# Deciding Differential Privacy for Programs with Finite Inputs and Outputs 

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#### Abstract

Differential privacy is a de facto standard for statistical computations over databases that contain private data. Its main and rather surprising strength is to guarantee individual privacy and yet allow for accurate statistical results. Thanks to its mathematical definition, differential privacy is also a natural target for formal analysis. A broad line of work develops and uses logical methods for proving privacy. A more recent and complementary line of work uses statistical methods for finding privacy violations. Although both lines of work are practically successful, they elide the fundamental question of decidability.

This paper studies the decidability of differential privacy. We first establish that checking differential privacy is undecidable even if one restricts to programs having a single Boolean input and a single Boolean output. Then, we define a non-trivial class of programs and provide a decision procedure for checking the differential privacy of a program in this class. Our procedure takes as input a program $P$ parametrized by a privacy budget $\epsilon$ and either establishes the differential privacy for all possible values of $\epsilon$ or generates a counter-example. In addition, our procedure works for both to $\epsilon$-differential privacy and $(\epsilon, \delta)$-differential privacy. Technically, the decision procedure is based on a novel and judicious encoding of the semantics of programs in our class into a decidable fragment of the first-order theory of the reals with exponentiation. We implement our procedure and use it for (dis)proving


privacy bounds for many well-known examples, including randomized response, histogram, report noisy max and sparse vector.

## 1 Introduction

Differential privacy [19] is a gold standard for the privacy of statistical computations. Differential privacy ensures that running the algorithm on any two "adjacent" databases yields two "approximately" equal distributions, where two databases are adjacent if they differ in a single element, and two distributions are approximately equivalent if their distance is small w.r.t. some metric specified by privacy parameter $\epsilon$ and error parameter $\delta$. Thus, differential privacy delivers a very strong form of individual privacy. Yet, and somewhat surprisingly, it is possible to develop differentially private algorithms for many tasks. Moreover, the algorithms are useful, in the sense that their results have reasonable accuracy. However, designing differentially private algorithms is difficult, and the privacy analysis can be error-prone, as witnessed by the example of the sparse vector technique.

This difficulty has motivated the development of formal approaches for analyzing differentially private algorithms (see [6] for a survey and the related work section of this paper). Broadly, two successful lines of work have emerged. The first line of work develops sound proof systems to establish differential privacy and uses these proof systems to prove the privacy of well-known and intricate examples [32, 22, 7, 5, 4, 4, 35, 17, 1, 34]. The second line of work searches for counter-examples to demonstrate the violation of differential privacy [18, 9]. Unfortunately, both lines of work elide the question of decidability. As previous experience in formal verification suggests, understanding decidable fragments of a problem not only help advance our theoretical knowledge, but can form the basis of practical tools when combined with ideas like abstraction and composition.

The goal of this paper is, therefore, to study the decision problem for differential privacy, and to make a first attempt at delineating the decidability/undecidability boundary. As a first contribution, we show that, as expected, checking differential privacy is computationally undecidable. Our undecidability result holds even if one restricts to programs having a single Boolean input and a single Boolean output. Given the undecidability result, we then consider the task of identifying a rich class of programs, that encompasses many known examples, for which checking differential privacy nonetheless is decidable. We impose two desiderata:

1. the class of programs must include programs with real-valued vari-
ables, and more generally, with variables over infinite domains. This requirement is critical for the method to cover a broad class of differential privacy algorithms;
2. the programs themselves are parametrized by the privacy parameter $\epsilon$ (throughout the paper, we assume that the error parameter $\delta$ is a function of $\epsilon$ ), and the decision procedure should decide privacy for all possible instances of the privacy parameter $\epsilon$. This requirement is motivated by the fact, supported by practice, that differential privacy algorithms are typically parametrized by $\epsilon$, and well-designed algorithms are private not only for a single value of $\epsilon$, but typically for all positive values of $\epsilon$.

We focus our attention on programs whose input and output spaces are finite. Note that such programs need not be finite-state, as per our first requirement, they could use program variables ranging over infinite (even uncountable) domains to carry out the computation. We introduce a class of programs, called DiPWhile, which are probabilistic while programs, for which the problem of checking differential privacy is decidable. We succeed in carefully balancing decidability and expressivity, by judiciously delineating the use of real-valued and integer-valued variables. Intuitively, the main restriction we impose is that these infinite-valued variables be used only to directly influence the program control-flow and not the data-flow that leads to the computation of the final output. More precisely, in an execution, the program output value depends only on the input, values sampled from user-defined distributions and the exponential mechanism, and branch conditions on the control flow path taken. The sampled values of real/integer variables affect only the branch conditions. Thus, the output values depend only on the branch conditions satisfied by the sampled real/integer variable values, but not on their actual sampled values. This restriction, though severe, turns out to capture many prominent differential privacy algorithms, including Report Noisy Max and Sparse Vector Technique (see Section 8 on experiments).

Key observations that enable us to establish decidability of DiPWhile programs are as follows. The first result is that the semantics of DiPWhileprograms can be defined using parametrized, finite-state Markov chains 1 . The fact that the semantics is definable using only finitely-many states is a surprising observation because our programs have both integer and real-

[^0]valued variables, and hence a naïve semantics yields uncountably many possible states. Our crucial insight here is that a precise semantics for DiPWhile-programs is possible without tracking the explicit values of the real and integer-valued variables. Since real and integer variables are intuitively used only in influencing control-flow, the semantics only tracks the symbolic relationships between the variables. Second, we show that the transition probabilities of the Markov chain are ratios of polynomial functions in $\epsilon$ and $e^{\epsilon}$, where $e$ is the Euler's constant; this was a difficult result to establish. These two observations together, allow us to reduce the problem of checking the differential privacy of DiPWhile-programs to the decidable fragment of the first-order theory of reals with exponentials, identified by McCallum and Weispfenning [29].

We leverage our decision procedure to build a stand-alone tool for checking $\epsilon$ - or $(\epsilon, \delta(\epsilon))$-differential privacy of mechanisms specified by DiPWhileprograms, for all values of $\epsilon$. We have implemented our decision procedure in a tool that we call DiPC (Differential Privacy Checker). Given DiPWhile-program, our tool constructs a sentence within the McCallumWeispfenning fragment of the theory of reals with exponentials. It then calls Mathematica ${ }^{\circledR}$ ) to check if the constructed sentence is true over the reals. Since our decision procedure is the first that can both prove differential privacy and detect its violation, we tried the tool on examples that known to be differentially private and those that are known to be not differentially private including variants of Sparse Vector, Report Noisy Max, and Histograms. DiPC successfully checked differential privacy for the former class of examples and produced counter-examples for the latter class. Our counter-examples are exact and are more compact than those discovered by prior tools.

As a contribution of independent interest, we also demonstrate how our method yields a theoretical complete under-approximation method for checking differential privacy of programs with infinite output sets. For such programs, it is possible to discretize the output domain into a finite domain, and to use the decision procedure to find privacy violations for the discretized algorithm (by post-processing, privacy violations for the discretized algorithms are also privacy violations for the original algorithm). The discretization yields a method for generating counter-examples for algorithms with infinite output sets.

We briefly contrast our results with prior work, and refer the reader to Section 9 for further details. Overall, we see our decidability results as complementary to prior works in checking differential privacy. In general, existing methods for proving or disproving differential privacy, although in-
herently incomplete due to the undecidability of checking differential privacy, are likely to be more efficient because they can trade-off efficiency for precision. However, the decision procedures for a sub-class of programs, like the one presented here, maybe more predictable - if a decision procedure fails to prove privacy, then it shall produce a counter-example that demonstrates that the algorithm is not differentially private. Moreover, counter-example search methods work for a fixed $(\epsilon)$ privacy parameter. As the counterexample methods are usually statistical, they may generate both false positives and false negatives. In contrast, our decision procedures work for all values for the privacy parameter and do not generate false positives or false negatives.

Contributions. We summarize our key contributions.

- We prove the undecidability of the problem of checking differential privacy of very simple programs, including those that have a single Boolean input and output. Though unsurprising, undecidability has not been previously established in any prior work.
- We prove the decidability of differential privacy for an interesting class of programs. Our method is fully automatic that can check both differential privacy and detect its violation by generating counter-examples. To the best of our knowledge, this is the first such result that encompasses sampling from integer and real-valued variables.
- We implement the decision procedure and evaluate our approach on private and non-private examples from the literature.

Due to lack of space, some proofs and other materials have been moved to an Appendix. The Appendix has been uploaded as an anonymous supplementary submission.

## 2 Primer on differential privacy

Differential privacy [19] is a rigorous definition and framework for private statistical data mining. In this model, a trusted curator with access to the database returns answers to queries made by possibly dishonest data analysts that do not have access to the database. The task of the curator is to return probabilistically noised answers, so that data analysts cannot distinguish between two databases that are adjacent, i.e. only differ in the value of a single individual. There are two common definitions: two
databases are adjacent if they are exactly the same except for the presence or absence of one record, or for the difference in one record. We abstract away from any particular definition of adjacency.

Henceforth, we denote the set of real numbers, rational numbers, natural numbers and integers by $\mathbb{R}, \mathbb{Q}, \mathbb{N}$, and $\mathbb{Z}$ respectively. The Euler constant shall be denoted by $e$. We assume given a set $\mathcal{U}$ of inputs, and a set $\mathcal{V}$ of outputs. A randomized function $P$ from $\mathcal{U}$ to $\mathcal{V}$ is a function that takes an input in $\mathcal{U}$ and returns a distribution over $\mathcal{V}$. For a measurable set $S \subseteq \mathcal{V}$, the probability that the output of $P$ on $u$ is in the set $S$ shall be denoted by $\operatorname{Prob}(P(u) \in S)$. In the case the output set is discrete, we use $\operatorname{Prob}(P(u)=v)$ as shorthand for $\operatorname{Prob}(P(u) \in\{v\})$.

We are now ready to define differential privacy. We assume that $\mathcal{U}$ is equipped with a binary symmetric relation $\Phi \subseteq \mathcal{U} \times \mathcal{U}$, which we shall call the adjacency relation. We say that $u_{1}, u_{2} \in \mathcal{U}$ are adjacent if $\left(u_{1}, u_{2}\right) \in \Phi$.

Definition 1. Let $\epsilon \geq 0$ and $0 \leq \delta \leq 1$. Let $\Phi \subseteq \mathcal{U} \times \mathcal{U}$ be an adjacency relation. Let $P$ be a randomized function with inputs from $\mathcal{U}$ and outputs in $\mathcal{V}$. We say that $P$ is $(\epsilon, \delta)$-differentially private with respect to $\Phi$ if for all measurable subsets $S \subseteq \mathcal{V}$ and $u, u^{\prime} \in \mathcal{U}$ such that $\left(u, u^{\prime}\right) \in \Phi$,

$$
\operatorname{Prob}(P(u) \in S) \leq e^{\epsilon} \operatorname{Prob}\left(P\left(u^{\prime}\right) \in S\right)+\delta
$$

As usual, we say that $P$ is $\epsilon$-differentially private iff it is $(\epsilon, 0)$-differentially private. If the output domain is discrete, it is equivalent to require that for all $v \in \mathcal{V}$ and $u, u^{\prime} \in \mathcal{U}$ such that $\left(u, u^{\prime}\right) \in \Phi$,

$$
\operatorname{Prob}(P(u)=v) \leq e^{\epsilon} \operatorname{Prob}\left(P\left(u^{\prime}\right)=v\right)
$$

Differential privacy is preserved by post-processing. Concretely, if $P$ is an $(\epsilon, \delta)$-differentially private computation from $\mathcal{U}$ to $\mathcal{V}$, and $h: \mathcal{V} \rightarrow \mathcal{W}$ is a deterministic function, then $h \circ P$ is an $(\epsilon, \delta)$-differentially private computation from $\mathcal{U}$ to $\mathcal{W}$. In the remainder, we shall exploit post-processing to connect differential privacy of randomized computations with infinite output spaces to differential privacy of their discretizations.

Laplace Mechanism. The Laplace mechanism [19] achieves differential privacy for numerical computations by adding random noise to outputs. Given $\epsilon>0$ and mean $\mu$, let $\operatorname{Lap}(\epsilon, \mu)$ be the continuous distribution whose probability density function (p.d.f.) is given by

$$
f_{\epsilon, \mu}(x)=\frac{\epsilon}{2} e^{-\epsilon|x-\mu|} .
$$

$\operatorname{Lap}(\epsilon, \mu)$ is said to be the Laplacian distribution with mean $\mu$ and scale parameter $\frac{1}{\epsilon}$. Consider a real-valued function $q: \mathcal{U} \rightarrow \mathbb{R}$. Assume that $q$ is $k$-sensitive w.r.t. an adjacency relation $\Phi$ on $\mathcal{U}$, i.e. for every pair of adjacent values $u_{1}$ and $u_{2},\left|q\left(u_{1}\right)-q\left(u_{2}\right)\right| \leq k$. Then the computation that maps $u$ to $\operatorname{Lap}\left(\frac{\epsilon}{k}, q(u)\right)$ is $\epsilon$-differentially private.

It is sometimes convenient to consider the discrete version of the Laplace distribution. Given $\epsilon>0$ and mean $\mu$, let $\operatorname{DLap}(\epsilon, \mu)$ be the discrete distribution on $\mathbb{Z}$, the set of integers, whose probability mass function (p.m.f.) is

$$
f_{\epsilon, \mu}(i)=\frac{1-e^{-\epsilon}}{1+e^{-\epsilon}} e^{-\epsilon|i-\mu|} .
$$

$\operatorname{DLap}(\epsilon, \mu)$ is said to be the discrete Laplacian distribution with mean $\mu$ and scale parameter $\frac{1}{\epsilon}$. The discrete Laplace mechanism achieves the same privacy guarantees as the continuous Laplace mechanism.

Exponential mechanism. The Exponential mechanism [30] is used for making non-numerical computations private. The mechanism takes as input a value $u$ from some input domain and a scoring function $F: \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{R}$ and outputs a discrete distribution over $\mathcal{V}$. Formally, given $\epsilon>0$ and $u \in \mathcal{U}$, the discrete distribution $\operatorname{Exp}(\epsilon, F, u)$ on $\mathcal{V}$ is given by the probability mass function:

$$
h_{\epsilon, F, u}(v)=\frac{e^{\epsilon F(u, v)}}{\sum_{v \in \mathcal{V}} e^{\epsilon F(u, v)}} .
$$

Suppose that the scoring function is $k$-sensitive w.r.t. some adjacency relation $\Phi$ on $\mathcal{U}$, i.e., for all for each pair of adjacent values $u_{1}$ and $u_{2}$ and $v \in \mathcal{V},\left|F\left(u_{1}, r\right)-F\left(u_{2}, r\right)\right| \leq k$. Then the exponential mechanism is $(2 k \epsilon, 0)$-differentially private w.r.t. $\Phi$.

## 3 Motivating Example

Before presenting the mathematical details of our results, let us informally introduce our method by showing how it would work on an illustrative example.

Sparse Vector Technique. Several differential privacy examples require that the randomized algorithms sampling from infinite support distributions (including continuous distributions). The Sparse Vector Technique (SVT) [20, 28] was designed to answer multiple $\Delta$-sensitive numerical queries in a differentially private fashion. The relevant information we want from
queries is, which amongst them are above a threshold $T$. The Sparse Vector Technique as given in Algorithm 1 is designed to identify the first $c$ queries that are above the threshold $T$ in an $\epsilon$-differentially private fashion.

```
Input: \(q[1: N]\)
Output: out \([1: N]\)
\(\mathrm{r}_{T} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2 \Delta}, T\right)\)
count \(\leftarrow 0\)
for \(i \leftarrow 1\) to \(N\) do
    \(\mathrm{r} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{4 c \Delta}, q[i]\right)\)
    \(\mathrm{b} \leftarrow \mathrm{r} \geq \mathrm{r}_{T}\)
    if \(b\) then
        out \([i] \leftarrow T\)
        count \(\leftarrow\) count +1
        if count \(\geq c\) then
            exit
        end
    else
        out \([i] \leftarrow \perp\)
    end
end
```

Algorithm 1: SVT algorithm (SVT1)
In the program, the integer $N$ represents the total number of queries, and the array $q$ of length $N$ represents the answers to queries. The array out represents the output array, $\perp$ represents False and $\top$ represents True. We assume that initially the constant $\perp$ is stored at each position in out. In the SVT technique, the $T$ answers account for most of the privacy cost, and we can only answer $c$ of them until we run out of the privacy budget [20, 35]. On the other hand, there is no restriction on the number of $\perp$ answers. Please observe that the SVT algorithm is parametrized by the privacy budget $\epsilon$. Thus, the SVT algorithm can be considered as representing a class of programs, one for each $\epsilon>0$.

Given $N$, the input set $\mathcal{U}$ in this context is the set of $N$ length vectors $q$, where the $k$ th element $q[k]$ represents the answer to the $k$ th query on the original database. The adjacency relation $\Phi$ on inputs is defined as follows: $q_{1}$ and $q_{2}$ are adjacent if and only if $\left|q_{1}[i]-q_{2}[i]\right| \leq 1$ for each $1 \leq i \leq N$.

Let us consider an instance of the SVT algorithm when $T=0, N=2$,
$\Delta=1$ and $c=1$. Let us assume that all array elements in $q$ come from the domain $\{0,1\}$. In this case, we have four possible inputs $[0,0],[0,1],[1,1]$, and $[1,0]$, and three possible outputs $[\perp, \perp],[\top, \perp]$, and $[\perp, T]$.

For example, the probability of outputting $[\perp, \top]$ on input $[0,1]$ can be computed as follows. Let $X_{T}$ be a random variable with Laplacian distribution $\operatorname{Lap}\left(\frac{\epsilon}{2}, 0\right), X_{1}$ be a random variable with Laplacian distribution $\operatorname{Lap}\left(\frac{\epsilon}{4}, 0\right)$ and $X_{2}$ be the random variable with Laplacian distribution $\operatorname{Lap}\left(\frac{\epsilon}{4}, 1\right)$. The probability of outputting $[\perp, \top]$ is the product of outputting of outputting $\perp$ first, which is $\operatorname{Prob}\left(X_{1}<X_{0}\right)$, and the conditional probability of outputting $\top$ given that $\perp$ is output, which is $\operatorname{Prob}\left(X_{2} \geq X_{0} \mid X_{1}<X_{0}\right)$. Note that we really require the second quantity to be conditional probability as the events $X_{1}<X_{0}$ and $X_{2} \geq X_{0}$ are not independent. This probability can be computed to be

$$
r_{1}(\epsilon)=\frac{24 e^{\frac{3 \epsilon}{4}}-1+8 e^{\frac{\epsilon}{4}}+21 e^{\frac{\epsilon}{2}}}{48 e^{\frac{3 \epsilon}{4}}} .
$$

Similarly, when the input is $[1,1]$ and the output is $[\perp, \top]$, the probability is given by

$$
r_{2}(\epsilon)=\frac{-22+32 e^{\frac{\epsilon}{4}}-3 \epsilon}{48 e^{\frac{\epsilon}{2}}} .
$$

Observe that $r_{1}(\epsilon)$ and $r_{2}(\epsilon)$ are functions of $\epsilon$, and hence the probabilities of outputting $[\perp, \top]$ on inputs $[0,1]$ and $[1,1]$ vary with $\epsilon$. Our immediate challenge is to automatically compute expressions like $r_{1}(\epsilon), r_{2}(\epsilon)$ from the given program, the adjacent inputs, and outputs. Note that this example involves sampling from continuous distributions and is a function of $\epsilon$. Nevertheless, we shall establish that (see Section 6 and Theorem (8) that for several programs, the former can be accomplished by interpreting the program as a finite-state DTMC whose transition probabilities are functions parameterized by $\epsilon$ even when the randomized choices involve infinite-support random variables. The set of programs that we identify (Section 66) is rich enough to model the most known differential privacy mechanisms when restricted to finite input and output sets.

Having computed such expressions, checking $\epsilon$-differential privacy requires one to determine if

$$
\begin{array}{ll}
\text { for all } & \epsilon>0 .\left(r_{1}(\epsilon) \leq e^{\epsilon} r_{2}(\epsilon)\right) \\
\text { and for all } & \epsilon>0 .\left(r_{2}(\epsilon) \leq e^{\epsilon} r_{1}(\epsilon)\right) .
\end{array}
$$

Note that the particular condition for the SVT example under consideration above is encodable as a first-order sentence with exponentials, and thus checking the formula for the example reduces to determining if such
a first-order sentence is valid for reals, with the standard interpretation of multiplication, addition, and exponentiation. Whether there is a decision procedure that can determine the truth of first-order sentences involving real arithmetic with exponentials, is a long-standing open problem. However, a decidable fragment of such an extended first-order theory has been identified by McCallum and Weispfenning [29]. The formula for the considered example lies in this fragment. Indeed, we can show that all the formulas for the SVT example lie in this fragment. This observation presents a challenge, namely, what guarantees do we have that checking differential privacy is reducible to this decidable fragment. Indeed, we shall establish that the set of formulas that arise from the class of programs with finite-state DTMC semantics in Theorem 8 also lead to formulas in the same decidable fragment.

Remark. Notice that if one can compute expressions for the probability producing individual outputs on a given input, we could also check $(\epsilon, \delta)$ differential privacy, instead of just $\epsilon$-differential privacy. The only change would be to account for $\delta$ in our constraints, and to consider all possible subsets of outputs, instead of just individual output values. Thus, the methods proposed here go beyond the scope of most automated approaches, which are restricted to vanilla $\epsilon$-differential privacy.

## 4 Preliminaries

In this section, we formally define the problem of differential privacy verification that we consider in this paper and also introduce the decidable fragment of real arithmetic with exponentiation that plays a crucial role in our decision procedure. The set of reals/positive reals/rationals/positive rationals shall be denoted by $\mathbb{R} / \mathbb{R}^{>0} / \mathbb{Q} / \mathbb{Q}^{>0}$ respectively.

### 4.1 The Computational Problem

As illustrated by the example in Section 3, a differential privacy mechanism is typically a randomized program $P_{\epsilon}$ parametrized by a variable $\epsilon$. Having a parameterized program $P_{\epsilon}$ captures the fact that the program's behavior depends on the privacy budget $\epsilon$, intending to guarantee that $P_{\epsilon}$ is $(f(\epsilon), g(\epsilon))$-differentially private, where $f$ and $g$ are some functions of $\epsilon$. The parameter $\epsilon$ is assumed to belong to some interval $I \subseteq \mathbb{R}^{>0}$ with rational end-points; usually, we take $\epsilon$ to just belong to the interval $(0, \infty)$. The program $P_{\epsilon}$ shall be assumed to terminate with probability 1 for every value of $\epsilon$ (in the appropriate interval).

The randomized program $P_{\epsilon}$ takes inputs from a set $\mathcal{U}$ and produces output in a set $\mathcal{V}$. In this paper, we shall assume that both $\mathcal{U}$ and $\mathcal{V}$ are $f_{i}$ nite sets that can be effectively enumerated. Despite our restriction to finite input and output sets, the computational problem of checking differential privacy is challenging (see Section 5.3). At the same time, the decidable subclass we identify (Section 6) is rich enough to model most differential privacy mechanisms when restricted to finite input and output sets. Extending our decidability results to subclasses of programs that have infinite input and output sets, is a non-trivial open problem at this time.

The computational problems we consider in this paper are as follows. Since our programs take inputs from a finite set $\mathcal{U}$, we assume that the adjacency relation $\Phi \subseteq \mathcal{U} \times \mathcal{U}$ is given as an explicit list of pairs. In general, when discussing $(\epsilon, \delta)$-differential privacy of some mechanism, the error parameter $\delta$ needs to be a function of $\epsilon$. To define the computational problem of checking differential privacy, the function $\delta: \mathbb{R}^{>0} \rightarrow[0,1]$ must be given as input. We, therefore, assume that this function $\delta$ has some finite representation; if $\delta$ is the constant $\delta_{0}$ (which is often the case), then we represent $\delta$ simply by the number $\delta_{0}$. There are two computational problems we consider in this paper.

Fixed Parameter Differential Privacy Given a program $P_{\epsilon}$ over inputs $\mathcal{U}$ and outputs $\mathcal{V}$, adjacency relation $\Phi \subseteq \mathcal{U} \times \mathcal{U}$, and positive rational numbers $\epsilon_{0}, \delta_{0}, t \in \mathbb{Q}^{>0}$, determine if $P_{\epsilon_{0}}$ is $\left(t \epsilon_{0}, \delta_{0}\right)$-differentially private with respect to $\Phi$.

Differential Privacy Given a program $P_{\epsilon}$ over inputs $\mathcal{U}$ and outputs $\mathcal{V}$, interval $I \subseteq \mathbb{R}^{>0}$ with rational end-points, $\delta: \mathbb{R}^{>0} \rightarrow[0,1]$, an adjacency relation $\Phi \subseteq \mathcal{U} \times \mathcal{U}$, and a rational number $t \in \mathbb{Q}^{>0}$, determine if $P_{\epsilon}$ is $(t \epsilon, \delta(\epsilon)$ )-differentially private with respect to $\Phi$ for every $\epsilon \in I$.

Observe that the Fixed Parameter Differential Privacy problem can be trivially reduced to the Differential Privacy problem by considering the singleton interval $I=\left[\epsilon_{0}, \epsilon_{0}\right]$ and $\delta(\epsilon)=\delta_{0}$, where the goal is to check fixed parameter differential privacy for constant privacy budget $\epsilon_{0}$ and error parameter $\delta_{0}$. Thus, an algorithm for checking Differential Privacy can be used to solve Fixed Parameter Differential Privacy. Unfortunately, the Fixed Parameter Differential Privacy problem is extremely challenging even when restricted to finite input and output sets - we show that it is undecidable (Section 5.3), and therefore, so is the Differential Privacy problem. We shall identify a class of programs (Section 6i) for which the Differential Privacy
problem (and therefore the Fixed Parameter Differential Privacy problem) is decidable.

When the differential privacy does not hold, we would like to output a counter-example.

Definition 2. A counter-example of $(\epsilon, \delta)$ differential privacy for $P_{\epsilon}$, with respect to an adjacency relation $\Phi$, a function $\delta: \mathbb{R}^{>0} \rightarrow[0,1]$ and a value $t \in \mathbb{Q}^{>0}$, is a quadruple (in, $\mathbf{i n}^{\prime}, O, \epsilon_{0}$ ) such that (in, in') $\in \Phi, O \subseteq \mathcal{V}$ and $\epsilon_{0}>0$ and

$$
\operatorname{Prob}\left(P_{\epsilon_{0}}(\mathbf{i n}) \in O\right)>e^{t \epsilon_{0}} \operatorname{Prob}\left(P\left(\mathbf{i n}^{\prime}\right) \in O\right)+\delta\left(\epsilon_{0}\right)
$$

When $\delta$ is the constant function 0 , then $O$ is $\{$ out $\}$ for some out $\in \mathcal{V}$.
Remark. For the rest of the paper, unless otherwise stated, we shall assume that the interval $I \subseteq \mathbb{R}^{>0}$ that contains the set of admissible $\epsilon \mathrm{S}$ is the interval $(0, \infty)$. In our paper, $\epsilon$ refers to the parameter in program $P_{\epsilon}$, and not the privacy budget. In our case, the privacy budget is $(t \epsilon)$. For example, some differential privacy algorithms $P_{\epsilon}$ are designed to satisfy ( $\frac{\epsilon}{2}, 0$ )-differential privacy, and so in this case $t$ would be $\frac{1}{2}$. In the standard differential privacy definition " $\epsilon$ " refers to the privacy budget and so $t$ does not appear. However, many theorems for differential privacy algorithms use " $\epsilon$ " as the program parameter, and then the privacy theorem is stated as the program being $(t \epsilon, \delta)$-differentially private. In most such cases, such a theorem is equivalent to saying that the program $P_{\epsilon}$ (obtained by replacing $\epsilon$ by $\frac{\epsilon}{t}$ ) is $\left(\epsilon, \delta\left(\frac{\epsilon}{t}\right)\right.$ )differentially private.

### 4.2 Reals with exponentials

As outlined in Section 3, our approach towards deciding differential privacy shall rely on reducing the question to the problem of checking the truth of a first-order sentence for the reals. Because of the definition of differential privacy, the constructed first-order sentence shall involve exponentials. It is a long-standing open problem whether there is a decision procedure for the first-order theory of reals with exponentials. However, some fragments of this theory are known to be decidable. In particular, there is a fragment identified by McCallum and Weispfenning [29], that we shall exploit in our results.

We will consider first-order formulas over a restricted signature and vocabulary. We will denote this collection of formulas as the language $\mathcal{L}_{\text {exp }}$. Formulas in $\mathcal{L}_{\text {exp }}$ are built using variables $\{\epsilon\} \cup\left\{x_{i} \mid i \in \mathbb{N}\right\}$, constant symbols 0,1 , unary function symbol $e^{(\cdot)}$ applied only to the variable $\epsilon$, binary
function symbols,,$+- \times$, and binary relation symbols $=,<$. The terms in the language are integral polynomials with rational coefficients over the variables $\{\epsilon\} \cup\left\{x_{i} \mid i \in \mathbb{N}\right\} \cup\left\{e^{\epsilon}\right\}$. Atomic formulas in the language are of the form $t=0$ or $t<0$ or $0<t$, where $t$ is a term. Quantifier free formulas are Boolean combinations of atomic formulas. Sentences in $\mathcal{L}_{\text {exp }}$ are formulas of the form

$$
Q \epsilon Q_{1} x_{1} \cdots Q_{n} x_{n} \psi\left(\epsilon, x_{1}, \ldots, x_{n}\right)
$$

where $\psi$ is a quantifier free formula, and $Q, Q_{i} \mathrm{~s}$ are quantifiers. In other words, sentences are formulas in prenex form, where all variables are quantified, and the outermost quantifier is for the special variable $\epsilon$.

The theory $\mathrm{Th}_{\text {exp }}$ is the collection of all sentences in $\mathcal{L}_{\text {exp }}$ that are valid in the structure $\left\langle\mathbb{R}, 0,1, e^{(\cdot)},+,-, \times,=,<\right\rangle$, where the interpretation for $0,1,+,-, \times$ is the standard one on reals, and $e$ is Euler's constant; notice that this is an extension of the first-order theory of reals. The crucial property about this theory is that it is decidable.

Theorem 3 (McCallum-Weispfenning [29]). Th $_{\text {exp }}$ is decidable.
Finally, our tractable restrictions (and our proofs of decidability) shall often utilize the notion of functions definable in $\mathrm{Th}_{\text {exp }}$; we, therefore, conclude this section with its formal definition.

Definition 4. A function $f:(0, \infty) \rightarrow \mathbb{R}$ is said to be definable in $T h_{\text {exp }}$, if there is a formula $\varphi_{f}(\epsilon, x)$ in $\mathcal{L}_{\text {exp }}$ with two free variables $(\epsilon$ and $x)$ such that

$$
\begin{aligned}
& \text { for all } a \in(0, \infty) . f(a)=b \text { iff } \\
& \quad\left\langle\mathbb{R}, 0,1, e^{(\cdot)},+,-, \times,=,<\right\rangle \models \varphi_{f}(\epsilon, x)[\epsilon \mapsto a, x \mapsto b]
\end{aligned}
$$

## 5 Program syntax and semantics

We consider randomized algorithms written as simple probabilistic while programs. We introduce the syntax of these programs, along with their "natural" semantics given using Markov kernels [15, 31]. We show that the problem of checking differential privacy is undecidable for these programs.

### 5.1 Syntax of Simple programs

We introduce a class of programs we call Simple. Programs in Simple are probabilistic while programs in which variables can be assigned values by drawing from distributions typically used in differential privacy algorithms.

Expressions ( $\mathrm{b} \in \mathcal{B}, \mathrm{x} \in \mathcal{X}, \mathrm{z} \in \mathcal{Z}, \mathrm{r} \in \mathcal{R}, d \in \mathrm{DOM}, i \in \mathbb{Z}, q \in \mathbb{Q}, g \in$ $\left.\mathcal{F}_{\text {Bool }}, f \in \mathcal{F}_{\text {DOM }}\right):$

$$
\begin{array}{ll}
B & ::=\text { true } \mid \text { false }|\mathrm{b}| \operatorname{not}(B) \mid B \text { and } B \mid B \text { or } B \mid g(\tilde{E}) \\
E & ::=d|\mathrm{x}| f(\tilde{E}) \\
Z & ::=\mathrm{z}|i Z| E Z|Z+Z| Z+i \mid Z+E \\
R & ::=\mathrm{r}|q R| E R|R+R| R+q \mid R+E
\end{array}
$$

Basic Program Statements $\left(a \in \mathbb{Q}^{>0}, \sim \in\{<,>,=, \leq, \geq\}, F\right.$ is a scoring function and choose is a user-defined distribution):

$$
\begin{aligned}
s::= & \mathrm{x} \leftarrow E|\mathrm{z} \leftarrow Z| \mathrm{r} \leftarrow R|\mathrm{~b} \leftarrow B| \mathrm{b} \leftarrow Z_{1} \sim Z_{2} \mid \\
& \mathrm{b} \leftarrow Z \sim E\left|\mathrm{~b} \leftarrow R_{1} \sim R_{2}\right| \mathrm{b} \leftarrow R \sim E \mid \\
& \mathrm{r} \leftarrow \operatorname{Lap}(a \epsilon, E)|\mathrm{z} \leftarrow \operatorname{DLap}(a \epsilon, E)| \\
& \mathrm{x} \leftarrow \operatorname{Exp}(a \epsilon, F(\tilde{\mathrm{x}}), E) \mid \mathrm{x} \leftarrow \text { choose }(a \epsilon, \tilde{E}) \mid \\
& \text { if } B \text { then } P \text { else } P \text { end } \mid \text { While } B \text { do } P \text { end } \mid \text { exit }
\end{aligned}
$$

Program Statements ( $\ell \in$ Labels)

$$
P::=\ell: s \mid \ell: s ; P
$$

Figure 1: BNF grammar for Simple. DOM is a finite discrete domain. $\mathcal{F}_{\text {Bool }},\left(\mathcal{F}_{\text {DOM }}\right.$ resp) are set of functions that output Boolean values (DOM respectively). $\mathcal{B}, \mathcal{X}, \mathcal{Z}, \mathcal{R}$ are the sets of Boolean variables, DOM variables, integer random variables and real random variables. Labels is a set of program labels. For a syntactic class $S, \tilde{S}$ denotes a sequence of elements from $S$. DiPWhile (see Section 6) is the subclass of programs in which the assignments to real and integer variables do not occur with the scope of a while statement.

Programs in Simple obey some syntactic restrictions; these syntactic restrictions are introduced to make it easier to describe the decidable fragment in Section 6. Despite these restrictions, the problem of checking differential privacy is undecidable for the language introduced here.

The formal syntax of Simple programs is shown in Figure 1. Programs have four types of variables: $B$ ool $=\{$ true, false $\}$; finite domain DOM 2 that we assume (without loss of generality) to be $\left\{-N_{\max }, \ldots 0,1, \ldots N_{\max }\right\}$, a finite subset of integers 3 ; reals $\mathbb{R}$; and integers $\mathbb{Z}$. The set of Boolean/DOM/ integer/real program variables are respectively denoted by $\mathcal{B} / \mathcal{X} / \mathcal{Z} / \mathcal{R}$. The set of Boolean/DOM/integer/real expressions is given by the non-terminal $B / E / Z / R$ in Figure 1. We now explain the rules for such expressions. Boolean expressions $(B)$ can be built using Boolean variables and constants, standard Boolean operations, and by applying functions from $\mathcal{F}_{\text {Bool }} . \mathcal{F}_{\text {Bool }}$ is assumed to be a collection of computable functions returning a Bool. We assume that $\mathcal{F}_{\text {Bool }}$ always contains a function $\mathrm{EQ}\left(x_{1}, x_{2}\right)$ that returns true iff $x_{1}$ and $x_{2}$ are equal. DOM expressions $(E)$ are similarly built from DOM variables, values in DOM, and applying functions from set of computable functions $\mathcal{F}_{\text {DOM }}$. Next, integer expressions $(Z)$ are built using multiplication and addition with integer constants and DOM expressions, and additions with other integer expressions. Finally, real expressions $(R)$ are built using multiplication and addition with rational constants and DOM expressions, and additions with other real-valued expressions. Notice that integer-valued expressions cannot be added or multiplied, in real-valued expressions; this syntactic restriction shall be useful later.

A program in Simple is a triple consisting of a set of (private) input variables, a set of (public) output variables, and a finite sequence of labeled statements (non-terminal $P$ in Figure (1). The private input variables and public output variables take values from the domain DOM. Thus, the set of possibles inputs/outputs $(\mathcal{U} / \mathcal{V})$, is identified with the set of valuations for input/output variables; a valuation over a set of variables $X^{\prime}=\left\{\mathrm{x}_{1}, \mathrm{x}_{2}, \ldots, \mathrm{x}_{m}\right\} \subseteq \mathcal{X}$ is a function from $X^{\prime}$ to DOM. Note that if we represent the set $X^{\prime}$ as a sequence $\mathrm{x}_{1}, \mathrm{x}_{2}, \ldots, \mathrm{x}_{m}$ then a valuation val over $X^{\prime}$ can be viewed as a sequence $\operatorname{val}\left(\mathrm{x}_{1}\right), \operatorname{val}\left(\mathrm{x}_{2}\right), \ldots, \operatorname{val}\left(\mathrm{x}_{m}\right)$ of DOM elements.

We assume every statement in our program is uniquely labeled from a set of labels called Labels. Basic program statements (non-terminal $s$ ) can either be assignments, conditionals, while loops, or exit. Statements other than as-

[^1]signments are self-explanatory. The syntax of assignments is designed to follow a strict discipline. Real and integer variables can either be assigned the value of real/integer expression or samples drawn using the Laplace or discrete Laplace mechanism. DOM variables are either assigned values of DOM expressions or values drawn either using an exponential mechanism $(\operatorname{Exp}(a \epsilon, F(\tilde{\mathrm{x}}), E))$ or a user-defined distribution (choose $(a \epsilon, \tilde{E}))$. For the exponential mechanism, we require that the scoring function $F$ be computable and return a rational value. Both of these restrictions are unlikely to be severe in practice. In the case of the user defined distribution, we demand that the probability with which a value $d$ in DOM is chosen (as a function of the privacy budget $\epsilon$ ), be definable in $\mathrm{Th}_{\text {exp }}$, and that there is an algorithm that on input $a, \tilde{d}, v$ returns the formula defining the probability of sampling $d \in \mathrm{DOM}$ from the distribution choose $(a \epsilon, \tilde{d})$ where $\tilde{d}$ is a sequence of values from DOM. This restriction is exploited in Section 6 to get decidability for a sub-fragment.

Finally, we consider assignments to Boolean variables. The interesting cases are those where the Boolean variable stores the result of the comparison of two expressions. The syntax does not allow for comparing real and integer expressions. This restriction is exploited later in Section 6 when the decidable fragment is identified. Finally, we will assume that in any execution, if a variable appears on the right side of an assignment statement, then it should have been assigned a value before. This assumption is not restrictive but is technically convenient when defining the semantics for programs.

### 5.2 Markov Kernel Semantics

We briefly sketch a "natural" semantics for Simple using Markov kernels. A key step in proving our decidability result is to define a semantics using finite-state (parametrized) DTMCs for the sub-fragment DiPWhile defined in Section 6. The DTMC semantics may not seem natural on first reading. The point of the semantics in this section is, therefore, to argue the correctness of our decision procedure on the basis of the equivalence of these two semantics for DiPWhile (Sections 6 and 7). Details for this section are given in Appendix A due of space constraints and because understanding this semantics is not critical to our decidability proof.

Given a fixed $\epsilon>0$, the states in the Markov kernel-based semantics for a program $P_{\epsilon}$ will be of the form ( $\ell, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}, h_{\mathbb{R}}$ ), where $\ell$ is the label of the statement of $P_{\epsilon}$ to be executed next, the functions $h_{\text {Bool }}$, $h_{\mathrm{DOM}}, h_{\mathbb{Z}}$ and $h_{\mathbb{R}}$ assign values to the Boolean, DOM, real and integer vari-
ables of the program $P_{\epsilon}$ respectively. Given an input state in, the initial state will correspond to one where DOM-valued input variables get the values given in in, and all other variables either get false or 0 , depending on their type. Observe that for a program $P_{\epsilon}$ with $k$ program statements, $i$ Boolean variables, $j$ DOM variables, $s$ integer variables, $t$ real variables a state $\left(\ell, h_{\text {Bool }}, h_{\text {DOM }}, h_{\mathbb{Z}}, h_{\mathbb{R}}\right)$ can be uniquely identified with an element of the set $D_{P_{\epsilon}}=\{1, \ldots, k\} \times \mathcal{F}_{\text {Bool }}^{i} \times \mathrm{DOM}^{j} \times \mathbb{Z}^{s} \times \mathbb{R}^{t}$. The "natural" Borel $\sigma$-algebra on $D_{P_{\epsilon}}$ induces a $\sigma$-algebra on the states of $P_{\epsilon}$.

The semantics of Simple programs can be defined as a Markov kernel over this $\sigma$-algebra on states. Intuitively, the Markov kernel $K_{\epsilon}$ corresponding to a program $P_{\epsilon}$ is such that for a state $s$ and a measurable set of states $C, K_{\epsilon}(s, C)$ is the probability of transitioning to a state in $C$ from $s$. The precise definition of this Markov kernel is in Appendix A.

Executions are just sequences of states, and the $\sigma$-field on executions is the product of the $\sigma$-field on states. The Markov kernel defines a probability measure on this $\sigma$-field. Given all these observations, we take $\operatorname{Prob}_{\text {natural }}\left(P_{\epsilon}(\mathbf{i n})=\right.$ out) to denote the probability (as defined by the Markov kernel of $P_{\epsilon}$ ) of the set of all executions that start in the initial state corresponding to in and end in an exit state with out as the valuation of output variables; the precise definition is in Appendix A. For the rest of the paper, we will assume that our programs terminate with probability 1.

### 5.3 Undecidability

The problem of checking differential privacy for Simple programs is undecidable.

Theorem 5. The Fixed Parameter Differential Privacy problem and the Differential Privacy problem for programs $P_{\epsilon}$ in Simple is undecidable.

The proof of Theorem 5 reduces the non-halting problem for deterministic 2-counter Minsky machines to the Fixed Parameter Differential Privacy problem. More precisely, we show that given a 2 -counter Minsky machine $\mathcal{M}$ (with no input), there is a program $P_{\epsilon}^{\mathcal{M}} \in$ Simple such that

- $P_{\epsilon}^{\mathcal{M}}$ has only one input $x_{\text {in }}$ and one output $x_{\text {out }}$ taking values in $\mathrm{DOM}=\{0,1\}$;
- $P_{\epsilon}^{\mathcal{M}}$ terminates with probability 1 for all $\epsilon \in \mathbb{R}^{>0}$;
- $P_{\epsilon}^{\mathcal{M}}$ is $(\epsilon, 0)$-differentially private with respect to the adjacency relation $\Phi=\{(0,1),(1,0)\}$ if and only if $\mathcal{M}$ does not halt.

This construction shows that Differential Privacy is undecidable. Undecidability of Fixed Parameter Differential Privacy is obtained by taking $\epsilon$ to be any constant rational number, say $\frac{1}{2}$. The formal details of the reduction are in Appendix B.

## 6 DiPWhile: A decidable class of programs

We now discuss a restricted class of programs, for which we can establish decidability of checking differential privacy. The class of programs that we consider are exactly those programs in Simple that satisfy the following restriction:

Bounded Assignments We do not allow assignments to real and integer variables within the scope of a while loop. This restriction ensures that assignments to such variables happen only a bounded number of times during execution. Thus, without loss of generality, we assume that real and integer variables are assigned at most once as a program with multiple assignments to a single real and variables can always be rewritten to an equivalent program with each assignment to a variable being an assignment to a fresh variable.

We refer to this restricted class as DiPWhile. The DiPWhile language is surprisingly expressive - many known randomized algorithms for differential privacy can be encoded. We give an example of such encodings in DiPWhile. We omit labels of program statements unless they are needed.

Example 6. Algorithm 2 shows how SVT can be encoded in our language with $T=0, \Delta=1, N=2, c=1$. In the example we are modeling $\perp$ by 0 and $T$ by 1 . Though for-loops are not part of our program syntax, they can modeled as while loops, or if bounded (like here), they can be unrolled.

Appendix $C$ shows how sampling from the standard exponential distribution can be encoded in DiPWhile. Other examples that can be encoded in our language (and for which the decision procedure applies) include randomized response, the private smart sum algorithm [10] with finite discretization of the output space (See 7.1), and private vertex cover [24].

The decidability of checking differential privacy for DiPWhile shall rely on two observations. First, the semantics of DiPWhile programs can also be defined as finite-state discrete-time Markov chains (DTMC), albeit with transition probabilities parameterized by $\epsilon$. This observation is surprising because DiPWhile programs have real and integer values variables, and so

## Input: $q_{1}, q_{2}$ <br> Output: out ${ }_{1}$, out $_{2}$

$1 T \leftarrow 0 ;$
2 out $_{1} \leftarrow 0$;
3 out $_{2} \leftarrow 0$;
$\mathbf{r}_{T} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2}, T\right) ;$
$5 \mathrm{r}_{1} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{4}, q_{1}\right)$;
$6 \mathrm{~b} \leftarrow \mathrm{r}_{1} \geq \mathrm{r}_{T}$;
7 if $b$ then

```
            out }\mp@subsup{\mp@code{1}}{\leftarrow1}{\leftarrow
```

    else
        \(r_{2} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{4}, q_{2}\right) ;\)
        \(\mathrm{b} \leftarrow \mathrm{r}_{2} \geq \mathrm{r}_{T}\);
        if \(b\) then
            out \(_{2} \leftarrow 1\)
        end
    end
    13 exit
Algorithm 2: SVT for 1 -sensitive queries with $N=2, c=1$ and $T=0$. The numbers at the beginning of a line indicate the label of the statement.


Figure 2: Partial DTMC semantics of Algorithm 2 showing the steps when lines 9 and 10 are executed. $q_{1}$ and $q_{2}$ are assumed to have values $u$ and $v$, respectively. Only values of assigned program variables is shown. Third line in state shows parameters for the real values that were sampled. Last line shows the accumulated set of Boolean conditions that hold on the path.
the natural semantics has uncountably many states (See Section 5.2). The key insight in establishing this observation is that an equivalent semantics of DiPWhile programs can be defined without explicitly tracking the values of real and integer-valued variables. Second, all the transition probabilities arising in our semantics are definable in $\mathrm{Th}_{\text {exp }}$. These two observations allow us to to establish decidability of checking differential privacy of DiPWhile programs. The rest of the section is devoted to establishing these observations. We start by formally defining parametrized DTMCs.

### 6.1 Parameterized DTMCs

Definition 7. A parametrized $D T M C$ is a pair $\mathcal{D}=(Z, \Delta)$, where $Z$ is a (countable) set of states, and $\Delta: Z \times Z \rightarrow\left(\mathbb{R}^{>0} \rightarrow[0,1]\right)$ is the probabilistic transition function. For any pair of states $z, z^{\prime}, \Delta$ returns a function from $\mathbb{R}^{>0}$ to $[0,1]$, such that for every $\epsilon>0, \sum_{z^{\prime} \in Z} \Delta\left(z, z^{\prime}\right)(\epsilon)=1$. We shall call $\Delta\left(z, z^{\prime}\right)$ as the probability of transitioning from $z$ to $z^{\prime}$.

A definable parametrized DTMC is a parametrized DTMC $\mathcal{D}=(Z, \Delta)$ such that for every pair of states $z, z^{\prime} \in Z$, the function $\Delta\left(z, z^{\prime}\right)$ is definable in $T h_{\text {exp }}$.

A parametrized DTMC associates with each (finite) sequence of states $\rho=z_{0}, z_{1}, \ldots z_{m}$, a function $\operatorname{Prob}(\rho): \mathbb{R}^{>0} \rightarrow[0,1]$ that given an $\epsilon>0$, returns the probability of the sequence $\rho$ when the parameter's value is fixed to $\epsilon$, i.e., $\operatorname{Prob}(\rho)(\epsilon)=\prod_{i=0}^{m-1} \Delta\left(z_{i}, z_{i+1}\right)(\epsilon)$. For a state $z_{0}$ and a set of states $Z^{\prime} \subseteq Z$, once again we have a function that given a value $\epsilon$ for the parameter, returns the probability of reaching $Z^{\prime}$ from $z_{0}$. This can be formally defined as $\operatorname{Prob}\left(z_{0}, Z^{\prime}\right)(\epsilon)=\sum_{\rho \in z_{0}\left(Z \backslash Z^{\prime}\right)^{*} Z^{\prime}} \operatorname{Prob}(\rho)(\epsilon)$. In other words, $\operatorname{Prob}\left(z_{0}, Z^{\prime}\right)(\epsilon)$ is the sum of the probability of all sequences starting in $z_{0}$, ending in $Z^{\prime}$, such that no state except the last is in $Z^{\prime}$.

### 6.2 Parametrized DTMC semantics of DiPWhile

The parametrized DTMC semantics of a DiPWhile program $P_{\epsilon}$ shall be denoted as $\llbracket P_{\epsilon} \rrbracket$. We describe $\llbracket P_{\epsilon} \rrbracket$ informally here and defer the formal definition to Appendix D. As mentioned above, the key insight in defining the semantics of a DiPWhile program as a finite-state, parametrized DTMC, is that the actual values of real and integer variables need not be tracked. A state of $\llbracket P_{\epsilon} \rrbracket$ is going to be a tuple of the form $\left(\ell, f_{\text {Bool }}, f_{\mathrm{DOM}}, f_{\text {int }}, f_{\text {real }}, C\right)$ where $\ell$ is the label of the statement of $P_{\epsilon}$ to be executed next. $\llbracket P_{\epsilon} \rrbracket$ is an abstraction of the set of all concrete states that are compatible with it.

The partial functions $f_{\text {Bool }}$ and $f_{\text {DOM }}$ assign values to the Bool and DOM variables, respectively; this is just like in the natural semantics.

Let us now look at the partial function $f_{\text {real }}$. Intuitively, $f_{\text {real }}$ is supposed to be the "valuation" for the real variables. But instead of mapping each variable to a concrete value in $\mathbb{R}$, we shall instead map it into a finite set. To understand this mapping, let us recall that in DiPWhile, a real variable is assigned only once in a program. Further, such an assignment either assigns the value of a linear expression over program variables, or a value sampled using a Laplace mechanism. In the former case, $f_{\text {real }}$ maps a variable to the linear expression it is assigned; and in the latter case, the value of the parameters of the Laplace mechanism used in sampling. In the latter case, since the first parameter is always of the form $a \epsilon$, we need to note only $a$ in the mapping. Notice that the range of $f_{\text {real }}$ is now a finite set as $P_{\epsilon}$ contains only a finite number of linear expressions, and the parameters of sampled Laplacian take values from the finite set DOM. Similarly, the partial function $f_{\text {int }}$ maps each integer variable to either the linear expression it is assigned or the parameters of the sampled discrete Laplace mechanism. The last state component $C$ is the set of Boolean conditions on real and integer variables that hold along the path thus far; this shall become clearer when we describe the transitions. Since the Boolean conditions must be Boolean expressions in the program or their negation, $C$ is also a finite set. These observations show that $\llbracket P_{\epsilon} \rrbracket$ has finitely many states. Intuitively, a state of $\llbracket P_{\epsilon} \rrbracket$ is an abstraction of the set of all concrete states that respect the Boolean conditions in $C$ and the constraints imposed by assignments of real and integer expressions to real and integer variables, respectively.

We now sketch how the state is updated in $\llbracket P_{\epsilon} \rrbracket$. Updates to DOM variables shall be as expected - it shall be a probabilistic transition if the assignment samples using an exponential mechanism or a user-defined distribution, and it shall be a deterministic step updating $f_{\text {DOM }}$ otherwise. Assignments to real variables are always deterministic steps that change the function $f_{\text {real }}$. Thus, even if the step samples using the Laplace mechanism, in the semantics, it shall be modeled as a deterministic step where $f_{\text {real }}$ is updated by storing the parameters of the distribution. Similarly, all integer assignments are deterministic steps as well.

The assignment of a Boolean expression to a Boolean variable is as expected - we update the valuation $f_{\text {Bool }}$ to reflect the assignment. The unexpected case is $\mathrm{b} \leftarrow R_{1} \sim R_{2}$ when a boolean variable gets assigned the result of the comparison of two real expressions; the case of comparing two integer expressions is similar. In this case, if the probability of $C$ holding is 0 , then our construction will ensure that this state is not reachable with
non-zero probability. Otherwise, we transition to a state where $R_{1} \sim R_{2}$ is added to $C$ with probability equal to the probability that ( $R_{1} \sim R_{2}$ ) holds conditioned on the fact that $C$ holds, and with the remaining probability, we shall transition to the state where $\neg\left(R_{1} \sim R_{2}\right)$ is added to $C$. Thus, Boolean assignments which compare integer and real variables are modeled by probabilistic transitions. Finally, branches and while loop conditions are deterministic steps, with the value of the Boolean variable (of the condition) in $f_{\text {Bool }}$ determining the choice of the next statement.

Let $\operatorname{Prob}_{D T M C}\left(P_{\epsilon}(\mathbf{i n})=\right.$ out $)$ denote the probability that $P_{\epsilon}$ outputs value out on the input in under the DTMC semantics. This is just the probability of reaching an exit state with out as valuation of output variables from the initial state with in as the valuation of input variables. We can show that this probability is the same as the probability $\operatorname{Prob}_{\text {natural }}\left(P_{\epsilon}(\mathbf{i n})=\right.$ out $)$ obtained by the natural semantics discussed above. The informal ideas outlined above are fleshed out to give a precise mathematical definition and presented in Appendix D.

It is worth noting how key syntactic restrictions in DiPWhile programs play a role in defining its semantics. The first restriction is that integer and real variables are not assigned in the scope of a while loop. This restriction is critical to ensure that the DTMC $\llbracket P_{\epsilon} \rrbracket$ is finite-state. Since we track distribution parameters and linear expressions for such variables, this restriction ensures that we only remember a bounded number of these. Second, DiPWhile disallows a comparison between real and integer expressions in its syntax. Recall that such comparison steps result in a probabilistic transition, where we compute the probability of the comparison holding conditioned on the properties in $C$ holding. It is unclear if a closed-form expression for such probabilities can be computed when integer and real random variables are compared. Hence such comparisons are disallowed.

Probabilistic transitions in our semantics arise due to two reasons. First are assignments to DOM variables that sample according to either the exponential or a user-defined distribution. The resulting probabilities are easily seen to be definable in $\mathrm{Th}_{\text {exp. }}$. The second is due to comparisons between real and integer expressions. We can prove that in this case also, the resulting probabilities are definable in $\mathrm{Th}_{\text {exp }}$; this proof is non-trivial and deferred to Appendix E. All these observations together give us the following theorem.

Theorem 8. For any DiPWhile program $P_{\epsilon}, \llbracket P_{\epsilon} \rrbracket$ is a finite, definable, parametrized DTMC that is computable.
Example 9. The parametrized DTMC semantics of Algorithm 2is partially shown in Figure 2. We show only the transitions corresponding to executing
lines 9 and 10 of the algorithm, when $q_{1}=u$ and $q_{2}=v$ initially; here $u, v \in$ $\{\perp, \top\}$. The multiple lines in a given state give the different components of the state. The first two lines give the assignment to Bool and DOM variables, the third line gives values to the integer/real variables, and the last line is the Boolean conditions that hold along a path. Since 9 and 10 are in the else-branch, the condition $r_{1}<r_{T}$ holds. Notice that values to real variables are not explicit values, but rather the parameters used when they were sampled. Finally, observe that probabilistic branching takes place when line 10 is executed, where the value of $b$ is taken to be the result of comparing $r_{2}$ and $r_{T}$. The numbers $p$ and $q$ correspond to the probability that the conditions in a branch hold, given the parameters used to sample the real variables and conditioned on the event that $r_{1}<r_{T}$.

## 7 Checking differential privacy for DiPWhile programs

We shall now establish that the problem of checking differential privacy for DiPWhile programs is decidable. The proof relies on the characterization of the semantics of a DiPWhile program as a finite, definable, parameterized DTMC (See Theorem8). An important observation about a finite, definable, parametrized DTMC is that the probability of reaching a given set of states $Z^{\prime}$ from a given state $z_{0}$ is both definable and computable.

Lemma 10. For any finite-state, definable, parametrized DTMC $\mathcal{D}=$ $(Z, \Delta)$, any state $z_{0} \in Z$ and set of states $Z^{\prime} \subseteq Z$, the function $\operatorname{Prob}\left(z_{0}, Z^{\prime}\right)$ is definable in $\mathrm{Th}_{\text {exp }}$. Moreover, there is an algorithm that computes the formula defining $\operatorname{Prob}\left(z_{0}, Z^{\prime}\right)$.

The proof of Lemma 10 exploits the connection between reachability probabilities in DTMCs and linear programming [33, 2]; details are in Appendix F The main result of the paper now follows from Theorem 8 and Lemma 10,

Theorem 11. The Fixed Parameter Differential Privacy and Differential Privacy problems are decidable for DiPWhile programs $P_{\epsilon}$, rational numbers $t \in \mathbb{Q}^{>0}$ and definable functions $\delta(\epsilon)$. Furthermore, if $P_{\epsilon}$ is not $(t \epsilon, \delta)$ differentially private for some rational number $t$ and admissible value of $\epsilon$ then we can compute a counter-example.

Proof. Let in and out be arbitrary valuations to input and output variables, respectively. Observe that the function $\epsilon \mapsto \operatorname{Prob}\left(P_{\epsilon}(\mathbf{i n})=\mathbf{o u t}\right)$ is nothing
but $\operatorname{Prob}\left(z_{0}, Z^{\prime}\right)$ in $\llbracket P_{\epsilon} \rrbracket$, where $z_{0}$ is the initial state corresponding to valuation in, and $Z^{\prime}$ is the set of all terminating states that have valuation out for output variables. Since $\llbracket P_{\epsilon} \rrbracket\left(\right.$ Theorem [8) and $\operatorname{Prob}\left(z_{0}, Z^{\prime}\right)($ Lemma 10) are computable, we can construct a formula $\varphi_{\text {in,out }}\left(\epsilon, x_{\text {in }, \text { out }}\right)$ of $\mathcal{L}_{\text {exp }}$ that defines the function $\epsilon \mapsto \operatorname{Prob}\left(P_{\epsilon}(\right.$ in $)=$ out $)$.

Let $\varphi_{\delta}\left(\epsilon, x_{\delta}\right)$ be the formula defining the function $\delta$. Let $t=\frac{p}{q}$ where $p, q$ are natural numbers. Consider the sentence

$$
\begin{aligned}
& \psi=\forall \epsilon . \forall z \cdot\left[\forall x_{\text {in }, \text { out }}\right]_{\text {in } \in \mathcal{U}, \text { out } \in \mathcal{V}} . \forall x_{\delta} \\
&\left((\epsilon>0) \wedge\left(e^{p \epsilon}=z^{q}\right) \wedge(z>0) \wedge \varphi_{\delta}\left(\epsilon, x_{\delta}\right)\right. \\
&\left.\bigwedge_{\text {in } \in \mathcal{U}, \mathbf{\text { out } \in \mathcal { V }}} \varphi_{\text {in }, \mathbf{\text { out }}}\left(\epsilon, x_{\text {in }, \text { out }}\right)\right) \\
& \rightarrow\left(\bigwedge_{\left(\mathbf{i n}_{1}, \mathbf{i n}_{2}\right) \in \Phi, O \subseteq \mathcal{V}}\right. \\
&\left.\left.\sum_{\text {out } \in O} x_{\mathbf{i n}_{1}, \text { out }}<z \sum_{\text {out } \in O} x_{\mathbf{i n}_{2}, \text { out }}+x_{\delta}\right)\right)
\end{aligned}
$$

It is easy to see $P_{\epsilon}$ is $(t \epsilon, \delta(\epsilon))$ differentially private for all $\epsilon$ iff $\psi$ is true over the reals. In the syntax of $\mathcal{L}_{\text {exp }}$, we cannot take $q$ th roots of $e$; therefore, we introduce the variable $z$, which enables us to write the constraints using only $e^{a \epsilon}$, where $a \in \mathbb{N}$. Notice that $\psi$ belongs to $\mathcal{L}_{\text {exp }}$ if we convert it to prenex form. Decidability, therefore, follows from the decidability of $\mathrm{Th}_{\text {exp }}$.

If $P_{\epsilon}$ is not differentially private, then the sentence $\psi$ does not hold. The decision procedure for $\mathrm{Th}_{\text {exp }}$ will, in this case, return an $\epsilon_{0}$ that witnesses the privacy violation of $P_{\epsilon}$. Using $\epsilon_{0}$, the counter-example (in, $\mathbf{i n}^{\prime}, O, \epsilon_{0}$ ) can be easily constructed by enumerating in, in ${ }^{\prime}$ and $O$.

An easy consequence of Theorem 11 is that differential privacy is decidable for the subclass of program in Simple that do not have integer and real-valued variables. Let Finite DiPWhile denote this set of programs (See Appendix $G$ for the formal syntax of Finite DiPWhile). Observe that due to the presence of While, Finite DiPWhile programs may still have unbounded length executions (including infinite executions).

Corollary 12. The Fixed Parameter Differential Privacy and Differential Privacy problems are decidable for Finite DiPWhile programs $P_{\epsilon}$, rational numbers $t \in \mathbb{Q}^{>0}$ and definable functions $\delta(\epsilon)$.

We observe that our methods can be employed to analyze larger classes of programs (than just those in DiPWhile). For example, a sufficient condition to ensure the decidability is to consider programs with the property that, for each input, the probability distribution on the outputs is definable in $\mathrm{Th}_{\text {exp }}$ (See Appendix G.1). We conclude the section by showing how our procedure is useful when reasoning about integer and real-valued outputs.

Remark. We sketch here how the proofs of Theorem 11 changes when the set of admissible $\epsilon$ is taken to be an interval $I$ with rational end-points. Let $P_{\epsilon}, t$ and $\delta(\epsilon)$ be as in the proof of Theorem [11. When $\epsilon$ is restricted to an interval $I$, we will require the user-definable distributions to be definable in $\mathrm{Th}_{\text {exp }}$ only on the interval $I$. As in the proof of Theorem 11, we can construct a formula $\varphi_{\text {in,out }}\left(\epsilon, x_{\text {in,out }}\right)$ of $\mathcal{L}_{\exp }$ that defines the function $\epsilon \mapsto$ $\operatorname{Prob}\left(P_{\epsilon}(\mathbf{i n})=\right.$ out $)$. For simplicity, consider the case when $I$ be the interval $[r, s]$. Consider the sentence $\psi_{I}$ that is obtained from $\psi$ in the proof of Theorem 11 by replacing the subformula $(\epsilon>0)$ by $(a \leq \epsilon) \wedge(\epsilon \leq b)$. Then $P_{\epsilon}$ is $(t \epsilon, \delta(\epsilon))$ will be differentially private for all $\epsilon \in I$ iff $\psi_{I}$ is true over the reals.

### 7.1 Finite discretization of infinite output spaces

Our decision procedure assumes that the output space is finite. In several examples, the program outputs are reals or unbounded integers (and combinations thereof). Nevertheless, we argue that our decision procedure is useful for the verification of differential privacy in this case also. In particular, our method provides an under-approximation technique for checking the differential privacy of programs with infinite outputs. Our approach in such cases is to discretize the output space into finitely many intervals.

We illustrate this for the special case when a program $P$ outputs the value of one real random variable, say r . Now, suppose that we modify $P$ to output a finite discretized version of r as follows. Let seq $=a_{0}<a_{1}<\ldots a_{n}$ be a sequence of rationals and let $\operatorname{Disc}_{\text {seq }}(x)$ be equal to $a_{0}$ if $x \leq a_{0}$, equal to $a_{i}(0<i<n)$ if $a_{i-1}<x \leq a_{i}$, and equal to $a_{n}$ if $x>a_{n-1}$.

Consider the program $P_{\text {Disc,seq }}$ that instead of outputting $r$, outputs Disc $_{\text {seq }}(r)$. It is easy to see that if $P$ is differentially private then so must be $P_{\text {Disc,seq }}$. Therefore, if $P_{\text {Disc,seq }}$ is not differentially private then we can conclude that $P$ is not differentially private. Thus, if our procedure finds a counter-example for $P_{\text {Disc,seq }}$, then it also has proved that the program $P$ is not differentially private. Our method is, therefore, an under-approximation technique for checking the differential privacy of $P$. In fact, it is a complete under-approximation method in the sense that $P$ is differentially private iff for each possible seq, $P_{\text {Disc,seq }}$ is differentially private.

## 8 Experimental evaluation

We implemented a simplified version of the algorithm, presented earlier, for proving/disproving differential privacy of DiPWhile programs. Our tool

| Algorithm | Runtime <br> $(\mathrm{T} 1 / \mathrm{T} 2)$ | $\epsilon$-Diff. <br> Pri- <br> vate |
| :---: | :---: | :---: |
| SVT | $0 \mathrm{~s} / 825 \mathrm{~s}$ | $\checkmark$ |
| SVT2 | $0 \mathrm{~s} / 768 \mathrm{~s}$ | $\boldsymbol{\checkmark}$ |
| SVT5 | $0 \mathrm{~s} / 2 \mathrm{~s}$ | $\boldsymbol{x}$ |
| NMax4 | $1 \mathrm{~s} / 58 \mathrm{~s}$ | $\boldsymbol{x}$ |
| Rand2 | $0 \mathrm{~s} / 0 \mathrm{~s}$ | $\boldsymbol{x}$ |

Table 1: Runtime for 3 queries for each algorithm searching over adjacency pairs and all $\epsilon_{i} 0$, with parameters being $[\mathrm{c}=1, \Delta=1$, $\mathrm{DOM}=\{-1,0,1\}$, seq $=(-1<0<1)]$. For SVT, we also have $T=0$.

DiPC [3] handles loop-free programs, i.e., acyclic programs. Programs with bounded loops (with constant bounds) can be handled by unrolling loops. The tool takes in an input program $P_{\epsilon}$ parametrized by $\epsilon$ and an adjacency relation, and either proves $P_{\epsilon}$ to be differentially private for all $\epsilon$ or returns a counter-example. The tool can also be used to check differential privacy for a given, fixed $\epsilon$, or to check for $k \epsilon$-differential privacy for some constant $k$. DiPC is implemented in C++ and uses Wolfram Mathematica $\circledR$ ®. It works in two phases - in the first phase, a Mathematica $®$ ®script is produced with commands for all the output probability computations and the subsequent inequality checks and in the second phase, the generated script is run on Mathematica. Details about the tool and its design can be found in Appendix $\mathbf{H}$.

We used various examples to measure the effectiveness of our tool. These include SVT [28, 21, Noisy Maximum [18, Noisy Histogram [18] and Randomized Response [20] and their variants. Detailed descriptions of these algorithms and their variants can be found in Appendix H. 1.

We ran all the experiments on an octa-core Intel®Core i7-8550U @ 1.8 gHz CPU with 8 GB memory. The running times reported are the average of 3 runs of the tool. In the tables, T1 refers to the time needed by the C++ phase to generate the Mathematica scripts, and T2 refers to the time used by Mathematica to check the scripts. Due to space constraints, we report only a small fraction of our experiments; full details of all our experiments can be found in Appendix H ,

Salient observations about our experiments are follows.

1. DiPC successfully proves algorithms to be differentially private and finds counter-examples to demonstrate a violation of privacy in reasonable time. Table 1 shows the running time of DiPC on some examples for 3 queries. We chose to use 3 queries because for algorithms

| Algo | $-\mathrm{Q}-$ | Output | Input 1 | Input 2 | $\epsilon$ | Runtime <br> $(\mathrm{T} 1 / \mathrm{T} 2)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| SVT5 | 2 | $[\perp \mathrm{~T}]$ | $[-10]$ | $[-1-1]$ | 27 | $0 \mathrm{~s} / 2 \mathrm{~s}$ |
| NMax3 | 3 | -1, seq $=(-1<0<1)$ | $[-1-1-1]$ | $\left[\begin{array}{lll}0 & 0\end{array}\right]$ | 27 | $0 \mathrm{~s} / 310 \mathrm{~s}$ |
| NMax4 | 1 | 0, seq $=(-1<0<1)$ | $[-1]$ | $[0]$ | 27 | $0 \mathrm{~s} / 2 \mathrm{~s}$ |
| Rand2 | 1 | $[\perp]$ | $[\perp]$ | $[\top]$ | $9 / 34$ | $0 \mathrm{~s} / 0 \mathrm{~s}$ |

Table 2: Smallest Counter-example found for each non-differentially private algorithm, searching over all adj. pairs and $\epsilon>0$, with parameters being $[\mathrm{c}=1, \Delta=1, \mathrm{DOM}=\{-1,0,1\}]$
that are not private, counter-examples can be found with 3 queries.
2. The time to generate Mathematica scripts is significantly smaller than the time taken by Mathematica to check the scripts (i.e., $\mathrm{T} 1 \ll \mathrm{~T} 2$ ). Further, most of the time spent by Mathematica is for computing output probabilities; the time to perform comparison checks for adjacent inputs was relatively small. Thus, programs that do not use real variables (Rand2 in Table 1, for example) can be analyzed more quickly.
3. For algorithms that are not differentially private, DiPC can automatically identify the pair of inputs, output, and $\epsilon$ for which privacy is violated. Table 2, shows the results for the smallest counter-example found by DiPC for some examples. Further, counter-examples found by DiPC are much smaller, in terms of queries, than those found in [18]; the number of queries needed in the counter-examples in [18] for NMax3, NMax4, and SVT5 were 5, 5, and 10, respectively, as opposed to 3,1 , and 2 found by DiPC.
4. DiPC is the first automated tool that can check $(\epsilon, \delta)$-differential privacy. To evaluate this feature, we tested DiPC on a version of SVT, Sparse [21], which is manually proven to be ( $\frac{\epsilon}{2}, \delta_{\text {svt }}$ )-differentially private for any number of queries in [21 by using advanced composition theorems. Here $\delta_{\text {svt }}$ is a second parameter in the algorithm. In our experiments, we tested $\left(\frac{\epsilon}{2}, \delta_{\text {svt }}\right)$-differential privacy of Sparse with fixed values of $\delta_{\text {svt }}$ for $c=1,2$ and 3 queries, validating the result in [21]. As we were dealing with only 3 queries, we also managed to obtain better bounds on the error parameter.

## 9 Related work

The main thread of related work has focused on formal systems for proving that an algorithm is differentially private. Such systems are helpful be-
cause they rule out the possibility of mistakes in privacy analyses. Starting from Reed and Pierce [32], several authors [22, 17] have proposed linear (dependent) type systems for proving differential privacy. However, it is not possible to verify some of the most advanced examples, such as a sparse vector or vertex cover, using these type systems. Moreover, type-checking and type-inference for linear (dependent) types are challenging. For example, the type checking problem for DFuzz, a language for differential privacy, is undecidable [16]. Barthe et al [7, 5, 4] develop several program logics based on probabilistic couplings for reasoning about differential privacy. These logics have been used successfully to analyze many classic examples from the literature, including the sparse vector technique. However, these logics are limited: they cannot disprove privacy; extensions may be required for specific examples; building proofs is challenging. The last issue has been addressed by a series of works that provide automated methods for proving differential privacy automatically. Zhang and Kifer [35] introduce randomness alignments as an alternative to couplings and build a dependent type system that tracks randomness alignments. Automation is then achieved by type inference. Albarghouthi and Hsu [1] propose coupling strategies, which rely on a fine-grained notion of variable approximate coupling, which draws inspiration both from approximate couplings and randomness alignment. They synthesize coupling strategies by considering an extension of Horn clauses with probabilistic coupling constraints and developing algorithms to solve such constraints. Recently Wang et al 34 develop an improved method based on the idea of shadow executions. Their approach is able to verify Sparse Vector and many other challenging examples efficiently. However, these methods are limited to vanilla $\epsilon$-differential privacy and do not accommodate bounds that are obtained by advanced composition (since $\delta \neq 0$ ).

In an independent line of work, Chatzikokolakis, Gebler and Palamidessi [11] consider the problem of differential privacy for Markov chains. Later, Liu, Wang, and Zhang [27] develop a probabilistic model checking approach for verifying differential privacy properties. Their approach is based on modeling differential private programs as Markov chains. Their encoding is more direct than ours (i.e. it assumes that a finite-state Markov chain is given), and they do not provide a decision procedure with real and integer variables. Furthermore, the DTMCs are not parameterized by $\epsilon$. Chistikov and Murawski and Purser [13, 14 propose an elegant method based on skewed Kantorovich distance for checking approximate differential privacy of Markov chains.

The dual problem is to find violations of differential privacy automati-
cally. This is useful to help privacy practitioners discover potential problems early in the development cycle. Two recent and concurrent works by Ding et al [18] and Bischel et al [9] develop automated methods for finding privacy violations. Ding et al. propose an approach that combines purely statistical methods based on hypothesis testing and symbolic execution. Bischel et al. develop an approach based on a combination of optimization methods and language-specific techniques for computing differentiable approximations of privacy estimations. Both methods are fully automated. However, both methods can only be used for concrete numerical values of the privacy budget $\epsilon$.

Gaboardi et. al [23] study the complexity of deciding differential privacy for randomized Boolean circuits. Their results are proved by reduction to majority problems and are incomparable with ours: the only probabilistic choices in [23] are fair coin tosses and $e^{\epsilon}$ is taken to be a fixed rational number.

## 10 Conclusions

We showed that the problem checking differential privacy is in general undecidable, identified an expressive sub-class of programs (DiPWhile) for which the problem is decidable, and presented the results of analyzing many known differential privacy algorithms using our tool DiPC which implements a decision procedure for DiPWhile programs. Advantages of DiPC include the ability to automatically, both prove algorithms to be private for all $\epsilon>0$, and find counter-examples to demonstrate privacy violations. In addition DiPC can check bounds that are based on concentration inequalities, in particular bounds that use advanced composition theorems. Such bounds are out of reach of most other tools that prove privacy or search for counterexamples.

In the future, it would be interesting to extend this work to handle programs with input/output variables that take values in infinite domains, and parametrized privacy algorithms that work for an unbounded number of input and output variables. Another important problem is developing decision procedures that can prove tight accuracy bounds, and detect violations of accuracy bounds. We also plan to investigate extending the decision procedure to cover algorithms that are currently out of the scope of our decision procedure such as the multiplicative weights and iterative database construction [26, 25, and those involving Gaussian distributions.

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## A Semantics of Simple

In this section, we give the semantics of our Simple language. This semantics will be given as a set of computations and a probability space on the set of computations. Recall that we have assumed that in each computation, a reference to a variable is preceded (sometime earlier) by an assignment to the variable.

For the rest of this section, let us fix a Simple program $P_{\epsilon}$ and an $\epsilon>0$. We let $\mathrm{L}_{\mathrm{P}_{\epsilon}}$ denote the set of labels appearing in $P_{\epsilon}$. The set of Boolean variables, DOM variables (including input/output variables), integer variables and reals variables occurring in $P_{\epsilon}$ shall be denoted by $\mathcal{B}_{P_{\epsilon}}, \mathcal{X}_{P_{\epsilon}}, \mathcal{Z}_{P_{\epsilon}}$ and $\mathcal{R}_{P_{\epsilon}}$ respectively.

In order to define the semantics of $P_{\epsilon}$, we will use an auxiliary function next that given a label, identifies the label of the statement to be executed next. Observe that for most program statements, the next statement to be executed is unique. However, for if and While statements, the next statement depends on the value of a Boolean expression. We will define next $(\ell)$ to be a set of pairs of the form $\left(\ell^{\prime}, c\right)$, where $c$ is a Boolean condition on the variables of $P_{\epsilon}$, with the understanding that $\ell^{\prime}$ is the label of the next statement to be executed if $c$ currently holds. Thus, for a label $\ell$, next $(\ell)$ will either be $\left\{\left(\ell^{\prime}\right.\right.$, true $\left.)\right\}$ or $\left\{\left(\ell_{1}, c\right),\left(\ell_{2}, \neg c\right)\right\}$. We do not give a precise definition of next $(\cdot)$, but we will use it when defining the semantics.

States. States of $P_{\epsilon}$ will be of the form

$$
\left(\ell, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}, h_{\mathbb{R}}\right)
$$

Informally, $\ell \in \mathrm{L}_{\mathrm{P}_{\epsilon}}$ is the label of the statement to be executed, $h_{\text {Bool }}$, $h_{\mathrm{DOM}}, h_{\mathbb{Z}}$, and $h_{\mathbb{R}}$ are functions assigning "values" to program variables (of appropriate type). More specifically, we have $h_{\text {Bool }}: \mathcal{B}_{P_{\epsilon}} \rightarrow$ \{true, false $\}$, $h_{\mathrm{DOM}}: \mathcal{X}_{P_{\epsilon}} \rightarrow \mathrm{DOM}, h_{\mathbb{Z}}: \mathcal{Z}_{P_{\epsilon}} \rightarrow \mathbb{Z}$ and $h_{\mathbb{R}}: \mathcal{R}_{P_{\epsilon}} \rightarrow \mathbb{R}$. We let $S$ denote the set of all states. We define a discrete state $d s$ to be a tuple $\left(\ell, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}\right)$ where $\ell, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}$ are as defined above. Note that a discrete state does not specify values to variables in $\mathcal{R}_{P_{\epsilon}}$. For a state $s$ and an expression $e$ which is a Boolean, real or an integer expression, we let $\operatorname{Val}(s, e)$ denote the value obtained by evaluating $e$ in the state $s$. Note that if $e$ is a boolean expression, $\operatorname{Val}(s, e)$ is either True or False. We also define the value of a comparison between two expressions as follows. For a comparison expression $e_{1} \sim e_{2}, \operatorname{Val}\left(s, e_{1} \sim e_{2}\right)=\operatorname{True}$ if $\operatorname{Val}\left(s, e_{1}\right) \sim \operatorname{Val}\left(s, e_{2}\right)$ holds, otherwise $\operatorname{Val}\left(s, e_{1} \sim e_{2}\right)=$ False. The value of a DOM expression $e$, its
value in state $s=\left(\ell, h_{\text {Bool }}, h_{\text {DOM }}, h_{\mathbb{Z}}, h_{\mathbb{R}}\right)$ will be denoted by $h_{\text {DOM }}(e)$. For a sequence of DOM expressions DOM $\tilde{e}=e_{1}, \ldots, e_{m}, h_{\text {DOM }}(\tilde{e})$ will denote the sequence $h_{\text {DOM }}\left(e_{1}\right), \ldots, h_{\text {DOM }}\left(e_{m}\right)$.

Measurable sets of states. Let $\mathcal{R}_{P_{\epsilon}}=\left\{r_{1}, \ldots, r_{t}\right\}$. With each vector $u=\left(u_{1}, \ldots, u_{t}\right) \in \mathbb{R}^{t}$, we associate a unique function $h_{\mathbb{R}}^{u}: \mathcal{R}_{P_{\epsilon}} \rightarrow \mathbb{R}$ such that $h_{\mathbb{R}}^{u}\left(r_{i}\right)=u_{i}$ for $1 \leq i \leq t$. Given a discrete state $d s=\left(\ell, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}\right)$ and a Borel set $D \subseteq \mathbb{R}^{t}$, we let $\llbracket(d s, D) \rrbracket=\left\{\left(\ell, h_{\text {Bool }}, h_{\text {DOM }}, h_{\mathbb{Z}}, h_{\mathbb{R}}^{u}\right) \mid u \in D\right\}$. Now, we define $\mathcal{E}$, the set of measurable sets of states, to be the $\sigma$-algebra generated by the sets of states of the form $\llbracket(d s, D) \rrbracket$ where $d s$ is a discrete state and $D \subseteq \mathbb{R}^{t}$ is a Borel set.

Markov Kernel $K_{\epsilon}$. We give the single step semantics of the program $P_{\epsilon}$ as a Markov kernel from the measure space $(S, \mathcal{E})$ to itself. Formally, $K_{\epsilon}: S \times$ $\mathcal{E} \rightarrow \mathbb{R}$, where $K_{\epsilon}(s, C)$ gives the probability that the next state of $P_{\epsilon}$ is in $C$ given that its current state is $s$. We fix the state $s=\left(\ell, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}, h_{\mathbb{R}}\right)$ and the set $C \in \mathcal{E}$ of states, and define the value of $K_{\epsilon}(s, C)$ based on the following cases.

DOM assignments. Let $\operatorname{next}(\ell)=\left\{\left(\ell^{\prime}\right.\right.$, true $\left.)\right\}$ and let $\times$ be the variable being assigned in $\ell$. There are two cases to consider. First, consider the case where x is assigned a value of a DOM expression $e$. In this case, $K_{\epsilon}(s, C)=1$ if $\left(\ell^{\prime}, h_{\text {Bool }}, h_{\text {DOM }}\left[\mathrm{x} \mapsto h_{\text {DOM }}(e)\right], h_{\mathbb{Z}}, h_{\mathbb{R}}\right) \in C$; otherwise $K_{\epsilon}(s, C)=0$. The second case is when x is assigned a random value according to $\operatorname{Exp}(a \epsilon, F(\tilde{\mathrm{x}}), e)$ or choose $(a \epsilon, \tilde{e})$. For $d \in \mathrm{DOM}$, let $\operatorname{prob}(d)$ be the probability of $d$ based on the distribution; note, that these probabilities will depend on the value of $h_{\mathrm{DOM}}(e)$ and $h_{\mathrm{DOM}}(\tilde{e})$. Then, $K_{\epsilon}(s, C)=\sum_{d \in D} \operatorname{prob}(d)$ where $D=$ $\left\{d \mid\left(\ell^{\prime}, h_{\text {Bool }}, h_{\mathrm{DOM}}[\mathrm{x} \mapsto d], h_{\mathbb{Z}}, h_{\mathbb{R}}\right) \in C\right\}$. Note that the right hand sum is zero if $D=\emptyset$.

Integer assignments. Let next $(\ell)=\left\{\left(\ell^{\prime}\right.\right.$, true $\left.)\right\}$ and let $\mathbf{z}$ be the variable being assigned in $\ell$. Again there are two cases to consider. First, consider the case where z is assigned a value of an integer expression $e$. In this case, $K_{\epsilon}(s, C)=1$ if $\left(\ell^{\prime}, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}[\mathbf{z} \mapsto \operatorname{Val}(s, e)], h_{\mathbb{R}}\right) \in C$; otherwise $K_{\epsilon}(s, C)=0$. Next, consider the case when z is assigned a random value according to $\operatorname{DLap}(a \epsilon, e)$. For $j \in \mathbb{Z}$, let $\operatorname{prob}(j)$ be the probability assigned to the integer $j$ by the distribution given by $\operatorname{DLap}\left(a \epsilon, h_{\text {DOM }}(e)\right)$. Then, $K_{\epsilon}(s, C)=\sum_{j \in D} \operatorname{prob}(j)$ where $D=\left\{j \mid\left(\ell^{\prime}, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}[\mathbf{z} \mapsto\right.\right.$ $\left.\left.j], h_{\mathbb{R}}\right) \in C\right\}$. Note that the right hand sum is zero if $D=\emptyset$.

Real assignments. Let $\operatorname{next}(\ell)=\left\{\left(\ell^{\prime}\right.\right.$, true $\left.)\right\}$ and let $r$ be the variable being assigned in $\ell$. Again there are two cases to consider. First, consider the case where $r$ is assigned a value of a real expression $e$. In this case, $K_{\epsilon}(s, C)=$ 1 if $\left(\ell^{\prime}, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}, h_{\mathbb{R}}[\mathrm{r} \mapsto \operatorname{Val}(s, e)]\right) \in C$; otherwise $K_{\epsilon}(s, C)=0$. In the second case, $r$ is assigned a random value according to $\operatorname{Lap}(a \epsilon, e)$. In this case, $K_{\epsilon}(s, C)=\operatorname{Prob}(D)$ where $D=\left\{r \in \mathbb{R} \mid\left(\ell^{\prime}, h_{\text {Bool }}, h_{\text {DOM }}, h_{\mathbb{Z}}, h_{\mathbb{R}}[r \mapsto\right.\right.$ $r]) \in C\}$ and $\operatorname{Prob}(D)$ is the probability given to set $D$ by the distribution $\operatorname{Lap}\left(a \epsilon, h_{\text {DOM }}(e)\right)$. Observe that $D \subseteq \mathbb{R}$ is a Borel set.

Boolean assignments. Again let next $(\ell)=\left\{\left(\ell^{\prime}\right.\right.$, true $\left.)\right\}$ and let b be the variable being assigned in $\ell$ and $e$ the expression being assigned. Now, $K_{\epsilon}(s, C)=1$ if $\left(\ell^{\prime}, h_{\text {Bool }}[\mathrm{b} \mapsto \operatorname{Val}(s, e)], h_{\text {DOM }}, h_{\mathbb{Z}}, h_{\mathbb{R}}\right) \in C$; otherwise $K_{\epsilon}(s, C)=$ 0 .
if statement. In this case, $\operatorname{next}(\ell)=\left\{\left(\ell_{1}, c\right),\left(\ell_{2}, \neg c\right)\right\}$ for some Boolean condition $c$. If either $\operatorname{Val}(s, c)=$ true and $\left(\ell_{1}, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}, h_{\mathbb{R}}\right) \in C$ or $\operatorname{Val}(s, c)=$ false and $\left(\ell_{2}, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}, h_{\mathbb{R}}\right) \in C$ then $K_{\epsilon}(s, C)=1$, otherwise $K_{\epsilon}(s, C)=0$.

While statement. Again let $\operatorname{next}(\ell)=\left\{\left(\ell_{1}, c\right),\left(\ell_{2}, \neg c\right)\right\}$. This case is identical to the case of if statement, and so is skipped.
exit statement. In this case, $K_{\epsilon}(s, C)=1$ if $s \in C$; otherwise $K_{\epsilon}(s, C)=$ 0 .

Probability Spaces on finite executions. For each $i>0$, we define a probability space $\Phi_{i}=\left(S^{i}, \Sigma_{i}, \phi_{i}\right)$ capturing the set of finite executions of length $i S^{i}$, the class $\Sigma_{i}$ of measurable sets of executions of length $i$ and a probability measure $\phi_{i}$, as follows. Let $\vec{C}=\left(C_{1}, \ldots, C_{i}\right)$ be a sequence of measurable sets where, for $1 \leq j \leq i, C_{j} \in \mathcal{E}$. For each such $\vec{C}$, let $\operatorname{Exec}(\vec{C})=\left\{\left(s_{1}, s_{2}, \ldots, s_{i}\right) \mid s_{j} \in C_{j}, 1 \leq j \leq i\right\}$. The set $\Sigma_{i}$ of measurable sets of finite executions of length $i$, is the $\sigma$-algebra generated by the sets of executions $\operatorname{Exec}(\vec{C})$ where $\vec{C}$ is a vector of measurable sets as specified above. Essentially, $\left(S^{i}, \Sigma_{i}\right)$ is the measurable space obtained by taking the product of $(S, \mathcal{E}), i$ times. The probability measure $\phi_{i}$ is defined by first fixing an initial state and using the Markov kernel $K_{\epsilon}$ as follows.

Initial State and initial distribution. For an integrable function $g$ with respect to a measure space $(X, \Sigma, \mu)$, let $\int_{X_{1}} g \mu(\mathrm{~d} x)$ denote the integral of
function $g$ with respect to measure $\mu$ over a measurable set $X_{1} \in \Sigma$. Let $\ell_{\text {in }}$ be the label of the first statement of $P_{\epsilon}$. Let $h_{B o o l}^{\mathrm{in}}, h_{\mathbb{Z}}^{\mathrm{in}}$, and $h_{\mathbb{R}}^{\mathrm{in}}$ be functions such that $h_{B o o l}^{\text {in }}$ assigns false to every variable in $\mathcal{B}_{P_{\epsilon}}$, and $h_{\mathbb{Z}}^{\text {in }}, h_{\mathbb{R}}^{\text {in }}$ assign value zero to every variable in $\mathcal{Z}_{P_{\epsilon}}, \mathcal{R}_{P_{\epsilon}}$ respectively. An initial state of $P_{\epsilon}$ will be of the form ( $\left.\ell_{\mathbf{i n}}, h_{B o o l}^{\mathrm{in}}, h_{\mathrm{DOM}}^{\mathrm{in}}, h_{\mathbb{Z}}^{\mathrm{in}}, h_{\mathbb{R}}^{\mathrm{in}}\right)$, where $h_{\mathrm{DOM}}^{\mathrm{in}}$ assigns the given values to input variables and assigns zero to all other variables in $\mathcal{X}_{P_{\epsilon}}$ (recall that all input variables are in $\mathcal{X}_{P_{\epsilon}}$ ); the values given to the input variables by $h_{\text {DOM }}^{\text {in }}$ will be the "initial input value". We fix a unique initial state $s_{\text {init }}$. Let $\phi_{\text {init }}$ be a distribution on the measure space $(S, \mathcal{E})$ such that for any $C^{\prime} \in \mathcal{E}, \phi_{\text {init }}\left(C^{\prime}\right)=1$ if $s_{\text {init }} \in C^{\prime}$; otherwise, $\phi_{\text {init }}\left(C^{\prime}\right)=0$. Now, $\phi_{i}$ is the unique probability measure defined by the Markov kernel $K_{\epsilon}$ with initial distribution $\phi_{\text {init }}$ such that for each sequence of measurable sets $\vec{C}=\left(C_{1}, \ldots, C_{i}\right), \phi_{i}(\operatorname{Exec}(\vec{C}))$ is

$$
\int_{C_{1}} \int_{C_{2}} \cdots \int_{C_{i}} 1 K_{\epsilon}\left(x_{i-1}, \mathrm{~d} x_{i}\right) \cdots K_{\epsilon}\left(x_{1}, \mathrm{~d} x_{2}\right) \phi_{\text {init }}\left(\mathrm{d} x_{1}\right)
$$

where $\mathbf{1}$ is the constant function that takes 1 everywhere. Please see 15 for additional details.

We let $\operatorname{Prob}_{\text {natural }}\left(P_{\epsilon}(\mathbf{i n})=\right.$ out $)$ denote the probability that $P_{\epsilon}$ outputs value out on the input in. We define this probability as follows. Let $\alpha=$ $\left(s_{1}, \ldots, s_{i}\right) \in S^{i}$ be an execution. We say that $\alpha$ is a required execution if $\alpha$ is a terminating execution with output out, i.e., it satisfies the following two conditions: (i) $s_{i}=\left(\ell, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}, h_{\mathbb{R}}\right)$ where $\ell$ is the label of ext statement and valuation of output variables is out; (ii) if $j<i$ and $s_{j}=$ $\left(\ell^{\prime}, f_{\text {Bool }}^{\prime}, f_{\mathrm{DOM}}^{\prime}, f_{\text {int }}^{\prime}, f_{\mathbb{R}}^{\prime}\right)$ then $\ell^{\prime}$ is not the label of ext statement. For each $i>0$, let $R e q_{i}$ be the set of all required executions in $S^{i}$. It is easy to see that, for each $i>0, R e q_{i} \in \Sigma_{i}$ and no execution in $R e q_{i}$ is a prefix of an execution in $R e q_{i+1}$. We define $\operatorname{Prob}_{\text {natural }}\left(P_{\epsilon}(\mathbf{i n})=\mathbf{o u t}\right)=\sum_{i>0} \phi_{i}\left(R e q_{i}\right)$.

## B Undecidability of checking differential privacy of Simple programs

In this section, we will prove Theorem [5. That is, we will show that both Fixed Parameter Differential Privacy and Differential Privacy are undecidable.

Proof. Recall that a 2-counter Minsky Machine is tuple $\mathcal{M}=\left(Q, q_{s}, q_{f}, \Delta_{i n c}^{1}, \Delta_{\text {inc }}^{2}, \Delta_{j z d e c}^{1}, \Delta_{j z d e c}^{2}\right)$ where

- $Q$ is a finite set of control states.
- $q_{s} \in Q$ is the initial state.
- $q_{f} \in Q$ is the final state.
- $\Delta_{\text {inc }}^{i} \subseteq Q \times Q$ is the increment of counter $i$ for $i=1,2$.
- $\Delta_{j z d e c}^{i} \subseteq Q \times Q \times Q$ is the conditional jump of counter $i$ for $i=1,2$.
$\mathcal{M}$ is said to be deterministic if from each state $q$, there is at most one transition out of $q$. The semantics of $\mathcal{M}$ is defined in terms of a transition system $\left(\operatorname{Conf},\left(q_{s}, 0,0\right), \rightarrow\right)$ where $\operatorname{Conf}=Q \times \mathbb{N} \times \mathbb{N}$ is the set of configurations, $\left(q_{s}, 0,0\right)$ is the initial configuration and $\rightarrow$ is defined as follows:

| $(q, i, j) \rightarrow\left(q^{\prime}, i+1, j\right)$ | if $\left(q, q^{\prime}\right) \in \Delta_{\text {inc }}^{1}$, |
| :--- | :--- |
| $(q, i, j) \rightarrow\left(q^{\prime}, i, j+1\right)$ | if $\left(q, q^{\prime}\right) \in \Delta_{i n c}^{2}$, |
| $(q, i, j) \rightarrow\left(q^{\prime}, i, j\right)$ | if $i=0$ and $\left(q, q^{\prime}, q^{\prime \prime}\right) \in \Delta_{j z d e c}^{1}$, |
| $(q, i, j) \rightarrow\left(q^{\prime \prime}, i-1, j\right)$ | if $i \neq 0$ and $\left(q, q^{\prime}, q^{\prime \prime}\right) \in \Delta_{j z d e c}^{1}$, |
| $(q, i, j) \rightarrow\left(q^{\prime}, i, j\right)$ | if $j=0$ and $\left(q, q^{\prime}, q^{\prime \prime}\right) \in \Delta_{j z d e c}^{2}$, |
| $(q, i, j) \rightarrow\left(q^{\prime \prime}, i, j-1\right)$ | if $j \neq 0$ and $\left(q, q^{\prime}, q^{\prime \prime}\right) \in \Delta_{j z d e c}^{2}$. |

A sequence of configurations $s_{0}, s_{1}, \ldots s_{k}$ is said to be a computation of $\mathcal{M}$ is $s_{0}=\left(q_{s}, 0,0\right)$ and $s_{i} \rightarrow s_{i+1}$ for $i=0,1, \ldots k-1$. A computation $s_{0}, s_{1}, \ldots s_{k}$ is said to be a terminating computation of $\mathcal{M}$ if $s_{k}=\left(q_{f}, i, j\right)$ for some $i, j \in \mathbb{N}$.

We show that given a 2 -counter Minsky Machine $\mathcal{M}$, there is a program $P_{\epsilon}^{\mathcal{M}} \in$ Simple such that for each $\epsilon>0$,
(a) $P_{\epsilon}^{\mathcal{M}}$ has only one input $\mathrm{x}_{\text {in }}$ and only one output $\mathrm{x}_{\text {out }}$ taking values in $\mathrm{DOM}=\{0,1\}$.
(b) $P_{\epsilon}^{\mathcal{M}}$ terminates with probability 1.
(c) $P_{\epsilon}^{\mathcal{M}}$ is $(\epsilon, 0)$-differentially private with respect to the adjacency relation $\Phi=\{(0,1),(1,0)\}$ if and only if $\mathcal{M}$ does not halt.

Given a 2 -counter Machine $\mathcal{M}, P_{\epsilon}^{\mathcal{M}}$ is constructed as follows. Without loss of generality, let $Q=\left\{q_{1}, \ldots, q_{m}\right\}$ and let $q_{1}$ be the initial state and $q_{m}$ be the final state. We will model a state in $Q$ using $m$ Boolean variables $\mathrm{b}_{1}, \ldots, \mathrm{~b}_{m}$. If the current state is $q_{i}$ then $\mathrm{b}_{i}$ will be set to true and all other variables will be set to false. The counters will be modeled using real variables as follows. Initially a real variable $r_{0}$ will be sampled from Laplacian distribution. If $r_{0} \leq 0$, we will exit the program. Otherwise, we will initialize two real variables $r_{1}, r_{2}$ to be $r_{0} . r_{1}, r_{2}$ will model the counters as follows. If the first (second respectively) counter is going to hold natural number $i$ then $\mathrm{r}_{1}=(i+1) \mathrm{r}_{0}\left(\mathrm{r}_{2}=(i+1) \mathrm{r}_{0}\right.$ respectively). Incrementing

```
Input: \(x_{\text {in }}\)
Output: \(\times_{\text {out }}\)
\(\mathrm{x}_{\text {out }} \leftarrow 0\)
\(\mathrm{r}_{0} \leftarrow \operatorname{Lap}(\epsilon, 0)\)
\(\mathrm{b}_{\text {test }} \leftarrow \mathrm{r}_{0}>0\)
if \(b_{\text {test }}\) then
    \(r_{\text {number_steps }} \leftarrow \operatorname{Lap}(\epsilon, 0)\)
    \(r_{\text {cur_step }} \leftarrow r_{0}\)
    \(\mathrm{b}_{\text {continue }} \leftarrow \mathrm{r}_{\text {number_steps }}>\mathrm{r}_{\text {curr_step }}\)
    \(\mathrm{b}_{1} \leftarrow\) true
    \(\mathrm{b}_{2} \leftarrow\) false
    ...
    \(\mathrm{b}_{m} \leftarrow\) false
    \(r_{1} \leftarrow r_{0}\)
    \(r_{2} \leftarrow r_{0}\)
    while \(b_{\text {continue }}\) do
        \(s_{1}\)
        \(\vdots\)
        \(s_{n}\)
        \(\mathrm{b}_{1} \leftarrow \mathrm{~b}_{1}^{\text {next }}\)
            ..
            \(\mathrm{b}_{m} \leftarrow \mathrm{~b}_{\mathrm{m}}^{\text {next }}\)
            \(\mathrm{r}_{1} \leftarrow \mathrm{r}_{1}^{\text {next }}\)
            \(r_{2} \leftarrow r_{2}^{\text {next }}\)
            \(r_{\text {curr_step }} \leftarrow r_{0}+r_{\text {curr_step }}\)
            \(\mathrm{b}_{\text {continue }} \leftarrow \mathrm{r}_{\text {number_steps }}>\mathrm{r}_{\text {curr_step }}\)
    end
    if \(\left(b_{m}\right.\) and \(\left.E Q\left(x_{\mathbf{i n}}, 1\right)\right)\) then
        \(\mid x_{\text {out }} \leftarrow 1\)
        end
    end
    exit
```

Algorithm 3: Program $P_{\epsilon}^{\mathcal{M}}$ simulating 2-counter machine $\mathcal{M}$
the first counter (second respectively) counter is achieved by adding $r_{0}$ to $r_{1}$ ( $r_{2}$ respectively). Decrementing the first counter (second respectively) counter is achieved by sibtracting $r_{0}$ from $r_{1}$ ( $r_{2}$ respectively). For encoding the transition relations $\Delta_{i n c}^{1}, \Delta_{i n c}^{2}, \Delta_{j z d e c}^{1}$ and $\Delta_{j z d e c}^{2}$, we use variables $b_{1}^{\text {next }}, \ldots, b_{m}^{\text {next }}, r_{1}^{\text {next }}, r_{2}^{\text {next }}$ to compute the next configuration as expected. For example, the transition $\left(q_{i}, q_{j}, q_{k}\right) \in \Delta_{j z d e c}^{1}$ can be encoded using conditional statements as follows:

$$
\begin{aligned}
& \mathrm{b}_{i, j, k} \leftarrow \mathrm{r}_{1}=\mathrm{r}_{0} \\
& \text { if }\left(\mathrm{b}_{i, j, k} \text { and } b_{i}\right) \\
& \text { then } b_{j}^{\text {next }} \leftarrow \text { true; } b_{1}^{\text {next }} \leftarrow \text { false } ; \ldots \mathrm{b}_{j-1}^{\text {next }} \leftarrow \text { false; } \\
& \quad \mathrm{b}_{j+1}^{\text {next }} \leftarrow \text { false; } \ldots ; \mathrm{b}_{m}^{\text {next }} \leftarrow \text { false } \\
& \text { else } r_{1}^{\text {next }} \leftarrow \mathrm{r}_{1}-r_{0} ; b_{k}^{\text {next }} \leftarrow \text { true; } b_{1}^{\text {next }} \leftarrow \text { false; } \ldots \\
& \quad b_{k-1}^{\text {next }} \leftarrow \text { false; } b_{k+1}^{\text {next }} \leftarrow \text { false; } \ldots ; b_{m}^{\text {next }} \leftarrow \text { false }
\end{aligned}
$$

Let $s_{1}, s_{2}, \ldots, s_{n}$ be the statements encoding the transition relation. Consider the program $P_{\epsilon}^{\mathcal{M}}$ given in Algorithm 3. The program $P_{\epsilon}^{\mathcal{M}}$ initially samples $r_{0}$ from a continuous Laplacian distribution. If the sampled value is $\leq 0$ then it outputs 0 . Otherwise, it starts simulating $\mathcal{M}$. In order to make sure that the program terminates, we sample another real variable $r_{\text {number_steps }}$ and simulate $k$ steps of the program where $k$ is the smallest number such that $k r_{0}>r_{\text {number_steps }}$.

At the end of the simulation, if the halting state is reached and the input is 1 then it outputs 1 . Otherwise, it outputs 0 .

Clearly, $P_{\epsilon}^{\mathcal{M}}$ satisfies properties (a) and (b) above. That the program $P_{\epsilon}^{\mathcal{M}}$ has property (c) above follows from the following observations:

1. If $\mathcal{M}$ does not halt then $P_{\epsilon}^{\mathcal{M}}$ outputs 0 with probability 1 .
2. If $\mathcal{M}$ halts then $P_{\epsilon}^{\mathcal{M}}$ outputs 1 with non-zero probability on input 1 and outputs 1 with zero probability on input 0 .

This shows that Fixed Parameter Differential Privacy is undecidable. Undecidability of Fixed Parameter Differential Privacy is obtained by taking $\epsilon_{0}$ to be any constant rational number, say $\frac{1}{2}$.

## C DiPWhile encoding of exponential distribution

Example 13. Given $\epsilon>0$ and offset, let $\operatorname{Lap}^{+}(\epsilon$, offset) be the continuous distribution whose probability density function (p.d.f.) is given by

$$
f_{\epsilon, \mu}(x)= \begin{cases}\epsilon e^{-\epsilon(x-\text { offset })} & \text { if } x \geq \text { offset } \\ 0 & \text { otherwise }\end{cases}
$$

Observe that the one-sided Laplacian distribution $\operatorname{Lap}^{+}(\epsilon, 0)$ is the standard exponential distribution. Our language is expressive enough to encode onesided Laplacians as follows. Consider the sequence of statements:

$$
\begin{aligned}
& X \leftarrow \operatorname{Lap}(\epsilon, 0) \\
& b \leftarrow X \leq 0 ; \\
& \text { if } b \text { then } Y \leftarrow X \text { else } Y \leftarrow(-1) X \text { end; } \\
& Z \leftarrow Y+\text { offset }
\end{aligned}
$$

The effect of the sequence of statements is that $Z$ has the one-sided Laplacian distribution $\mathrm{Lap}^{+}(\epsilon$, offset $)$.

## D Formal DTMC Semantics of DiPWhile programs

We define formally $\llbracket P_{\epsilon} \rrbracket$, the DTMC semantics of an DiPWhile program $P_{\epsilon}$. Let us recall some key restrictions in DiPWhile programs. The first restriction is that real and integer-valued variables are never assigned within the scope of a while statement. Hence, they are assigned only a bounded number of times, and therefore, without loss of generality, we can assume that they are assigned a value exactly once. Second, real valued expressions are never compared against integer valued expressions.

Let us fix some basic notation. Partial functions from $A$ to $B$ will be denoted as $A \hookrightarrow B$. The value of $f: A \hookrightarrow B$ on $a \in A$, will be denoted as $f(a)$. Two partial functions $f$ and $g$ will be equal (denoted $f \simeq g$ ) if for every element $a$, either $f$ and $g$ are both undefined, or $f(a)=f(b)$. If $f: A \hookrightarrow B, a \in A$ and $b \in B$, then $f[a \mapsto b]$ denotes the partial function that agrees with $f$ on all elements of $A$ except $a$; on $a, f[a \mapsto b](a)=b$.

In the rest of this section let us fix a DiPWhile program $P_{\epsilon}$. L will denote the set of labels appearing in $P_{\epsilon}$. A valuation val for DOM variables is a function that assigns a value in DOM to variables in $\mathcal{X}$; we will denote set of all such valuations by $\mathrm{V}_{\text {DOM }}$. Given a valuation val $\in \mathrm{V}_{\text {DOM }}$ and a real expression $e, \operatorname{val}(e)$ denotes the real expression that results from substituting
all the DOM variables appearing in $e$ by their value in val. Similarly, for an integer expression, $\operatorname{val}(e)$ is the partial evaluation of $e$ with respect to val. Finally, for a comparison $e_{1} \sim e_{2}$ between two expressions $e_{1}$ and $e_{2}$, again we will define $\operatorname{val}\left(e_{1} \sim e_{2}\right)$ to be $\operatorname{val}\left(e_{1}\right) \sim \operatorname{val}\left(e_{2}\right)$. Let us denote the set of integer expressions, real expressions, and Boolean comparisons, appearing on the right hand side of assignments in $P_{\epsilon}$ by $P_{Z}, P_{R}$, and $P_{B}$, respectively. Three sets of expressions will be used in defining the semantics, and they are as follows.

$$
\begin{aligned}
& \mathrm{zExp}=\left\{\text { val }(e) \mid \text { val } \in \mathrm{V}_{\text {DOM }}, e \in P_{Z}\right\} \\
& \mathrm{rExp}=\left\{\text { val }(e) \mid \text { val } \in \mathrm{V}_{\text {DOM }}, e \in P_{R}\right\} \\
& \mathrm{bExp}=\left\{\text { val }(e) \mid \text { val } \in \mathrm{V}_{\text {DOM }}, e \in P_{B}\right\}
\end{aligned}
$$

Thus, $z E x p$, $r$ Exp, and bExp are partially evaluated expression appearing on the right hand side of assignments in $P_{\epsilon}$. Notice that the sets L, zExp, rExp, and bExp are all finite. Finally, let Const be the set of rational constants appearing as coefficient of $\epsilon$ of Laplace and discrete Laplace assignments in $P_{\epsilon} ;$ again Const is finite.

In order to define the semantics of $P_{\epsilon}$, we will use an auxiliary function next that given a label, identifies the label of the statement to be executed next. Observe that for most program statements, the next statement to be executed is unique. However, for if and While statements, the next statement depends on the value of a Boolean expression. We will define next $(\ell)$ to be a set of pairs of the form $\left(\ell^{\prime}, c\right)$ with the understanding that $\ell^{\prime}$ is the next label if $c$ holds. Thus, for a label $\ell$, next $(\ell)$ will either be $\left\{\left(\ell^{\prime}\right.\right.$, true $\left.)\right\}$ or $\left\{\left(\ell_{1}, c\right),\left(\ell_{2}, \neg c\right)\right\}$. We do not give a precise definition of next $(\cdot)$, but we will use it when defining the semantics.

The semantics of $P_{\epsilon}$ will given as a finite-state, parametrized DTMC $\llbracket P_{\epsilon} \rrbracket$. To define the parametrized DTMC $\llbracket P_{\epsilon} \rrbracket$, we need to define the states and the transitions.

States. States of $\llbracket P_{\epsilon} \rrbracket$ will be of the form

$$
\left(\ell, f_{\text {Bool }}, f_{\mathrm{DOM}}, f_{\text {int }}, f_{\text {real }}, C\right)
$$

Informally, $\ell \in \mathrm{L}$ is the label of the statement to be executed, $f_{\text {Bool }}, f_{\mathrm{DOM}}$, $f_{\text {int }}$, and $f_{\text {real }}$ are partial functions assigning "values" to program variables (of appropriate type), and $C$ is a collection of inequalities among program variables that hold on the current computational path. Both $f_{\text {Bool }}$ and $f_{\text {DOM }}$ are valuations for the appropriate set of variables, and so we have $f_{\text {Bool }}: \mathcal{B} \hookrightarrow\{$ true, false $\}$ and $f_{\mathrm{DOM}}: \mathcal{X} \hookrightarrow \mathrm{DOM}$. For real and integer
variables, instead of tracking exact values, we will track the expressions used in assignments and parameters of (discrete) Laplace mechanisms used in random assignments. Therefore, we have $f_{\text {int }}: \mathcal{Z} \hookrightarrow \mathrm{zExp} \cup($ Const $\times$ DOM $)$ and $f_{\text {real }}: \mathcal{R} \hookrightarrow \mathrm{rExp} \cup($ Const $\times \mathrm{DOM})$. Finally, $C \subseteq \mathrm{bExp} \cup\{\neg e \mid e \in \mathrm{bExp}\}$. It follows immediately that the set of states of $\llbracket P_{\epsilon} \rrbracket$ is finite.

Well-Formed States. The functions $f_{*}$ (for $* \in\{$ Bool, DOM, int, real $\}$ ) assign values to program variables that have been assigned during the computation thus far. Since we assume variables in DiPWhile program are defined before they are used, if a variable $z^{\prime}$ appears in $f_{\text {int }}(z) \in z E x p$, then $f_{\text {int }}\left(z^{\prime}\right)$ must be defined. A similar condition holds for real variables. The comparisons in $C$ are also relationships that must hold on the current path, and so all variables participating in it must be defined. If a state satisfies these consistency properties between $f_{\text {int }}, f_{\text {real }}$, and $C$, we will say it is wellformed. All reachable states in $\llbracket P_{\epsilon} \rrbracket$ will be well-formed. So when we define transitions we will assume that the states are well-formed.

Initial States. Let $\ell_{\text {in }}$ be the label of the first statement $P_{\epsilon}$. Let $C^{\text {in }}=\emptyset$, and let $f_{B o o l}^{\mathrm{in}}, f_{\text {int }}^{\mathrm{in}}$, and $f_{\text {real }}^{\mathrm{in}}$ be partial functions with an empty domain. An initial state of $\llbracket P_{\epsilon} \rrbracket$ will be of the form $\left(\ell_{\mathbf{i n}}, f_{\text {Bool }}^{\mathrm{in}}, f_{\mathrm{DOM}}^{\mathrm{in}}, f_{\mathrm{int}}^{\mathrm{in}}, f_{\text {real }}^{\mathrm{in}}, C^{\mathbf{i n}}\right)$, where $f_{\text {DOM }}^{\text {in }}$ is defined only on the input variables; the values given to these variables by $f_{\mathrm{DOM}}^{\mathrm{in}}$ will be the "initial input value".

We will now define the semantics of transitions in $\llbracket P_{\epsilon} \rrbracket$. Let us fix a state $z=\left(\ell, f_{\text {Bool }}, f_{\text {DOM }}, f_{\text {int }}, f_{\text {real }}, C\right)$. Transitions out of $z$ will be defined based on the effect of executing the statement labeled $\ell$, and so its definition will depend on this statement. We handle each case below.

DOM assignments. Let next $(\ell)=\left\{\left(\ell^{\prime}\right.\right.$, true $\left.)\right\}$ and let $\times$ be the variable being assigned in $\ell$. There are two cases to consider. First, consider the case where x is assigned a value for a DOM expression $e$. In this case, $\llbracket P_{\epsilon} \rrbracket$ will transition to

$$
\left(\ell^{\prime}, f_{\text {Bool }}, f_{\mathrm{DOM}}\left[\mathrm{x} \mapsto f_{\mathrm{DOM}}(e)\right], f_{\text {int }}, f_{\text {real }}, C\right)
$$

with probability 1 . The second case is when $\times$ is assigned a random value according to $\operatorname{Exp}(a \epsilon, F(\tilde{\mathrm{x}}), e)$ or choose $(a \epsilon, \tilde{e})$. For $d \in \operatorname{DOM}$, let $\operatorname{prob}(d)$ be the probability of $d$ (as a function of $\epsilon$ ) based on the distribution; note, that these probabilities will depend on the value of $f_{\mathrm{DOM}}(e)$ and $f_{\mathrm{DOM}}(\tilde{e})$. Then, $\llbracket P_{\epsilon} \rrbracket$ will transition to

$$
\left(\ell^{\prime}, f_{\text {Bool }}, f_{\mathrm{DOM}}[\mathrm{x} \mapsto d], f_{\mathrm{int}}, f_{\text {real }}, C\right)
$$

with probability $\operatorname{prob}(d)$.
Integer assignments. Let $\operatorname{next}(\ell)=\left\{\left(\ell^{\prime}\right.\right.$, true $\left.)\right\}$ and let $\mathbf{z}$ be the variable being assigned in $\ell$. Again there are two cases to consider. First, consider the case where $\mathbf{z}$ is assigned a value for an integer expression $e$. In this case, $\llbracket P_{\epsilon} \rrbracket$ will transition to

$$
\left(\ell^{\prime}, f_{\text {Bool }}, f_{\mathrm{DOM}}, f_{\text {int }}\left[\mathrm{z} \mapsto f_{\mathrm{DOM}}(e)\right], f_{\text {real }}, C\right)
$$

with probability 1. Next, if $\mathbf{z}$ is assigned a random value according to $\operatorname{DLap}(a \epsilon, e)$, then $\llbracket P_{\epsilon} \rrbracket$ transitions to

$$
\left(\ell^{\prime}, f_{\text {Bool }}, f_{\mathrm{DOM}}, f_{\text {int }}\left[\mathrm{z} \mapsto\left(a, f_{\mathrm{DOM}}(e)\right)\right], f_{\text {real }}, C\right)
$$

with probability 1 . Notice that we have a deterministic transition even if the assignment samples from a discrete Laplace. The effect of choosing randomly a value will get accounted for during Boolean assignments.

Real assignments. Let $\operatorname{next}(\ell)=\left\{\left(\ell^{\prime}\right.\right.$, true $\left.)\right\}$ and let r be the variable being assigned in $\ell$. First, if $\mathbf{z}$ is assigned a value for a real expression $e$, $\llbracket P_{\epsilon} \rrbracket$ will transition to

$$
\left(\ell^{\prime}, f_{\text {Bool }}, f_{\mathrm{DOM}}, f_{\text {int }}, f_{\text {real }}\left[\mathrm{r} \mapsto f_{\mathrm{DOM}}(e)\right], C\right)
$$

with probability 1 . If $\mathbf{z}$ is assigned a random value according to $\operatorname{Lap}(a \epsilon, e)$, then $\llbracket P_{\epsilon} \rrbracket$ transitions to

$$
\left(\ell^{\prime}, f_{\text {Bool }}, f_{\mathrm{DOM}}, f_{\text {int }}, f_{\text {real }}\left[\mathrm{r} \mapsto\left(a, f_{\mathrm{DOM}}(e)\right)\right], C\right)
$$

with probability 1 . Again sampling according to Laplace is modeled deterministically.

Boolean assignments. Again let next $(\ell)=\left\{\left(\ell^{\prime}\right.\right.$, true $\left.)\right\}$ and let b be the variable being assigned in $\ell$. When b is assigned the value of Boolean expression $e, \llbracket P_{\epsilon} \rrbracket$ transitions to

$$
\left(\ell^{\prime}, f_{\text {Bool }}\left[\mathrm{b} \mapsto f_{\text {Bool }}(e)\right], f_{\mathrm{DOM}}, f_{\text {int }}, f_{\text {real }}, C\right)
$$

with probability 1 . The interesting case is when b is assigned the result of comparing expressions $e_{1} \sim e_{2}$. If the probability of all conditions in $C$ holding is 0 , then let $p_{1}$ be 0 . Otherwise, let $p_{1}$ denote the probability of $f_{\text {DOM }}\left(e_{1}\right) \sim f_{\text {DOM }}\left(e_{2}\right)$ holding given all conditions in $C$ hold; notice that this
probability depends on the functions $f_{\text {int }}$ and $f_{\text {real }}$ that store the parameters to various random sampling steps. Now $\llbracket P_{\epsilon} \rrbracket$ will transition to

$$
\left(\ell^{\prime}, f_{\text {Bool }}[\mathrm{b} \mapsto \text { true }], f_{\text {DOM }}, f_{\text {int }}, f_{\text {real }}, C \cup\left\{f_{\text {DOM }}\left(e_{1}\right) \sim f_{\mathrm{DOM}}\left(e_{2}\right)\right\}\right)
$$

with probability $p_{1}$, and it will transition to

$$
\begin{aligned}
& \left(\ell^{\prime}, f_{\text {Bool }}[\mathrm{b} \mapsto \text { false }], f_{\mathrm{DOM}}, f_{\text {int }}, f_{\text {real }},\right. \\
& \left.\quad C \cup\left\{\neg\left(f_{\mathrm{DOM}}\left(e_{1}\right) \sim f_{\mathrm{DOM}}\left(e_{2}\right)\right)\right\}\right)
\end{aligned}
$$

with probability $1-p_{1}$. Thus, the effect of the probabilistic sampling steps for integer and real variables gets accounted for when the result of a comparison is assigned to a Boolean variable.
if statement. In this case, $\operatorname{next}(\ell)=\left\{\left(\ell_{1}, c\right),\left(\ell_{2}, \neg c\right)\right\}$. If $f_{\text {Bool }}(c)=$ true then we transition to

$$
\left(\ell_{1}, f_{\text {Bool }}, f_{\mathrm{DOM}}, f_{\text {int }}, f_{\text {real }}, C\right)
$$

with probability 1 . On the other hand, if $f_{\text {Bool }}(c)=$ false then transition to

$$
\left(\ell_{2}, f_{\text {Bool }}, f_{\mathrm{DOM}}, f_{\text {int }}, f_{\text {real }}, C\right)
$$

with probability 1.
While statement. Again let $\operatorname{next}(\ell)=\left\{\left(\ell_{1}, c\right),\left(\ell_{2}, \neg c\right)\right\}$. This case is identical to the case of if statement, and so is skipped.
exit statement. In this case we stay in state $z$ with probability 1 .

Equivalence of the two semantics. Let in be a valuation over input variables and out be a valuation over output variables. We let $\operatorname{Prob}_{D T M C}\left(P_{\epsilon}(\mathbf{i n})=\right.$ out) denote the probability that $P_{\epsilon}$ outputs value out, on the input in, under the DTMC semantics. This probability is defined to be the probability of reaching a state of the form $\left(\ell, f_{\text {Bool }}, f_{\mathrm{DOM}}, f_{\text {int }}, f_{\text {real }}, C\right)$ where $\ell$ is the label of an exit statement and $f_{\text {DOM }}$ assigns the values given by out to output variables, from an initial state in which the values of the input variables is given by in, in the DTMC $\llbracket P_{\epsilon} \rrbracket$. The following theorem states the equivalence of the natural semantics given in Appendix A to that of the DTMC semantics for DiPWhile programs.

Theorem 14. For every $\epsilon>0$ and DiPWhile program $P_{\epsilon}$, and for every pair of evaluations in, out to the input and output variables respectively, $\operatorname{Prob}_{D T M C}\left(P_{\epsilon}(\mathbf{i n})=\right.$ out $)=\operatorname{Prob}_{\text {natural }}\left(P_{\epsilon}(\mathbf{i n})=\right.$ out $)$.
Proof Sketch. Let us fix an $\epsilon>0$ and a program $P_{\epsilon}$. Then $\llbracket P_{\epsilon} \rrbracket$ can be considered as a (non-paramaterized) DTMC. For any path $\rho=z_{1}, \ldots, z_{i}$ in the DTMC $\llbracket P_{\epsilon} \rrbracket$, let $\operatorname{prob}(\rho)$ denote the product of the probabilities of all the transitions in $\rho$. We call $\rho$ an initialized path if it starts with an initial state, and a proper path if $\operatorname{prob}(\rho)>0$. For any initialized path $\rho$ of $\llbracket P_{\epsilon} \rrbracket$, let $\operatorname{prob}_{\mathrm{DOM}}(\rho)$ be the product of all the transitions in $\rho$ that result from random assignments to DOM variables, $\operatorname{prob}_{\mathbb{Z}}(\rho)$ be the product of the probabilities that result from a comparison between integer variables and $\operatorname{prob}_{\mathbb{R}}(\rho)$ be the product of the probabilities that result from a comparison between real variables. It is easy to see that

$$
\operatorname{prob}(\rho)=\operatorname{prob}_{\mathrm{DOM}}(\rho) \operatorname{prob}_{\mathbb{R}}(\rho) \operatorname{prob}_{\mathbb{Z}}(\rho)
$$

We recall some of the notation as defined in Appendix © Let $S$ be the set of states of $P_{\epsilon}$ in the natural semantics. A state $s \in S$ is a tuple ( $\ell, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}, h_{\mathbb{R}}$ ) denoting the label of a statement to be executed, and the values of Boolean, DOM, integer and real variables of $P_{\epsilon}$. A discrete state of $P_{\epsilon}, d s$, is a tuple ( $\ell, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}$ ) specifying the label of the statement and the values of Boolean, DOM and integer variables of $P_{\epsilon}$. For a state $s=\left(\ell, h_{\text {Bool }}, h_{\text {DOM }}, h_{\mathbb{Z}}, h_{\mathbb{R}}\right)$, let $\operatorname{disc}(s)$ be the discrete state $\left(\ell, h_{\text {Bool }}, h_{\mathrm{DOM}}, h_{\mathbb{Z}}\right)$. A discrete state $d s$ is said to be initial if $d s=\operatorname{disc}\left(s_{\text {init }}\right)$ where $s_{\text {init }}$ is the initial state of $S$.

A discrete execution $\beta=d s_{1}, \ldots, d s_{i}$ of $P_{\epsilon}$ is a sequence of discrete states. The discrete execution $\beta=d s_{1}, \ldots, d s_{i}$ is an initialized if $d s_{1}$ is the initial discrete state. For a discrete execution $\beta$ as given above, let $\operatorname{ext}(\beta)=\left\{\left(s_{1}, \ldots, s_{i}\right) \mid d s_{j}=\operatorname{disc}\left(s_{j}\right), 1 \leq j \leq i\right\}$. It is easy to see that, for any discrete execution $\beta$ of length $i, \operatorname{ext}(\beta)$ is in $\Sigma_{i}$ (see Appendix A) , i.e., is measurable. For a discrete computation $\beta$, of length $i>0$, let $\operatorname{pr}(\beta)=\phi_{i}(\operatorname{ext}(\beta))$ where $\phi_{i}$ is the probability function defined on the measure space $\left(S^{i}, \Sigma_{i}\right)$ in Appendix A. If $\operatorname{pr}(\beta)>0$ then we call $\beta$ a proper discrete execution of $P_{\epsilon}$.

Consider an initialized proper discrete execution $\beta$ of length $i$, as given above, where $d s_{j}=\left(\ell_{j}, h_{\mathrm{DOM}}^{j}, h_{\text {Bool }}^{j}, h_{\mathbb{Z}}^{j}\right)$ for $1 \leq j \leq i$. It can be shown that there exists a unique initialized path $\rho_{\beta}=z_{0}, \ldots, z_{i}$ in the DTMC $\llbracket P_{\epsilon} \rrbracket$ corresponding to $\beta$ such that for each $j$,

1. the state $z_{j}=\left(\ell_{j}, f_{\mathrm{DOM}}^{j}, f_{\text {Bool }}^{j}, f_{\text {int }}^{j}, f_{\text {real }}^{j}, C_{j}\right)$ for some appropriate $f_{\mathrm{DOM}}^{j}, f_{\text {Bool }}^{j}, f_{\text {int }}^{j}, f_{\text {real }}^{j}$ and $C_{j}$, and
2. $f_{\text {Bool }}^{j}(\mathrm{~b})=h_{\text {Bool }}^{j}(\mathrm{~b})\left(f_{\mathrm{DOM}}^{j}(\mathrm{x})=h_{\text {DOM }}^{j}(\mathrm{x})\right.$ respectively $)$ whenever $f_{\text {Bool }}^{j}(\mathrm{~b})$ ( $f_{\text {DOM }}^{j}(\mathrm{x})$ respectively) is defined.

Let $H$ be the function mapping initialized proper discrete executions of $P_{\epsilon}$ to corresponding initialized paths in $\llbracket P_{\epsilon} \rrbracket$, as specified above.

For an initialized proper discrete execution $\beta$ of length $i$ as above, we define a number $p_{j}$ for each $j \leq i$ as follows. For $d s_{j}=\left(\ell_{j}, h_{\mathrm{DOM}}^{j}, h_{\text {Bool }}^{j}, h_{\mathbb{Z}}^{j}\right)$, let $\widehat{d s}{ }_{j}=\left(\ell_{j}, h_{\mathrm{DOM}}^{j}, h_{\text {Bool }}^{j}, h_{\mathbb{Z}}^{j}, h_{\mathbb{R}}^{\mathrm{in}}\right)$ where $h_{\mathbb{R}}^{\mathrm{in}}$ is the function that maps each real variable of $P_{\epsilon}$ to 0 . Let $K_{\epsilon}$ be the Markov kernel as defined in Appendix A. If $j>1$ and $\ell_{j-1}$ is the label of an assignment to an integer variable that samples from a discrete Laplacian variable then $p_{j}=K_{\epsilon}\left(\widehat{d s_{j-1}},\left\{\widehat{d s_{j}}\right\}\right)$, otherwise $p_{j}=1$. Let $\mathrm{pr}_{\mathbb{Z}}(\beta)=p_{1} \ldots p_{i}$. It can be shown using the definition of measure $\phi_{i}$ on $\Sigma_{i}$ (See Section (A) that

$$
\operatorname{pr}(\beta)=\operatorname{prob}_{\mathrm{DOM}}(H(\beta)) \operatorname{prob}_{\mathbb{R}}(H(\beta)) \operatorname{pr}_{\mathbb{Z}}(\beta) .
$$

Furthermore, $\operatorname{prob}(H(\beta))>0$ if $\operatorname{pr}(\beta)>0$.
Now consider any initialized proper path $\rho=z_{1}, \ldots, z_{i}$ in $\llbracket P_{\epsilon} \rrbracket$. From the above observations, it can be shown that $\operatorname{prob}(\rho)=\sum_{u \in H^{-1}(\rho)} \phi_{i}(\operatorname{ext}(\beta))$. Now, the theorem follows from this observation and the definitions of $\operatorname{Prob}_{D T M C}\left(P_{\epsilon}(\mathbf{i n})\right.$ $=$ out $)$ and $\operatorname{Prob}_{\text {natural }}\left(P_{\epsilon}(\mathbf{i n})=\right.$ out $)$.

Complexity. Now, we bound the size of the state space of DTMC $\llbracket P_{\epsilon} \rrbracket$ as follows. Let $m=|\mathrm{DOM}|=2 N_{\max }+1$ and $m^{\prime}$ be the length of $P_{\epsilon}$. Let $n_{1}, n_{2}, n_{3}, n_{4}$, respectively, be the number DOM variables, boolean variables, integer variables, and real variables occurring in $P_{\epsilon}$. In a state $s$ of $M_{P}$, the number of possible values for $f_{\text {dom }}$ is $\leq m^{n_{1}}$, the number of possible values for $f_{\text {bool }}$ is $\leq 2^{n_{2}}$. The number of possible values for $f_{\text {int }}$ can be bounded as follows. An integer variable can be assigned a Laplacian distribution whose parameters are pairs of the form $(a \epsilon, e)$ where $e$ is an expression over variables in $U \cap \mathcal{X}$; the number of such pairs is $\leq m_{1} m^{n_{1}}$ where $m_{1}$ is the number of values of $a$ in $P$ and $m^{n_{1}}$ is the bound on the number of values of $e$. An integer variable can also be assigned a linear expression over integer variables with coefficients that are integer constants or expressions over DOM variables; the number of such linear combinations is $\leq m_{2} m^{n_{1}}$ where $m_{2}$ is the number of such expressions appearing in $P$. Since, $m_{1}+m_{2} \leq m^{\prime}$, we see that number of values that an integer variable can be mapped to is $\leq m^{\prime} m^{n_{1}}$. Hence the number of possible values for $f_{\text {int }}$ is $\leq\left(m^{\prime} m^{n_{1}}\right)^{n_{3}}$. By a similar reasoning we observe that the number of possible values for $f_{\text {real }}$ is $\leq\left(m^{\prime} m^{n_{1}}\right)^{n_{4}}$. Now we bound the number of values for $C$ as follows. The only
places where comparisons appear are on the right hand sides of assignments to boolean variables. In each such assignment we have comparisons over linear expressions of integer and real variables ; such comparisons also have integer constants and DOM variables appearing in them. Since the number of integer constants is $\leq m^{\prime}$ and the number of valuations to DOM variables $\leq m^{n_{1}}$, we get that the number of possible comparions is $\leq m^{\prime} m^{n_{1}}$. Since $C$ is a subset of such comparisons, the number of possible values for $C$ is $\leq 2^{\left(m^{\prime} m^{n_{1}}\right)}$. Now, the number of states is bounded by the product of possible values to each component of a state, which is seen to be $O\left(2^{n} \cdot n^{n_{1}+n_{2}+n_{3}+n_{4}}\right)$ where $n=m^{\prime} m^{n_{1}}$.

## E DiPWhile programs are finite, definable, parametrized DTMCs

We show the proof of Theorem 8, namely that for any DiPWhile program $P_{\epsilon}, \llbracket P_{\epsilon} \rrbracket$ is a finite, definable, parametrized DTMC.

Proof. From our definition of the DTMC semantics (Appendix D), it follows that $\llbracket P_{\epsilon} \rrbracket$ is a finite parameterized DTMC. We now show that it is definable also. In order to show this, we have to show that the transition probabilities of $\llbracket P_{\epsilon} \rrbracket$ are definable. Observe that, by definition, the transition probabilities of choose $(a \epsilon, \tilde{E})$ construct are definable. The other probabilistic transitions arise as a result of comparison between random variables of the same sort or from using the exponential mechanism. These transition probabilities turn out to be from a special class of definable functions. We define this form next.

Definition 15. Let $p(\epsilon)=\sum_{i=1}^{m} a_{i} \epsilon^{n_{i}} e^{\epsilon q_{i}}$ where each $a_{i}$ is a rational number, $n_{i}$ is a natural number and $q_{i}$ is a non-negative rational number. We shall call all such expressions pseudo-polynomials in $\epsilon$. Given a real number $b>0$ and a pseudo-polynomial $p(\epsilon), p(b)$ is the real number obtained by substituting $b$ for $\epsilon$. The ratio of two pseudo-polynomials in $\epsilon, \frac{p_{1}(\epsilon)}{p_{2}(\epsilon)}$, shall be called a pseudo-rational function in $\epsilon$ if $p_{2}(b) \neq 0$ for all real $b>0$. Given a real number $b>0$ and a pseudo-rational function $r t(\epsilon)=\frac{p_{1}(\epsilon)}{p_{2}(\epsilon)}, r t(b)$ is defined to be $\frac{p_{1}(b)}{p_{2}(b)}$.

Observe that a pseudo-rational function $r t$ defines a function $f_{r t}$ from the set of strictly positive reals to the set of reals. We will henceforth confuse $f_{r t}$ with $r t$. Pseudo-rational functions are easily seen to be closed under addition and multiplication.

Corollary 16. Each pseudo-rational function $r t$ is definable in the theory Thexp .

Proof. Let $r t(\epsilon)=\frac{\sum_{i=1}^{m} a_{i} \epsilon^{n} e^{\epsilon \epsilon q_{i}}}{\sum_{i=1}^{m \prime} a_{i}^{\prime} \epsilon^{n_{i}^{\prime}} e^{\epsilon \epsilon_{i}^{\prime}}}$. Let $N$ be the least common multiple of all denominators of $q_{i}, q_{i}^{\prime}$. Let $p_{i}=q_{i} N$ and $p_{i}^{\prime}=q_{i}^{\prime} N$. Let $a$ be the least common multiple of all denominators of $a_{i}, a_{i}^{\prime}$. Let $b_{i}=a a_{i}$ and $b_{i}^{\prime}=a a_{i}^{\prime}$. It is easy to see that $r t$ is definable by the formula $\phi(x)$ :

$$
\phi(x) \equiv \forall z .\left(\left(x \sum_{i=1}^{m^{\prime}} b_{i}^{\prime} \epsilon^{n_{i}^{\prime}} z^{p_{i}^{\prime}}=\sum_{i=1}^{m} b_{i} \epsilon^{n_{i}} z^{p_{i}}\right) \wedge\left(z^{N}=e^{\epsilon}\right) \wedge(z>0)\right) .
$$

Note that in the above formula, $z$ is the $N$ th root of $\epsilon$.
Now, it follows from our restriction on our scoring functions, namely that they take values in rationals, that the transition probabilities in exponential mechanism are pseudo-rational functions that can be computed.

Let us now consider the case of comparison between random variables. Let state $=\left(\ell, f_{\text {Bool }}, f_{\text {DOM }}, f_{\text {int }}, f_{\text {real }}, C\right)$ of $\llbracket P_{\epsilon} \rrbracket$ be a state of $\llbracket P_{\epsilon} \rrbracket$. Recall that when we compare random variables in state, we add a new linear comparison $e$ to $C$. Further, in order to compute transition probabilities, we compute the conditional probability that the set of linear comparison $C \cup e$ is true given that $C$ is true. For this, it suffices to show that we can compute the probability that the set of linear comparisons $C$ is true and the probability $C \cup e$ is true. We make the following observations:

- Since every random variable must be defined before it is used, we can simplify $C$ and $C \cup e$ to only refer to program variables that were used in random assignments.
- All our random assignments sample from independent random variables. Since we never compare integer and real random variables, it suffices to compute the probability that a system of linear comparisons over integers with integer coefficients hold and the probability that a system of linear comparisons over reals with rational coefficients hold. We will now show that these probabilities can be computed and are pseudo-rational functions.
- In order to compute the probability that a system of linear comparisons over reals with rational coefficients hold, we only need to consider systems of linear inequalities. Clearly any equality $u_{1}=u_{2}$ can be written as two inequalities, $u_{1} \leq u_{2}$ and $u_{2} \leq u_{1}$. If a comparison
in $C$ is $u_{1} \neq u_{2}$ then we can consider the systems $C_{1}=\left(C \backslash\left\{u_{1} \neq\right.\right.$ $\left.\left.u_{2}\right\}\right) \cup\left\{u_{1}<u_{2}\right\}$ and $C_{2}=\left(C \backslash\left\{u_{1} \neq u_{2}\right\}\right) \cup\left\{u_{2}<u_{1}\right\}$, compute probabilities of $C_{1}$ and $C_{2}$ separately and add them up to compute the probability that $C$ holds. Thus, without loss of generality we can assume that $C$ consists of only linear inequalities.

Probability of system of linear inequalities over integers. Let $\bar{Z}=$ $\left(Z_{1}, \ldots, Z_{n}\right)$ be a discrete random variable taking values in $\mathbb{Z}^{n}$. Consider a finite system of linear inequalities $C$ with integer coefficients and with $n$ unknowns $Z_{1}, \ldots, Z_{n}$. A solution of $C$ is a tuple $\bar{b}=\left(b_{1}, \ldots, b_{n}\right) \in \mathbb{Z}^{n}$ such that all inequalities in $C$ are satisfied when each $Z_{j} \in C$ is replaced by $b_{j}$. Let $\operatorname{sol}(C) \subseteq \mathbb{Z}^{n}$ denote the set of all solutions of $C$. The probability that $\bar{Z}$ satisfies $C$ is said to be the probability of the event $E=\{\bar{Z}=$ $\bar{b} \mid \bar{b}$ is a solution of $C\}$. We denote this probability by $\operatorname{Prob}(\bar{Z} \models C)$. We have the following:

Lemma 17. Let $C$ be a finite system of linear inequalities with integer coefficients and with $n$ unknowns $Z_{1}, \ldots, Z_{n}$. Let $Z_{j}=\operatorname{DLap}\left(a_{j} \epsilon, \mu_{1}\right), \ldots$, $Z_{n}=\operatorname{DLap}\left(a_{n} \epsilon, \mu_{n}\right)$ be mutually independent discrete Laplacians such that for each $1 \leq j \leq n, a_{j}$ is a strictly positive rational number and $\mu_{j}$ is an integer. Let $\bar{Z}=\left(Z_{1}, \ldots, Z_{n}\right)$. There is a pseudo-rational function $r t_{\bar{Z}, C}$ in $\epsilon$ such that $\operatorname{Prob}(\bar{Z} \models C)=r t_{\bar{Z}, C}$. The function $r t_{\bar{Z}, C}$ can be computed from $C,\left(a_{1}, \mu_{1}\right), \ldots,\left(a_{n}, \mu_{n}\right)$.

Proof. For, each $1 \leq j \leq n$, consider $Y_{j}=\operatorname{DLap}\left(a_{j} \epsilon, 0\right)$. It is easy to see that $Z_{j}$ has the same distribution as $Y_{j}+\mu_{j}$. Now consider the system of inequalities $C^{\prime}$ in which each $Z_{j}$ is replaced by $Y_{j}+\mu_{j}$. Let $\bar{Y}=\left(Y_{1}, \ldots, Y_{n}\right)$. It is easy to see that $\operatorname{Prob}(\bar{Z} \models C)=\operatorname{Prob}\left(\bar{Y} \models C^{\prime}\right)$. This observation implies that it suffices to prove the Lemma in the special case that each $\mu_{j}=0$. Thus, for the rest of the proof we assume that each $\mu_{j}=0$.

Now, consider a set pos $\subseteq\{1, \ldots, n\}$. Let $C_{\text {pos }}$ be the system of inequalities $C \cup\left\{Z_{j} \geq 0 \mid j \in \operatorname{pos}\right\} \cup\left\{Z_{j}<0 \mid j \notin \operatorname{pos}\right\}$. It is easy to see that the set of solutions of $C$ is the disjoint union $\cup_{\{\operatorname{pos} \subseteq 1, \ldots, n\}} C_{\text {pos }}$. Thus, it suffices to the prove that for each pos $\subseteq\{1, \ldots, n\}, \operatorname{Prob}\left(\bar{Z} \models C_{\text {pos }}\right)$ is a pseudo-rational function that can be computed.

Consider the system of inequalities $C_{\text {pos }}^{\prime}$ obtained from $C_{\text {pos }}$ by replacing each $Z_{j}$ by $Y_{j}$ for $j \in$ pos and by $-Y_{j}$ for $j \notin$ pos. Let $Y=\left(Y_{1}, \ldots, Y_{n}\right)$. From the fact that Laplacians are symmetric distributions, it follows each $Y_{j}$ has the same distribution as $Z_{j}$. Thus, $\operatorname{Prob}\left(\bar{Z} \models C_{\text {pos }}\right)=\operatorname{Prob}\left(\bar{Y} \models C_{\text {pos }}^{\prime}\right)$. Observe that the set of solutions of $C_{\text {pos }}^{\prime}$ are a subset of $\mathbb{N}^{n}$. Without loss
of generality, we can also assume that the terms in each inequality of $C_{\text {pos }}^{\prime}$ are rearranged so that the constant terms in $C_{\text {pos }}^{\prime}$ and the coefficients of the variables $Y_{j}$ are natural numbers, ie, non-negative integers.

Therefore, $C_{\text {pos }}^{\prime}$ is a system of linear inequalities with natural number coefficients. We are interested in solutions of $C_{\text {pos }}^{\prime}$ over natural numbers. For such system of inequalities, the set of solutions can be written as a disjoint union of simple linear sets [12]; a set $S \subseteq \mathbb{N}^{n}$ is said to be linear if there are tuples $\bar{b}_{0}, \bar{p}_{1}, \ldots, \bar{p}_{m} \in \mathbb{N}^{n}$ such that $S=\left\{\bar{b}_{0}+\sum_{i=1}^{m} k_{i} \bar{p}_{i} \mid\right.$ for each i, $\left.k_{i} \in \mathbb{N}\right\}$ and simple if each $\bar{b} \in S$ has a unique representation as a sum $\bar{b}_{0}+\sum_{i=1}^{m} k_{i} \bar{p}_{i} . \bar{b}_{0}$ is said to be the offset of $S$ and $\bar{p}_{1}, \ldots, \bar{p}_{m}$ the periods of $S$. From the fact that the set of solutions of $C_{\text {pos }}^{\prime}$ can be written as a disjoint union of simple linear sets, it follows that it suffices to show that $\operatorname{Prob}(\bar{Y} \in S \mid S$ is simple linear) is a pseudo-rational function in $\epsilon$. In order to show this we need a couple of additional notations.

For two $n$-tuples $\bar{x}=\left(x_{1}, \ldots, x_{n}\right)$ and $\bar{y}=\left(y_{1}, \ldots, y_{n}\right), \bar{x} \cdot \bar{y}$ will denote the sum $\sum_{j=1}^{n} x_{j} y_{j}$. Secondly, we will denote the tuple $\left(a_{1}, \ldots, a_{n}\right)$ by $\bar{a}$.

Fix a simple semilinear set $S$. Let $\bar{b}_{0}$ be its offset and $\bar{p}_{1}, \ldots, \bar{p}_{m}$ its periods. Let $\kappa=\prod_{i=1}^{n} \frac{1-e^{-a_{i} \epsilon}}{1+e^{-a_{i} \epsilon}}$. From the fact that each $\bar{b} \in S$ has a unique representation as a sum $\bar{b}_{0}+\sum_{i=1}^{m} k_{i} \bar{p}_{i}$, it follows that

$$
\begin{aligned}
\operatorname{Prob}(\bar{Y} \in S) & =\sum_{k_{1} \in \mathbb{N}} \cdots \sum_{k_{m} \in \mathbb{N}} \operatorname{Prob}\left(\bar{Y}=\bar{b}_{0}+\sum_{i=1}^{m} k_{i} \bar{p}_{i}\right) \\
& =\sum_{k_{1} \in \mathbb{N}} \cdots \sum_{k_{m} \in \mathbb{N}} \kappa e^{-\epsilon\left(\bar{b}_{0} \cdot \bar{a}+k_{1} \bar{p}_{1} \cdot \bar{a}+\cdots+k_{m} \bar{p}_{m} \cdot \bar{a}\right)} \\
& =\kappa\left(e^{-\epsilon \bar{\epsilon}_{0} \cdot \bar{a}}\right) \\
& \left(\sum_{k_{1}} e^{-\epsilon k_{1} \bar{p}_{1} \cdot \bar{a}}\right) \cdots\left(\sum_{k_{m} \in \mathbb{N}} e^{-\epsilon k_{m} \bar{p}_{1} \cdot \bar{a}}\right) \\
& =\kappa\left(e^{-\epsilon \bar{b}_{0} \cdot \bar{a}}\right)\left(\frac{1}{1-e^{-\epsilon \bar{p}_{1} \cdot \bar{a}}}\right) \cdots\left(\frac{1}{1-e^{-\epsilon \bar{\epsilon}_{m} \cdot \bar{a}}}\right)
\end{aligned}
$$

The latter is clearly a pseudo-rational function in $\epsilon$.
Probability of system of linear inequalities over reals. Let $\bar{R}=$ $\left(R_{1}, \ldots, R_{n}\right)$ be a continuous random variable taking values in $\mathbb{R}^{n}$. Consider a finite system of linear inequalities $C$ with rational coefficients and with $n$ unknowns $R_{1}, \ldots, R_{n}$. As in the case of discrete random variables, we can define $\operatorname{sol}(C) \subseteq \mathbb{R}^{n}$, the set of solutions, and $\operatorname{Prob}(\bar{R} \models C)$, the probability that $\bar{R}$ satisfies C. We have the following result.

Lemma 18. Let $C$ be a finite system of linear inequalities with rational coefficients and with $n$ unknowns $R_{1}, \ldots, R_{n}$. Let $R_{1}=\operatorname{Lap}\left(a_{1} \epsilon, \mu_{1}\right), \ldots$, $R_{n}=\operatorname{Lap}\left(a_{n} \epsilon, \mu_{n}\right)$ be mutually independent Laplacian doistributions such that for each $1 \leq j \leq n, a_{j}$ is a strictly positive rational number and $\mu_{j}$ is a rational number. Let $\bar{R}=\left(R_{1}, \ldots, R_{n}\right)$. There is a pseudo-rational
function $r t_{\bar{R}, C}$ in $\epsilon$ such that $\operatorname{Prob}(\bar{R} \models C)=r t_{\bar{R}, C}$. The function $r t_{\bar{R}, C}$ can be computed from $C,\left(a_{1}, \mu_{1}\right), \ldots,\left(a_{n}, \mu_{n}\right)$.

Proof. As in the proof of Lemma 17, it suffices to consider the case when each $\mu_{i}=0$ and to show that the probability measure of the set $S o l=$ $\operatorname{sol}(C) \cap\left\{\left(b_{1}, \ldots, b_{n}\right) \mid b_{i} \in \mathbb{R}^{>0}\right\}$ is a computable pseudo-rational function.

Since $\bar{R}$ is continuous, we can also assume that each inequality is of the form $\leq$. This is because the measure of any set in $\mathbb{R}^{n}$ that satisfies a linear equation over $n$ unknowns $R_{1}, \ldots, R_{n}$ is 0 . There are computable finite sets $S_{1}, \ldots, S_{m}$ such that (See [8])

1. $S o l=S_{1} \cup \ldots S_{m}$,
2. The measure of the $S_{i} \cap S_{j}$ is 0 for $i \neq j$, and
3. Each $S_{i}$ is a positive repetitive polyhedra. $S \subseteq\left(\mathbb{R}^{>0}\right)^{n}$ is said to be a positive repetitive polyhedra if there are constants $h_{0}^{-}, h_{0}^{+}$and functions $h_{1}^{-}\left(x_{1}\right), h_{1}^{+}\left(x_{1}\right), h_{2}^{-}\left(x_{1}, x_{2}\right), h_{2}^{+}\left(x_{1}, x_{2}\right), \ldots, h_{n-1}^{-}\left(x_{1}, x_{2}, \ldots x_{n-1}\right)$, $h_{n-1}^{+}\left(x_{1}, x_{2}, \ldots x_{n-1}\right)$ such that

- $S_{i}=\begin{aligned} & \left\{\left(x_{1}, \ldots, x_{n}\right) \mid h_{0}^{-} \leq x_{1} \leq h_{0}^{+}, \ldots,\right. \\ & \left.h_{n-1}^{-}\left(x_{1}, \ldots x_{n-1}\right) \leq x_{n} \leq h_{n-1}^{+}\left(x_{1}, \ldots x_{n-1}\right)\right\} .\end{aligned}$
- $h_{0}^{-}$is a rational number $\geq 0$.
- $h_{0}^{+}$is either $\infty$ or a rational number.
- For each $1 \leq j \leq n, h_{j}^{-}$is a linear function in its arguments. In the latter case, $h_{j}^{-}$has rational coefficients.
- For each $1 \leq j \leq n, h_{j}^{+}$is either $\infty$ or a linear function in its arguments. $h_{j}^{+}$has rational coefficients in the latter case.
- For each $1 \leq j \leq n, h_{j}^{-} \neq h_{j}^{+}$.

Thanks to conditions (1) and (2) above, it suffices to show that for any positive repetitive polyhedra $S$, the probability measure of the event $\{\bar{R}=$ $\bar{b} \mid \bar{b} \in S\}$ is a pseudo-rational function.

Fix $S$ and let $h_{0}^{-}, h_{0}^{+}, h_{1}^{-}, h_{1}^{+}, \ldots, h_{n-1}^{-}, h_{n-1}^{+}$be as above, The measure of the event $\{\bar{R}=\bar{b} \mid \bar{b} \in S\}$ can be computed using the nested integral

$$
F=\int_{h_{0}^{-}}^{h_{0}^{+}} f_{a_{1}}\left(x_{1}\right) \int_{h_{1}^{-}}^{h_{1}^{+}} f_{a_{2}}\left(x_{2}\right) \cdots \int_{h_{n-1}^{-}}^{h_{n-1}^{+}} f_{a_{n}}\left(x_{n}\right) d x_{n} \cdots d x_{1}
$$

where $f_{a_{i}}\left(x_{i}\right)=\frac{a_{i} \epsilon}{2} e^{-a_{i} \epsilon x_{i}}$ is the pdf of $R_{i}$ (we always have that $x_{i} \geq 0$ ) and the arguments of $h_{i}^{+}, h_{i}^{-}$are omitted for readability.

For $1 \leq j \leq n$, let $I_{j}$ be the nested integral

$$
I_{j}=\int_{h_{j-1}^{-}}^{h_{j-1}^{+}} f_{a_{j}}\left(x_{j}\right) \cdots \int_{h_{n-1}^{-}}^{h_{n-1}^{+}} f_{a_{n}}\left(x_{n}\right) d x_{n} \cdots d x_{j}
$$

We claim by induction on $k=n-j$ that $I_{j}$ is a finite sum of terms of the form

$$
a \epsilon^{m} e^{b \epsilon}\left(x_{1}^{m_{1}} e^{\epsilon b_{1} x_{1}}\right) \ldots\left(x_{j-1}^{m_{j-1}} e^{\epsilon b_{j-1} x_{j-1}}\right)
$$

where $a, b, b_{1}, \ldots, b_{j-i}$ are rational numbers (including negative numbers), $m$ is an integer, and $m_{1}, \ldots, m_{j-1}$ are natural numbers. We will assume that the sum is always presented in simplest form, namely, that all cancellations have already taken place in the sum.

Clearly the claim is true when $k=0$. Suppose that the claim is true for $k=k_{0}$. Let $j_{0}=n-k_{0}$. Suppose

$$
w=a \epsilon^{m} e^{b \epsilon}\left(x_{1}^{m_{1}} e^{\epsilon b_{1} x_{1}}\right) \ldots\left(x_{j_{0}-1}^{m_{j_{0}-1}} e^{\epsilon b_{j_{0}-1} x_{j_{0}-1}}\right)
$$

is a summand in $I_{j_{0}}$. Let $k=k_{0}+1$ and $j=n-k=n-k_{0}-1=j_{0}-1$.
Consider the indefinite integral

$$
\begin{aligned}
& J=\int f_{a_{j_{0}-1}} w d x_{j_{0}-1} \\
&\left.=\int \frac{a_{j_{0}-1} \epsilon}{} e^{-a_{j_{0}-1} \epsilon x_{j_{0}-1}} w d x_{j_{0}-1}{ }_{m_{j_{0}-2}} e^{\epsilon b_{j_{0}-2} x_{j_{0}-2}}\right) \\
&=\frac{a_{j_{0}-1}}{2} \epsilon^{m+1} e^{b \epsilon}\left(x_{1}^{m_{1}} e^{\epsilon b_{1} x_{1}}\right) \ldots\left(x_{j_{0}-2} e^{2}\right) \\
& \quad \int x_{j_{0}-1}^{m_{j_{0}-1}} e^{\epsilon\left(b_{j_{0}-1}-a_{j_{0}-1}\right) x_{j_{0}-1}} d x_{j_{0}-1}
\end{aligned}
$$

Let

$$
J^{\prime}=\int x_{j_{0}-1}^{m_{j_{0}-1}} e^{\epsilon\left(b_{j_{0}-1}-a_{j_{0}-1}\right) x_{j_{0}-1}} d x_{j_{0}-1}
$$

Now, if $b_{j_{0}-1}-a_{j_{0}-1}=0$ then

$$
J^{\prime}=\frac{x_{j_{0}-1}^{m_{j_{0}-1}+1}}{m_{j_{0}-1+1}} .
$$

If $b_{j_{0}-1}-a_{j_{0}-1} \neq 0$ then by doing a change of variables $t=\left(b_{j_{0}-1}-\right.$ $\left.a_{j_{0}-1}\right) \epsilon x_{j_{0}-1}$, it is not too hard to show that

$$
J^{\prime}=\sum_{k=0}^{m_{j_{0}-1}} c_{k} \epsilon^{t_{k}} x_{j_{0}-1}^{k} e^{\epsilon\left(b_{j_{0}-1}-a_{j_{0}-1}\right) x_{j_{0}-1}}
$$

where $c_{k}$ is a rational number and $t_{k}$ an integer for each $k$.

Thus, the indefinite integeral $J$ is a sum, each of whose terms is of the form

$$
a^{\prime} \epsilon^{m^{\prime}} e^{b^{\prime} \epsilon}\left(x_{1}^{m_{1}^{\prime}} e^{\epsilon b_{1}^{\prime} x_{1}}\right) \ldots\left(x_{j_{0}-1}^{m_{j_{0}-1}^{\prime}} e^{\epsilon b_{j_{0}-1}^{\prime} x_{j_{0}-1}}\right)
$$

If $h_{j_{0}-2}^{-}$and $h_{j_{0}-2}^{+}$are linear functions, we get immediately that $I_{j}=$ $\int_{h_{j_{0}-2}}^{h_{j_{0}-2}^{+}} f_{a_{j_{0}-1}} w d x_{j_{0}-1}$ is of the right form. The induction step follows in this case.

If $h_{j_{0}-2}^{+}=\infty$, and each $b_{j}^{\prime}$ in a summand of $J$ is strictly negative, then it is also easy to see that the induction step follows. Apriori, it seems that there might be a problem when $b_{j}^{\prime} \geq 0$ as in this case, $I_{j}$ will evaluate to either $\infty$ or $-\infty$. This, however, will contradict the fact that the nested integral $F$ defines probability of an event (and hence is bounded above by 1). Thus, if $h_{j_{0}-2}^{+}=\infty$ then $b_{j}$ must be strictly negative.

The claim immediately implies that the measure of the set $S o l=\operatorname{sol}(C) \cap$ $\left\{\left(b_{1}, \ldots, b_{n}\right) \mid b_{i} \in \mathbb{R}^{>0}\right\}$ is a pseudo-rational function.

## F Reachability in Parametrized DTMCs

In this section we will prove Lemma 10. Let us first recall how reachability probabilities are computed in (non-parametrized) finite-state DTMCs. Recall that a (non-parametrized) DTMC is a pair $(Q, \delta)$ where $Q$ is a finite set of states, and $\delta: Q \times Q \rightarrow[0,1]$ is such that for every $q \in Q$, $\sum_{q^{\prime} \in Q} \delta\left(q, q^{\prime}\right)=1$. So in a DTMC the transition probabilities are fixed, and are not functions of a parameter. The probability of reaching a set of states $Q^{\prime} \subseteq Q$ from a state $q_{0}$ is computed by solving a more general problem, namely, the problem of computing the probability of reaching $Q^{\prime}$ from each state $q \in Q$. Let the variable $x_{q}$ denote the probability of reaching $Q^{\prime}$ from state $q$. One simple observation is that if $q \in Q^{\prime}$ then $x_{q}=1$. Second, if $Q_{0}$ denotes the set of all states from which $Q^{\prime}$ is not reachable in the underlying graph (i.e., one where we ignore the probabilities and just have edges for all transitions that are non-zero), then $x_{q}=0$ if $q \in Q_{0}$. Now the set $Q_{0}$ can be computed by performing a simple graph search on the underlying graph. For states $q \notin\left(Q^{\prime} \cup Q_{0}\right)$, we could write $x_{q}$ as $x_{q}=\sum_{q^{\prime} \in Q} \delta\left(q, q^{\prime}\right) x_{q^{\prime}}$. This gives us the following system of linear equations.

$$
\begin{array}{ll}
x_{q}=1 & \text { if } q \in Q^{\prime} \\
x_{q}=0 & \text { if } q \in Q_{0} \\
x_{q}=\sum_{q^{\prime} \in Q} \delta\left(q, q^{\prime}\right) x_{q^{\prime}} & \text { otherwise }
\end{array}
$$

The above system of linear equations can be shown to have a unique solution, with the solution giving the probability of reaching $Q^{\prime}$ from each state $q$.

Now let us consider a parametrized DTMC $\mathcal{D}=(Z, \Delta)$. Let $\varphi_{z z^{\prime}}$ be a $\mathcal{L}_{\text {exp }}$ formula that defines the function $\Delta\left(z, z^{\prime}\right)$. Recall that in the algorithm outlined in the previous paragraph, one crucial step is to compute the set of states that have probability 0 of reaching the target set. This requires knowing the underlying graph of the DTMC, i.e., knowing which transitions have probability 0 and which ones have probability $>0$. In a parametrized DTMC this is challenging because the probability of transitions depends on the value of $\epsilon$, and our goal is to compute the reachability probability as a function of $\epsilon$. We will overcome this challenge by "guessing" the underlying graph.

Let $C \subseteq Z \times Z$. We will construct a formula $\varphi_{C}$ that will capture the constraints that reachablity probabilities need to satisfy under the assumption that the probability of edges in $C$ is 0 , and those outside $C$ is $>0$. Based on the assumption that $C$ is exactly the set of 0 probability edges, we can compute the set $Z_{0}^{C}$ of states that cannot reach $Z^{\prime}$. The formula $\varphi_{C}$ will have variables that will have the following intuitive interpretations $-p_{z z^{\prime}}$ the probability of transitioning from $z$ to $z^{\prime} ; x_{z}$ the probability of reaching $Z^{\prime}$ from state $z$.

$$
\begin{aligned}
\varphi_{C}= & \bigwedge_{\left(z, z^{\prime}\right) \in C}\left(p_{z z^{\prime}}=0\right) \wedge \bigwedge_{\left(z, z^{\prime}\right) \notin C}\left(p_{z z^{\prime}}>0\right) \wedge \bigwedge_{z \in Z^{\prime}}\left(x_{z}=1\right) \\
& \wedge \bigwedge_{z \in Z_{0}^{C}}\left(x_{z}=0\right) \wedge \bigwedge_{z \notin\left(Z^{\prime} \cup Z_{0}^{C}\right)}\left(x_{z}=\sum_{z^{\prime}} p_{z z^{\prime}} x_{z^{\prime}}\right)
\end{aligned}
$$

Notice that $\varphi_{C}$ is a formula in $\mathcal{L}_{\text {exp }} \cdot \varphi_{C}$ can be used to construct the formula we want. To construct the formula $\varphi_{z_{0} Z^{\prime}}$ that characterizes the probability of reaching $Z^{\prime}$ from $z_{0}$, we need to account for two things. First, we need to ensure that $p_{z z^{\prime}}$ is indeed the probability of transitioning from $z$ to $z^{\prime}$. Second, we need to account for the fact that we don't know the exact set of edges with probability 0 . Based on these observations, we can define $\varphi_{z_{0}, Z^{\prime}}$ as follows.

$$
\varphi_{z_{0} Z^{\prime}}=\left[\exists x_{z}\right]_{z \neq z_{0}}\left[\exists p_{z z^{\prime}}\right]_{z, z^{\prime} \in Z} \bigwedge_{z, z^{\prime} \in Z} \varphi_{z z^{\prime}}\left(\epsilon, p_{z z^{\prime}}\right) \wedge\left(\bigvee_{C \subseteq Z \times Z} \varphi_{C}\right)
$$

In the above definition of $\varphi_{z_{0} Z^{\prime}}$ all variables except $x_{z_{0}}$ (and $\epsilon$ ) are existentially quantified. Notice, that $\varphi_{z_{0} Z^{\prime}}$ is in $\mathcal{L}_{\text {exp }}$ provided we pull all the quantifiers to get it in prenex form. Given that $Z_{0}^{C}$ can be effectively constructed for any set $C$, the above formula can also be computed for any parametrized DTMC $\mathcal{D}$.

Expressions ( $\mathrm{b} \in \mathcal{B}, \mathrm{x} \in \mathcal{X}, d \in \mathrm{DOM}, g \in \mathcal{F}_{\text {Bool }}, f \in \mathcal{F}_{\mathrm{DOM}}$ ):

$$
\begin{aligned}
& B::=\text { true } \mid \text { false }|\mathrm{b}| \operatorname{not}(B) \mid B \text { and } B \mid B \text { or } B \mid g(\tilde{E}) \\
& E::=d|\mathrm{x}| f(\tilde{E})
\end{aligned}
$$

Basic Program Statements $\left(a \in \mathbb{Q}^{>0}, \sim \in\{<,>,=, \leq, \geq\}, F\right.$ is a scoring function and choose is a user-defined distribution):

$$
\begin{aligned}
& \qquad \begin{aligned}
s::= & \mathrm{x} \leftarrow E|\mathrm{~b} \leftarrow B| \mathrm{x} \leftarrow \operatorname{Exp}(a \epsilon, F(\tilde{\mathrm{x}}), E) \mid \\
& \times \leftarrow \text { choose }(a \epsilon, \tilde{E}) \mid \text { if } B \text { then } P \text { else } P \text { end } \mid \\
& \text { While } B \text { do } P \text { end } \mid \text { exit }
\end{aligned} \\
& \text { Program Statements }(\ell \in \text { Labels }) \\
& \\
& P::=\ell: s \mid \ell: s ; P
\end{aligned}
$$

Figure 3: BNF grammar for Finite DiPWhile. DOM is a finite discrete domain. $\mathcal{F}_{\text {Bool }},\left(\mathcal{F}_{\text {DOM }}\right.$ resp $)$ are set of functions that output Boolean values (DOM respectively). $\mathcal{B}, \mathcal{X}$ are the sets of Boolean variables, and DOM variables, respectively. Labels is a set of program labels. For a syntactic class $S, \tilde{S}$ denotes a sequence of elements from $S$.

## G Syntax of Finite DiPWhile programs

The syntax of Finite DiPWhile programs is presented in Figure 3.

## G. 1 A general semantic class of programs

Our methods imply decidability of checking differential privacy for a large semantic class of programs (which include DiPWhile.) A sufficient condition to ensure the decidability of checking differential privacy is to consider programs with the property that for each input, the probability distribution on the outputs is definable in $T h_{\text {exp }}$ :

Definition 19. A parametrized program $P_{\epsilon}$ with inputs $\mathcal{U}$ and outputs $\mathcal{V}$ is said to identify a definable distribution on $\mathcal{V}$ if for each in $\in \mathcal{U}$ and out $\in \mathcal{V}$ the function $\epsilon \mapsto \operatorname{Prob}\left(P_{\epsilon}(\right.$ in $)=$ out $)$ is definable in $\mathrm{Th}_{\text {exp }}$.

A parametrized program $P_{\epsilon}$ with inputs $\mathcal{U}$ and outputs $\mathcal{V}$ is said to effectively identify a definable distribution on $\mathcal{V}$ if there is an algorithm $\mathcal{A}$
such that for each in $\in \mathcal{U}$ and out $\in \mathcal{V}, \mathcal{A}$ outputs a formula $\varphi_{\text {in,out }}(\epsilon, x)$ in $\mathcal{L}_{\text {exp }}$ that defines the function $\epsilon \mapsto \operatorname{Prob}\left(P_{\epsilon}(\mathbf{i n})=\mathbf{o u t}\right)$.

We can conclude by a proof similar to the proof Theorem 11 .
Theorem 20. The Fixed Parameter Differential Privacy and Differential Privacy problems are decidable for programs $P_{\epsilon}$ that effectively identify a definable distribution, rationals $t \in \mathbb{Q}^{>} 0$ and definable functions $\delta$ (in the case of the Differential Privacy problem). Furthermore, if $P_{\epsilon}$ is not $(t \epsilon, \delta)$ differentially private for some admissible value of $\epsilon$ then we can compute a counter-example.

## H Detailed Experimental Results

We implemented a simplified version of the algorithm, presented earlier, for proving/disproving differential privacy of DiPWhile programs. Our tool DiPC [3] handles loop-free programs, i.e., acyclic programs. Programs with bounded loops (with constant bounds) can be handled by unrolling loops. The tool takes in an input program $P_{\epsilon}$ parametrized by $\epsilon$, and either proves $P_{\epsilon}$ to be differentially private for all $\epsilon$ or returns a counter-example. The tool can also be used to check differential privacy for a given, fixed $\epsilon$, or to check for $k \epsilon$-differential privacy for some constant $k$. The design of the tool will be discussed in detail in Section H.2.

## H. 1 Examples

We used various examples to measure the effectiveness of our tool. These include SVT [28, 21, Noisy Maximum [18], Noisy Histogram [18] and Randomized Response [20]. Pseudocodes for all variants of these examples that we tried are given in this section for completeness. Though the pseudo-codes don't strictly adhere to the syntax of DiPWhile programs, they can easily be rewritten to fit the syntax.

Sparse Vector Technique (SVT) We looked at six different variants of the Sparse Vector Technique (SVT). Algorithms addressed as SVT1-6, are Algorithms 1-6 in [28], respectively. In these programs, the array $q$ represents the input queries. The array out represents the output array, $\perp$ represents False and $T$ represents True. In all our experiments, we set the threshold $T=0$. SVT1 was previously introduced in this paper as Algorithm $\square$ on page 8, The adjacency relation $\Phi$ we used is given by
$\left(q_{1}, q_{2}\right) \in \Phi$ if and only if $\left|q_{1}[i]-q_{2}[i]\right| \leq 1$ for all $i$. While SVT1 and SVT2 are differentially private, the other four variants are not. We will present counter-examples for all four of these variants in Section H.3. The pseudocode for the six variants of SVT are given in Figures 4 and 5.
(SVT1) First Instantiation of SVT (SVT2) Second Instantiation of SVT

```
Input: \(q[1: N]\)
Output: out \([1: N]\)
\(\mathbf{r}_{T} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2 \Delta}, T\right)\)
count \(\leftarrow 0\)
    for \(i \leftarrow 1\) to \(N\) do
        \(r \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{4 c \Delta}, q[i]\right)\)
        \(\mathrm{b} \leftarrow \mathrm{r} \geq \mathrm{r}_{T}\)
        if \(b\) then
            out \([i] \leftarrow \top\)
            count \(\leftarrow\) count +1
            if count \(\geq c\) then
                l exit
            end
        else
            out \([i] \leftarrow \perp\)
        end
    end
```

```
```

Input: $q[1: N]$

```
```

Input: $q[1: N]$
Output: out $[1: N]$
Output: out $[1: N]$
$\mathbf{r}_{T} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2 c \Delta}, T\right)$
$\mathbf{r}_{T} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2 c \Delta}, T\right)$
count $\leftarrow 0$
count $\leftarrow 0$
for $i \leftarrow 1$ to $N$ do
for $i \leftarrow 1$ to $N$ do
$\mathrm{r} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{4 c \Delta}, q[i]\right)$
$\mathrm{r} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{4 c \Delta}, q[i]\right)$
$\mathrm{b} \leftarrow \mathrm{r} \geq \mathrm{r}_{T}$
$\mathrm{b} \leftarrow \mathrm{r} \geq \mathrm{r}_{T}$
if $b$ then
if $b$ then
out $[i] \leftarrow T$,
out $[i] \leftarrow T$,
$\mathrm{r}_{T} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2 c \Delta}, T\right)$
$\mathrm{r}_{T} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2 c \Delta}, T\right)$
count $\leftarrow$ count +1
count $\leftarrow$ count +1
if count $\geq c$ then
if count $\geq c$ then
exit
exit
end
end
else
else
out $[i] \leftarrow \perp$
out $[i] \leftarrow \perp$
end
end
end

```
```

end

```
```

Figure 4: Sparse Vector Technique Algorithms

Noisy Maximum Noisy maximum algorithms are a differentially private way to compute different statistical measures for a given set of queries. Algorithms addressed as NMax1-4 are Algorithms 5-8, respectively, in [18. Algorithms NMax1 and NMax2 are mechanisms to compute the index of the query with maximum value after adding a Laplacian (or exponential) noise. Inputs $Q_{1}$ and $Q_{2}$ are considered adjacent iff $\left|Q_{1}[i]-Q_{2}[i]\right| \leq 1$ for all $i$. Under this relation, Algorithms NMax1 and NMax2 are both $\epsilon$-differentially private. Algorithms NMax3 and NMax4 are variants to print the maximum value instead of the index. These variants are shown to be not differentially private in Section H.3. The pseudocode for these algorithms can be found in Figure 6
(SVT3) Third Instantiation of SVT

```
Input: \(q[1: N]\)
Output: out \([1: N]\)
\(\mathbf{r}_{T} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2 \Delta}, T\right)\)
count \(\leftarrow 0\)
for \(i \leftarrow 1\) to \(N\) do
    \(r \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2 c \Delta}, q[i]\right)\)
    \(\mathbf{b} \leftarrow \mathrm{r} \geq \mathrm{r}_{T}\)
    if \(b\) then
                out \([i] \leftarrow \operatorname{Disc}_{\text {seq }}(r)\)
                count \(\leftarrow\) count +1
                if count \(\geq c\) then
                exit
            end
    else
        out \([i] \leftarrow \perp\)
    end
end
```

(SVT5) Fifth Instantiation of SVT

```
Input: \(q[1: N]\)
Output: out \([1: N]\)
\(\mathrm{r}_{T} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2 \Delta}, T\right)\)
for \(i \leftarrow 1\) to \(N\) do
    \(r \leftarrow q[i]\)
    \(\mathrm{b} \leftarrow \mathrm{r} \geq \mathrm{r}_{T}\)
    if \(b\) then
        out \([i] \leftarrow \top\)
        else
            out \([i] \leftarrow \perp\)
        end
end
```

```
Input: \(q[1: N]\)
Output: out \([1: N]\)
\(\mathbf{r}_{T} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{4 \Delta}, T\right)\)
count \(\leftarrow 0\)
for \(i \leftarrow 1\) to \(N\) do
        \(\mathrm{r} \leftarrow \operatorname{Lap}\left(\frac{3 \epsilon}{4 \Delta}, q[i]\right)\)
        \(\mathrm{b} \leftarrow \mathrm{r} \geq \mathrm{r}_{T}\)
        if \(b\) then
                out \([i] \leftarrow T\)
        count \(\leftarrow\) count +1
        if count \(\geq c\) then
            | exit
        end
    else
        out \([i] \leftarrow \perp\)
        end
end
```

(SVT6) Sixth Instantiation of SVT

```
Input: \(q[1: N]\)
```

Input: $q[1: N]$
Output: out $[1: N]$
Output: out $[1: N]$
$\mathrm{r}_{T} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2 \Delta}, T\right)$
$\mathrm{r}_{T} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2 \Delta}, T\right)$
for $i \leftarrow 1$ to $N$ do
for $i \leftarrow 1$ to $N$ do
$\mathrm{r} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2 \Delta}, q[i]\right)$
$\mathrm{r} \leftarrow \operatorname{Lap}\left(\frac{\epsilon}{2 \Delta}, q[i]\right)$
$\mathrm{b} \leftarrow \mathrm{r} \geq \mathrm{r}_{T}$
$\mathrm{b} \leftarrow \mathrm{r} \geq \mathrm{r}_{T}$
if $b$ then
if $b$ then
out $[i] \leftarrow \top$
out $[i] \leftarrow \top$
else
else
out $[i] \leftarrow \perp$
out $[i] \leftarrow \perp$
end
end
end

```
end
```

Figure 5: Sparse Vector Technique Algorithms

Histogram Algorithms Histogram algorithms also target computing statistical measures on queries in a differentially private manner. Algorithms referred to as Hist1-2 here are Algorithms 9-10 in [18. Algorithm Hist1
(NMax1) Correct Noisy Max with Laplacian Noise

```
Input: \(q[1: N]\)
Output: out
NoisyVector \(\leftarrow[]\)
for \(i \leftarrow 1\) to \(N\) do
        NoisyVector \([\mathrm{i}] \leftarrow\)
        \(\operatorname{Lap}\left(\frac{\epsilon}{2}, q[i]\right)\)
end
out \(\leftarrow \operatorname{argmax}(\) NoisyVector)
```

(NMax2) Correct Noisy Max with Exponential Noise

```
Input: \(q[1: N]\)
Output: out
```

NoisyVector $\leftarrow[]$
for $i \leftarrow 1$ to $N$ do
NoisyVector $[\mathrm{i}] \leftarrow$
$\operatorname{Lap}^{+}\left(\frac{\epsilon}{2}, q[i]\right)$
end
out $\leftarrow \operatorname{argmax}($ NoisyVector $)$
(NMax3) Incorrect Noisy Max with (NMax4) Incorrect Noisy Max with

Laplacian Noise

```
Input: q[1:N]
Output: out
NoisyVector \leftarrow []
for }i\leftarrow1\mathrm{ to }N\mathrm{ do
        NoisyVector[i]}
        Lap(\frac{\epsilon}{2},q[i])
end
out }
    Disc
```

Laplacian Noise

```
Input: q[1:N]
Output: out
```

    NoisyVector \(\leftarrow[]\)
    for \(i \leftarrow 1\) to \(N\) do
        NoisyVector \([\mathrm{i}] \leftarrow\)
        \(\mathrm{Lap}^{+}\left(\frac{\epsilon}{2}, q[i]\right)\)
    end
    out \(\leftarrow\)
    Disc \(_{\text {seq }}(\max (\) Noisy Vector \())\)
    Figure 6: Noisy Max Algorithms
and Hist2 are variants of noisy maximum, where we return the histogram, instead of the maximum. Under the above adjacency relation where $Q_{1}$ and $Q_{2}$ are adjacent if $\left|Q_{1}[i]-Q_{2}[i]\right| \leq 1$ for all $i$, both these variants are not $\epsilon$-differentially private. However, if we consider an alternative definition for the adjacency relation, where $Q_{1}$ and $Q_{2}$ are adjacent iff $\sum_{i}\left(\mid Q_{1}[i]-\right.$ $\left.Q_{2}[i] \mid\right) \leq 1$, then Hist1] is $\epsilon$-differentially private but Hist2] still is not. All experiments listed in Section H. 3 for Algorithms NMax1 and NMax2 were run using the second adjacency relation. The pseudocode for these algorithms can be found in Figure 7
(Hist1) Noisy Histogram
(Hist2) Noisy Histogram, Wrong Scale

```
Input: \(q[1: N]\)
Output: out \([1: N]\)
NoisyVector \(\leftarrow[]\)
for \(i \leftarrow 1\) to \(N\) do
        NoisyVector[i] \(\leftarrow\)
            \(\operatorname{Lap}(\epsilon, q[i])\)
end
out \(\leftarrow\) Disc \(_{\text {seq }}\) (NoisyVector)
```

```
Input: \(q[1: N]\)
Output: out \([1: N]\)
NoisyVector \(\leftarrow[]\)
for \(i \leftarrow 1\) to \(N\) do
    NoisyVector \([\mathrm{i}] \leftarrow\)
    \(\operatorname{Lap}\left(\frac{1}{\epsilon}, q[i]\right)\)
end
out \(\leftarrow\) Disc \(_{\text {seq }}\) (NoisyVector)
```

Figure 7: Noisy Histogram Algorithms

Randomized Response All the previous algorithms use the Laplace mechanism. Randomized Response [20], on the other hand, uses discrete probabilities. In this algorithm (henceforth called Rand1), given a set of Boolean input queries, we flip each input query with a probability of $\frac{e^{\epsilon}-1}{2}$ and output the resulting outcome . We also consider a non-private version (called Rand2) where the input query is flipped with probability $\frac{1-\epsilon}{2}$. The pseudocodes can be found in Figure 8.
(Rand1) Differentially Private Random-
ized Response
(Rand2) Non-Differentially Private Randomized Response

```
Input: \(q[1: N]\)
```

Input: $q[1: N]$
Output: out $[1: N]$
Output: out $[1: N]$
for $i \leftarrow 1$ to $N$ do
for $i \leftarrow 1$ to $N$ do
out[i] $\leftarrow$
out[i] $\leftarrow$
$\begin{cases}q[i] & \text { with prob }=\frac{e^{\epsilon}}{1+e^{\epsilon}} \\ \neg q[i] & \text { with prob }=\frac{1}{1+e^{\epsilon}}\end{cases}$
$\begin{cases}q[i] & \text { with prob }=\frac{e^{\epsilon}}{1+e^{\epsilon}} \\ \neg q[i] & \text { with prob }=\frac{1}{1+e^{\epsilon}}\end{cases}$
end

```
end
```

```
Input: \(q[1: N]\)
    Output: out \([1: N]\)
    for \(i \leftarrow 1\) to \(N\) do
        out \([\mathrm{i}] \leftarrow\)
        \(\begin{cases}q[i] & \text { with prob }=\frac{1+\epsilon}{2} \\ \neg q[i] & \text { with prob }=\frac{1-\epsilon}{2}\end{cases}\)
    end
```

Figure 8: Randomized Response Algorithms

Sparse Sparse is a variant of SVT that is discussed in [21]. Our reason for considering this example is to demonstrate our tool's ability to handle $(\epsilon, \delta)$-differential privacy (see Section H.4). Pseudocode for this algorithm is provided in Section H.4.

## H. 2 Tool Design

Given a program and an adjacency relation, DiPC outputs true if the program is differentially private and outputs a counter-example if it is not. The tool works in two phases. In the first phase, the tool parses the program, computes symbolic expressions that capture the output distribution, and identify inequalities that must hold for differential privacy. The symbolic expressions for the probability computation, and the logical constraints that must hold, are written in a Wolfram Mathematica $\circledR$ )script. In the second phase, Mathematica is run to perform the symbolic computations and check the results.

The computation of the output distribution proceeds in a manner consistent with the decision procedure outlined in the proof of Theorem 11. Recall that the parametrized DTMC semantics, the state tracks constraints that must hold between different real variables. These constraints can be tracked by maintaining a partial order between the variables. One of the engineering challenges we experienced was in the computation of the probability of the partial order holding, given the parameters used during sampling. The "Probability []" command in Mathematica was very slow and inefficient. Instead we decided to convert the partial order into a set of total orders, and compute the probability of each total order through integration.

For example, to compute the probability of $x_{1}<x_{2}<x_{3} \ldots<x_{n}$, where variable $x_{i}$ has p.d.f $D_{i}$, we would first compute the probability $P\left(x_{n}>\right.$ $x)=\int_{x}^{\infty} D_{n}(y) d y$. We then compute the probabilities $P\left(x_{n}>x_{n-1}>x\right)=$ $\int_{x}^{\infty} P\left(x_{n}>y\right) D_{n-1}(y) d y, P\left(x_{n}>x_{n-1}>x_{n-2}>x\right)=\int_{x}^{\infty} P\left(x_{n}>x_{n-1}>\right.$ y) $D_{n-2}(y) d y$ and so on. Once we have computed $P\left(x_{n}>x_{n-1}>\ldots>x_{1}>\right.$ $x)$, we can compute $P\left(x_{n}>x_{n-1}>\ldots>x_{1}\right)=\operatorname{Lim}_{x \rightarrow-\infty} P\left(x_{n}>x_{n-1}>\right.$ $\left.\ldots>x_{1}>x\right)$. Additionally, we try to optimize the above process by splitting the partial order into connected components and computed probability for each component. We also deal with constant assignments to real variables by slightly modifying the integration method.

## H. 3 Experimental Results

We ran all the experiments on an octa-core Intel $®$ Core i7-8550U @ 1.8 gHz CPU with 8GB memory. The tool is implemented in C++ and uses Wolfram Mathematica $\circledR$. As mentioned in Section $H .2$, the tool works in two phases - in the first phase, a Mathematica script is produced with commands for all the output probability computations and the subsequent inequality checks and in the second phase, the generated script is run on Mathematica. In all the following tables, we refer the times of the Script Generation Phase (i.e. Phase 1) as T1 and that of the Script Validation Phase (i.e. Phase 2) as T2.

Unless stated otherwise, all the experiments were run with the parameters $c=1, \Delta=1$ and discretization parameter seq $=(-1<0<1)$ wherever applicable. The range of input query values was $\mathrm{DOM}=\{-1,0,1\}$ in all the experiments. The running times in all experiments were averaged over 3 runs of the tool.

Table 3 shows the runtime of our tool for all the listed algorithms with 3 queries. We chose to use 3 queries because counter-examples for most of the programs which were not differentially private could be found with 3 queries; the only exception being SVT3, Majority of the time is taken for running the Mathematica code. We also observed that most of the time spent by Mathematica was in computing the output probability; the time to perform the inequality checks for adjacent inputs was relatively smaller. Consequently, programs which do not use real variables are much faster to run. Results in the table also show that the time taken for disproving Differential Privacy is lower than the time for proving Differential Privacy on average. This is because the tool terminates on finding a counter-example. On the other hand, to prove differential privacy the tool has to check all inequalities.

Table 5 lists the smallest counter-example found for each non differentially private algorithm. Given a program and an adjacency relation, the tool automatically finds an $\epsilon$, the pair of adjacent inputs, and the output value that demonstrate the violation of differential privacy. All four columns in the table were output by the tool. Further, we observe that the counterexamples found were much smaller, in number of queries, compared to those found in [18]. For example, algorithms NMax3 and NMax4 counter-examples need just 3 and 1 queries respectively, compared to the 5 queries required in 18. Similarly, algorithm SVT5 has a counter-example with just 2 queries, as compared to the 10 queries.

To study the performance of the tool as the number of queries increases,

| Algorithm | Runtime <br> $(\mathrm{T} 1 / \mathrm{T} 2)$ | $\epsilon$-Diff. <br> Pri- <br> vate |
| :---: | :---: | :---: |
| SVT1 | $0 \mathrm{~s} / 825 \mathrm{~s}$ | $\checkmark$ |
| SVT2 | $0 \mathrm{~s} / 768 \mathrm{~s}$ | $\checkmark$ |
| SVT3 | $0 \mathrm{~s} / 3816 \mathrm{~s}$ | $\boldsymbol{\checkmark}$ |
| SVT4 | $0 \mathrm{~s} / 269 \mathrm{~s}$ | $\boldsymbol{x}$ |
| SVT5 | $0 \mathrm{~s} / 2 \mathrm{~s}$ | $\boldsymbol{x}$ |
| SVT6 | $0 \mathrm{~s} / 661 \mathrm{~s}$ | $\boldsymbol{x}$ |
| NMax1 | $0 \mathrm{~s} / 197 \mathrm{~s}$ | $\boldsymbol{\checkmark}$ |
| NMax2 | $0 \mathrm{~s} / 59 \mathrm{~s}$ | $\boldsymbol{\checkmark}$ |
| NMax3 | $0 \mathrm{~s} / 310 \mathrm{~s}$ | $\boldsymbol{x}$ |
| NMax4 | $1 \mathrm{~s} / 58 \mathrm{~s}$ | $\boldsymbol{x}$ |
| Hist1 | $0 \mathrm{~s} / 1450 \mathrm{~s}$ | $\boldsymbol{\checkmark}$ |
| Hist2 | $0 \mathrm{~s} / 55 \mathrm{~s}$ | $\boldsymbol{x}$ |
| Rand1 | $0 \mathrm{~s} / 0 \mathrm{~s}$ | $\boldsymbol{x}$ |
| Rand2 | $0 \mathrm{~s} / 0 \mathrm{~s}$ | $\boldsymbol{x}$ |

Table 3: Runtime for 3 queries for each algorithm searching over adjacency pairs and all $\epsilon_{i} 0$, with parameters being $[\mathrm{c}=1, \Delta=1$, DOM $=\{-$ $1,0,1\}$, seq $=(-1<0<1)]$. For SVT, we also have $T=0$.

| $\mathrm{Q}-$ | c | $\epsilon$ | Runtime (T1/T2) |  |
| :---: | :---: | :---: | :---: | :--- | :--- |
|  |  |  | Fixed $\epsilon$ | General |
| 1 | 1 | 1.0 | $0 \mathrm{~s} / 7 \mathrm{~s}$ | $0 \mathrm{~s} / 16 \mathrm{~s}$ |
| 1 | 1 | 0.5 | $0 \mathrm{~s} / 8 \mathrm{~s}$ | $0 \mathrm{~s} / 16 \mathrm{~s}$ |
| 2 | 1 | 1.0 | $0 \mathrm{~s} / 43 \mathrm{~s}$ | $0 \mathrm{~s} / 113 \mathrm{~s}$ |
| 2 | 1 | 0.5 | $0 \mathrm{~s} / 46 \mathrm{~s}$ | $0 \mathrm{~s} / 113 \mathrm{~s}$ |
| 2 | 2 | 1.0 | $0 \mathrm{~s} / 95 \mathrm{~s}$ | $0 \mathrm{~s} / 155 \mathrm{~s}$ |
| 2 | 2 | 0.5 | $0 \mathrm{~s} / 113 \mathrm{~s}$ | $0 \mathrm{~s} / 155 \mathrm{~s}$ |
| 3 | 1 | 1.0 | $0 \mathrm{~s} / 307 \mathrm{~s}$ | $0 \mathrm{~s} / 825 \mathrm{~s}$ |
| 3 | 1 | 0.5 | $0 \mathrm{~s} / 265 \mathrm{~s}$ | $0 \mathrm{~s} / 825 \mathrm{~s}$ |
| 3 | 2 | 1.0 | $0 \mathrm{~s} / 541 \mathrm{~s}$ | $0 \mathrm{~s} / 1202 \mathrm{~s}$ |
| 3 | 2 | 0.5 | $0 \mathrm{~s} / 572 \mathrm{~s}$ | $0 \mathrm{~s} / 1202 \mathrm{~s}$ |
| 4 | 1 | 1.0 | $0 \mathrm{~s} / 1772 \mathrm{~s} / 0 \mathrm{~s} / 4727 \mathrm{~s}$ |  |
| 4 | 1 | 0.5 | $0 \mathrm{~s} / 1832 \mathrm{~s} / 0 \mathrm{~s} / 4727 \mathrm{~s}$ |  |
| 4 | 2 | 1.0 | $1 \mathrm{~s} / 2904 \mathrm{~s}$ | $0 \mathrm{~s} / 6715 \mathrm{~s}$ |
| 4 | 2 | 0.5 | $1 \mathrm{~s} / 3295 \mathrm{~s}$ | $0 \mathrm{~s} / 6715 \mathrm{~s}$ |

Table 4: Runtimes of SVT1 over different query length and counts, searching over all adjacency pairs and fixed $\epsilon$, with parameters being $[\Delta=1, T=0, \mathrm{DOM}=\{-1,0,1\}]$.

| Algo | - Q - | Output | Input 1 | Input 2 | $\epsilon$ | $\begin{aligned} & \hline \begin{array}{l} \text { Runtime } \\ \text { (T1/T2) } \end{array} \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| SVT3 | 5 | $[\perp \perp \perp \perp 0]$, seq $=(0<1)$ | $\left[\begin{array}{lllll}-1 & -1 & -1 & -1 & -1\end{array}\right]$ | $\left[\begin{array}{lllll}0 & 0 & 0 & 0 & 0\end{array}\right]$ | 27 | 18s/5042 |
| SVT4 | 2 | $[\perp$ T] | [-1 0] | [0-1] | 27/50 | 0s/81s |
| SVT5 | 2 | $[\perp \mathrm{T}]$ | $\left[\begin{array}{ll}-1 & 0\end{array}\right]$ | $\left[\begin{array}{lll}-1 & -1\end{array}\right]$ | 27 | 0s/2s |
| SVT6 | 3 | $[\perp \perp \mathrm{T}]$ | $\left[\begin{array}{lll}-1 & -1 & 0\end{array}\right]$ | $\left[\begin{array}{lll}0 & 0 & -1\end{array}\right]$ | 67/92 | 0s/661s |
| NMax3 | 3 | -1, seq $=(-1<0<1)$ | $\left[\begin{array}{lll}-1 & -1 & -1\end{array}\right]$ | $\left[\begin{array}{lll}0 & 0 & 0\end{array}\right]$ | 27 | 0s/310s |
| NMax4 | 1 | 0, seq $=(-1<0<1)$ | [-1] | [0] | 27 | 0s/2s |
| Hist2 | 1 | $[-1]$, seq $=(-1<0<1)$ | [-1] | [0] | 9/34 | 0s/3s |
| Rand2 | 1 | [ $\perp$ ] | [ $\perp$ ] | [T] | 9/34 | 0s/0s |

Table 5: Smallest Counter-example found for each non-differentially private algorithm, searching over all adj. pairs and $\epsilon>0$, with parameters being $[\mathrm{c}=1, \Delta=1, \mathrm{DOM}=\{-1,0,1\}]$

| $-\mathrm{Q}-$ | c | Runtimg <br> $(\mathrm{T} 1 / \mathrm{T} 2)$ |
| :---: | :---: | :---: |
| 1 | 1 | $0 \mathrm{~s} / 16 \mathrm{~s}$ |
| 2 | 1 | $0 \mathrm{~s} / 113 \mathrm{~s}$ |
| 2 | 2 | $0 \mathrm{~s} / 155 \mathrm{~s}$ |
| 3 | 1 | $0 \mathrm{~s} / 825 \mathrm{~s}$ |
| 3 | 2 | $0 \mathrm{~s} / 1202 \mathrm{~s}$ |
| 4 | 1 | $0 \mathrm{~s} / 4727 \mathrm{~s}$ |
| 4 | 2 | $0 \mathrm{~s} / 6715 \mathrm{~s}$ |

Table 6: Runtimes of SVT1 over different query length and counts, searching over all adjacency pairs and all $\epsilon_{i} 0$, with parameters being $[\Delta=1, T=0, \mathrm{DOM}=\{-1,0,1\}]$

| \#Queries | 1 Pair <br> Run- <br> time <br> $(\mathrm{T} 1 / \mathrm{T} 2)$ | General <br> Run- <br> time <br> $(\mathrm{T} 1 / \mathrm{T} 2)$ | $\epsilon$-Diff. <br> Private |
| :---: | :---: | :---: | :---: |
| 1 | $0 \mathrm{~s} / 15 \mathrm{~s}$ | $0 \mathrm{~s} / 25 \mathrm{~s}$ | $\checkmark$ |
| 2 | $0 \mathrm{~s} / 40 \mathrm{~s}$ | $0 \mathrm{~s} / 192 \mathrm{~s}$ | $\mathbf{\checkmark}$ |
| 3 | $0 \mathrm{~s} / 100 \mathrm{~s}$ | $0 \mathrm{~s} / 1562 \mathrm{~s}$ | $\mathbf{\checkmark}$ |
| 4 | $0 \mathrm{~s} / 199 \mathrm{~s}$ | $1 \mathrm{~s} / 1051 \mathrm{~s}$ | $\mathbf{\checkmark}$ |
| 5 | $0 \mathrm{~s} / 141 \mathrm{~s}$ | $18 \mathrm{~s} / 5044 \mathrm{~s}$ | $\boldsymbol{x}$ |

Table 7: Runtimes of SVT3 over different query lengths, searching over a single adj. pair ([00...] [11...]) and all $\epsilon>0$, with parameters being $[c=1, T=0, \Delta=1, \quad \mathrm{DOM}=\{-$ $1,0,1\}$, seq $=(0 ; 1)]$
we analyzed SVT1 for various number of queries. The running times along with the number of queries and the value for $c$ is shown in Table 6. The table shows that the tool can handle a reasonable number of queries.

In all the experiments so far, the value of $\epsilon$ was not fixed. So DiPC had to either prove privacy for all $\epsilon$ or find an $\epsilon$ where privacy is violated. Many automated tools are designed only to disprove differential privacy for a fixed $\epsilon$. We tried the performance of the tool on SVT1 for a fixed $\epsilon$. The results are reported in Table 4. As can be seen by comparing the numbers in Tables 6 and 4, fixing $\epsilon$ makes the problem easier to handle.

Finally, we wanted to explore the scalability of our tool when we checking
differential privacy for a single pair of adjacent inputs. In Table 7 we have the results when a non differentially private algorithm, namelySVT3 was run with a single adjacency pair ( $[00 \ldots] \sim[11 \ldots]$ ), while varying number of queries. We notice that the running times is significantly lower in this case. Another interesting observation is that the time taken for 5 queries is lower than the time for 4 queries. This is because with 5 queries, the tool successfully finds a counter-example and terminates before checking the remaining inequalities.

## H. 4 ( $\epsilon, \delta$ )-Differential Privacy

DiPC can also verify $(\epsilon, \delta)$-differential privacy. Algorithm 4 (taken from [21), referred to henceforth as Sparse, was used to evaluate DiPC's performance in this case. This algorithm has been manually proven to be $\left(\frac{\epsilon}{2}, \delta_{\text {svt }}\right)$ differentially private for any number of queries in [21] by using advanced composition theorems.

When $c=1$ and $\delta_{\text {svt }}=e^{-\frac{1}{32}}$, this algorithm is identical to Algorithm SVT1, where parameters $c$ and $\Delta$ are replaced by parameter $\sigma$. This algorithm is, therefore, $\epsilon$-differentially private. Further, our tool proves that the algorithm is not $\frac{\epsilon}{2}$-differentially private. Thanks to the advanced composition theorem, we can show that the resulting algorithm is $\left(\frac{\epsilon}{2}, e^{-\frac{1}{32}}\right)$ differentially private. The tool also shows that for all $\epsilon>0$, the algorithm is $\left(\frac{\epsilon}{2}, e^{-2}\right)$-differentially private for $c=1$ for queries of length 3 with $\mathrm{DOM}=\{0,1\}$ and $T=0$ (observe that $e^{-\frac{1}{32}}>e^{-2}$ ). Additionally, we get a counter-example for ( $\left(\frac{\epsilon}{2}, e^{-2.125}\right)$-differential privacy.

When $c=2$ and $\delta_{\text {svt }}=e^{-\frac{1}{64}}$, Sparse differs from SVT1 since in this case we also need to choose $\mathbf{r}_{T}$ again after outputting a $T$. The resulting program is $\left(\frac{\epsilon}{2}, e^{-1 / 64}\right)$-differentially private thanks to the advanced composition theorem. DiPC confirms that for queries of length 3 , the resulting program is infact $\left(\frac{\epsilon}{2}, e^{-2}\right)$-differentially private with $\operatorname{DOM}=\{0,1\}$ and $T=0$. Further, DiPC also demonstrates that the resulting program is not $\left(\frac{\epsilon}{2}, e^{-2.5}\right)$ differentially private.

Here we are able to check the correctness of Sparse automatically, for values of $c=1,2$ and for the above given values of $\delta_{\text {svt }}$ and for all $\epsilon>0$. To the best of our knowledge, our approach is the first method to automatically check this. These results are summarized in Table 8,

```
Input: \(q[1: N]\)
Output: out \([1: N]\)
\(\sigma \leftarrow \frac{\epsilon}{2 \sqrt{32 c \ln \frac{1}{\delta_{\text {svt }}}}}\)
\(\mathrm{r}_{T} \leftarrow \operatorname{Lap}(\sigma, T)\)
count \(\leftarrow 0\)
for \(i \leftarrow 1\) to \(N\) do
    \(r \leftarrow \operatorname{Lap}\left(\frac{\sigma}{2}, q[i]\right)\)
    \(\mathrm{b} \leftarrow \mathrm{r} \geq \mathrm{r}_{T}\)
    if \(b\) then
        out \([i] \leftarrow \mathrm{T}\),
                \(\mathrm{r}_{T} \leftarrow\)
                \(\operatorname{Lap}(\sigma, T)\)
                count \(\leftarrow\)
                count +1
            if count \(\geq c\)
                    then
                I exit
        end
        else
            out \([i] \leftarrow \perp\)
        end
end
```

Algorithm 4: Sparse algorithm

| $c$ | $\delta_{\text {svt }}$ | $\delta$ | Runtime (T1/T2) | $\left(\frac{\epsilon}{2}, \delta\right)$-Diff. Privacy |
| :---: | :---: | :---: | :---: | :---: |
| 1 | $e^{-\frac{1}{32}}$ | 0 | $0 \mathrm{~s} / 48 \mathrm{~s}$ | $\boldsymbol{X}$ |
| 1 | $e^{-\frac{1}{32}}$ | $e^{-3}$ | $0 \mathrm{~s} / 142 \mathrm{~s}$ | $\boldsymbol{x}$ |
| 1 | $e^{-\frac{1}{32}}$ | $e^{-2.125}$ | $0 \mathrm{~s} / 146 \mathrm{~s}$ | $\boldsymbol{X}$ |
| 1 | $e^{-\frac{1}{32}}$ | $e^{-2}$ | $0 \mathrm{~s} / 161 \mathrm{~s}$ | $\boldsymbol{\checkmark}$ |
| 2 | $e^{-\frac{1}{64}}$ | 0 | $0 \mathrm{~s} / 72 \mathrm{~s}$ | $\boldsymbol{x}$ |
| 2 | $e^{-\frac{1}{64}}$ | $e^{-3}$ | $0 \mathrm{~s} / 187 \mathrm{~s}$ | $\boldsymbol{x}$ |
| 2 | $e^{-\frac{1}{64}}$ | $e^{-2.5}$ | $0 \mathrm{~s} / 182 \mathrm{~s}$ | $\boldsymbol{X}$ |
| 2 | $e^{-\frac{1}{64}}$ | $e^{-2}$ | $0 \mathrm{~s} / 288 \mathrm{~s}$ | $\boldsymbol{\checkmark}$ |

Table 8: DiPCresult for $\left(\frac{\epsilon}{2}, \delta\right)$-Diff. Privacy of SPARSE (Algorithm (4) with 3 queries, searching over all adj. pairs and $\epsilon>0$, with parameters being $[T=0, \mathrm{DOM}=\{0,1\}]$


[^0]:    ${ }^{1}$ A parametrized Markov chain is a Markov chain whose transition probabilities are a function of the privacy budget.

[^1]:    ${ }^{2}$ Though not necessary to distinguish between Booleans and finite domains, having such a distinction makes our future technical development easier.
    ${ }^{3}$ Our decidability results also hold if DOM is taken to be a finite subset of the rationals.

