $V$ occurs in at most $m$ bags. For a given tree decomposition $\mathcal{T}=(T, \chi)$ with $T=(N, A, r)$, and an element $x \in \bigcup_{t \in N} \chi(t)$, we denote by $\mathcal{T}[x]$ the result $\mathcal{T}^{\prime}$ of restricting $\mathcal{T}$ to nodes, whose bags contain $x$. Formally, $\mathcal{T}^{\prime}:=\left(T^{\prime}, \chi^{\prime}\right)$, where $T^{\prime}:=\left(N^{\prime}, A^{\prime}, r^{\prime}\right), N^{\prime}:=\{t \mid t \in N, x \in \chi(t)\}, A^{\prime}:=A \cap\left(N^{\prime} \times N^{\prime}\right)$, and for each $t \in N^{\prime}, \chi^{\prime}(t)=\chi(t)$. Finally, $r^{\prime} \in N^{\prime}$ is the first node reachable from $r$. The literature distinguishes so-called nice tree decompositions, which can be computed in linear time without increasing the width [45]. For our purposes, the following relaxed variant of almost nice tree decompositions suffices.

Definition 3. Given an integer $c \in \mathbb{N}$. A tree decomposition $\mathcal{T}=(T, \chi)$, where $T=(N, \cdot, r)$, is called almost $c$-nice, if for each node $t \in N$ with $\operatorname{cld}(t)=\left\{t_{1}, \ldots, t_{s}\right\}$, the following conditions are true (i) $s \leq 2$ and (ii) $\left|\chi(t) \backslash \bigcup_{i=1}^{i=s} \chi\left(t_{i}\right)\right| \leq c$.

In order to use tree decompositions for QBFs, we need a graph representation of Boolean formulas [60]. The primal graph $P_{F}$ of a Boolean formula $F$ in CNF or DNF has the variables $\operatorname{var}(F)$ of $F$ as vertices and an edge $\{x, y\}$ if there exists a term or clause $f \in F$ such that $x, y \in \operatorname{var}(f)$, respectively. For a QBF $Q$, we identify its primal graph with the primal graph of its matrix, i.e., let $P_{Q}:=P_{\text {matrix }(Q)}$.

Example 4. Figure 1 illustrates the primal graph $P_{Q}$ of the $Q B F$ from Example 2 and two tree decompositions of $P_{Q}$ of width 2. The graph $P_{Q}$ has treewidth 2 , since the vertices $w, x, y$ are completely connected and hence width 2 is optimal 45].

Definition 5. Let $\mathcal{T}=(T, \chi)$ be a tree decomposition of a graph $G$. A labeled tree decomposition (LTD) $\mathcal{T}$ of a Boolean formula $F$ in $C N F$ or DNF is a tuple $\mathcal{T}=(T, \chi, \delta)$


Figure 2: Simplified illustration of a certain tree decomposition $\mathcal{T}^{\prime}=\left(T, \chi^{\prime}\right)$ of $P_{R(Q)}$ (yielded ${ }^{3}$ by reduction $R$ ), and its relation to tree decomposition $\mathcal{T}=(T, \chi)$ of $P_{Q}$. Each bag $\chi^{\prime}\left(t_{i}\right)$ of a node $t_{i}$ of $\mathcal{T}^{\prime}$ contains variable $x_{i}$ for any variable $x$ introduced in $\chi\left(t_{i}\right)$ and $\lceil\log (\operatorname{width}(\mathcal{T}))\rceil$ many (green) pointer variables $p_{i}$ selecting one variable in $\chi\left(t_{i}\right)$ of $\mathcal{T}$. Squiggly red arrows indicate the propagation between pointers $p_{i}, p_{j}$ and ensure consistency. In particular, although truth values for variable $a$ are "guessed" using $a_{1}$ and $a_{4}$ (and "propagated" via blue squiggly arrows to corresponding pointers $p_{1}$ and $p_{4}$, respectively), these red arrows ensure via pointers $p_{1}, p_{3}, p_{4}, p_{5}$ that truth values for $a_{1}$ and $a_{4}$ coincide.
where $(T, \chi)$ is a tree decomposition of $P_{F}$ with $T=(N, \cdot, r)$, and $\delta: N^{\prime} \rightarrow F$ with $N^{\prime} \subseteq N$ is a bijective mapping from TD nodes to clauses or terms in $F$ such that for every $t \in N^{\prime}$, $\operatorname{var}(\delta(t)) \subseteq \chi(t)$.

Observation 6. Given an almost c-nice tree decomposition $\mathcal{T}$ of a primal graph $P_{F}$ for a given 3-CNF or 3-DNF formula $F$ of width $k$. Then, one can easily create a labeled almost c-nice tree decomposition $\mathcal{T}^{\prime}$ of $P_{F}$ with a linear number of nodes in the number of nodes in $\mathcal{T}$ such that $\operatorname{width}\left(\mathcal{T}^{\prime}\right)=\operatorname{width}(\mathcal{T})$.

Example 7. Consider again Figure 1 (right). Observe that $\mathcal{T}_{2}$ is a labeled almost 3-nice tree decomposition of $P_{Q}$, where labeling function $\delta$ sets $\delta\left(t_{i}\right)=d_{i}$, for $1 \leq i \leq 4$.

## 3 Decomposition-Guided Compression

Next, we present our approach to transform a given input instance of treewidth $k$ into an instance of exponentially smaller treewidth ("compression") compared to the original treewidth $k$. Thereby, we trade the compression of the parameter for the cost of additional computation power required to solve the compressed instance. For the canonical QSat problem, we require an increased quantifier rank.

First, we introduce the reduction $R$ that takes an instance $Q$ of $\ell$-QSAT, and computes a corresponding tree decomposition $\mathcal{T}$ of the primal graph $P_{Q}$, where width $(\mathcal{T})=\operatorname{tw}\left(P_{Q}\right)$. Then, it returns a compressed instance $R(Q)$ of $(\ell+1)$-QSAT of treewidth $\mathcal{O}(\log ($ width $(\mathcal{T})))$. The reduction $R$, which is guided by TD $\mathcal{T}$, yield $\$^{3}$ a new compressed tree decomposition $\mathcal{T}^{\prime}$ of $P_{R(Q)}$ of width $\mathcal{O}(\log (\operatorname{width}(\mathcal{T})))$. For that it is crucial to balance introducing copies of variables (redundancy) and saving treewidth (structural dependency), such that, intuitively, we can still evaluate $R(Q)$ given the limitation of treewidth $\mathcal{O}(\log (\boldsymbol{w i d t h}(\mathcal{T})))$. To keep this balance, we can only analyze in a bag in $\mathcal{T}^{\prime}$ a constant number of elements of the corresponding original bag of $\mathcal{T}$. Still, considering $\log (\operatorname{width}(\mathcal{T}))$ many elements in a bag at once allows us to represent one "pointer" to address at most width $(\mathcal{T})$ many elements of each bag of $\mathcal{T}$ and, consequently, the restriction to $\mathcal{O}(\log (\operatorname{width}(\mathcal{T})))$ many elements in a bag at once enables constantly many such pointers. To give a first glance at the idea

[^3]of the reduction $R$, Figure 2 provides an intuition and illustrates a tree decomposition $\mathcal{T}$ of $P_{Q}$ together with a corresponding compressed tree decomposition $\mathcal{T}^{\prime}$ of $P_{R(Q)}$, whose bags contain pointers to original bags of $\mathcal{T}$. Actually, we can encode the propagation of information from one bag of $\mathcal{T}^{\prime}$ to its parent bag with the help of these pointers. Thereby, we ensure that information is consistent and this consistency can be preserved, even though we guess in $R$ truth values for copies of the same variable in $Q$ independently. Note that these "local" pointers for each bag are essential to achieve treewidth compression.

Below, we discuss the reduction $R$ in more detail. Then, Section 3.2 provides a description of a general methodology for establishing lower bounds for problems parameterized by treewidth. In more detail, equipped with our lower bound results for QSAt, we propose reductions from QSAT as a general toolkit for proving lower bounds assuming ETH. Further, we discuss several showcases to illustrate this methodology.

### 3.1 The Reduction

The formula $R(Q)$ constructed by $R$ mainly consists of three interacting parts. In the presentation, we refer to them as guess, check, and propagate part.

- "Guess" $(\mathcal{G})$ : Contains clauses responsible for guessing truth values of variables occurring in the original $\mathrm{QBF} Q$.
- "Check" $(\mathcal{C K})$ : These clauses ensure that there is at least one 3-DNF term in $Q$ that is satisfied, thereby maintaining 3 pointers for each node as discussed above.
- "Propagate" $(\mathcal{P})$ : These clauses ensure consistency using a pointer for each node of the tree decomposition.

We commence with the formal description of $R$. Given a $\mathrm{QBF} Q$ of the form $Q:=$ $Q_{1} V_{1} \cdot Q_{2} V_{2} \cdots \forall V_{\ell} . D$, where $D$ is in 3 -DNF such that the quantifier blocks are alternating, i.e., quantifiers of quantifier blocks with even indices are equal, which are different from those of blocks with odd indices. Further, assume a labeled almost $c$-nice tree decomposition $\mathcal{T}=(T, \chi, \delta)$, where $T=(N, \cdot \cdot \cdot)$ of the primal graph $P_{Q}$ of $D$, which always exists by Observation 6. Notice that by Definition 5 for all terms $d \in D$, the inverse function $\delta^{-1}(d)$ is well-defined. Further, actually $R$ can deal with open QBFs, i.e, QBF $Q$ does not necessarily have to be closed. Open formulas are needed later to simplify the correctness proof of Section 4.

We use the following sets of variables. Let $\operatorname{NodeI}(x):=\{t \mid t \in N, x \in \chi(t) \backslash$ $\left.\left(\cup_{t_{i} \in \operatorname{cld}(t)} \chi\left(t_{i}\right)\right)\right\}$ be the set of nodes, where a given element $x$ is introduced. For a set $V \subseteq$ $\operatorname{var}(D)$ of variables, we denote by $\operatorname{VarI}(V):=\left\{x_{t} \mid x \in V, t \in \operatorname{NodeI}(x)\right\}$ the set of fresh variables generated for each original variable $x$ and node $t$, where $x$ is introduced. Later, we need to distinguish whether the set $V_{i}$ of variables is universally or existentially quantified. Universal quantification requires to shift for each $x \in V_{i}$ all but one representative of $\left\{x_{t} \mid t \in \operatorname{NodeI}(x)\right\}$ to the next existential quantifier block $Q_{i+1}$. The representative variable that is not shifted is denoted by rep $(x)$. In particular, given a quantifier block $Q_{2}$, its variables $V_{2}$ and the variables $V_{1}$ of the preceding quantifier block, we define: $\operatorname{VarI}\left(Q_{2}, V_{2}, V_{1}\right):=\left\{x_{t} \mid x \in V_{2}, t \in \operatorname{NodeI}(x), Q_{2}=\exists\right\} \cup\left\{x_{t} \mid x \in V_{2}, t \in\right.$ $\left.\operatorname{NodeI}(x), x_{t}=\operatorname{rep}(x), Q_{2}=\forall\right\} \cup\left\{x_{t} \mid x \in V_{1}, t \in \operatorname{NodeI}(x), x_{t} \neq \operatorname{rep}(x), Q_{2}=\exists\right\}$. We denote by VarSat $:=\left\{s a t_{t}\right.$, sat $\left.{ }_{\leq t} \mid t \in N\right\}$ the set of fresh decision variables responsible for storing for each node $t \in N$ whether any term at $t$ or at any node below $t$ is satisfied, respectively. Finally, we denote by $\operatorname{Var} B:=\left\{b_{t}^{0}, \ldots, b_{t}^{[\log (|\chi(t)|)]-1} \mid t \in N\right\}$, and $\operatorname{Var} B V:=\left\{v_{t} \mid t \in N\right\}$ the set of fresh variables for each node $t \in N$ that will be used to address particular elements of the corresponding bags (pointer as depicted in Figure 2 in binary representation), and to assign truth values for these elements, respectively.

Overall, the variables in $\operatorname{VarB}$ allow us to guide the evaluation of formula $D$ along the tree decomposition $\mathcal{T}$. For checking 3-DNF terms, we need the same functionality three more times, resulting in the sets VarB3 $:=\left\{b_{t, j}^{0}, \ldots, b_{t, j}^{\log (|\chi(t)|+1)\rceil-1} \mid t \in N, 1 \leq j \leq 3\right\}$ that additionally may refer to a special fresh element nil (therefore the +1 in the exponent in definition of VarB3), and VarBV3 $:=\left\{v_{t, j} \mid t \in N, 1 \leq j \leq 3\right\}$ of fresh variables. Notice that the construction is designed in such a way that the focus lies only on certain elements of the bag (one at a time, and independent of other elements within the same bag). In the end, this ensures that the treewidth of our reduced instance is only logarithmic in the original treewidth of the primal graph of $D$. Reduction $R(Q)$ creates $Q^{\prime}:=$

$$
\begin{aligned}
& Q_{1} \operatorname{VarI}\left(Q_{1}, V_{1}, \emptyset\right) \cdot Q_{2} \operatorname{VarI}\left(Q_{2}, V_{2}, V_{1}\right) \cdots \forall \operatorname{VarI}\left(\forall, V_{\ell}, V_{\ell-1}\right), \operatorname{Var} B . \\
& \exists \operatorname{VarI}\left(\exists, \emptyset, V_{\ell}\right), \operatorname{Var} B V, \operatorname{VarBV3}, \operatorname{VarB3}, \operatorname{VarSat} . C,
\end{aligned}
$$

where $C$ is a CNF formula consisting of guess, check and propagate parts, i.e., sets $\mathcal{G}, \mathcal{C K}$, and $\mathcal{P}$ of clauses, respectively.

Example 8. Consider again $Q$ from Example Q, The resulting instance $R(Q)$ looks as follows assuming that $\operatorname{rep}(y)=y_{t_{1}}$, where $C$ consists of a guess, check and, propagate part.

$$
\begin{aligned}
& \underbrace{v_{t_{1}}, \ldots, v_{t_{5}},}_{\text {VarBV }} \underbrace{v_{t_{1}, 1}, v_{t_{1}, 2}, v_{t_{1}, 3}, v_{t_{2}, 1, \ldots, v_{t_{5}, 3}}}_{\text {VarBV }} \\
& \underbrace{b_{1,1}^{0}, b_{t_{1}, 1}^{1}, b_{t_{1}, 2}^{0}, \ldots, b_{t_{5}, 3}^{1},}_{\text {VarB3 }} \underbrace{s a t_{t_{1}}, \ldots, s a t_{t_{5}}, s a t_{\leq t_{1}}, \ldots, s a t_{\leq t_{5}}}_{\text {VarSat }} . C
\end{aligned}
$$

In the following, we define sets $\mathcal{G}, \mathcal{C} \mathcal{K}$, and $\mathcal{P}$ of clauses. To this end, we require for the pointers a bit-vector (binary) representation of the elements in a bag of $\mathcal{T}$, and a mapping that assigns bag elements to its corresponding binary representation. In particular, we assume an arbitrary, but fixed total order $\prec$ of elements of a bag $\chi(t)$ of any given node $t \in N$. With $\prec$, we can then assign each element $x$ in $\chi(t)$ its unique (within the bag) induced ordinal number $o(t, x)$. This ordinal number $o(t, x)$ is expressed in binary. For that we need precisely $\lceil\log (|\chi(t)|)\rceil$ many bit-variables $B:=\left\{b_{t}^{0}, \ldots, b_{t}^{[\log (|\chi(t)|)\rceil-1}\right\}$. We denote by $\llbracket x \rrbracket_{t}$ the (consistent) set of literals over variables in $B$ that encode (in binary) the ordinal number $o(t, x)$ of $x \in \chi(t)$ in $t$, such that whenever a literal $b_{t}^{i}$ or $\neg b_{t}^{i}$ is contained in the set $\llbracket x \rrbracket_{t}$, the $i$-th bit in the unique binary representation of $o(t, x)$ is 1 or 0 , respectively. Analogously, for $1 \leq j \leq 3$ we denote by $\llbracket x \rrbracket_{t, j}$ the (consistent) set of literals over variables in $B_{j}:=\left\{b_{t, j}^{0}, \ldots, b_{t, j}^{[\log (|\chi(t)|+1)]-1}\right\}$ that either binary-encode the ordinal number $o(t, x)$ of $x \in \chi(t)$ in $t$, or these literals binary-encode number $\max _{y \in \chi(t)} o(t, y)+1$ for $x=$ nil.
The guess part $\mathcal{G}$. The clauses in $\mathcal{G}$, which we denote as implications, are defined as follows.

$$
\begin{array}{ll}
x_{t} \wedge \bigwedge_{b \in \llbracket x]_{t}} b \longrightarrow v_{t} & \text { for each } x_{t} \in \operatorname{VarI}(\operatorname{var}(D)) \\
\neg x_{t} \wedge \bigwedge_{b \in \llbracket x]_{t}} b \longrightarrow \neg v_{t} & \text { for each } x_{t} \in \operatorname{VarI}(\operatorname{var}(D)) \tag{2}
\end{array}
$$

Intuitively, this establishes that whenever a certain variable $x_{t}$ for an introduced variable $x \in$ $\chi(t)$ is assigned to true (false) and all the corresponding literals in $\llbracket x \rrbracket_{t}$ of the binary representation of $o(t, x)$ are satisfied (i.e, $x$ is "selected"), then also $v_{t} \in \operatorname{VarBV}$ of node $t$ has to be set to true (false).

Analogously, set $\mathcal{G}$ further contains the following clauses:

$$
\begin{array}{cl}
x_{t} \wedge \bigwedge_{b \in \llbracket x \rrbracket_{t, j}} b \longrightarrow v_{t, j} & \text { for each } x_{t} \in \operatorname{VarI}(\operatorname{var}(D)), 1 \leq j \leq 3 \\
\neg x_{t} \wedge \bigwedge_{b \in \llbracket x \rrbracket_{t, j}} b \longrightarrow \neg v_{t, j} & \text { for each } x_{t} \in \operatorname{VarI}(\operatorname{var}(D)), 1 \leq j \leq 3 \tag{4}
\end{array}
$$

Example 9．Consider formula $C$ from Example 8．Let $1 \leq j \leq 3$ ．Further，assume the following mapping of bag contents to bit－vector assignments．For any variable $a \in \operatorname{var}(D)$ with $t \in \operatorname{NodeI}(a)$ and for $a=$ nil with $t \in N$ ，we arbitrarily fix the total ordering $\prec$ and have $\llbracket a \rrbracket_{t}$ and $\llbracket a \rrbracket_{t, j}$ as follows．

| $a$ |  | ， |  | Th | $\left\{\left\|\llbracket a \rrbracket_{t}\right\| \quad \llbracket a \rrbracket_{t, j}^{\imath}\right.$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | 【a】 $\rrbracket_{t}$ | 【a］ |  | 【a $]_{t, j}$ |  |
|  | $\left\{b_{t}^{0}, \neg b_{t}^{1}\right\}$ | $\left\{\neg b_{t, j}^{0}, \neg b_{1}\right.$ |  | \} |  |
|  | $\left\{\neg b_{t}, b_{t}^{t}\right\}$ | $\left\{\neg b_{t, j}^{0}, b_{t, j}^{1}\right\}$ |  |  |  |
| ${ }^{y}$ | $\left\{b_{t}^{0}, \neg b_{t}^{1}\right\}$ | $\left\{b_{t, j}^{0}, \neg b_{t, j}^{1, j}\right\}$ |  | $\left.\left\lvert\, \begin{array}{l}\left\{\rightarrow b_{t, j}^{0}, b_{t, j}^{1}\right\} \\ \left\{b_{0}^{0}, b^{1} b^{1}\right.\end{array}\right.\right\}$ | $\left\{b_{t}^{0}\right\} \mid\left\{\neg b_{t, j}^{0}, b_{t, j}^{1}\right\}$ |
| nil ${ }^{z}$ | － | $\left\{b_{t, j}^{0}\right.$ | $\left\{b_{t}^{o}, \neg b_{t}^{1}\right\}$ | $\left.\left\lvert\, \begin{array}{l}\left\{b_{t,}^{0}, \neg b_{t, j}^{0}\right\} \\ \left\{b_{t, j}^{0}, b_{t, j}^{t}\right\}\end{array}\right.\right\}$ | $\left.b_{t, j}^{0}, \neg b_{t, j}^{1}\right\}$ |

The guess part of $C$ contains for example for variable $w \in \operatorname{var}(D)$ the following clauses．

$$
\begin{array}{ll}
w_{t_{1}} \wedge \neg b_{t_{1}}^{0} \wedge \neg b_{t_{1}}^{1} \longrightarrow v_{t_{1}}, & \neg w_{t_{1}} \wedge \neg b_{t_{1}}^{0} \wedge \neg b_{t_{1}}^{1} \longrightarrow \neg v_{t_{1}} \\
w_{t_{3}} \wedge \neg b_{t_{3}}^{0} \wedge \neg \neg b_{t_{3}}^{1} \longrightarrow v_{t_{3}}, & \neg w_{t_{3}} \wedge \neg b_{t_{3}}^{0} \wedge \neg \neg b_{t_{3}}^{1} \longrightarrow \neg v_{t_{3}} .
\end{array}
$$

Thereby，whenever we guess a certain truth value for $w_{t_{1}}\left(w_{t_{3}}\right)$ it is ensured that there is a certain bit－vector，namely $\llbracket w \rrbracket_{t_{1}}\left(\llbracket w \rrbracket_{t_{3}}\right)$ such that $v_{t_{1}}\left(v_{t_{3}}\right)$ has to be set to the same truth value．Analogously，clauses of the form（3）and（4）are in $\mathcal{G}$ ．

The check part $\mathcal{C K}$ ．In the following，we assume an arbitrary，but fixed total order of the（three）literals of each（3－DNF）term $d \in D$ ．We refer to the first，second，and third literal of $d$ by $\operatorname{tlit}(d, 1), \operatorname{tlit}(d, 2)$ ，and $\operatorname{tlit}(d, 3)$ ，respectively．Analogously， $\operatorname{tvar}(d, 1), \operatorname{tvar}(d, 2)$ ， and $\operatorname{tvar}(d, 3)$ refers to the variable of the first，second，and third literal，respectively．Further， for a given term $d \in D$ and $1 \leq j \leq 3$ ，let $\operatorname{bv}(d, t, j)$ denote $v_{t, j}$ if $\operatorname{tlit}(d, j)$ is a variable， and $\neg v_{t, j}$ otherwise．Set $\mathcal{C K}$ contains the following clauses：

$$
\begin{array}{ll}
s a t_{\leq t} \longrightarrow s a t_{\leq t_{1}} \vee \cdots \vee s a t_{\leq t_{s}} \vee \text { sat }_{t} & \text { for each } t \in N, \\
& \operatorname{cld}(t)=\left\{t_{1}, \ldots, t_{s}\right\} \tag{5}
\end{array}
$$

Informally speaking，for any node $t$ this ensures the propagation of whether we satisfied at least one term directly in node $t$ ，or in any descendant of $t$ ．

In order to check whether a particular term is satisfied，we add for each term $d \in D$ clauses encoding the implication $s a t_{\delta^{-1}(d)} \longrightarrow \bigwedge_{1 \leq j \leq 3}\left[\bigwedge_{b \in \llbracket \operatorname{tvar}(d, j) \rrbracket_{\delta^{-1}(d), j}} b \wedge \mathrm{bv}\left(d, \delta^{-1}(d), j\right)\right]$ as follows：

$$
\begin{array}{ll}
\operatorname{sat}_{\delta^{-1}(d)} \longrightarrow b & \text { for each } d \in D, 1 \leq j \leq 3, \\
& b \in \llbracket \operatorname{tvar}(d, j) \rrbracket_{\delta^{-1}(d), j} \\
\operatorname{sat}_{\delta^{-1}(d)} \longrightarrow \operatorname{bv}\left(d, \delta^{-1}(d), j\right) & \text { for each } d \in D, 1 \leq j \leq 3 \tag{7}
\end{array}
$$

Finally，we add $s a t_{\leq r}$ for root $r$ ，and $\neg s a t_{t}$ for each node $t$ in $N \backslash \bigcup_{d \in D}\left\{\delta^{-1}(d)\right\}$ since these nodes are not used for checking satisfiability of any term．

$$
\begin{array}{ll}
s a t_{\leq r} \\
\neg s a t_{t} & \text { for each } t \in N \backslash \bigcup_{d \in D}\left\{\delta^{-1}(d)\right\} \tag{9}
\end{array}
$$

Example 10．Consider again formula Crom Example 8．We discuss clauses of the check part for node $t_{2}=\delta^{-1}\left(d_{2}\right)$ and root node $t_{5}$ ．Thereby，we encode satisfiability of term $d_{2}=\neg w \wedge \neg x \wedge y$ assuming $\operatorname{tlit}\left(d_{2}, 1\right)=\neg w, \operatorname{tlit}\left(d_{2}, 2\right)=\neg x$ ，and $\operatorname{tlit}\left(d_{2}, 3\right)=y$ ．

$$
\begin{aligned}
& s a t_{\leq t_{2}} \longrightarrow s a t_{\leq_{1}} \vee \text { sat }_{t_{2}}, \\
& s a t_{t_{2}} \longrightarrow \neg b_{t_{2}, 1}^{0}, \quad s a t_{t_{2}} \longrightarrow \neg b_{t_{2}, 1}^{1}, \quad s a t_{t_{2}} \longrightarrow \neg v_{t_{2}, 1}, \\
& s a t_{t_{2}} \longrightarrow \neg b_{t_{2}, 2}^{0}, \quad s a t_{t_{2}} \longrightarrow b_{t_{2}, 2}^{1}, \quad s a t_{t_{2}} \longrightarrow \neg v_{t_{2}, 2}, \\
& s a t_{t_{2}} \longrightarrow b_{t_{2}, 3}^{0}, \quad s a t_{t_{2}} \longrightarrow \neg b_{t_{2}, 3}^{1}, \quad s a t_{t_{2}} \longrightarrow v_{t_{2}, 3}, \\
& s a t_{\leq t_{5}} \longrightarrow s a t_{\leq t_{2}} \vee s a t_{\leq t_{4}} \vee s a t_{t_{5}}, \quad s a t_{\leq t_{5}}, \quad \neg s a t_{t_{5}}
\end{aligned}
$$

The propagate part $\mathcal{P}$. The sets $\mathcal{G}$ and $\mathcal{C K}$ contain clauses responsible for guessing truth values and checking that at least one term of the original formula $D$ is satisfied accordingly. In particular, the guess of truth values for $\operatorname{var}(D)$ happens at different tree decomposition nodes "independently", whereas checking whether at least one term $d \in D$ is satisfied is achieved in exactly one tree decomposition node $\delta^{-1}(d)$. Intuitively, in order to ensure that these independent guesses of truth values for $\operatorname{var}(D)$, are consistent, clauses in $\mathcal{P}$ make use of the connectedness condition of TDs in order to guide the comparison of these independent guesses along the TD. More precisely, for each tree decomposition node $t \in N$, every node $t_{i} \in \operatorname{cld}(t)$, and every variable $x \in \chi(t) \cap \chi\left(t_{i}\right)$ that both nodes $t$ and $t_{i}$ have in common, the set $\mathcal{P}$ contains clauses:

$$
\begin{array}{ll}
v_{t} \wedge \bigwedge_{b \in \llbracket x \rrbracket_{t}} b \wedge \bigwedge_{b \in \llbracket x \rrbracket_{t_{i}}} b \longrightarrow v_{t_{i}} & \text { for each } t \in N, t_{i} \in \operatorname{cld}(t), \\
& x \in \chi(t) \cap \chi\left(t_{i}\right) \\
\neg v_{t} \wedge \bigwedge_{b \in \llbracket x \rrbracket_{t}} b \wedge \bigwedge_{b \in \llbracket x \rrbracket_{t_{i}}} b \longrightarrow \neg v_{t_{i}} & \text { for each } t \in N, t_{i} \in \operatorname{cld}(t), \\
x \in \chi(t) \cap \chi\left(t_{i}\right) \tag{11}
\end{array}
$$

Further, for each clause $d \in D$, every node $t_{i}$ in $\operatorname{cld}\left(\delta^{-1}(d)\right)$, and $1 \leq j \leq 3$ such that $\operatorname{tvar}(d, j) \in \chi\left(t_{i}\right)$, set $\mathcal{P}$ contains:

$$
\begin{array}{ll}
\bigwedge_{b^{\prime} \in \llbracket \operatorname{tvar}(d, j) \rrbracket_{t, j}} b^{\prime} \longrightarrow b & \text { for each } d \in D \text { with } 1 \leq j \leq 3, \\
& t=\delta^{-1}(d), t_{i} \in \operatorname{cld}(t), \operatorname{var}(d, j) \in \chi\left(t_{i}\right), \\
& b \in \llbracket \operatorname{tvar}(d, j) \rrbracket_{t_{i}, j} \\
v_{t, j} \longleftrightarrow v_{t_{i}, j} & \text { for each } t \in N, t_{i} \in \operatorname{cld}(t), 1 \leq j \leq 3 \tag{13}
\end{array}
$$

Vaguely speaking, this construction ensures that whenever a bag element (using VarB3) or a truth value (using $\operatorname{Var} B V 3$ ) is "selected" in node $t$, we also have to select the same (if exists) below in children of $t$.

Example 11. Consider once more $C$ from Example 8. We illustrate the propagate part for node $t_{4}=\delta^{-1}\left(d_{4}\right)$ and variable $w$ assuming that $w=\operatorname{tvar}\left(d_{4}, 1\right)$. Observe that $d_{4}=w \wedge y \wedge z$, and $w \in \chi\left(t_{4}\right) \cap \chi\left(t_{3}\right)$.

$$
\begin{aligned}
& v_{t_{4}} \wedge \underbrace{\neg b_{t_{4}}^{0} \wedge \neg b_{t_{4}}^{1}}_{\llbracket w \rrbracket_{t_{4}}} \wedge \underbrace{\neg b_{t_{3}}^{0} \wedge \neg b_{t_{3}}^{1}}_{\llbracket w \rrbracket_{t_{3}}} \longrightarrow v_{t_{3}}, \\
& \neg v_{t_{4}} \wedge \underbrace{\neg b_{t_{4}}^{0} \wedge \neg b_{t_{4}}^{1}}_{\llbracket w \rrbracket_{t_{4}}} \wedge \underbrace{\neg b_{t_{3}}^{0} \wedge \neg b_{t_{3}}^{1}}_{\llbracket w \rrbracket_{t_{3}}} \longrightarrow \neg v_{t_{3}}, \\
& \neg b_{t_{3}, 1}^{0} \wedge \neg b_{t_{3}, 1}^{1} \longrightarrow \neg b_{t_{4}, 1}^{0}, \quad \neg b_{t_{3}, 1}^{0} \wedge \neg b_{t_{3}, 1}^{1} \longrightarrow \neg b_{t_{4}, 1}^{1}, \\
& v_{t_{4}, 1} \longleftrightarrow v_{t_{3}, 1}, \quad v_{t_{4}, 2} \longleftrightarrow v_{t_{3}, 2}, \quad v_{t_{4}, 3} \longleftrightarrow v_{t_{3}, 3}
\end{aligned}
$$

Remark 12. Recalling Figure 2, we would like to highlight the relation between elements of the figure and variables or clauses of reduction $R$ introduced above. Blue elements $a_{1}, b_{2}, a_{4}, b_{4}$ represent "introduce variables" VarI $(\operatorname{var}(D))$ and the blue squiggly arrows visualize the guess part $\mathcal{G}$. Green elements $p_{1}, p_{2}, p_{3}, p_{4}, p_{5}$ represent "pointer variables" VarB and VarB3 and the green squiggly arrows point to elements of tree decomposition $\mathcal{T}$. Finally, red squiggly arrows visualize the propagate part $\mathcal{P}$. (The check part $\mathcal{C K}$ is not explicitly visualized.)

Converting $C$ to 3-CNF formula $C^{\prime}$. Observe that by similar arguments (cf., [46]) one can transform using an additional reduction $R^{\prime}$ the CNF formula $C$ of the QBF $R(Q)$ into 3-CNF, resulting in $Q^{\prime \prime}=R^{\prime}(R(Q))$ such that $\operatorname{tw}\left(P_{Q^{\prime \prime}}\right) \leq \operatorname{tw}\left(P_{R(Q)}\right)+2$. To this end, one has to perform the following standard reduction (cf., [46]): As long as there exists a clause $c \in C$ consisting of more than 3 literals, we introduce a fresh existentially quantified variable $v$, remove $c$ from $C$ and replace it with two new clauses. The first new clause contains $v$ and two literals of $c$, while the second clause contains $\neg v$ and the remaining literals of $c$. Note that this standard reduction $R^{\prime}$ does not affect satisfiability, and can be
done such that it causes only constant increase of the treewidth (cf., Lemma 23 and 46]). Observe that by construction the same argument actually holds for pathwidth.

### 3.2 Methodology for Lower Bounds

The reduction discussed in the previous subsection allows us to establish our main result, which is the following theorem.

Theorem 13 (QBF lower bound). Given an arbitrary $Q B F$ of the form $Q=Q_{1} V_{1} \cdot Q_{2} V_{2} \cdot Q_{3} V_{3}$ $\cdots Q_{\ell} V_{\ell} . F$ where $\ell \geq 1$, and $F$ is a 3-CNF formula (if $Q_{\ell}=\exists$ ), or $F$ is a 3-DNF formula (if $Q_{\ell}=\forall$ ). Then, unless ETH fails, $Q$ cannot be solved in time $\operatorname{tow}(\ell, o(k)) \cdot \operatorname{poly}(|\operatorname{var}(F)|)$, where $k$ is the treewidth of the primal graph $P_{Q}$.

In the following, we first use this theorem to establish a full methodology to obtain lower bound results for bounded treewidth and then provide a proof for the theorem in the next section. The result for $\ell=2$ (cf., [46]) has already been applied as a strategy to show lower bound results for problems in artificial intelligence, as for example abstract argumentation, abduction, circumscription, and projected model counting, that are hard for the second level of the polynomial hierarchy when parameterized by treewidth [28, 30, 47. With the generalization to an arbitrary quantifier rank in Theorem 13, one can obtain lower bounds for variants of these problems and even more general problems on the third level or higher levels of the polynomial hierarchy.
Methodology. This motivates our methodology to show lower bounds for problems parameterized by treewidth. To this end, we make use of a stricter notion of fpt-reductions, which linearly preserves the parameter. Given functions $f, g: \mathbb{N} \rightarrow \mathbb{N}$, where $g$ is linear, and an $f$-bounded fpt-reduction $r$ using $g$. Then, we call $r$ an $f$-bounded fptl-reduction using $g$. Next, we discuss the methodology for proving lower bounds of a problem P for treewidth consisting of the following.

1. Graph Representation: Pick a graph representation $G(I)$ for a given instance $I$ of problem P.
2. Quantifier Rank: Fix a quantifier rank $\ell$ such that there is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ with $f(k) \in \mathcal{O}(\operatorname{tow}(\ell, k))$ and aim for establishing lower bound $\operatorname{tow}(\ell, \Omega(k)) \cdot \operatorname{poly}(\|I\|)$.
3. Establish Reduction: Establish an $f$-bounded fptl-reduction from an arbitrary QBF $Q$ of quantifier rank $\ell$ parameterized by treewidth of the primal graph of $Q$ to an instance $I$ of P parameterized by treewidth as well.
4. Conclude lower bound: Then, by applying Theorem 13 conclude that unless ETH fails, an arbitrary instance $I$ of problem P cannot be solved in time $\operatorname{tow}(\ell, o(k)) \cdot \operatorname{poly}(\|I\|)$ where $k=\operatorname{tw}(G(I))$.

We can generalize this to "non-canonical" lower bounds. To this end, one aims in Step 2 for a lower bound of the form tow $\left(\ell, \Omega\left(g^{-1}(k)\right) \cdot \operatorname{poly}(\|I\|)\right.$ for some function $g: \mathbb{N} \rightarrow \mathbb{N}$ such that $g^{-1}$ is well-defined, and $f(k) \in \mathcal{O}\left(\operatorname{tow}\left(\ell, g^{-1}(k)\right)\right)$. Then, in Step 3 one needs to establish an $f$-bounded fpt-reduction using $g$ accordingly, in order to conclude in Step 4 that under ETH an arbitrary instance $I$ of P cannot be solved in time $\operatorname{tow}\left(\ell, o\left(g^{-1}(k)\right)\right) \cdot \operatorname{poly}(\|I\|)$, where $k=\operatorname{tw}(G(I))$.

With the help of this methodology one can show lower bounds $f(k)$ for certain problems P , parameterized by treewidth, by reducing from the canonical $\ell$-QSAT problem parameterized by treewidth $k$ as well. Thus, one avoids directly using ETH via tedious reductions from


Table 1: Runtime lower bounds (under ETH) for selected problems, where $I$ denotes an instance of problem P and $k$ refers to the treewidth (" $\nabla{ }_{t}$ "), pathwidth (" $\nabla$ "), or 2local pathwidth (" $\mathbf{\nabla}$ ") of the corresponding (primal) graph of $I$. Results known from the literature are marked by " $\nabla$ " and " $\nabla$ ". By " $\nabla$ ", we indicate that the result holds due to lower bound advancements and the methodology described in this paper. We obtain results for " $\mathbf{\nabla}$ ", with known lower bound ( $" \nabla$ ", " $\nabla$ "), by the existing lower bound proof together with our methodology for 2-local pathwidth. Bounds are asymptotically tight unter ETH; corresponding upper bounds (e.g., [4, 10, 28, 31, 39, 42, 47, 50, 60]) are out of scope. For definitions, we refer to the problem compendium in an online self-archived version.

Sat, which involves problem-tailored constructions of instances of P whose treewidth is $\ell$-fold logarithmic in the number of variables or clauses of the given Sat formula.

Note that the methodology naturally extends to pathwidth, since the result of Theorem 13 easily extends to (2-local) pathwidth by construction of our reduction $R$, which works for any tree decomposition including the special case of path decompositions. Formal details will be provided in Section 4 on correctness in Corollary 27, followed by further consequences of Theorem 13 ,

Showcases. Table 1 gives a brief overview of selected problems and their respective runtime lower bounds under ETH. Then, the proof of Theorem 14 below serves as an example for applying the methodology, showing that Theorem 13 also allows for quite general results on projection. Note that this bounds are tight under ETH.

Theorem 14. Given an open $Q B F$ of the form $Q=Q_{1} V_{1} \cdot Q_{2} V_{2} . Q_{3} V_{3} \cdots Q_{\ell} V_{\ell} . F$ where $\ell \geq$ 1 , and $F$ is a 3-CNF formula (if $Q_{\ell}=\exists$ ), or a 3-DNF formula (if $Q_{\ell}=\forall$ ). Then, under ETH, PQSAT is indeed harder than deciding validity of $Q[\iota]$ for any assignment $\iota: \operatorname{fvar}(Q) \rightarrow$ $\{0,1\}$. In particular, assuming ETH, PQSAT cannot be solved in time tow $(\ell+1, o(k))$. $\operatorname{poly}(|\operatorname{var}(F)|)$, where $k$ is the pathwidth of the primal graph $P_{Q}$.
Proof. Assume towards a contradiction that under ETH one can solve projected model counting of $Q$ in time tow $(\ell+1, o(k)) \cdot \operatorname{poly}(|\operatorname{var}(F)|)$. In the following, we define an fptlreduction $r$ from QSat to the decision variant PQSat-at-least- $u$ of PQSAT, where a given open QBF $Q$ is a yes instance if and only if the solution (count) to PQSAT of $Q$ is at least $u$.

In particular, we transform a closed QBF $Q^{\prime}=Q_{0} V_{0} \cdot Q_{1} V_{1} \cdot Q_{2} V_{2} \cdot Q_{3} V_{3} \cdots Q_{\ell} V_{\ell} \cdot F$, where $k$ is the pathwidth of $P_{Q}$ to an instance $Q=Q_{1} \cdot V_{1} \cdot Q_{2} V_{2} \cdot Q_{3} V_{3} \cdots Q_{\ell} V_{\ell} \cdot F$ of PQSAT-at-least- $u$, where $\operatorname{fvar}(Q)=V_{0}$, and we set $u:=1$ if $Q_{0}=\exists$ and $u:=2^{\left|V_{0}\right|}$, otherwise. The reduction is indeed correct, since $Q^{\prime}$ is a yes-instance of QSAT if and only if $Q=r\left(Q^{\prime}\right)$
is a yes-instance of PQSat-at-least- $u$. Then, one can solve $Q^{\prime}$ of quantifier rank $\ell+1$ in time tow $(\ell+1, o(k)) \cdot \operatorname{poly}(|\operatorname{var}(F)|)$, which contradicts Theorem 13 and Corollary 27, $\square$

Corollary 15. Assuming ETH, an instance $Q$ of the problem $\# \Sigma_{\ell}$ SAT or $\# \Pi_{\ell}$ SAT cannot be solved in time $\operatorname{tow}(\ell+1, o(k)) \cdot \operatorname{poly}(|\operatorname{var}(\operatorname{matrix}(Q))|)$, where $k$ is the pathwidth of $P_{Q}$.

Next, we provide further examples (listed in Table 1) of the applicability of our methodology. We provide brief definitions of the problems discussed below in an online self-archived version.

Proposition 16 (cf., [28]). Unless ETH fails, PASP for given ASP program $\Pi$ and a set $P \subseteq \operatorname{var}(\Pi)$ of projection variables cannot be solved in time tow $(3, o(k)) \cdot \operatorname{poly}(|\Pi|)$, where $k$ is the pathwidth of the primal grap $\prod^{\prod}$ of $\Pi$.

Proof (Idea). Fptl-reduction from $\forall \exists \forall$-SAT to PASP, both parameterized by the pathwidth of its primal graph.

Proposition 17 (cf., [31]). Let $\mathcal{S} \in\{$ pref, semi-st, stage $\}$ and $F$ be an argumentation framework. Unless ETH fails, we cannot solve the problem \# $\operatorname{PCRED}_{\mathcal{S}}$ in time tow $(3, o(k))$. poly $(\|F\|)$ where $k$ is the pathwidth of $F$ (underlying graph).

Proof (Idea). Fptl-reduction from $\forall \exists \forall$-Sat parameterized by pathwidth of primal graph, to \#PCred ${ }_{\text {SEM }}$ (parameterized by pathwidth of the underlying graph).

Theorem 18. Unless ETH fails, \#PPAP for given instance $(T, H, M)$ and set $P$ of projection variables cannot be solved in time $\operatorname{tow}(3, o(k)) \cdot \operatorname{poly}(|\operatorname{var}(T)|)$, where $k$ is the pathwidth of the primal graph of $T$.

Proof (Idea). Fptl-reduction from $\forall \exists \forall$-Sat to the decision variant of \#PPAP, either from scratch or by lifting existing reduction [47] from $\exists \forall$-Sat.

Theorem 19. Given an epistemic program $\Pi$ and a variable $a \in \operatorname{var}(\Pi)$. Then, unless ETH fails, deciding the problem Candidate World View Check cannot be solved in time $\operatorname{tow}(3, o(k)) \cdot \operatorname{poly}(|\Pi|)$, and the problem World View Check for a cannot be solved in time $\operatorname{tow}(4, o(k)) \cdot \operatorname{poly}(|\Pi|)$, where $k$ is the pathwidth of the primal graph of $\Pi$.

Proof (Idea). Fptl-reduction from $\exists \forall \exists$-Sat, or $\exists \forall \exists \forall$-Sat (parameterized by pathwidth of primal graph), respectively. Actually the reductions from the literature for showing $\Sigma_{3}^{P}$ hardness and $\Sigma_{4}^{P}$-hardness [61 form fptl-reductions.

Note that our work focuses on lower bounds. However, the corresponding upper bounds for treewidth can be established by reductions to $\ell$-QSAT to obtain asymptotically tight results under ETH, see Table 1 .

## 4 Correctness, Compression and Runtime

In the following, we show correctness and properties of our reduction presented in Section 3.1 Therefore, we assume a given QBF $Q:=Q_{1} V_{1} \cdot Q_{2} V_{2} \cdots \forall V_{\ell} \cdot D$, where $D$ is in 3-DNF. Further, let $\mathcal{T}=(T, \chi, \delta)$ such that $T=(N, \cdot, r)$ be a labeled almost $c$-nice tree decomposition of primal graph $P_{Q}$ of width $k$. The reduced instance is addressed by $R(Q)$, where reduction $R$ is defined as in Section 3.1. The resulting QBF of quantifier rank $\ell+1$ is referred to by $R^{\prime}(R(Q))$ and its matrix in 3-CNF is given by $C=\operatorname{matrix}\left(R^{\prime}(R(Q))\right)$.

[^4]To simplify presentation, we introduce the following definitions. Let $d \in D$ be a term, $t \in N$ be a node of the tree decomposition, and $1 \leq j \leq 3$, then bit-term $(d, t):=$ $\bigcup_{1 \leq j \leq 3}\left[\left[\operatorname{tvar}(d, j) \rrbracket_{t, j} \cup\{\operatorname{bv}(d, t, j)\}\right]\right.$. Further, given an assignment $\alpha: \operatorname{var}(D) \rightarrow\{0,1\}$, we define a function local(•) that produces new assignments to copies of the variables, therefore let local $(\alpha):=\left\{x_{t} \mapsto \alpha(x) \mid x \in \operatorname{dom}(\alpha), t \in \operatorname{NodeI}(x)\right\}$ denote the matching assignment of the corresponding guess variables. Further, for a set $\mathcal{S}$ of literals and an assignment $\iota$, we say assignment $\iota$ respects $\mathcal{S}$, if $\left(\bigwedge_{l \in \mathcal{S}} l\right)[\iota]$ evaluates to true.
Correctness. Next, we establish correctness of reduction $R$.
Lemma 20. Let $\kappa$ be any assignment of at least two variables $x_{t}, x_{t^{\prime}} \in \operatorname{VarI}(\operatorname{var}(D))$ such that $\kappa\left(x_{t}\right) \neq \kappa\left(x_{t^{\prime}}\right)$ for nodes $t, t^{\prime} \in \operatorname{NodeI}(x)$ with $x \in \operatorname{var}(D)$. Then, $R(Q)[\kappa]$ is invalid.

Proof. We construct an assignment $\kappa^{\prime}$ which extends $\kappa$ and sets certain variables in $\operatorname{VarB}$. Then, we show that $R(Q)\left[\kappa^{\prime}\right]$ is invalid, which suffices since $\operatorname{Var} B$ is universally quantified. The construction of $\kappa^{\prime}$ is as follows: for every $b \in \llbracket x \rrbracket_{t^{\prime \prime}}$ and every $t^{\prime \prime} \in N$ where $x \in \chi\left(t^{\prime \prime}\right)$, we set $\kappa^{\prime}(b):=1$, if $b$ is a variable; and $\kappa^{\prime}(b):=0$, otherwise. Assume towards contradiction that there is an assignment $\kappa^{\prime \prime}: \operatorname{VarI}(\operatorname{var}(D)) \cup \operatorname{Var} B \cup \operatorname{Var} B V \rightarrow\{0,1\}$ that extends $\kappa^{\prime}$ such that $R(Q)\left[\kappa^{\prime \prime}\right]$ is valid. In particular, the assignment $\kappa^{\prime \prime}$ sets variables in $\operatorname{Var} B V$ such that every clause in the assigned variables of $R(Q)$, in particular, parts $\mathcal{G}$ and $\mathcal{P}$, is valid under the assignment $\kappa^{\prime \prime}$. By Condition (ii) of the definition of a tree decomposition (connectedness), $\mathcal{T}[x]$ induces a connected tree as well. In consequence, irrelevant of how $\kappa^{\prime \prime}$ assigns variable $v_{r^{\prime}}$ for the root node $r^{\prime}$ of $\mathcal{T}[x]$, the clauses in Formulas (10) and (11) enforce that exactly the same truth value $v=\kappa^{\prime \prime}\left(v_{n}\right)$ has to be set for any node $n \in \operatorname{NodeI}(x)$. Then, $\kappa^{\prime \prime}\left(v_{t}\right)=v$ and $\kappa^{\prime \prime}\left(v_{t^{\prime}}\right)=v$ holds. By part $\mathcal{G}$ of our reduction, more precisely, Formulas (1) and (2), we conclude that both $\kappa^{\prime \prime}\left(x_{t}\right)=v$ and $\kappa^{\prime \prime}\left(x_{t^{\prime}}\right)=v$, which contradicts that $\kappa\left(x_{t}\right) \neq \kappa\left(x_{t^{\prime}}\right)$.

Lemma 21. Given an assignment $\iota: \operatorname{VarI}(\operatorname{var}(D)) \cup \operatorname{VarB} \cup \operatorname{VarBV} \rightarrow\{0,1\}$. Then, for any assignment $\kappa: \operatorname{Var}(\operatorname{var}(D)) \cup \operatorname{VarB} \cup \operatorname{VarB} V \cup \operatorname{VarB3} \cup \operatorname{VarBV3} \rightarrow\{0,1\}$ that extends $\iota, R(Q)[\kappa]$ is invalid, if (a) there is no term $d_{i} \in D$ with $t=\delta^{-1}\left(d_{i}\right)$ such that $\kappa$ respects bit-term $\left(d_{i}, t\right)$. Now assume that there is $d_{i} \in D$ with $\kappa$ respecting bit-term $\left(d_{i}, \delta^{-1}\left(d_{i}\right)\right)$, then $R(Q)[\kappa]$ is also invalid, if (b) $\kappa\left(v_{t, j}\right) \neq \kappa\left(x_{t^{\prime}}\right)$, where $x=\operatorname{tvar}\left(d_{i}, j\right)$ for some $1 \leq j \leq 3$ and $t^{\prime} \in \operatorname{NodeI}(x)$.

Proof. Assume towards a contradiction that (a) is not the case, i.e., there is no $d_{i} \in D$ such that $\kappa$ respects bit-term $\left(d_{i}, t\right)$ for $t=\delta^{-1}\left(d_{i}\right)$ and still $R(Q)[\kappa]$ is valid. Observe that by $R(Q)$, in particular, by construction of the check part $\mathcal{C K}$ of $R, \kappa\left(s a t_{\leq r}\right)=1$ by (8) and therefore $\kappa\left(s a t_{t}\right)=1$ by (5) for at least one node $t \in N$ has to be set in $\kappa$. This, however, implies by (9) that $t=\delta^{-1}\left(d_{i}\right)$ for some $d_{i} \in D$. In consequence, by construction of (6) and (7), $\kappa$ respects bit-term $\left(d_{i}, t\right)$, where $t=\delta^{-1}\left(d_{i}\right)$, contradicting the assumption.

Towards contradicting (b), assume that there is $d_{i} \in D$ with $t=\delta^{-1}\left(d_{i}\right)$ and $x=$ $\operatorname{tvar}\left(d_{i}, j\right)$ as well as $t^{\prime} \in \operatorname{NodeI}(x)$ such that $\kappa\left(v_{t, j}\right) \neq \kappa\left(x_{t^{\prime}}\right)$ and still $R(Q)[\kappa]$ is valid. Observe that for any two nodes $t^{\prime \prime}, t^{\prime \prime \prime} \in \mathcal{T}[x], \kappa$ respects $\llbracket x \rrbracket_{t^{\prime \prime}, j}$ and $\llbracket x \rrbracket_{t^{\prime \prime \prime}, j}$ by 12 p ) and connectedness of $\mathcal{T}[x]$. Further, for any $t^{\prime \prime}, t^{\prime \prime \prime} \in \mathcal{T}[x], \kappa\left(v_{t^{\prime \prime}, j}\right)=\kappa\left(v_{t^{\prime \prime \prime}, j}\right)$ by (13). Then, since (3) and (4) ensure that $\kappa\left(x_{t^{\prime}}\right)=\kappa\left(v_{t^{\prime}, j}\right)$, ultimately by connectedness of $\mathcal{T}[x], \kappa\left(v_{t, j}\right)=$ $\kappa\left(x_{t^{\prime}}\right)$ holds.

Theorem 22 (Correctness). Let $Q$ be a QBF of the form $Q=Q_{1} V_{1} \cdot Q_{2} V_{2} \cdots \forall V_{\ell} \cdot D$ where $D$ is in $D N F$. Then, for any assignment $\alpha: \operatorname{fvar}(Q) \rightarrow\{0,1\}$, we have $Q[\alpha]$ is valid if and only if $R(Q)\left[\alpha^{\prime}\right]$ is valid, where assignment: $\operatorname{fvar}(R(Q)) \rightarrow\{0,1\}$ is such that $\alpha^{\prime}=\operatorname{local}(\alpha)$.

Proof. Let $\mathcal{T}=(T, \chi, \delta)$ be the labeled tree decomposition that is computed when constructing $R$, where $T=(N, A, r)$. We proceed by induction on the quantifier rank $\ell$.
Base case. Assume $\ell=1$.
" $\Longrightarrow$ ": Let $\alpha$ be an assignment to the free variables of $Q$ for which $Q[\alpha]$ is valid. Further, let $\alpha^{\prime}:=\operatorname{local}(\alpha)$. We show that $R(Q)\left[\alpha^{\prime}\right]$ is valid as well. Let therefore $\iota$ be an arbitrarily chosen assignment to the variables in $V_{1}$. Since $\ell=1$, we have $Q_{1}=\forall$. We define an assignment $\kappa: \operatorname{Var} I\left(\forall, V_{1}, \emptyset\right) \rightarrow\{0,1\}$ such that $\kappa\left(x_{t}\right):=\iota(x)$ for every $x_{t} \in \operatorname{Var} I\left(\forall, V_{1}, \emptyset\right)$ with $t \in N$ and $x \in \operatorname{var}(D)$. Next, we define an assignment $\kappa^{\prime}: \operatorname{Var} I\left(\forall, V_{1}, \emptyset\right) \cup \operatorname{VarI}(\exists$, $\left.\emptyset, V_{1}\right) \rightarrow\{0,1\}$ that extends $\kappa$ and sets $\kappa^{\prime}\left(x_{t^{\prime}}\right):=\iota(x)$ for every $x_{t^{\prime}} \in \operatorname{VarI}\left(\exists, \emptyset, V_{1}\right)$ with $t^{\prime} \in N$. Assignment $\kappa^{\prime}$ has by construction the same truth value for each of the copies $x_{t}$ of $x$, which is needed for $R(Q)\left[\alpha^{\prime} \cup \kappa\right]$ to be valid in order to not contradict Lemma 20 .

Then, we construct an assignment $\kappa^{\prime \prime}$, which extends $\kappa^{\prime}$ by the variables in VarB3, $\operatorname{Var} B V 3$, and VarSat. By construction of $\iota$ and since $Q[\alpha]$ is valid, $Q[\alpha \cup \iota]$ is valid, which is the same as $D[\alpha \cup \iota]$ is valid. In consequence, as $D$ is in DNF, there is at least one term $d \in D$ such that $d[\alpha \cup \iota]$ is valid. Depending on the term $d$, we assign the variables in $\operatorname{VarB3}$, $\operatorname{Var} B V 3$, and VarSat with assignment $\kappa^{\prime \prime}$. By Definition 5 , there is a unique node $t=\delta^{-1}(d)$ in the labeled tree decomposition for the term $d$. Then, we set $\kappa^{\prime \prime}\left(s a t_{\leq t}\right):=\kappa^{\prime \prime}\left(s a t_{t}\right):=1$. For every ancestor $t^{\prime}$ of $t \in N$, we assign $\kappa^{\prime \prime}\left(s a t_{\leq t^{\prime}}\right):=1$. For every node $s \in N$ that is not an ancestor of $t$, we set $\kappa^{\prime \prime}\left(s a t_{\leq s}\right):=0$. Finally, for every node $u$, where $u \neq t$, we set $\kappa^{\prime \prime}\left(s a t_{u}\right):=0$. For every node $t \in N$ and $1 \leq j \leq 3$ with $\operatorname{tvar}(d, j) \notin \chi(t)$, we set $\kappa^{\prime \prime}$ such that it respects $\llbracket n i l \rrbracket_{t, j} \cup\{\operatorname{bv}(d, t, j)\}$. Finally, for every node $t$ and $1 \leq j \leq 3$ with $\operatorname{tvar}(d, j) \in \chi(t)$, we set $\kappa^{\prime \prime}$ such that it respects bit-term $(d, t)$.

It remains to prove that for every assignment $\beta: \operatorname{Var} B \rightarrow\{0,1\}$, there is an assignment $\zeta$ : $\operatorname{Var} B V \rightarrow\{0,1\}$ for which $R(Q)\left[\alpha^{\prime} \cup \kappa^{\prime \prime} \cup \beta \cup \zeta\right]$ is valid. For every variable $x \in \chi(t)$, if for every node $t \in N$, assignment $\beta$ respects $\llbracket x \rrbracket_{t}$, then we set $\zeta\left(v_{t}\right):=(\alpha \cup \iota)(x)$. Otherwise, $\zeta\left(v_{t}\right):=0$, since we can assign any truth value here. By construction of $\alpha^{\prime}$ and $\kappa^{\prime \prime}$, clauses in Formulas (3) and (4) are satisfied of $\mathcal{G}$. Every clause of $\mathcal{C K}$ of $R(Q)$ is satisfied by construction of $\kappa^{\prime \prime} \backslash \kappa^{\prime}$ (and also by $\kappa^{\prime \prime}$ ) and $\zeta$. Clauses in Formulas (12) and (13) of $\mathcal{P}$ are satisfied by $\kappa^{\prime \prime} \backslash \kappa^{\prime}$. Further, clauses in Formulas (1) and (2) of $\mathcal{G}$ are satisfied because of $\beta, \kappa^{\prime \prime}, \alpha^{\prime}$, and $\zeta$. Finally, the clauses in Formulas (10) and (11) of $\mathcal{P}$ are satisfied by construction of $\beta, \zeta$, and $\kappa^{\prime}$.
$" \Longleftarrow ":$ Let $\alpha$ be an assignment to the free variables of $Q$ for which $Q[\alpha]$ is invalid. We show that if QBF $Q[\alpha]$ is invalid, then $R(Q)\left[\alpha^{\prime}\right]$ is invalid as well. Since $Q[\alpha]$ is invalid, $Q[\alpha \cup \iota]$ is invalid for any assignment $\iota: V_{1} \rightarrow\{0,1\}$. Assume towards a contradiction that $R(Q)\left[\alpha^{\prime}\right]$ is valid. We define an assignment $\kappa:=\operatorname{local}(\iota)$, which is $\kappa: \operatorname{VarI}\left(\forall, V_{1}, \emptyset\right) \cup \operatorname{VarI}\left(\exists, \emptyset, V_{1}\right) \rightarrow$ $\{0,1\}$ such that $\kappa\left(x_{t}\right):=\iota(x)$ for every $x_{t} \in \operatorname{VarI}\left(\forall, V_{1}, \emptyset\right) \cup \operatorname{VarI}\left(\exists, \emptyset, V_{1}\right)$ with $t \in N$ and $x \in \operatorname{var}(D)$. Observe that by Lemma 20, $\kappa$ is the only remaining option to obtain valid $R(Q)[\alpha]$. As a result, since $R(Q)\left[\alpha^{\prime}\right]$ is claimed valid, $R(Q)\left[\alpha^{\prime} \cup \kappa\right]$ is valid as well. In consequence, by Lemma 21 Statement (a), there has to exist an extension $\kappa^{\prime}$ of $\alpha^{\prime} \cup \kappa$ such that for some $d \in D, \kappa^{\prime}$ respects bit-term $(d, t)$, where $t=\delta^{-1}(d)$. By Lemma 21 Statement (b), for $1 \leq j \leq 3$ and every node $t^{\prime} \in \operatorname{NodeI}(y)$, where $y:=\operatorname{tvar}(d, j)$, we have $\kappa^{\prime}\left(v_{t, j}\right)=\kappa^{\prime}\left(y_{t^{\prime}}\right)$. However, by construction of $\kappa^{\prime}$ and connectedness of $\mathcal{T}[y]$, then $(\alpha \cup \iota)$ respects $d$. In consequence, this contradicts our assumption that $Q[\alpha \cup \iota]$ is invalid.
Induction step $(\ell>1)$ : We assume that the theorem holds for a given $\ell-1$ and it remains to prove that it then holds for $\ell$.
$" \Longrightarrow "$ : We proceed by case distinction on the first quantifier, i.e., (Case 1) $Q_{1}=\exists$ and (Case 2) $Q_{1}=\forall$. Thereby, we show that if $Q[\alpha]$ is valid and has quantifier rank $\ell$, then $R(Q)\left[\alpha^{\prime}\right]$ is valid as well.
(Case 1) $Q_{1}=\exists$ : Since $Q[\alpha]$ is valid, we can construct at least one assignment $\iota: V_{1} \rightarrow$ $\{0,1\}$ such that $Q[\alpha \cup \iota]$ is valid. By induction hypothesis, since the $\mathrm{QBF} Q[\alpha \cup \iota]$ has
quantifier rank $\ell-1$, and is valid, there are $\alpha^{\prime}, \iota^{\prime}$ such that $R(Q)\left[\alpha^{\prime} \cup \iota^{\prime}\right]$ is valid as well. In particular, by induction hypothesis, $\alpha^{\prime}=\operatorname{local}(\alpha), \iota^{\prime}=\operatorname{local}(\iota)$ and therefore $R(Q)\left[\alpha^{\prime}\right]$ is valid as well.
(Case 2) $Q_{1}=\forall$ : Since $Q[\alpha]$ is valid, for any assignment $\iota$ of $V_{1}$, we obtain that $Q[\alpha \cup \iota]$ is valid. In the following, we denote by $R^{\prime \prime}(Q)$ the QBF that is obtained from $R(Q)$, where variables in $\operatorname{VarI}\left(\exists, \emptyset, V_{1}\right)$ do not appear in the scope of a quantifier, i.e., these variables, while existentially quantified in $R(Q)$, are free variables in $R^{\prime \prime}(Q)$. By induction hypothesis, since the $\mathrm{QBF} Q[\alpha \cup \ell]$ has quantifier rank $\ell-1$, and is valid, there are $\alpha^{\prime}, \iota^{\prime}$ with $\alpha^{\prime}=\operatorname{local}(\alpha), \iota^{\prime}=\operatorname{local}(\iota)$, such that $R^{\prime \prime}(Q)\left[\alpha^{\prime} \cup \iota^{\prime}\right]$ is valid as well. Then, since $\iota$ was chosen arbitrarily, for every assignment $\kappa$ of variables in $\operatorname{dom}\left(\iota^{\prime}\right) \cap \operatorname{VarI}\left(Q_{1}, V_{1}, \emptyset\right)$, there is (by Lemma 20, since $R^{\prime \prime}(Q)\left[\alpha^{\prime} \cup \iota^{\prime}\right]$ is valid) an assignment $\kappa^{\prime}$ of variables in $\operatorname{dom}\left(\iota^{\prime}\right) \cap \operatorname{VarI}\left(\exists, \emptyset, V_{1}\right)$ such that $R(Q)\left[\alpha^{\prime} \cup \kappa \cup \kappa^{\prime}\right]$ is valid. In consequence, $R(Q)\left[\alpha^{\prime}\right]$ is valid as well.
" $\Longleftarrow$ ": Again, we proceed by case distinction in order to show that if $Q[\alpha]$ is invalid and has quantifier rank $\ell, R(Q)\left[\alpha^{\prime}\right]$ is invalid as well.
(Case 1) $Q_{1}=\exists$ : Since $Q[\alpha]$ is invalid, for every assignment $\iota$ of variables $V_{1}$ we have that $Q[\alpha \cup \ell]$ is also invalid. By induction hypothesis, since the $\mathrm{QBF} Q[\alpha \cup \iota]$ has quantifier rank $\ell-1$, and is invalid, there are assignments $\alpha^{\prime}$ and $\iota^{\prime}$ such that $R(Q)\left[\alpha^{\prime} \cup \iota^{\prime}\right]$ is invalid as well, where $\alpha^{\prime}=\operatorname{local}(\alpha), \iota^{\prime}=\operatorname{local}(\iota)$. Therefore $R(Q)\left[\alpha^{\prime}\right]$ is invalid, since $\iota$ was chosen arbitrarily and by Lemma $20 \iota^{\prime}$ covers all relevant cases, where $R(Q)\left[\alpha^{\prime}\right]$ could be valid.
(Case 2) $Q_{1}=\forall$ : Since $Q[\alpha]$ is invalid, there is at least one assignment $\iota$ of variables $V_{1}$, such that $Q[\alpha \cup \ell]$ is invalid. By induction hypothesis, since QBF $Q[\alpha \cup \iota]$ has quantifier rank $\ell-1$, and is invalid, there are assignments $\alpha^{\prime}$ and $\iota^{\prime}$ with $\alpha^{\prime}=\operatorname{local}(\alpha), \iota^{\prime}=\operatorname{local}(\iota)$, such that $R^{\prime \prime}(Q)\left[\alpha^{\prime} \cup \iota^{\prime}\right]$ is invalid ( $R^{\prime \prime}$ defined above), either. By Lemma 20, even for an assignment $\iota^{\prime \prime}$ that restricts $\iota^{\prime}$ to variables in $\operatorname{dom}\left(\iota^{\prime}\right) \cap \operatorname{VarI}\left(Q_{1}, V_{1}, \emptyset\right)$, there cannot be an assignment $\kappa$ to variables in $\operatorname{dom}\left(\iota^{\prime}\right) \cap \operatorname{VarI}\left(\exists, \emptyset, V_{1}\right)$ such that $R(Q)\left[\alpha^{\prime} \cup \iota^{\prime \prime} \cup \kappa\right]$ is valid. In consequence, $R(Q)\left[\alpha^{\prime}\right]$ is invalid as well.

Compression and Runtime. After having established the correctness of reduction $R$, we move on to showing that this reduction indeed compresses the treewidth of the resulting QBF $R(Q)$, as depicted in Figure 2. In particular, we prove this claim by constructing a tree decomposition $\mathcal{T}^{\prime}$ of the primal graph of $R(Q)$ and show its relation to labeled almost $c$-nice tree decomposition $\mathcal{T}$ of $Q$, where $\operatorname{width}(\mathcal{T})=\operatorname{tw}\left(P_{Q}\right)$. Then, we discuss runtime properties of the reduction.

Lemma 23 (Compression). The reduction $R$ exponentially decreases treewidth. In particular, $R^{\prime}(R(Q))$ constructs a QBF such that the treewidth of the primal graph of $R^{\prime}(R(Q))$ is 12 . $\lceil\log (k+1)\rceil+7 c+6$, where $k$ is the treewidth of $P_{Q}$ and $c \leq k$.

Proof. Assume a labeled almost $c$-nice tree decomposition $\mathcal{T}=(T, \chi, r)$ of $P_{Q}$ of width $k$, where $T=(N, E)$. From this we will construct a tree decomposition $\mathcal{T}^{\prime}=\left(T, \chi^{\prime}, r\right)$ of the primal graph of $R(Q)$. For each tree decomposition node $t \in N$ with $\operatorname{cld}(t)=\left\{t_{1}, \ldots, t_{s}\right\}$, we set its bag $\chi^{\prime}(t):=\left\{b \mid x \in \chi(t), b \in \llbracket x \rrbracket_{t} \cup \llbracket x \rrbracket_{t_{1}} \cup \cdots \cup \llbracket x \rrbracket_{t_{s}}\right\} \cup\{b \mid x \in \chi(t), 1 \leq j \leq 3, b \in$ $\left.\llbracket x \rrbracket_{t, j} \cup \llbracket x \rrbracket_{t_{1}, j} \cup \cdots \cup \llbracket x \rrbracket_{t_{s}, j}\right\} \cup\left\{x_{t^{\prime}} \mid x_{t^{\prime}} \in \operatorname{VarI}(V), t^{\prime}=t\right\} \cup \bigcup_{t^{\prime} \in\left\{t, t_{1}, \ldots, t_{s}\right\}}\left\{v_{t^{\prime}, 1}, v_{t^{\prime}, 2}, v_{t^{\prime}, 3}, v_{t^{\prime}}\right.$, $\left.s a t_{t^{\prime}}, s a t_{\leq t^{\prime}}\right\}$. Observe that all the properties of tree decompositions are satisfied. In particular, connectedness is not destroyed since the only elements that are shared among (at most two) different tree decompositions nodes are in $\operatorname{Var} B, \operatorname{Var} B V$ and in $\operatorname{VarB3}$, and VarBV3.

Each bag $\chi^{\prime}(t)$ contains bit-vectors $\llbracket x \rrbracket_{t}, \llbracket x \rrbracket_{t_{1}}, \ldots, \llbracket x \rrbracket_{t_{s}}$ for each $x \in \chi(t)$, resulting in at most $3 \cdot\lceil\log (k)\rceil$ many elements, since each node can have at most $s=2$ many children. Further, each bag additionally consists of bit-vectors $\llbracket x \rrbracket_{t, j}, \llbracket x \rrbracket_{t_{1}, j, \ldots,} \llbracket x \rrbracket_{t_{s}, j}$ for
each $x \in \chi(t) \cup\{n i l\}$, where $1 \leq j \leq 3$, which are at most $3 \cdot 3 \cdot\lceil\log (k+1)\rceil$ many elements. In total everything sums up to at most $12 \cdot\lceil\log (k+1)\rceil+7 c+6$ many elements per node, since $\left|\left\{x_{t^{\prime}} \mid x_{t^{\prime}} \in \operatorname{VarI}(V), t=t^{\prime}\right\}\right| \leq c$, and moreover $\mid \bigcup_{t^{\prime} \in\left\{t, t_{1}, \ldots, t_{s}\right\}}\left\{v_{t^{\prime}, 1}, v_{t^{\prime}, 2}, v_{t^{\prime}, 3}, v_{t^{\prime}}\right.$, sat $_{t^{\prime}}$, $\left.s a t_{\leq t^{\prime}}\right\} \mid \leq 6 \cdot(c+1)$ due to $\mathcal{T}$ being labeled and almost $c$-nice. Note that the treewidth of $R^{\prime}(R(Q))$ only marginally increases, since there are at most $\mathcal{O}(k \cdot\lceil\log (k)\rceil)$ many clauses in each bag of $\chi^{\prime}(t)$ for any node $t \in N$, each of size at most $\mathcal{O}(\lceil\log (k)\rceil)$. However, the fresh variables, that were introduced during the 3-CNF reduction only turn up in at most two new clauses (that is, they have degree two in the primal graph). Further, the construction can be controlled in such a way, that each new clause consists of at most two fresh variables. In consequence, one can easily modify $\mathcal{T}^{\prime}$, by adding at most $\mathcal{O}\left(k \cdot\lceil\log (k)\rceil^{2}\right)$ many intermediate nodes for each node $t \in N$, such that the width of $\mathcal{T}^{\prime}$ is at most $12 \cdot\lceil\log (k+1)\rceil+7 c+6$.

Theorem 24 (Runtime). Given a $Q B F Q$, where $D=$ matrix $(Q)$, $k$ is the treewidth of the primal graph of $Q$. Then, constructing $R^{\prime}(R(Q))$ takes time $\mathcal{O}\left(2^{k^{4}} \cdot\|D\| \cdot c\right)$, where $c \leq k$.

Proof. First, we construct [5] a tree decomposition of the primal graph of $Q$ of width $k$ in time $2^{\mathcal{O}\left(k^{3}\right)} \cdot|\operatorname{var}(D)|$, consisting of at most $\mathcal{O}\left(2^{k^{3}} \cdot|\operatorname{var}(D)|\right)$ many nodes. Then, we compute a labeled almost $c$-nice tree decomposition in time $\mathcal{O}\left(k^{2} \cdot 2^{k^{3}} \cdot(\|D\|)\right.$ [45, Lemma 13.1.3] without increasing the width $k$, resulting in decomposition $\mathcal{T}=(T, \chi)$, where $T=(N, E, r)$ of the primal graph of $Q$. Note that thereby the number of nodes is at most $\mathcal{O}\left(k \cdot 2^{k^{3}} \cdot\|D\|\right)$. The reduction $R(Q)$ then uses at most $\mathcal{O}\left(k \cdot 2^{k^{3}} \cdot\|D\| \cdot c\right)$ many variables in $\operatorname{Var} I(V)$ since in almost $c$-nice tree decompositions one node "introduces" at most $c$ variables. The other sets of variables used in $R$ are bounded by $\mathcal{O}\left(\lceil\log (k+1)\rceil \cdot k^{2} \cdot 2^{k^{3}} \cdot\|D\|\right)$. Overall, there are $\mathcal{O}\left(\lceil\log (k+1)\rceil \cdot k^{2} \cdot 2^{k^{3}} \cdot\|D\| \cdot c\right)$ many clauses constructed by $R(Q)$. Hence, the claim follows, since $R^{\prime}(R(Q))$ runs in time $\mathcal{O}\left(\lceil\log (k+1)\rceil^{2} \cdot k^{2} \cdot 2^{k^{3}} \cdot\|D\| \cdot c\right) \subseteq \mathcal{O}\left(2^{k^{4}} \cdot\|D\| \cdot c\right)$.

Proof of the main result. We are in position to prove the main result of this work. To this end, we show that the lower bounds are closed under negation and restate Theorem 13.

Lemma 25. Assume a given closed $Q B F$ of the form $Q=Q_{1} V_{1} \cdot Q_{2} V_{2} \cdot Q_{3} V_{3} \cdots Q_{\ell} V_{\ell} \cdot F$, where $\ell \geq 1$ and $F$ is in $C N F$ if $Q_{\ell}=\exists$, and $F$ is in $D N F$ if $Q_{\ell}=\forall$. Under ETH, one cannot solve $Q$ in time $\operatorname{tow}(\ell, o(k)) \cdot \operatorname{poly}(|\operatorname{var}(F)|)$ if and only if one cannot solve the negation $\neg Q$ in the same time.

Proof. Assume towards a contradiction that $\neg Q$ can be solved in time tow $(\ell, o(k))$. poly $(|\operatorname{var}(F)|)$ under ETH. But then, since inverting the result can be achieved in constant time, under ETH we can solve $Q$ in time $\operatorname{tow}(\ell, o(k)) \cdot \operatorname{poly}(|\operatorname{var}(F)|)$. Hence, we arrive at a contradiction.

Theorem 13 (QBF lower bound). Given an arbitrary $Q B F$ of the form $Q=Q_{1} V_{1} \cdot Q_{2} V_{2} \cdot Q_{3} V_{3}$ $\cdots Q_{\ell} V_{\ell} \cdot F$ where $\ell \geq 1$, and $F$ is a 3-CNF formula (if $Q_{\ell}=\exists$ ), or $F$ is a 3-DNF formula (if $Q_{\ell}=\forall$ ). Then, unless ETH fails, $Q$ cannot be solved in time $\operatorname{tow}(\ell, o(k)) \cdot \operatorname{poly}(|\operatorname{var}(F)|)$, where $k$ is the treewidth of the primal graph $P_{Q}$.

Proof. We assume that $Q$ is closed, i.e., for $Q$ we have $\operatorname{fvar}(Q)=\emptyset$. We show the theorem by induction on the quantifier rank $\ell$. For the induction base, where $\ell=1$, the result follows from the ETH in case of $Q_{\ell}=\exists$ since $k \leq|\operatorname{var}(F)|$. If $Q_{\ell}=\forall$, by Lemma 25 , the result follows. Note that for the case of $\ell=2$, the result has already been shown 46] as well.

For the induction step, we assume that the theorem holds for given $Q$ of quantifier rank $\ell \geq 1$, where $Q_{\ell}=\forall$, the treewidth of primal graph $P_{Q}$ is $k$, and $F$ is in 3-DNF. We show that then the theorem also holds for quantifier rank $\ell+1$. Towards a contradiction, we assume that in general we can solve any QBF $Q^{\prime}$ of quantifier rank $\ell+1$, in time tow $(\ell+$
$\left.1, o\left(\operatorname{tw}\left(P_{Q^{\prime}}\right)\right)\right) \cdot \operatorname{poly}\left(\left|\operatorname{var}\left(C^{\prime}\right)\right|\right)$, where $C^{\prime}=\operatorname{matrix}\left(Q^{\prime}\right)$. We compute a labeled almost $c$-nice TD $\mathcal{T}$ of $P_{Q}$ of width $k$, where $c$ is in $\mathcal{O}(\log (k))$. We proceed by case distinction on the last quantifier $Q_{\ell+1}^{\prime}$ of $Q^{\prime}$.
(Case 1) $Q_{\ell+1}^{\prime}=\exists$ : Let $Q^{\prime}=R^{\prime}(R(Q)), C^{\prime}=$ matrix $\left(Q^{\prime}\right)$ be the matrix of $Q^{\prime}$, and $k^{\prime}$ be the treewidth of the primal graph of $C^{\prime}$. Observe that $Q^{\prime}$ has quantifier rank $\ell+1$ and is of the required form. By Lemma 23, $k^{\prime}=12 \cdot\lceil\log (k+1)\rceil+7 c+6$. As a result, since $R$ is an fpt-reduction (including time for computing $\mathcal{T}$ ) according to Theorem 24, one can solve $Q^{\prime}$ in time $\operatorname{tow}(\ell+1, o(12 \cdot\lceil\log (k+1)\rceil+7 c+6)) \cdot \operatorname{poly}\left(\left|\operatorname{var}\left(C^{\prime}\right)\right|\right)=\operatorname{tow}(\ell+1, o(\log (k)))$. $\operatorname{poly}\left(\left|\operatorname{var}\left(C^{\prime}\right)\right|\right)$. Therefore, by Theorem 22 we can solve $Q$ in time tow $(\ell, o(k)) \cdot \operatorname{poly}(|\operatorname{var}(F)|)$, which contradicts the induction hypothesis.
(Case 2) $Q_{\ell+1}^{\prime}=\forall$ : By Lemma 25 one can decide in time tow $(\ell+1, o(k)) \cdot \operatorname{poly}\left(\left|\operatorname{var}\left(C^{\prime}\right)\right|\right)$ whether $Q^{\prime}$ is valid if and only if we can decide in time tow $(\ell+1, o(k)) \cdot \operatorname{poly}\left(\left|\operatorname{var}\left(C^{\prime}\right)\right|\right)$ whether $\neg Q^{\prime}$ is valid. Note that after bringing $\neg Q^{\prime}$ into prenex normal form, the last quantifier is $\exists$. Therefore, the remainder of this case is (Case 1). Hence, we have established the second case and this concludes the proof.

Further Consequences. We generalize Theorem 13 to the incidence graph. The incidence graph of a formula $F$ in CNF or DNF is the bipartite graph, which has as vertices the variables and clauses (terms) of $F$ and an edge $v c$ between every variable $v$ and clause (term) $c$ whenever $v$ occurs in $c$ in $F$ [60]. We obtain the following.

Corollary 26. Given an arbitrary $Q B F Q$ of quantifier rank $\ell \geq 1$. Then, under ETH one cannot solve $Q$ in time $\operatorname{tow}(\ell, o(k)) \cdot \operatorname{poly}(|\operatorname{var}(\operatorname{matrix}(Q))|)$, where $k$ is the treewidth of the incidence graph of matrix $(Q)$.

Proof. The claim follows from Theorem 13, since, in general, the treewidth $k$ of the incidence graph of $Q$ is bounded [29, 60] by treewidth $k^{\prime}$ of $P_{Q}$, i.e., $k \leq k^{\prime}+1$. As a result, if the weaker lower bound of this corollary did not hold, Theorem 13 would be violated.

Corollary 27 (2-Local Pathwidth bound). Given an arbitrary $Q B F$ of the form $Q=$ $Q_{1} V_{1} \cdot Q_{2} V_{2} \cdot Q_{3} V_{3} \cdots Q_{\ell} V_{\ell} . F$, where $\ell \geq 1$ and $F$ in 3 -CNF (if $Q_{\ell}=\exists$ ) or 3-DNF (if $Q_{\ell}=\forall$ ). Then, unless ETH fails, $Q$ cannot be solved in time $\operatorname{tow}(\ell, o(k)) \cdot \operatorname{poly}(|\operatorname{var}(F)|)$, where $k$ is the 2-local pathwidth of graph $P_{Q}$.

Proof. First, we show the claim for pathwidth by induction, which can be easily established for base case $\ell=1$. For the case of $\ell=2$, related work [46] holds only for treewidth. However, the cases for $\ell \geq 2$ follow from the proof of Theorem 13, since every path decomposition is also a tree decomposition, and the proofs of lemmas and theorems used intermediately only rely on an arbitrary tree decomposition. To be more concrete, the proof of Theorem 13 relies on Lemma 23, whose proof shows compression for any tree decomposition, which hence also works for any PD as well. Similarly, Theorem 24 also holds for pathwidth, since a PD of fixed pathwidth can be computed [5] even in time $\mathcal{O}\left(2^{k^{2}} \cdot|\operatorname{var}(F)|\right)$, and since computation of labeled almost $c$-nice decompositions works anologously for PDs. Further, the remainder of the proof holds for the thereby obtained PD, since the construction works for any TD. Finally, Lemma 25 holds independently of the parameter. As a result, reductions $R^{\prime}$ and $R$ used by Theorem 13 indeed are sufficient for PDs.

The proof above can be lifted to the case of 2-local pathwidth by observing that the 2local pathwidth of $P_{R^{\prime}(R(Q))}$ is also bounded by $12 \cdot\lceil\log (k+1)\rceil+7 c+6$ (cf. Lemma $\left.\sqrt{23}\right)$, since each variable of $P_{R^{\prime}(R(Q))}$ occurs at most twice in the constructed (path) decomposition $\mathcal{T}^{\prime}$ of $R^{\prime}(R(Q))$.

Remark 28. We remark that reduction $R$ can be generalized to finite, non-Boolean domains (QCSP, e.g., [27]). For given variables $V$ of a $Q C S P$ formula $Q$, the variables in $\operatorname{VarI}(V)$, $\operatorname{VarBV}$, and VarBV3 have to be made non-Boolean, whereas the other variables used in $R$ stay Boolean. Consequently, one obtains similar results as in related work [46], but for quantifier rank $\ell \geq 3$. We conject that under $E T H$, validity of QCSPs $Q$ over domain $\mathcal{D}$ of quantifier rank $\ell$, where $k=\mathrm{pw}\left(P_{Q}\right)$, cannot be decided in time $\operatorname{tow}\left(\ell-1,|\mathcal{D}|^{o(k)}\right)$. $\operatorname{poly}(|\operatorname{var}(Q)|)$.

Finally, we establish a corollary that improves a result from the literature. To this end, we denote for given positive number $n$ by $\log ^{*}(n)$ the smallest value $i$ such that tow $(i, 1) \geq n$. A known result [1, Corollary 1] states $\Sigma_{\ell}^{P}$-hardness for instances $Q$ of $\left(4 \cdot \log ^{*}(\mid \operatorname{var}(\right.$ matrix $\left.(Q)) \mid)\right)$ QSAT, wheras here we establish para- $\Sigma_{\ell}^{P}$-hardness for instances $Q^{\prime}$ of $\left(\log ^{*}\left(\mid \operatorname{var}\left(\right.\right.\right.$ matrix $\left.\left.\left.\left(Q^{\prime}\right)\right) \mid\right)\right)$ QSat. This is possible by applying our established reduction $R$, which is rather fine-grained since it only increases quantifier rank by one, and it works indeed for any QBF, and not just for a certain classes of QBFs in contrast to the known result. As a consequence, whenever a new class of $\ell$-QBFs with a certain treewidth or pathwidth guarantee was discovered, which is still $\Sigma_{\ell}^{P}$-hard, one immediately obtains para- $\Sigma_{\ell}^{P}$-hardness by using reduction $R$. Then, one could potentially further improve quantifier alternations by applying reduction $R$, which is (asymptotically) tight under ETH.

Corollary 29. Given any integer $\ell \geq 1$. Then, deciding QSAT is para- $\Sigma_{\ell}^{P}$-hard when parameterized by (2-local) pathwidth of the primal graph $P_{Q}$ for input QBFs of the form $Q=Q_{1} V_{1} \cdot Q_{2} V_{2} . Q_{3} V_{3} \cdots Q_{\ell+\log ^{*}(|\operatorname{var}(F)|)} V_{\ell+\log ^{*}(|\operatorname{var}(F)|)} \cdot F$, where is $F$ in $3-C N F\left(\right.$ if $\left.Q_{\ell}=\exists\right)$ or 3-DNF (if $Q_{\ell}=\forall$ ).

Proof. Given a closed QBF of the form $Q^{\prime}=Q_{1} V_{1}^{\prime} \cdot Q_{2} V_{2}^{\prime} \cdot Q_{3} V_{3}^{\prime} \cdots Q_{\ell} V_{\ell}^{\prime} \cdot F^{\prime}$, where $\ell \geq 1$ and $F^{\prime}$ is in 3-CNF if $Q_{\ell}=\exists$, and $F^{\prime}$ is in 3-DNF if $Q_{\ell}=\forall$, and $k^{\prime}$ is the 2-local pathwidth of $P_{Q^{\prime}}$. Then, we apply our reduction $R$ followed by $R^{\prime}$ on $Q^{\prime}$ and iteratively apply $R$ and $R^{\prime}$. We repeat this step exactly $\log ^{*}\left(k^{\prime}\right)$ many times and refer to the final result by $Q^{\prime \prime}$. Note that the solutions to problem QSat on $Q^{\prime}$ and $Q^{\prime \prime}$ are equivalent by Theorem 22. Then, the resulting 2-local pathwidth $k^{\prime \prime}$ of $P_{Q^{\prime \prime}}$ is in $\mathcal{O}(1)$ by Lemma 23, i.e., parameter $k^{\prime \prime}$ is constant. Hence, since $Q^{\prime}$ is hard for $\Sigma_{\ell}^{P}$, also $Q^{\prime \prime}$ is hard for $\Sigma_{\ell}^{P}$, and $Q^{\prime \prime}$ is para- $\Sigma_{\ell}^{P}$-hard since $k^{\prime \prime}$ is a constant. Observe that $k^{\prime} \leq\left|\operatorname{var}\left(F^{\prime}\right)\right| \leq\left|\operatorname{var}\left(\operatorname{matrix}\left(Q^{\prime \prime}\right)\right)\right|$. As a result, QSAT for QBFs of the form $Q$ above is hard for para- $\Sigma_{\ell}^{P}$.

## 5 Conclusion

In this work, we presented a lower bound for deciding the validity (QSAT) of quantified Boolean formulas (QBFs). Thereby, we have significantly extended the current state-of-theart of this line of research: So far, lower bound results under ETH for QSAT parameterized by treewidth were not available for all levels of the polynomial hierarchy. The generalization of this result in Theorem 13 does not only cover QBFs, parameterized by treewidth and an arbitrary quantifier rank, but solves a natural question for a well-known problem in complexity theory. Interestingly, the result confirms the (asymptotic) optimality of the algorithm by Chen [10] for solving QSAt and thereby answers a longstanding open question. Indeed, one cannot expect to solve QSAT of quantifier rank $\ell$ significantly better than in time $\Omega^{*}(\operatorname{tow}(\ell, k))$ in the treewidth $k$. The proof of this result relies on a novel reduction approach that makes use of a fragile balance between redundancy and structural dependency (captured by treewidth) and uses tree decompositions as a "guide" in order to achieve exponential compression of the parameter treewidth. We encode dynamic programming
on tree decompositions and obtain a technique for compressing treewidth. Note that both our technique and the results naturally carry over to path decompositions and (2-local) pathwidth.

Given the nature of our reduction, we observe that the reduction might also serve in reducing treewidth in practice. In particular, solvers based on tree decompositions such as the QBF solver dynQBF [9] could benefit from significantly reduced treewidth; at the cost of increased quantifier rank by one. Since dynQBF is capable [49] of solving instances up to treewidth 80 with quantifier rank more than two, slightly increasing the quantifier rank might be in practice a good trade-off for decreasing the treewidth significantly.

Another advantage of our reduction is that it gives rise to a versatile methodology for showing lower bounds for arbitrary problems (depending on the ETH, parameterized by treewidth) by reduction from $\ell$-QSAT, parameterized by treewidth as well. Thereby we avoid tedious reductions from SAT (directly using ETH), which involves problem-tailored gadgets to construct instances whose treewidth is $\ell$-fold logarithmic in the number of variables or clauses of the given Sat formula. Further, we have listed a number of showcases to illustrate the applicability of this approach to natural problems that are beyond the second level of the polynomial hierarchy. As a by-product we have established that the canonical problems $\# \Sigma_{\ell}$ SAT and $\# \Pi_{\ell}$ SAT of projected model counting applied to QBFs when parameterized by treewidth always come at the price of an additional level of exponentiality in the treewidth (compared to $\ell$-QSAt).

One direction for future work is to explore further problems parameterized by treewidth and to establish tightness of the so far existing upper bounds. Another important direction is to work out techniques and showcases for "non-canonical" lower bounds, where fptlreductions are not sufficient and using a customized function $g$ is necessary. Hence, our goal is to continue this line of research in order to use this toolkit for problems that do not exhibit (e.g., [48]) canonical runtimes, where fptl-reductions suffice. We hope this work will foster research and new insights on lower bounds.

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## A Appendix

## A. 1 Counting Problems

A witness function is a function $\mathcal{W}: \Sigma^{*} \rightarrow 2^{\Sigma^{*}}$ that maps an instance $I \in \Sigma^{*}$ to a finite subset of $\Sigma^{\prime *}$. We call the set $\mathcal{W}(I)$ the witnesses. A parameterized counting problem $L: \Sigma^{*} \times \mathbb{N} \rightarrow \mathbb{N}_{0}$ is a function that maps a given instance $I \in \Sigma^{*}$ and an integer $k \in \mathbb{N}$ to the cardinality of its witnesses $|\mathcal{W}(I)|$. Let C be a decision complexity class, e.g., P. Then, $\# \cdot \mathrm{C}$ denotes the class of all counting problems whose witness function $\mathcal{W}$ satisfies (i) there is a function $f: \mathbb{N}_{0} \rightarrow \mathbb{N}_{0}$ such that for every instance $I \in \Sigma^{*}$ and every $W \in \mathcal{W}(I)$ we have $|W| \leq f(\|I\|)$ and $f$ is computable in time $\mathcal{O}($ poly $(\|I\|))$ and (ii) for every instance $I \in \Sigma^{*}$ the decision problem $\mathcal{W}(I)$ belongs to the complexity class C. Then, $\# \cdot \mathrm{P}$ is the complexity class consisting of all counting problems associated with decision problems in NP. Let $L$ and $L^{\prime}$ be counting problems with witness functions $\mathcal{W}$ and $\mathcal{W}^{\prime}$. A parsimonious reduction from $L$ to $L^{\prime}$ is a polynomial-time reduction $r: \Sigma^{*} \rightarrow \Sigma^{\prime *}$ such that for all $I \in \Sigma^{*}$, we have $|\mathcal{W}(I)|=\left|\mathcal{W}^{\prime}(r(I))\right|$. It is easy to see that the counting complexity classes \#. C defined above are closed under parsimonious reductions. It is clear for counting problems $L$ and $L^{\prime}$ that if $L \in \# \cdot \mathrm{C}$ and there is a parsimonious reduction from $L^{\prime}$ to $L$, then $L^{\prime} \in \# \cdot \mathrm{C}$.

## A. 2 Problem Compendium

Projected Model Counting (PQSat). Durand et al. 19] have shown that the canonical problems $\# \Sigma_{\ell} S A T$, and $\# \Pi_{\ell}$ SAT are $\# \cdot \Sigma_{\ell}^{\mathrm{P}}$-complete, and $\# \cdot \Pi_{\ell}^{\mathrm{P}}$-complete (both via parsimonious reductions), respectively. Note that both problems are special cases of PQSAT as defined in Section 2, Further, \#SAT is the problem $\# \Sigma_{0} S A T$, and PMC is the problem $\# \Sigma_{1}$ Sat.

Problem: \# $\Sigma_{\ell}$ SAT [19]
Input: Open QBF $Q=\exists V_{1} . \forall V_{2} \cdots Q_{\ell} V_{\ell} F$, where $F$ is a Boolean formula in 3-DNF if $Q_{\ell}=\forall$ (and 3-CNF if $Q_{\ell}=\exists$ ).
Output: $\quad$ Number of assignments $\iota: \operatorname{fvar}(Q) \rightarrow\{0,1\}$ such that $Q[\iota]$ is valid.

## Problem: $\# \Pi_{\ell}$ SAt [19]

Input: Open QBF $Q=\forall V_{1} \cdot \exists V_{2} \cdots Q_{\ell} V_{\ell} F$, where $F$ is a Boolean formula in 3-DNF if $Q_{\ell}=\forall$ (and 3-CNF if $Q_{\ell}=\exists$ ).
Output: $\quad$ Number of assignments $\iota: \operatorname{fvar}(Q) \rightarrow\{0,1\}$ such that $Q[\iota]$ is valid.

Abstract Argumentation. We consider the Argumentation Framework by Dung [18]. An argumentation framework $(A F)$ is a directed graph $F=(A, R)$ where $A \neq \emptyset$ is a finite set of arguments and $R \subseteq A \times A$ a pair of arguments representing direct attacks of arguments. In argumentation, we interest in computing so-called extensions, which are subsets $S \subseteq A$ of the arguments that meet certain properties according to certain semantics as given below. An argument $s \in S$, is called defended by $S$ in $F$ if for every $\left(s^{\prime}, s\right) \in R$, there exists $s^{\prime \prime} \in S$ such that $\left(s^{\prime \prime}, s^{\prime}\right) \in R$. The family $\operatorname{def}_{F}(S)$ is defined by $\operatorname{def}_{F}(S):=\{s \mid s \in A, s$ is defended by $S$ in $F\}$. We say $S \subseteq A$ is conflict-free in $S$ if $(S \times S) \cap R=\emptyset ; S$ is admissible (adm) in $F$ if (i) $S$ is conflict-free in $F$, and (ii) every $s \in S$ is defended by $S$ in $F$. Assume an admissible set $S$. Then, (iiia) $S$ is preferred (pref) in $F$, if there is no $S^{\prime} \supset S$ that is admissible in $F$; (iiib) $S$ is semi-stable (semi-st) in $F$ if there is
no admissible set $S^{\prime} \subseteq A$ in $F$ with $S_{R}^{+} \subsetneq\left(S^{\prime}\right)_{R}^{+}$where $S_{R}^{+}:=S \cup\{a \mid(b, a) \in R, b \in S\}$; a conflict-free set $S$ is stage in $F$ if there is no conflict-free set $S^{\prime} \subseteq A$ in $F$ with $S_{R}^{+} \subsetneq\left(S^{\prime}\right)_{R}^{+}$. Let ALL abbreviate the set \{pref, semi-st, stage\}. For a semantics $\mathcal{S} \in$ ALL, $\mathcal{S}(F)$ denotes the set of all extensions of semantics $\mathcal{S}$ in $F$.

Let $\mathcal{S} \in$ ALL be an abstract argumentation semantic, $F=(A, R)$ be an argumentation framework, and $a \in A$ an argument. The credulous reasoning problem Cred $\mathcal{S}_{\mathcal{S}}$ asks to decide whether there is an $\mathcal{S}$-extension of $F$ that contains the (credulous) argument $a$. The skeptical reasoning problem $\operatorname{SkEP}_{\mathcal{S}}$ asks to decide whether all $\mathcal{S}$-extensions of $F$ contain the argument $a$.

$$
\begin{array}{ll}
\text { Problem: } & \text { Projected Credulous Counting \#PCred }{ }_{\mathcal{S}} \text { [31] } \\
\text { Input: } & \mathcal{S} \in \text { ALL, argumentation framework } F, \text { set } P \text { of projection } \\
& \text { arguments, argument } a \in A . \\
\text { Output: } & \begin{array}{l}
\text { Number of } \mathcal{S} \text {-extensions, where } a \text { is credulously accepted, re- } \\
\\
\\
\text { stricted to } P, \text { in more detail, }|\{S \cap P \mid S \in \mathcal{S}(F), a \in S\}| .
\end{array}
\end{array}
$$

Propositional Abduction. The Propositional Abduction Problem (PAP) consists of a tuple ( $T, H, M$ ), where $T$ is a Boolean formula (theory) in CNF, $H \subseteq \operatorname{var}(T)$ is a set of hypothesis, and $M \subseteq \operatorname{var}(T)$ forms a set of manifestations. A set $S \subseteq H$ is a solution to the problem $(T, H, M)$ if there is an assignment $\iota: \operatorname{var}(T) \rightarrow\{0,1\}$ such that $(T \cup S)[\iota]$ evaluates to true, and for every assignment $\iota: \operatorname{var}(T) \rightarrow\{0,1\}$, where $(T \cup S)[\iota]$ evaluates to true, also $M[\iota]$ evaluates to true. Given a $\operatorname{PAP}(T, H, M)$. Then, let $\mathcal{S}(T, H, M)$ denote the set of solutions to this problem.

Problem: Projected Propositional Abduction Counting \#PPAP Input: $\quad$ PAP problem $(T, H, M)$, set $P \subseteq \operatorname{var}(T)$ of projection variables.
Output: Number of solutions to ( $T, H, M$ ) where only the number of combinations with respect to $P$ are of interest, i.e., $|\{S \cap P \mid S \in \mathcal{S}(T, H, M)\}|$.

Answer Set Programming (ASP). We follow standard definitions of Boolean disjunctive ASP. For comprehensive foundations, we refer to introductory literature [8, 43]. Let $\ell, m, n$ be non-negative integers such that $\ell \leq m \leq n, a_{1}, \ldots, a_{n}$ be Boolean variables. A program $\Pi$ is a set of rules of the form $a_{1} \vee \cdots \vee a_{\ell} \leftarrow a_{\ell+1}, \ldots, a_{m}, \neg a_{m+1}, \ldots, \neg a_{n}$. For a rule $r$, we let $H_{r}:=\left\{a_{1}, \ldots, a_{\ell}\right\}, B_{r}^{+}:=\left\{a_{\ell+1}, \ldots, a_{m}\right\}, B_{r}^{-}:=\left\{a_{m+1}, \ldots, a_{n}\right\}$, and $\operatorname{var}(r):=H_{r} \cup B_{r}^{+} \cup B_{r}^{-}$. Consequently, $\operatorname{var}(\Pi):=\bigcup_{r \in \Pi} \operatorname{var}(r)$.

An interpretation $I \subseteq \operatorname{var}(\Pi)$ is a set of variables. I satisfies a rule $r$ if $\left(H_{r} \cup B_{r}^{-}\right) \cap I \neq \emptyset$ or $B_{r}^{+} \backslash I \neq \emptyset . I$ is a model of $\Pi$ if it satisfies all rules of $\Pi$. The Gelfond-Lifschitz (GL) reduct of $\Pi$ under $I$ is the program $\Pi^{I}$ obtained from $\Pi$ by first removing all rules $r$ with $B_{r}^{-} \cap I \neq \emptyset$ and then removing all $\neg z$ where $z \in B_{r}^{-}$from the remaining rules $r$ [36]. $I$ is an answer set of a program $\Pi$ if $I$ is a subset-minimal model of $\Pi^{I}$.

```
Problem: Answer Set Consistency (ASPCons)
Input: Program П.
Question: Does there exist an answer set of program \Pi!?
```

```
Problem: Answer Set Counting (#ASP)
Input: Program \Pi.
Output: Number of answer sets of \Pi.
```

Problem: Projected Answer Set Counting (PASP) [2]
Input: $\quad$ Program $\Pi$, set $P \subseteq \operatorname{var}(\Pi)$ of projection variables.
Output: Number of answer sets projected to the set $P$, which is, $\mid\{S \cap P \mid S$ is an answer set of $\Pi\} \mid$.

Epistemic Logic Programming (ELP). Let $\ell, m, n$ be non-negative integers such that $\ell \leq m \leq n, a_{1}, \ldots, a_{n}$ be Boolean variables. Then, an epistemic program $\Pi$ is a set of epistemic rules of the form $a_{1} \vee \cdots \vee a_{\ell} \leftarrow l_{\ell+1}, \ldots, l_{m}, \neg l_{m+1}, \ldots, \neg l_{n}$., where $l_{i}$ for $\ell+1 \leq i \leq n$ is either variable $a_{i}$, or an epistemic literal potentially using epistemic negation not of the form not $a_{i}$ or not $\neg a_{i}$. Let $\mathrm{EP}(\Pi)$ be the set of epistemic literals occurring in $\Pi$. Given a guess $\sigma \subseteq \operatorname{EP}(\Pi)$. The epistemic reduct $\Pi^{\sigma}$ corresponds to the program $\Pi$, where each occurrence of not $l \in \sigma$ is replaced by truth constant $T$, and each occurrence of not $l \in \mathrm{EP}(\Pi) \backslash \sigma$ is replaced by $\neg l$. Note that $\Pi^{\sigma}$ corresponds to an ASP program, assuming that $T$ is in any answer set and potential double negation ( $\neg \neg$ ) is treated [25] accordingly ${ }^{5}$. Then, given guess $\sigma \subseteq \operatorname{EP}(\Pi)$, the collection $\mathcal{A}$ of all the answer sets of $\Pi^{\sigma}$ is a candidate world view of $\Pi$ w.r.t. $\sigma$, if (i) $\mathcal{A} \neq \emptyset$, (ii) for each not $l \in \sigma$ there is an answer set $A \in \mathcal{A}$ where $A \cup\{l\}$ is unsatisfiable ( $A$ is viewed as a formula of facts), and (iii) for each not $l \in \mathrm{EP}(\Pi) \backslash \sigma$ and each answer set $A \in \mathcal{A}, A \cup\{l\}$ is satisfiable. Further, set $\mathcal{A}$ of all the answer sets of $\Pi$ is referred to by world view of $\Pi$ w.r.t. $\sigma$, if there is no set $\sigma^{\prime}$ that is a candidate world view of $\Pi$ w.r.t. $\sigma^{\prime}$ such that $\sigma \subsetneq \sigma^{\prime} \subseteq \mathrm{EP}(\Pi)$.

| Problem: | Candidate World View Check [61] |  |
| :--- | :--- | :--- | :--- |
| Input: | Epistemic program $\Pi$, variable $a \in \operatorname{var}(\Pi)$. |  |
| Question: | Does there exist a guess $S \quad \subseteq \quad \operatorname{EP}(\Pi)$ that in- |  |
|  | duces a candidate world view, more precisely, |  |
|  | $\left\{S \quad \mid \quad S \quad \subseteq \quad \operatorname{EP}(\Pi)\right.$, set $\mathcal{A}$ of answer sets of $\Pi^{S}$ |  |
|  | is a candidate world view $\} \neq \emptyset ?$ |  |

## Problem: World View Check [61]

Input: $\quad$ Epistemic program $\Pi$, variable $a \in \operatorname{var}(\Pi)$.
Question: Does there exist a guess $S \subseteq \operatorname{EP}(\Pi)$ that induces a world view, where $a$ is true, i.e., $\{S \mid S \subseteq$ $\mathrm{EP}(\Pi)$, set $\mathcal{A}$ of answer sets of $\Pi^{S}$ is a world view, $a \in$ $A$ for any $A \in \mathcal{A}\} \neq \emptyset$ ?

Problem: \#Projected Guesses to World Views
Input: $\quad$ Epistemic program $\Pi$, variable $a \in \operatorname{var}(\Pi)$, set $P \subseteq \operatorname{var}(\Pi)$ of projection variables.
Output: $\quad$ Number of distinct epistemic literal sets $\mathrm{EP}(\Pi)$ projected to $P$ that lead to world views, where $a$ is true, i.e., $\mid\{S \cap\{$ not $p$, not $\neg p \mid$ $p \in P\} \quad \mid \quad S \in \mathrm{EP}(\Pi)$, the set $\mathcal{A}$ of answer sets of $\Pi^{S}$ is a world view, $a \in A$ for any $\left.A \in \mathcal{A}\right\} \mid$.

[^5]
[^0]:    *johannes.fichte@tu-dresden.de
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[^1]:    ${ }^{1}$ Function $\operatorname{tow}(\ell, k)$ is a tower of iterated exponentials of 2 of height $\ell$ with $k$ on top. More precisely, for integer $k$, we define tow : $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ by $\operatorname{tow}(1, k)=2^{k}$ and $\operatorname{tow}(\ell+1, k)=2^{\operatorname{tow}(\ell, k)}$ for all $\ell \in \mathbb{N}$.

[^2]:    ${ }^{2}$ Pathwidth is similar to treewidth, but admits only certain tree decompositions, whose tree is just a path.

[^3]:    ${ }^{3}$ For the sake of readability, we defer the discussion of formal details on the construction of $\mathcal{T}^{\prime}$ to the proof of Lemma 23 .

[^4]:    ${ }^{4}$ For a definition of the primal graph of a program, we refer to 42.

[^5]:    ${ }^{5}$ Note that there are several ways to treat potential occurrences of double negation, which are out of the scope of this work.

