Deciding Differential Privacy for Programs with Finite Inputs and Outputs

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Abstract

Differential privacy is a *de facto* standard for statistical computations over databases that contain private data. Its main and rather surprising strength is to guarantee individual privacy and yet allow for accurate statistical results. Thanks to its mathematical definition, differential privacy is also a natural target for formal analysis. A broad line of work develops and uses logical methods for proving privacy. A more recent and complementary line of work uses statistical methods for finding privacy violations. Although both lines of work are practically successful, they elide the fundamental question of decidability.

This paper studies the decidability of differential privacy. We first establish that checking differential privacy is undecidable even if one restricts to programs having a single Boolean input and a single Boolean output. Then, we define a non-trivial class of programs and provide a decision procedure for checking the differential privacy of a program in this class. Our procedure takes as input a program P parametrized by a privacy budget ϵ and either establishes the differential privacy for all possible values of ϵ or generates a counter-example. In addition, our procedure works for both to ϵ -differential privacy and (ϵ, δ) -differential privacy. Technically, the decision procedure is based on a novel and judicious encoding of the semantics of programs in our class into a decidable fragment of the first-order theory of the reals with exponentiation. We implement our procedure and use it for (dis)proving privacy bounds for many well-known examples, including randomized response, histogram, report noisy max and sparse vector.

1 Introduction

Differential privacy [19] is a gold standard for the privacy of statistical computations. Differential privacy ensures that running the algorithm on any two "adjacent" databases yields two "approximately" equal distributions, where two databases are adjacent if they differ in a single element, and two distributions are approximately equivalent if their distance is small w.r.t. some metric specified by privacy parameter ϵ and error parameter δ . Thus, differential privacy delivers a very strong form of individual privacy. Yet, and somewhat surprisingly, it is possible to develop differentially private algorithms for many tasks. Moreover, the algorithms are useful, in the sense that their results have reasonable accuracy. However, designing differentially private algorithms is difficult, and the privacy analysis can be error-prone, as witnessed by the example of the sparse vector technique.

This difficulty has motivated the development of formal approaches for analyzing differentially private algorithms (see [6] for a survey and the related work section of this paper). Broadly, two successful lines of work have emerged. The first line of work develops sound proof systems to establish differential privacy and uses these proof systems to prove the privacy of well-known and intricate examples [32, 22, 7, 5, 4, 35, 17, 1, 34]. The second line of work searches for counter-examples to demonstrate the violation of differential privacy [18, 9]. Unfortunately, both lines of work elide the question of decidability. As previous experience in formal verification suggests, understanding decidable fragments of a problem not only help advance our theoretical knowledge, but can form the basis of practical tools when combined with ideas like abstraction and composition.

The goal of this paper is, therefore, to study the decision problem for differential privacy, and to make a first attempt at delineating the decidability/undecidability boundary. As a first contribution, we show that, as expected, checking differential privacy is computationally undecidable. Our undecidability result holds even if one restricts to programs having a single Boolean input and a single Boolean output. Given the undecidability result, we then consider the task of identifying a rich class of programs, that encompasses many known examples, for which checking differential privacy nonetheless is decidable. We impose two desiderata:

1. the class of programs must include programs with real-valued vari-

ables, and more generally, with variables over infinite domains. This requirement is critical for the method to cover a broad class of differential privacy algorithms;

2. the programs themselves are parametrized by the privacy parameter ϵ (throughout the paper, we assume that the error parameter δ is a function of ϵ), and the decision procedure should decide privacy for all possible instances of the privacy parameter ϵ . This requirement is motivated by the fact, supported by practice, that differential privacy algorithms are typically parametrized by ϵ , and well-designed algorithms are private not only for a single value of ϵ , but typically for all positive values of ϵ .

We focus our attention on programs whose input and output spaces are finite. Note that such programs need not be finite-state, as per our first requirement, they could use program variables ranging over infinite (even uncountable) domains to carry out the computation. We introduce a class of programs, called DiPWhile, which are probabilistic while programs, for which the problem of checking differential privacy is decidable. We succeed in carefully balancing decidability and expressivity, by judiciously delineating the use of real-valued and integer-valued variables. Intuitively, the main restriction we impose is that these infinite-valued variables be used only to directly influence the program control-flow and not the data-flow that leads to the computation of the final output. More precisely, in an execution, the program output value depends only on the input, values sampled from user-defined distributions and the exponential mechanism, and branch conditions on the control flow path taken. The sampled values of real/integer variables affect only the branch conditions. Thus, the output values depend only on the branch conditions satisfied by the sampled real/integer variable values, but not on their actual sampled values. This restriction, though severe, turns out to capture many prominent differential privacy algorithms, including Report Noisy Max and Sparse Vector Technique (see Section 8 on experiments).

Key observations that enable us to establish decidability of DiPWhile programs are as follows. The first result is that the semantics of DiPWhile-programs can be defined using parametrized, *finite-state* Markov chains ¹. The fact that the semantics is definable using only finitely-many states is a surprising observation because our programs have both integer and real-

¹A parametrized Markov chain is a Markov chain whose transition probabilities are a function of the privacy budget.

valued variables, and hence a naïve semantics yields uncountably many possible states. Our crucial insight here is that a precise semantics for DiPWhile-programs is possible without tracking the explicit values of the real and integer-valued variables. Since real and integer variables are intuitively used only in influencing control-flow, the semantics only tracks the symbolic relationships between the variables. Second, we show that the transition probabilities of the Markov chain are ratios of polynomial functions in ϵ and e^{ϵ} , where e is the Euler's constant; this was a difficult result to establish. These two observations together, allow us to reduce the problem of checking the differential privacy of DiPWhile-programs to the decidable fragment of the first-order theory of reals with exponentials, identified by McCallum and Weispfenning [29].

We leverage our decision procedure to build a stand-alone tool for checking ϵ - or $(\epsilon, \delta(\epsilon))$ -differential privacy of mechanisms specified by DiPWhileprograms, for all values of ϵ . We have implemented our decision procedure in a tool that we call DiPC (Differential Privacy Checker). Given DiPWhile-program, our tool constructs a sentence within the McCallum-Weispfenning fragment of the theory of reals with exponentials. It then calls Mathematica® to check if the constructed sentence is true over the reals. Since our decision procedure is the *first* that can both prove differential privacy and detect its violation, we tried the tool on examples that known to be differentially private and those that are known to be not differentially private including variants of Sparse Vector, Report Noisy Max, and Histograms. DiPC successfully checked differential privacy for the former class of examples and produced counter-examples for the latter class. Our counter-examples are exact and are more compact than those discovered by prior tools.

As a contribution of independent interest, we also demonstrate how our method yields a theoretical complete under-approximation method for checking differential privacy of programs with infinite output sets. For such programs, it is possible to discretize the output domain into a finite domain, and to use the decision procedure to find privacy violations for the discretized algorithm (by post-processing, privacy violations for the discretized algorithms are also privacy violations for the original algorithm). The discretization yields a method for generating counter-examples for algorithms with infinite output sets.

We briefly contrast our results with prior work, and refer the reader to Section 9 for further details. Overall, we see our decidability results as complementary to prior works in checking differential privacy. In general, existing methods for proving or disproving differential privacy, although inherently incomplete due to the undecidability of checking differential privacy, are likely to be more efficient because they can trade-off efficiency for precision. However, the decision procedures for a sub-class of programs, like the one presented here, maybe more predictable — if a decision procedure fails to prove privacy, then it shall produce a counter-example that demonstrates that the algorithm is not differentially private. Moreover, counter-example search methods work for a fixed (ϵ) privacy parameter. As the counter-example methods are usually statistical, they may generate both false positives and false negatives. In contrast, our decision procedures work for all values for the privacy parameter and do not generate false positives or false negatives.

Contributions. We summarize our key contributions.

- We prove the undecidability of the problem of checking differential privacy of very simple programs, including those that have a single Boolean input and output. Though unsurprising, undecidability has not been previously established in any prior work.
- We prove the decidability of differential privacy for an interesting class of programs. Our method is fully automatic that can check both differential privacy *and* detect its violation by generating *counter-examples*. To the best of our knowledge, this is the first such result that encompasses sampling from integer and real-valued variables.
- We implement the decision procedure and evaluate our approach on private and non-private examples from the literature.

Due to lack of space, some proofs and other materials have been moved to an Appendix. The Appendix has been uploaded as an anonymous supplementary submission.

2 Primer on differential privacy

Differential privacy [19] is a rigorous definition and framework for private statistical data mining. In this model, a trusted curator with access to the database returns answers to queries made by possibly dishonest data analysts that do not have access to the database. The task of the curator is to return probabilistically noised answers, so that data analysts cannot distinguish between two databases that are adjacent, i.e. only differ in the value of a single individual. There are two common definitions: two databases are adjacent if they are exactly the same except for the presence or absence of one record, or for the difference in one record. We abstract away from any particular definition of adjacency.

Henceforth, we denote the set of real numbers, rational numbers, natural numbers and integers by $\mathbb{R}, \mathbb{Q}, \mathbb{N}$, and \mathbb{Z} respectively. The Euler constant shall be denoted by e. We assume given a set \mathcal{U} of inputs, and a set \mathcal{V} of outputs. A randomized function P from \mathcal{U} to \mathcal{V} is a function that takes an input in \mathcal{U} and returns a distribution over \mathcal{V} . For a measurable set $S \subseteq \mathcal{V}$, the probability that the output of P on u is in the set S shall be denoted by $\operatorname{Prob}(P(u) \in S)$. In the case the output set is discrete, we use $\operatorname{Prob}(P(u) = v)$ as shorthand for $\operatorname{Prob}(P(u) \in \{v\})$.

We are now ready to define differential privacy. We assume that \mathcal{U} is equipped with a binary symmetric relation $\Phi \subseteq \mathcal{U} \times \mathcal{U}$, which we shall call the *adjacency relation*. We say that $u_1, u_2 \in \mathcal{U}$ are *adjacent* if $(u_1, u_2) \in \Phi$.

Definition 1. Let $\epsilon \geq 0$ and $0 \leq \delta \leq 1$. Let $\Phi \subseteq \mathcal{U} \times \mathcal{U}$ be an adjacency relation. Let P be a randomized function with inputs from \mathcal{U} and outputs in \mathcal{V} . We say that P is (ϵ, δ) -differentially private with respect to Φ if for all measurable subsets $S \subseteq \mathcal{V}$ and $u, u' \in \mathcal{U}$ such that $(u, u') \in \Phi$,

$$\operatorname{Prob}(P(u) \in S) \leq e^{\epsilon} \operatorname{Prob}(P(u') \in S) + \delta$$

As usual, we say that P is ϵ -differentially private iff it is $(\epsilon, 0)$ -differentially private. If the output domain is discrete, it is equivalent to require that for all $v \in \mathcal{V}$ and $u, u' \in \mathcal{U}$ such that $(u, u') \in \Phi$,

$$\operatorname{Prob}(P(u) = v) \leq e^{\epsilon} \operatorname{Prob}(P(u') = v)$$

Differential privacy is preserved by post-processing. Concretely, if P is an (ϵ, δ) -differentially private computation from \mathcal{U} to \mathcal{V} , and $h : \mathcal{V} \to \mathcal{W}$ is a deterministic function, then $h \circ P$ is an (ϵ, δ) -differentially private computation from \mathcal{U} to \mathcal{W} . In the remainder, we shall exploit post-processing to connect differential privacy of randomized computations with infinite output spaces to differential privacy of their discretizations.

Laplace Mechanism. The Laplace mechanism [19] achieves differential privacy for numerical computations by adding random noise to outputs. Given $\epsilon > 0$ and mean μ , let $Lap(\epsilon, \mu)$ be the continuous distribution whose probability density function (p.d.f.) is given by

$$f_{\epsilon,\mu}(x) = \frac{\epsilon}{2} e^{-\epsilon |x-\mu|}$$

 $\operatorname{Lap}(\epsilon, \mu)$ is said to be the Laplacian distribution with mean μ and scale parameter $\frac{1}{\epsilon}$. Consider a real-valued function $q : \mathcal{U} \to \mathbb{R}$. Assume that qis k-sensitive w.r.t. an adjacency relation Φ on \mathcal{U} , i.e. for every pair of adjacent values u_1 and u_2 , $|q(u_1) - q(u_2)| \leq k$. Then the computation that maps u to $\operatorname{Lap}(\frac{\epsilon}{k}, q(u))$ is ϵ -differentially private.

It is sometimes convenient to consider the discrete version of the Laplace distribution. Given $\epsilon > 0$ and mean μ , let $\mathsf{DLap}(\epsilon, \mu)$ be the discrete distribution on \mathbb{Z} , the set of integers, whose probability mass function (p.m.f.) is

$$f_{\epsilon,\mu}(i) = \frac{1 - e^{-\epsilon}}{1 + e^{-\epsilon}} e^{-\epsilon|i-\mu|}$$

 $\mathsf{DLap}(\epsilon,\mu)$ is said to be the discrete Laplacian distribution with mean μ and scale parameter $\frac{1}{\epsilon}$. The discrete Laplace mechanism achieves the same privacy guarantees as the continuous Laplace mechanism.

Exponential mechanism. The Exponential mechanism [30] is used for making non-numerical computations private. The mechanism takes as input a value u from some input domain and a scoring function $F : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ and outputs a discrete distribution over \mathcal{V} . Formally, given $\epsilon > 0$ and $u \in \mathcal{U}$, the discrete distribution $\mathsf{Exp}(\epsilon, F, u)$ on \mathcal{V} is given by the probability mass function:

$$h_{\epsilon,F,u}(v) = \frac{e^{\epsilon F(u,v)}}{\sum_{v \in \mathcal{V}} e^{\epsilon F(u,v)}}$$

Suppose that the scoring function is k-sensitive w.r.t. some adjacency relation Φ on \mathcal{U} , i.e., for all for each pair of adjacent values u_1 and u_2 and $v \in \mathcal{V}$, $|F(u_1, r) - F(u_2, r)| \leq k$. Then the exponential mechanism is $(2k\epsilon, 0)$ -differentially private w.r.t. Φ .

3 Motivating Example

Before presenting the mathematical details of our results, let us informally introduce our method by showing how it would work on an illustrative example.

Sparse Vector Technique. Several differential privacy examples require that the randomized algorithms sampling from infinite support distributions (including continuous distributions). The Sparse Vector Technique (SVT) [20, 28] was designed to answer multiple Δ -sensitive numerical queries in a differentially private fashion. The relevant information we want from queries is, which amongst them are above a threshold T. The Sparse Vector Technique as given in Algorithm 1 is designed to identify the first c queries that are above the threshold T in an ϵ -differentially private fashion.

Input: q[1:N]**Output:** out[1:N] $\mathsf{r}_T \leftarrow \mathsf{Lap}(\frac{\epsilon}{2\Delta}, T)$ $count \leftarrow 0$ for $i \leftarrow 1$ to N do $\mathsf{r} \leftarrow \mathsf{Lap}(\tfrac{\epsilon}{4c\Delta}, q[i])$ $\mathsf{b} \leftarrow \mathsf{r} \geq \mathsf{r}_T$ if b then $out[i] \leftarrow \top$ $count \gets count + 1$ if $count \ge c$ then exit end else $out[i] \leftarrow \bot$ end end

Algorithm 1: SVT algorithm (SVT1)

In the program, the integer N represents the total number of queries, and the array q of length N represents the answers to queries. The array out represents the output array, \perp represents False and \top represents True. We assume that initially the constant \perp is stored at each position in out. In the SVT technique, the \top answers account for most of the privacy cost, and we can only answer c of them until we run out of the privacy budget [20, 35]. On the other hand, there is no restriction on the number of \perp answers. Please observe that the SVT algorithm is parametrized by the privacy budget ϵ . Thus, the SVT algorithm can be considered as representing a class of programs, one for each $\epsilon > 0$.

Given N, the input set \mathcal{U} in this context is the set of N length vectors q, where the kth element q[k] represents the answer to the kth query on the original database. The adjacency relation Φ on inputs is defined as follows: q_1 and q_2 are adjacent if and only if $|q_1[i] - q_2[i]| \leq 1$ for each $1 \leq i \leq N$.

Let us consider an instance of the SVT algorithm when T = 0, N = 2,

 $\Delta = 1$ and c = 1. Let us assume that all array elements in q come from the domain $\{0, 1\}$. In this case, we have four possible inputs [0, 0], [0, 1], [1, 1], and [1, 0], and three possible outputs $[\bot, \bot], [\top, \bot],$ and $[\bot, \top]$.

For example, the probability of outputting $[\bot, \top]$ on input [0, 1] can be computed as follows. Let X_T be a random variable with Laplacian distribution $\mathsf{Lap}(\frac{\epsilon}{2}, 0)$, X_1 be a random variable with Laplacian distribution $\mathsf{Lap}(\frac{\epsilon}{4}, 0)$ and X_2 be the random variable with Laplacian distribution $\mathsf{Lap}(\frac{\epsilon}{4}, 1)$. The probability of outputting $[\bot, \top]$ is the product of outputting of outputting \bot first, which is $Prob(X_1 < X_0)$, and the conditional probability of outputting \top given that \bot is output, which is $Prob(X_2 \ge X_0|X_1 < X_0)$. Note that we really require the second quantity to be conditional probability as the events $X_1 < X_0$ and $X_2 \ge X_0$ are <u>not</u> independent. This probability can be computed to be

$$r_1(\epsilon) = \frac{24e^{\frac{3\epsilon}{4}} - 1 + 8e^{\frac{\epsilon}{4}} + 21e^{\frac{\epsilon}{2}}}{48e^{\frac{3\epsilon}{4}}}.$$

Similarly, when the input is [1, 1] and the output is $[\bot, \top]$, the probability is given by

$$r_2(\epsilon) = \frac{-22 + 32e^{\frac{\epsilon}{4}} - 3\epsilon}{48e^{\frac{\epsilon}{2}}}$$

Observe that $r_1(\epsilon)$ and $r_2(\epsilon)$ are functions of ϵ , and hence the probabilities of outputting $[\bot, \top]$ on inputs [0, 1] and [1, 1] vary with ϵ . Our immediate challenge is to automatically compute expressions like $r_1(\epsilon), r_2(\epsilon)$ from the given program, the adjacent inputs, and outputs. Note that this example involves sampling from continuous distributions and is a function of ϵ . Nevertheless, we shall establish that (see Section 6 and Theorem 8) that for several programs, the former can be accomplished by interpreting the program as a finite-state DTMC whose transition probabilities are functions parameterized by ϵ even when the randomized choices involve infinite-support random variables. The set of programs that we identify (Section 6) is rich enough to model the most known differential privacy mechanisms when restricted to finite input and output sets.

Having computed such expressions, checking ϵ -differential privacy requires one to determine if

for all
$$\epsilon > 0. \ (r_1(\epsilon) \le e^{\epsilon} r_2(\epsilon))$$

and for all $\epsilon > 0. \ (r_2(\epsilon) \le e^{\epsilon} r_1(\epsilon))$

Note that the particular condition for the SVT example under consideration above is encodable as a first-order sentence with exponentials, and thus checking the formula for the example reduces to determining if such a first-order sentence is valid for reals, with the standard interpretation of multiplication, addition, and exponentiation. Whether there is a decision procedure that can determine the truth of first-order sentences involving real arithmetic with exponentials, is a long-standing open problem. However, a decidable fragment of such an extended first-order theory has been identified by McCallum and Weispfenning [29]. The formula for the considered example lies in this fragment. Indeed, we can show that all the formulas for the SVT example lie in this fragment. This observation presents a challenge, namely, what guarantees do we have that checking differential privacy is reducible to this decidable fragment. Indeed, we shall establish that the set of formulas that arise from the class of programs with finite-state DTMC semantics in Theorem 8 also lead to formulas in the same decidable fragment.

Remark. Notice that if one can compute expressions for the probability producing individual outputs on a given input, we could also check (ϵ, δ) -differential privacy, instead of just ϵ -differential privacy. The only change would be to account for δ in our constraints, and to consider all possible subsets of outputs, instead of just individual output values. Thus, the methods proposed here go beyond the scope of most automated approaches, which are restricted to vanilla ϵ -differential privacy.

4 Preliminaries

In this section, we formally define the problem of differential privacy verification that we consider in this paper and also introduce the decidable fragment of real arithmetic with exponentiation that plays a crucial role in our decision procedure. The set of reals/positive reals/rationals/positive rationals shall be denoted by $\mathbb{R}/\mathbb{R}^{>0}/\mathbb{Q}/\mathbb{Q}^{>0}$ respectively.

4.1 The Computational Problem

As illustrated by the example in Section 3, a differential privacy mechanism is typically a randomized program P_{ϵ} parametrized by a variable ϵ . Having a parameterized program P_{ϵ} captures the fact that the program's behavior depends on the privacy budget ϵ , intending to guarantee that P_{ϵ} is $(f(\epsilon), g(\epsilon))$ -differentially private, where f and g are some functions of ϵ . The parameter ϵ is assumed to belong to some interval $I \subseteq \mathbb{R}^{>0}$ with rational end-points; usually, we take ϵ to just belong to the interval $(0, \infty)$. The program P_{ϵ} shall be assumed to terminate with probability 1 for every value of ϵ (in the appropriate interval). The randomized program P_{ϵ} takes inputs from a set \mathcal{U} and produces output in a set \mathcal{V} . In this paper, we shall assume that both \mathcal{U} and \mathcal{V} are *finite* sets that can be effectively enumerated. Despite our restriction to finite input and output sets, the computational problem of checking differential privacy is challenging (see Section 5.3). At the same time, the decidable subclass we identify (Section 6) is rich enough to model most differential privacy mechanisms when restricted to finite input and output sets. Extending our decidability results to subclasses of programs that have infinite input and output sets, is a non-trivial open problem at this time.

The computational problems we consider in this paper are as follows. Since our programs take inputs from a finite set \mathcal{U} , we assume that the adjacency relation $\Phi \subseteq \mathcal{U} \times \mathcal{U}$ is given as an explicit list of pairs. In general, when discussing (ϵ, δ) -differential privacy of some mechanism, the error parameter δ needs to be a function of ϵ . To define the computational problem of checking differential privacy, the function $\delta : \mathbb{R}^{>0} \to [0, 1]$ must be given as input. We, therefore, assume that this function δ has some finite representation; if δ is the constant δ_0 (which is often the case), then we represent δ simply by the number δ_0 . There are two computational problems we consider in this paper.

- Fixed Parameter Differential Privacy Given a program P_{ϵ} over inputs \mathcal{U} and outputs \mathcal{V} , adjacency relation $\Phi \subseteq \mathcal{U} \times \mathcal{U}$, and positive rational numbers $\epsilon_0, \delta_0, t \in \mathbb{Q}^{>0}$, determine if P_{ϵ_0} is $(t\epsilon_0, \delta_0)$ -differentially private with respect to Φ .
- **Differential Privacy** Given a program P_{ϵ} over inputs \mathcal{U} and outputs \mathcal{V} , interval $I \subseteq \mathbb{R}^{>0}$ with rational end-points, $\delta : \mathbb{R}^{>0} \to [0, 1]$, an adjacency relation $\Phi \subseteq \mathcal{U} \times \mathcal{U}$, and a rational number $t \in \mathbb{Q}^{>0}$, determine if P_{ϵ} is $(t\epsilon, \delta(\epsilon))$ -differentially private with respect to Φ for every $\epsilon \in I$.

Observe that the Fixed Parameter Differential Privacy problem can be trivially reduced to the Differential Privacy problem by considering the singleton interval $I = [\epsilon_0, \epsilon_0]$ and $\delta(\epsilon) = \delta_0$, where the goal is to check fixed parameter differential privacy for constant privacy budget ϵ_0 and error parameter δ_0 . Thus, an algorithm for checking Differential Privacy can be used to solve Fixed Parameter Differential Privacy. Unfortunately, the Fixed Parameter Differential Privacy problem is extremely challenging even when restricted to finite input and output sets— we show that it is undecidable (Section 5.3), and therefore, so is the Differential Privacy problem. We shall identify a class of programs (Section 6) for which the Differential Privacy problem (and therefore the Fixed Parameter Differential Privacy problem) is decidable.

When the differential privacy does not hold, we would like to output a counter-example.

Definition 2. A counter-example of (ϵ, δ) differential privacy for P_{ϵ} , with respect to an adjacency relation Φ , a function $\delta : \mathbb{R}^{>0} \to [0, 1]$ and a value $t \in \mathbb{Q}^{>0}$, is a quadruple $(\mathbf{in}, \mathbf{in}', O, \epsilon_0)$ such that $(\mathbf{in}, \mathbf{in}') \in \Phi$, $O \subseteq \mathcal{V}$ and $\epsilon_0 > 0$ and

$$\mathsf{Prob}(P_{\epsilon_0}(\mathbf{in}) \in O) > e^{t\epsilon_0} \operatorname{Prob}(P(\mathbf{in}') \in O) + \delta(\epsilon_0)$$

When δ is the constant function 0, then O is $\{\mathbf{out}\}\$ for some $\mathbf{out} \in \mathcal{V}$.

Remark. For the rest of the paper, unless otherwise stated, we shall assume that the interval $I \subseteq \mathbb{R}^{>0}$ that contains the set of admissible ϵ s is the interval $(0, \infty)$. In our paper, ϵ refers to the parameter in program P_{ϵ} , and not the privacy budget. In our case, the privacy budget is $(t\epsilon)$. For example, some differential privacy algorithms P_{ϵ} are designed to satisfy $(\frac{\epsilon}{2}, 0)$ -differential privacy, and so in this case t would be $\frac{1}{2}$. In the standard differential privacy definition " ϵ " refers to the privacy budget and so t does not appear. However, many theorems for differential privacy algorithms use " ϵ " as the program parameter, and then the privacy theorem is stated as the program being $(t\epsilon, \delta)$ -differentially private. In most such cases, such a theorem is equivalent to saying that the program $P_{\frac{\epsilon}{t}}$ (obtained by replacing ϵ by $\frac{\epsilon}{t}$) is $(\epsilon, \delta(\frac{\epsilon}{t}))$ -differentially private.

4.2 Reals with exponentials

As outlined in Section 3, our approach towards deciding differential privacy shall rely on reducing the question to the problem of checking the truth of a first-order sentence for the reals. Because of the definition of differential privacy, the constructed first-order sentence shall involve exponentials. It is a long-standing open problem whether there is a decision procedure for the first-order theory of reals with exponentials. However, some fragments of this theory are known to be decidable. In particular, there is a fragment identified by McCallum and Weispfenning [29], that we shall exploit in our results.

We will consider first-order formulas over a restricted signature and vocabulary. We will denote this collection of formulas as the language \mathcal{L}_{exp} . Formulas in \mathcal{L}_{exp} are built using variables $\{\epsilon\} \cup \{x_i \mid i \in \mathbb{N}\}$, constant symbols 0, 1, unary function symbol $e^{(\cdot)}$ applied only to the variable ϵ , binary function symbols $+, -, \times$, and binary relation symbols =, <. The terms in the language are integral polynomials with rational coefficients over the variables $\{\epsilon\} \cup \{x_i | i \in \mathbb{N}\} \cup \{e^{\epsilon}\}$. Atomic formulas in the language are of the form t = 0 or t < 0 or 0 < t, where t is a term. Quantifier free formulas are Boolean combinations of atomic formulas. Sentences in \mathcal{L}_{exp} are formulas of the form

$$Q\epsilon Q_1 x_1 \cdots Q_n x_n \psi(\epsilon, x_1, \dots, x_n)$$

where ψ is a quantifier free formula, and Q, Q_i s are quantifiers. In other words, sentences are formulas in prenex form, where all variables are quantified, and the outermost quantifier is for the special variable ϵ .

The theory $\mathsf{Th}_{\mathsf{exp}}$ is the collection of all sentences in $\mathcal{L}_{\mathsf{exp}}$ that are valid in the structure $\langle \mathbb{R}, 0, 1, e^{(\cdot)}, +, -, \times, =, < \rangle$, where the interpretation for $0, 1, +, -, \times$ is the standard one on reals, and e is Euler's constant; notice that this is an extension of the first-order theory of reals. The crucial property about this theory is that it is decidable.

Theorem 3 (McCallum-Weispfenning [29]). Th_{exp} is decidable.

Finally, our tractable restrictions (and our proofs of decidability) shall often utilize the notion of functions *definable* in $\mathsf{Th}_{\mathsf{exp}}$; we, therefore, conclude this section with its formal definition.

Definition 4. A function $f : (0, \infty) \to \mathbb{R}$ is said to be *definable* in $\mathsf{Th}_{\mathsf{exp}}$, if there is a formula $\varphi_f(\epsilon, x)$ in $\mathcal{L}_{\mathsf{exp}}$ with two free variables (ϵ and x) such that

for all $a \in (0, \infty)$. f(a) = b iff $\langle \mathbb{R}, 0, 1, e^{(\cdot)}, +, -, \times, =, < \rangle \models \varphi_f(\epsilon, x)[\epsilon \mapsto a, x \mapsto b]$

5 Program syntax and semantics

We consider randomized algorithms written as simple probabilistic while programs. We introduce the syntax of these programs, along with their "natural" semantics given using Markov kernels [15, 31]. We show that the problem of checking differential privacy is undecidable for these programs.

5.1 Syntax of Simple programs

We introduce a class of programs we call Simple. Programs in Simple are probabilistic while programs in which variables can be assigned values by drawing from distributions typically used in differential privacy algorithms.
$$\begin{split} & \text{Expressions } (\mathbf{b} \in \mathcal{B}, \mathbf{x} \in \mathcal{X}, \mathbf{z} \in \mathcal{Z}, \mathbf{r} \in \mathcal{R}, d \in \text{DOM}, i \in \mathbb{Z}, q \in \mathbb{Q}, g \in \mathcal{F}_{Bool}, f \in \mathcal{F}_{\text{DOM}}): \\ & B \quad ::= \quad \text{true} \mid \text{false} \mid \mathbf{b} \mid not(B) \mid B \text{ and } B \mid B \text{ or } B \mid g(\tilde{E}) \\ & E \quad ::= \quad d \mid \mathbf{x} \mid f(\tilde{E}) \\ & Z \quad ::= \quad z \mid iZ \mid EZ \mid Z + Z \mid Z + i \mid Z + E \\ & R \quad ::= \quad \mathbf{r} \mid qR \mid ER \mid R + R \mid R + q \mid R + E \end{split}$$
Basic Program Statements $(a \in \mathbb{Q}^{>0}, \sim \in \{<, >, =, \leq, \geq\}, F \text{ is a scoring function and choose is a user-defined distribution}): \\ & s \quad ::= \quad \mathbf{x} \leftarrow E \mid \mathbf{z} \leftarrow Z \mid \mathbf{r} \leftarrow R \mid \mathbf{b} \leftarrow B \mid \mathbf{b} \leftarrow Z_1 \sim Z_2 \mid \\ & \mathbf{b} \leftarrow Z \sim E \mid \mathbf{b} \leftarrow R_1 \sim R_2 \mid \mathbf{b} \leftarrow R \sim E \mid \\ & \mathbf{r} \leftarrow \text{Lap}(a\epsilon, E) \mid \mathbf{z} \leftarrow \text{DLap}(a\epsilon, E) \mid \\ & \mathbf{x} \leftarrow \text{Exp}(a\epsilon, F(\tilde{\mathbf{x}}), E) \mid \mathbf{x} \leftarrow \text{choose}(a\epsilon, \tilde{E}) \mid \\ & \text{ if } B \text{ then } P \text{ else } P \text{ end } \mid \text{ While } B \text{ do } P \text{ end } \mid \text{ exit} \end{split}$ Program Statements $(\ell \in \text{Labels})$

Figure 1: BNF grammar for Simple. DOM is a finite discrete domain. \mathcal{F}_{Bool} , $(\mathcal{F}_{DOM}$ resp) are set of functions that output Boolean values (DOM respectively). $\mathcal{B}, \mathcal{X}, \mathcal{Z}, \mathcal{R}$ are the sets of Boolean variables, DOM variables, integer random variables and real random variables. Labels is a set of program labels. For a syntactic class S, \tilde{S} denotes a sequence of elements from S. DiPWhile (see Section 6) is the subclass of programs in which the assignments to real and integer variables do not occur with the scope of a while statement.

Programs in Simple obey some syntactic restrictions; these syntactic restrictions are introduced to make it easier to describe the decidable fragment in Section 6. Despite these restrictions, the problem of checking differential privacy is undecidable for the language introduced here.

The formal syntax of Simple programs is shown in Figure 1. Programs have four types of variables: $Bool = \{ true, false \};$ finite domain DOM² that we assume (without loss of generality) to be $\{-N_{\max}, \ldots, 0, 1, \ldots, N_{\max}\}$, a finite subset of integers ³; reals \mathbb{R} ; and integers \mathbb{Z} . The set of Boolean/DOM/ integer/real program variables are respectively denoted by $\mathcal{B}/\mathcal{X}/\mathcal{Z}/\mathcal{R}$. The set of Boolean/DOM/integer/real expressions is given by the non-terminal B/E/Z/R in Figure 1. We now explain the rules for such expressions. Boolean expressions (B) can be built using Boolean variables and constants, standard Boolean operations, and by applying functions from \mathcal{F}_{Bool} . \mathcal{F}_{Bool} is assumed to be a collection of *computable* functions returning a *Bool*. We assume that \mathcal{F}_{Bool} always contains a function $\mathsf{EQ}(x_1, x_2)$ that returns true iff x_1 and x_2 are equal. DOM expressions (E) are similarly built from DOM variables, values in DOM, and applying functions from set of computable functions \mathcal{F}_{DOM} . Next, integer expressions (Z) are built using multiplication and addition with integer constants and DOM expressions, and additions with other integer expressions. Finally, real expressions (R) are built using multiplication and addition with rational constants and DOM expressions, and additions with other real-valued expressions. Notice that integer-valued expressions cannot be added or multiplied, in real-valued expressions; this syntactic restriction shall be useful later.

A program in Simple is a triple consisting of a set of (private) input variables, a set of (public) output variables, and a finite sequence of labeled statements (non-terminal P in Figure 1). The private input variables and public output variables take values from the domain DOM. Thus, the set of possibles inputs/outputs $(\mathcal{U}/\mathcal{V})$, is identified with the set of valuations for input/output variables; a valuation over a set of variables $X' = \{x_1, x_2, \ldots, x_m\} \subseteq \mathcal{X}$ is a function from X' to DOM. Note that if we represent the set X' as a sequence x_1, x_2, \ldots, x_m then a valuation val over X'can be viewed as a sequence $val(x_1), val(x_2), \ldots, val(x_m)$ of DOM elements.

We assume every statement in our program is uniquely labeled from a set of labels called Labels. Basic program statements (non-terminal s) can either be assignments, conditionals, while loops, or exit. Statements other than as-

²Though not necessary to distinguish between Booleans and finite domains, having such a distinction makes our future technical development easier.

³Our decidability results also hold if **DOM** is taken to be a finite subset of the rationals.

signments are self-explanatory. The syntax of assignments is designed to follow a strict discipline. Real and integer variables can either be assigned the value of real/integer expression or samples drawn using the Laplace or discrete Laplace mechanism. DOM variables are either assigned values of DOM expressions or values drawn either using an exponential mechanism $(\text{Exp}(a\epsilon, F(\tilde{x}), E))$ or a user-defined distribution $(\text{choose}(a\epsilon, \tilde{E}))$. For the exponential mechanism, we require that the scoring function F be computable and return a rational value. Both of these restrictions are unlikely to be severe in practice. In the case of the user defined distribution, we demand that the probability with which a value d in DOM is chosen (as a function of the privacy budget ϵ), be definable in Th_{exp} , and that there is an algorithm that on input a, \tilde{d}, v returns the formula defining the probability of sampling $d \in \mathsf{DOM}$ from the distribution choose $(a\epsilon, \tilde{d})$ where \tilde{d} is a sequence of values from DOM. This restriction is exploited in Section 6 to get decidability for a sub-fragment.

Finally, we consider assignments to Boolean variables. The interesting cases are those where the Boolean variable stores the result of the comparison of two expressions. The syntax does not allow for comparing real and integer expressions. This restriction is exploited later in Section 6 when the decidable fragment is identified. Finally, we will assume that in any execution, if a variable appears on the right side of an assignment statement, then it should have been assigned a value before. This assumption is not restrictive but is technically convenient when defining the semantics for programs.

5.2 Markov Kernel Semantics

We briefly sketch a "natural" semantics for Simple using Markov kernels. A key step in proving our decidability result is to define a semantics using finite-state (parametrized) DTMCs for the sub-fragment DiPWhile defined in Section 6. The DTMC semantics may not seem natural on first reading. The point of the semantics in this section is, therefore, to argue the correctness of our decision procedure on the basis of the equivalence of these two semantics for DiPWhile (Sections 6 and 7). Details for this section are given in Appendix A due of space constraints and because understanding this semantics is not critical to our decidability proof.

Given a fixed $\epsilon > 0$, the states in the Markov kernel-based semantics for a program P_{ϵ} will be of the form $(\ell, h_{Bool}, h_{DOM}, h_{\mathbb{Z}}, h_{\mathbb{R}})$, where ℓ is the label of the statement of P_{ϵ} to be executed next, the functions h_{Bool} , h_{DOM} , $h_{\mathbb{Z}}$ and $h_{\mathbb{R}}$ assign values to the Boolean, DOM, real and integer variables of the program P_{ϵ} respectively. Given an input state **in**, the initial state will correspond to one where DOM-valued input variables get the values given in **in**, and all other variables either get **false** or 0, depending on their type. Observe that for a program P_{ϵ} with k program statements, i Boolean variables, j DOM variables, s integer variables, t real variables a state $(\ell, h_{Bool}, h_{DOM}, h_{\mathbb{Z}}, h_{\mathbb{R}})$ can be uniquely identified with an element of the set $D_{P_{\epsilon}} = \{1, \ldots, k\} \times \mathcal{F}^{i}_{Bool} \times \text{DOM}^{j} \times \mathbb{Z}^{s} \times \mathbb{R}^{t}$. The "natural" Borel σ -algebra on $D_{P_{\epsilon}}$ induces a σ -algebra on the states of P_{ϵ} .

The semantics of Simple programs can be defined as a Markov kernel over this σ -algebra on states. Intuitively, the Markov kernel K_{ϵ} corresponding to a program P_{ϵ} is such that for a state s and a measurable set of states $C, K_{\epsilon}(s, C)$ is the probability of transitioning to a state in C from s. The precise definition of this Markov kernel is in Appendix A.

Executions are just sequences of states, and the σ -field on executions is the product of the σ -field on states. The Markov kernel defines a probability measure on this σ -field. Given all these observations, we take $\text{Prob}_{natural}(P_{\epsilon}(\mathbf{in}) = \mathbf{out})$ to denote the probability (as defined by the Markov kernel of P_{ϵ}) of the set of all executions that start in the initial state corresponding to \mathbf{in} and end in an exit state with **out** as the valuation of output variables; the precise definition is in Appendix A. For the rest of the paper, we will assume that our programs terminate with probability 1.

5.3 Undecidability

The problem of checking differential privacy for Simple programs is undecidable.

Theorem 5. The Fixed Parameter Differential Privacy problem and the Differential Privacy problem for programs P_{ϵ} in Simple is undecidable.

The proof of Theorem 5 reduces the non-halting problem for deterministic 2-counter Minsky machines to the Fixed Parameter Differential Privacy problem. More precisely, we show that given a 2-counter Minsky machine \mathcal{M} (with no input), there is a program $P_{\epsilon}^{\mathcal{M}} \in \mathsf{Simple}$ such that

- P^M_ε has only one input x_{in} and one output x_{out} taking values in DOM = {0,1};
- $P_{\epsilon}^{\mathcal{M}}$ terminates with probability 1 for all $\epsilon \in \mathbb{R}^{>0}$;
- $P_{\epsilon}^{\mathcal{M}}$ is $(\epsilon, 0)$ -differentially private with respect to the adjacency relation $\Phi = \{(0, 1), (1, 0)\}$ if and only if \mathcal{M} does not halt.

This construction shows that Differential Privacy is undecidable. Undecidability of Fixed Parameter Differential Privacy is obtained by taking ϵ to be any constant rational number, say $\frac{1}{2}$. The formal details of the reduction are in Appendix B.

6 DiPWhile: A decidable class of programs

We now discuss a restricted class of programs, for which we can establish decidability of checking differential privacy. The class of programs that we consider are exactly those programs in Simple that satisfy the following restriction:

Bounded Assignments We do not allow assignments to real and integer variables within the scope of a while loop. This restriction ensures that assignments to such variables happen only a *bounded* number of times during execution. Thus, without loss of generality, we assume that real and integer variables are assigned *at most once* as a program with multiple assignments to a single real and variables can always be rewritten to an equivalent program with each assignment to a variable being an assignment to a fresh variable.

We refer to this restricted class as DiPWhile. The DiPWhile language is surprisingly expressive — many known randomized algorithms for differential privacy can be encoded. We give an example of such encodings in DiPWhile. We omit labels of program statements unless they are needed.

Example 6. Algorithm 2 shows how SVT can be encoded in our language with $T = 0, \Delta = 1, N = 2, c = 1$. In the example we are modeling \perp by 0 and \top by 1. Though for-loops are not part of our program syntax, they can modeled as while loops, or if bounded (like here), they can be unrolled.

Appendix C shows how sampling from the standard exponential distribution can be encoded in DiPWhile. Other examples that can be encoded in our language (and for which the decision procedure applies) include randomized response, the private smart sum algorithm [10] with finite discretization of the output space (See 7.1), and private vertex cover [24].

The decidability of checking differential privacy for DiPWhile shall rely on two observations. First, the semantics of DiPWhile programs can also be defined as finite-state discrete-time Markov chains (DTMC), albeit with transition probabilities parameterized by ϵ . This observation is surprising because DiPWhile programs have real and integer values variables, and so

Input: q_1, q_2 **Output:** out_1, out_2 1 $T \leftarrow 0;$ **2** $out_1 \leftarrow 0;$ **3** $out_2 \leftarrow 0;$ 4 $\mathbf{r}_T \leftarrow \mathsf{Lap}(\frac{\epsilon}{2}, T);$ 5 $\mathbf{r}_1 \leftarrow \mathsf{Lap}(\frac{\overline{\epsilon}}{4}, q_1);$ 6 b \leftarrow r₁ \geq r_T; $\boldsymbol{7}~\mathrm{if}~\boldsymbol{b}~\mathrm{then}$ $out_1 \leftarrow 1$ 8 else $\mathbf{r}_2 \leftarrow \mathsf{Lap}(\frac{\epsilon}{4}, q_2);$ 9 $\mathbf{b} \leftarrow \mathbf{r}_2 \geq \mathbf{r}_T;$ 10 if b then 11 $out_2 \leftarrow 1$ 12 ${\rm end}$ end 13 exit

Algorithm 2: SVT for 1-sensitive queries with N = 2, c = 1 and T = 0. The numbers at the beginning of a line indicate the label of the statement.



Figure 2: Partial DTMC semantics of Algorithm 2 showing the steps when lines 9 and 10 are executed. q_1 and q_2 are assumed to have values u and v, respectively. Only values of assigned program variables is shown. Third line in state shows parameters for the real values that were sampled. Last line shows the accumulated set of Boolean conditions that hold on the path.

the natural semantics has uncountably many states (See Section 5.2). The key insight in establishing this observation is that an equivalent semantics of DiPWhile programs can be defined without explicitly tracking the values of real and integer-valued variables. Second, all the transition probabilities arising in our semantics are definable in Th_{exp} . These two observations allow us to to establish decidability of checking differential privacy of DiPWhile programs. The rest of the section is devoted to establishing these observations. We start by formally defining *parametrized DTMCs*.

6.1 Parameterized DTMCs

Definition 7. A parametrized DTMC is a pair $\mathcal{D} = (Z, \Delta)$, where Z is a (countable) set of states, and $\Delta : Z \times Z \to (\mathbb{R}^{>0} \to [0, 1])$ is the probabilistic transition function. For any pair of states z, z', Δ returns a function from $\mathbb{R}^{>0}$ to [0, 1], such that for every $\epsilon > 0$, $\sum_{z' \in Z} \Delta(z, z')(\epsilon) = 1$. We shall call $\Delta(z, z')$ as the probability of transitioning from z to z'.

A definable parametrized DTMC is a parametrized DTMC $\mathcal{D} = (Z, \Delta)$ such that for every pair of states $z, z' \in Z$, the function $\Delta(z, z')$ is definable in Th_{exp}.

A parametrized DTMC associates with each (finite) sequence of states $\rho = z_0, z_1, \ldots z_m$, a function $\operatorname{Prob}(\rho) : \mathbb{R}^{>0} \to [0,1]$ that given an $\epsilon > 0$, returns the probability of the sequence ρ when the parameter's value is fixed to ϵ , i.e., $\operatorname{Prob}(\rho)(\epsilon) = \prod_{i=0}^{m-1} \Delta(z_i, z_{i+1})(\epsilon)$. For a state z_0 and a set of states $Z' \subseteq Z$, once again we have a function that given a value ϵ for the parameter, returns the probability of reaching Z' from z_0 . This can be formally defined as $\operatorname{Prob}(z_0, Z')(\epsilon) = \sum_{\rho \in z_0(Z \setminus Z')^* Z'} \operatorname{Prob}(\rho)(\epsilon)$. In other words, $\operatorname{Prob}(z_0, Z')(\epsilon)$ is the sum of the probability of all sequences starting in z_0 , ending in Z', such that no state except the last is in Z'.

6.2 Parametrized DTMC semantics of DiPWhile

The parametrized DTMC semantics of a DiPWhile program P_{ϵ} shall be denoted as $\llbracket P_{\epsilon} \rrbracket$. We describe $\llbracket P_{\epsilon} \rrbracket$ informally here and defer the formal definition to Appendix D. As mentioned above, the key insight in defining the semantics of a DiPWhile program as a finite-state, parametrized DTMC, is that the actual values of real and integer variables need not be tracked. A state of $\llbracket P_{\epsilon} \rrbracket$ is going to be a tuple of the form $(\ell, f_{Bool}, f_{DOM}, f_{int}, f_{real}, C)$ where ℓ is the label of the statement of P_{ϵ} to be executed next. $\llbracket P_{\epsilon} \rrbracket$ is an abstraction of the set of all concrete states that are compatible with it.

The partial functions f_{Bool} and f_{DOM} assign values to the *Bool* and DOM variables, respectively; this is just like in the natural semantics.

Let us now look at the partial function f_{real} . Intuitively, f_{real} is supposed to be the "valuation" for the real variables. But instead of mapping each variable to a *concrete* value in \mathbb{R} , we shall instead map it into a finite set. To understand this mapping, let us recall that in DiPWhile, a real variable is assigned only once in a program. Further, such an assignment either assigns the value of a linear expression over program variables, or a value sampled using a Laplace mechanism. In the former case, f_{real} maps a variable to the linear expression it is assigned; and in the latter case, the value of the parameters of the Laplace mechanism used in sampling. In the latter case, since the first parameter is always of the form $a\epsilon$, we need to note only a in the mapping. Notice that the range of f_{real} is now a finite set as P_{ϵ} contains only a finite number of linear expressions, and the parameters of sampled Laplacian take values from the finite set DOM. Similarly, the partial function f_{int} maps each integer variable to either the linear expression it is assigned or the parameters of the sampled discrete Laplace mechanism. The last state component C is the set of Boolean conditions on real and integer variables that hold along the path thus far; this shall become clearer when we describe the transitions. Since the Boolean conditions must be Boolean expressions in the program or their negation, C is also a finite set. These observations show that $[P_{\epsilon}]$ has finitely many states. Intuitively, a state of $[P_e]$ is an abstraction of the set of all concrete states that respect the Boolean conditions in C and the constraints imposed by assignments of real and integer expressions to real and integer variables, respectively.

We now sketch how the state is updated in $\llbracket P_{\epsilon} \rrbracket$. Updates to DOM variables shall be as expected — it shall be a probabilistic transition if the assignment samples using an exponential mechanism or a user-defined distribution, and it shall be a deterministic step updating f_{DOM} otherwise. Assignments to real variables are *always deterministic* steps that change the function f_{real} . Thus, even if the step samples using the Laplace mechanism, in the semantics, it shall be modeled as a deterministic step where f_{real} is updated by storing the parameters of the distribution. Similarly, all integer assignments are deterministic steps as well.

The assignment of a Boolean expression to a Boolean variable is as expected — we update the valuation f_{Bool} to reflect the assignment. The unexpected case is $\mathbf{b} \leftarrow R_1 \sim R_2$ when a boolean variable gets assigned the result of the comparison of two real expressions; the case of comparing two integer expressions is similar. In this case, if the probability of C holding is 0, then our construction will ensure that this state is not reachable with

non-zero probability. Otherwise, we transition to a state where $R_1 \sim R_2$ is added to C with probability equal to the probability that $(R_1 \sim R_2)$ holds conditioned on the fact that C holds, and with the remaining probability, we shall transition to the state where $\neg(R_1 \sim R_2)$ is added to C. Thus, Boolean assignments which compare integer and real variables are modeled by probabilistic transitions. Finally, branches and while loop conditions are deterministic steps, with the value of the Boolean variable (of the condition) in f_{Bool} determining the choice of the next statement.

Let $\operatorname{Prob}_{DTMC}(P_{\epsilon}(\mathbf{in}) = \mathbf{out})$ denote the probability that P_{ϵ} outputs value **out** on the input **in** under the DTMC semantics. This is just the probability of reaching an exit state with **out** as valuation of output variables from the initial state with **in** as the valuation of input variables. We can show that this probability is the same as the probability $\operatorname{Prob}_{natural}(P_{\epsilon}(\mathbf{in}) = \mathbf{out})$ obtained by the natural semantics discussed above. The informal ideas outlined above are fleshed out to give a precise mathematical definition and presented in Appendix D.

It is worth noting how key syntactic restrictions in DiPWhile programs play a role in defining its semantics. The first restriction is that integer and real variables are not assigned in the scope of a while loop. This restriction is critical to ensure that the DTMC $[P_{\epsilon}]$ is finite-state. Since we track distribution parameters and linear expressions for such variables, this restriction ensures that we only remember a bounded number of these. Second, DiPWhile disallows a comparison between real and integer expressions in its syntax. Recall that such comparison steps result in a probabilistic transition, where we compute the probability of the comparison holding conditioned on the properties in *C* holding. It is unclear if a closed-form expression for such probabilities can be computed when integer and real random variables are compared. Hence such comparisons are disallowed.

Probabilistic transitions in our semantics arise due to two reasons. First are assignments to DOM variables that sample according to either the exponential or a user-defined distribution. The resulting probabilities are easily seen to be definable in Th_{exp} . The second is due to comparisons between real and integer expressions. We can prove that in this case also, the resulting probabilities are definable in Th_{exp} ; this proof is non-trivial and deferred to Appendix E. All these observations together give us the following theorem.

Theorem 8. For any DiPWhile program P_{ϵ} , $\llbracket P_{\epsilon} \rrbracket$ is a finite, definable, parametrized DTMC that is computable.

Example 9. The parametrized DTMC semantics of Algorithm 2 is partially shown in Figure 2. We show only the transitions corresponding to executing

lines 9 and 10 of the algorithm, when $q_1 = u$ and $q_2 = v$ initially; here $u, v \in \{\perp, \top\}$. The multiple lines in a given state give the different components of the state. The first two lines give the assignment to *Bool* and DOM variables, the third line gives values to the integer/real variables, and the last line is the Boolean conditions that hold along a path. Since 9 and 10 are in the else-branch, the condition $r_1 < r_T$ holds. Notice that values to real variables are not explicit values, but rather the parameters used when they were sampled. Finally, observe that probabilistic branching takes place when line 10 is executed, where the value of *b* is taken to be the result of comparing r_2 and r_T . The numbers *p* and *q* correspond to the probability that the conditions in a branch hold, given the parameters used to sample the real variables and *conditioned* on the event that $r_1 < r_T$.

7 Checking differential privacy for DiPWhile programs

We shall now establish that the problem of checking differential privacy for DiPWhile programs is decidable. The proof relies on the characterization of the semantics of a DiPWhile program as a finite, definable, parameterized DTMC (See Theorem 8). An important observation about a finite, definable, parametrized DTMC is that the probability of reaching a given set of states Z' from a given state z_0 is both definable and computable.

Lemma 10. For any finite-state, definable, parametrized DTMC $\mathcal{D} = (Z, \Delta)$, any state $z_0 \in Z$ and set of states $Z' \subseteq Z$, the function $\mathsf{Prob}(z_0, Z')$ is definable in $\mathsf{Th}_{\mathsf{exp}}$. Moreover, there is an algorithm that computes the formula defining $\mathsf{Prob}(z_0, Z')$.

The proof of Lemma 10 exploits the connection between reachability probabilities in DTMCs and linear programming [33, 2]; details are in Appendix F. The main result of the paper now follows from Theorem 8 and Lemma 10.

Theorem 11. The Fixed Parameter Differential Privacy and Differential Privacy problems are decidable for DiPWhile programs P_{ϵ} , rational numbers $t \in \mathbb{Q}^{>0}$ and definable functions $\delta(\epsilon)$. Furthermore, if P_{ϵ} is not $(t\epsilon, \delta)$ differentially private for some rational number t and admissible value of ϵ then we can compute a counter-example.

Proof. Let **in** and **out** be arbitrary valuations to input and output variables, respectively. Observe that the function $\epsilon \mapsto \operatorname{Prob}(P_{\epsilon}(\mathbf{in}) = \mathbf{out})$ is nothing

but $\operatorname{Prob}(z_0, Z')$ in $\llbracket P_{\epsilon} \rrbracket$, where z_0 is the initial state corresponding to valuation in, and Z' is the set of all terminating states that have valuation **out** for output variables. Since $\llbracket P_{\epsilon} \rrbracket$ (Theorem 8) and $\operatorname{Prob}(z_0, Z')$ (Lemma 10) are computable, we can construct a formula $\varphi_{in,out}(\epsilon, x_{in,out})$ of \mathcal{L}_{exp} that defines the function $\epsilon \mapsto \operatorname{Prob}(P_{\epsilon}(in) = out)$.

Let $\varphi_{\delta}(\epsilon, x_{\delta})$ be the formula defining the function δ . Let $t = \frac{p}{q}$ where p, q are natural numbers. Consider the sentence

It is easy to see P_{ϵ} is $(t\epsilon, \delta(\epsilon))$ differentially private for all ϵ iff ψ is true over the reals. In the syntax of \mathcal{L}_{exp} , we cannot take *q*th roots of *e*; therefore, we introduce the variable *z*, which enables us to write the constraints using only $e^{a\epsilon}$, where $a \in \mathbb{N}$. Notice that ψ belongs to \mathcal{L}_{exp} if we convert it to prenex form. Decidability, therefore, follows from the decidability of Th_{exp} .

If P_{ϵ} is not differentially private, then the sentence ψ does not hold. The decision procedure for $\mathsf{Th}_{\mathsf{exp}}$ will, in this case, return an ϵ_0 that witnesses the privacy violation of P_{ϵ} . Using ϵ_0 , the counter-example ($\mathbf{in}, \mathbf{in}', O, \epsilon_0$) can be easily constructed by enumerating $\mathbf{in}, \mathbf{in}'$ and O.

An easy consequence of Theorem 11 is that differential privacy is decidable for the subclass of program in Simple that do not have integer and real-valued variables. Let Finite DiPWhile denote this set of programs (See Appendix G for the formal syntax of Finite DiPWhile). Observe that due to the presence of While, Finite DiPWhile programs may still have unbounded length executions (including infinite executions).

Corollary 12. The Fixed Parameter Differential Privacy and Differential Privacy problems are decidable for Finite DiPWhile programs P_{ϵ} , rational numbers $t \in \mathbb{Q}^{>0}$ and definable functions $\delta(\epsilon)$.

We observe that our methods can be employed to analyze larger classes of programs (than just those in DiPWhile). For example, a sufficient condition to ensure the decidability is to consider programs with the property that, for each input, the probability distribution on the outputs is definable in Th_{exp} (See Appendix G.1). We conclude the section by showing how our procedure is useful when reasoning about integer and real-valued outputs.

Remark. We sketch here how the proofs of Theorem 11 changes when the set of admissible ϵ is taken to be an interval I with rational end-points. Let P_{ϵ}, t and $\delta(\epsilon)$ be as in the proof of Theorem 11. When ϵ is restricted to an interval I, we will require the user-definable distributions to be definable in Th_{exp} only on the interval I. As in the proof of Theorem 11, we can construct a formula $\varphi_{in,out}(\epsilon, x_{in,out})$ of \mathcal{L}_{exp} that defines the function $\epsilon \mapsto \operatorname{Prob}(P_{\epsilon}(in) = out)$. For simplicity, consider the case when I be the interval [r, s]. Consider the sentence ψ_I that is obtained from ψ in the proof of Theorem 11 by replacing the subformula $(\epsilon > 0)$ by $(a \leq \epsilon) \land (\epsilon \leq b)$. Then P_{ϵ} is $(t\epsilon, \delta(\epsilon))$ will be differentially private for all $\epsilon \in I$ iff ψ_I is true over the reals.

7.1 Finite discretization of infinite output spaces

Our decision procedure assumes that the output space is finite. In several examples, the program outputs are reals or unbounded integers (and combinations thereof). Nevertheless, we argue that our decision procedure is useful for the verification of differential privacy in this case also. In particular, our method provides an under-approximation technique for checking the differential privacy of programs with infinite outputs. Our approach in such cases is to discretize the output space into finitely many intervals.

We illustrate this for the special case when a program P outputs the value of one real random variable, say r. Now, suppose that we modify P to output a finite discretized version of r as follows. Let $seq = a_0 < a_1 < \ldots a_n$ be a sequence of rationals and let $Disc_{seq}(x)$ be equal to a_0 if $x \le a_0$, equal to a_i (0 < i < n) if $a_{i-1} < x \le a_i$, and equal to a_n if $x > a_{n-1}$.

Consider the program $P_{\text{Disc,seq}}$ that instead of outputting r, outputs Disc_{seq}(r). It is easy to see that if P is differentially private then so must be $P_{\text{Disc,seq}}$. Therefore, if $P_{\text{Disc,seq}}$ is not differentially private then we can conclude that P is not differentially private. Thus, if our procedure finds a counter-example for $P_{\text{Disc,seq}}$, then it also has proved that the program P is not differentially private. Our method is, therefore, an under-approximation technique for checking the differential privacy of P. In fact, it is a *complete* under-approximation method in the sense that P is differentially private iff for each possible seq, $P_{\text{Disc,seq}}$ is differentially private.

8 Experimental evaluation

We implemented a simplified version of the algorithm, presented earlier, for proving/disproving differential privacy of DiPWhile programs. Our tool

Algorithm	Runtime (T1/T2)	ε-Diff. Pri- vate
SVT	0s/825s	~
SVT2	0s/768s	✓
SVT5	0s/2s	X
NMax4	1s/58s	X
Rand2	0s/0s	X

Table 1: Runtime for 3 queries for each algorithm searching over adjacency pairs and all $\epsilon_i 0$, with parameters being [c=1, Δ =1, DOM={-1,0,1}, seq = (-1 < 0 < 1)]. For SVT, we also have T=0.

DiPC [3] handles loop-free programs, i.e., acyclic programs. Programs with bounded loops (with constant bounds) can be handled by unrolling loops. The tool takes in an input program P_{ϵ} parametrized by ϵ and an adjacency relation, and either proves P_{ϵ} to be differentially private for all ϵ or returns a counter-example. The tool can also be used to check differential privacy for a given, fixed ϵ , or to check for $k\epsilon$ -differential privacy for some constant k. DiPC is implemented in C++ and uses Wolfram Mathematica(R). It works in two phases — in the first phase, a Mathematica(R)script is produced with commands for all the output probability computations and the subsequent inequality checks and in the second phase, the generated script is run on Mathematica. Details about the tool and its design can be found in Appendix H.

We used various examples to measure the effectiveness of our tool. These include SVT [28, 21], Noisy Maximum [18], Noisy Histogram [18] and Randomized Response [20] and their variants. Detailed descriptions of these algorithms and their variants can be found in Appendix H.1.

We ran all the experiments on an octa-core Intel®Core i7-8550U @ 1.8gHz CPU with 8GB memory. The running times reported are the average of 3 runs of the tool. In the tables, T1 refers to the time needed by the C++ phase to generate the Mathematica scripts, and T2 refers to the time used by Mathematica to check the scripts. Due to space constraints, we report only a small fraction of our experiments; full details of all our experiments can be found in Appendix H.

Salient observations about our experiments are follows.

1. DiPC successfully proves algorithms to be differentially private and finds counter-examples to demonstrate a violation of privacy in reasonable time. Table 1 shows the running time of DiPC on some examples for 3 queries. We chose to use 3 queries because for algorithms

Algo	—Q—	Output	Input 1	Input 2	ϵ	Runtime (T1/T2)
SVT5	2	$[\perp \top]$	[-1 0]	[-1 -1]	27	0s/2s
NMax3	3	-1, seq = $(-1 < 0 < 1)$	[-1 -1 -1]	$[0 \ 0 \ 0]$	27	0s/310s
NMax4	1	$0, \sec (-1 < 0 < 1)$	[-1]	[0]	27	0s/2s
Rand2	1	$[\perp]$	$[\perp]$	$[\top]$	9/34	0s/0s

Table 2: Smallest Counter-example found for each non-differentially private algorithm, searching over all adj. pairs and $\epsilon > 0$, with parameters being [c=1, Δ =1, DOM={-1,0,1}]

that are not private, counter-examples can be found with 3 queries.

- 2. The time to generate Mathematica scripts is significantly smaller than the time taken by Mathematica to check the scripts (i.e., $T1 \ll T2$). Further, most of the time spent by Mathematica is for computing output probabilities; the time to perform comparison checks for adjacent inputs was relatively small. Thus, programs that do not use real variables (Rand2 in Table 1, for example) can be analyzed more quickly.
- 3. For algorithms that are not differentially private, DiPC can automatically identify the pair of inputs, output, and ϵ for which privacy is violated. Table 2, shows the results for the smallest counter-example found by DiPC for some examples. Further, counter-examples found by DiPC are much smaller, in terms of queries, than those found in [18]; the number of queries needed in the counter-examples in [18] for NMax3, NMax4, and SVT5 were 5, 5, and 10, respectively, as opposed to 3, 1, and 2 found by DiPC.
- 4. DiPC is the first automated tool that can check (ϵ, δ) -differential privacy. To evaluate this feature, we tested DiPC on a version of SVT, Sparse [21], which is manually proven to be $(\frac{\epsilon}{2}, \delta_{svt})$ -differentially private for any number of queries in [21] by using advanced composition theorems. Here δ_{svt} is a second parameter in the algorithm. In our experiments, we tested $(\frac{\epsilon}{2}, \delta_{svt})$ -differential privacy of Sparse with fixed values of δ_{svt} for c = 1, 2 and 3 queries, validating the result in [21]. As we were dealing with only 3 queries, we also managed to obtain better bounds on the error parameter.

9 Related work

The main thread of related work has focused on formal systems for proving that an algorithm is differentially private. Such systems are helpful because they rule out the possibility of mistakes in privacy analyses. Starting from Reed and Pierce [32], several authors [22, 17] have proposed linear (dependent) type systems for proving differential privacy. However, it is not possible to verify some of the most advanced examples, such as a sparse vector or vertex cover, using these type systems. Moreover, type-checking and type-inference for linear (dependent) types are challenging. For example, the type checking problem for DFuzz, a language for differential privacy, is undecidable [16]. Barthe et al [7, 5, 4] develop several program logics based on probabilistic couplings for reasoning about differential privacy. These logics have been used successfully to analyze many classic examples from the literature, including the sparse vector technique. However, these logics are limited: they cannot disprove privacy; extensions may be required for specific examples; building proofs is challenging. The last issue has been addressed by a series of works that provide automated methods for proving differential privacy automatically. Zhang and Kifer [35] introduce randomness alignments as an alternative to couplings and build a dependent type system that tracks randomness alignments. Automation is then achieved by type inference. Albarghouthi and Hsu [1] propose coupling strategies, which rely on a fine-grained notion of variable approximate coupling, which draws inspiration both from approximate couplings and randomness alignment. They synthesize coupling strategies by considering an extension of Horn clauses with probabilistic coupling constraints and developing algorithms to solve such constraints. Recently Wang et al [34] develop an improved method based on the idea of shadow executions. Their approach is able to verify Sparse Vector and many other challenging examples efficiently. However, these methods are limited to vanilla ϵ -differential privacy and do not accommodate bounds that are obtained by advanced composition (since $\delta \neq 0$).

In an independent line of work, Chatzikokolakis, Gebler and Palamidessi [11] consider the problem of differential privacy for Markov chains. Later, Liu, Wang, and Zhang [27] develop a probabilistic model checking approach for verifying differential privacy properties. Their approach is based on modeling differential private programs as Markov chains. Their encoding is more direct than ours (i.e. it assumes that a finite-state Markov chain is given), and they do not provide a decision procedure with real and integer variables. Furthermore, the DTMCs are not parameterized by ϵ . Chistikov and Murawski and Purser [13, 14] propose an elegant method based on skewed Kantorovich distance for checking approximate differential privacy of Markov chains.

The dual problem is to find violations of differential privacy automati-

cally. This is useful to help privacy practitioners discover potential problems early in the development cycle. Two recent and concurrent works by Ding et al [18] and Bischel et al [9] develop automated methods for finding privacy violations. Ding et al. propose an approach that combines purely statistical methods based on hypothesis testing and symbolic execution. Bischel et al. develop an approach based on a combination of optimization methods and language-specific techniques for computing differentiable approximations of privacy estimations. Both methods are fully automated. However, both methods can only be used for concrete numerical values of the privacy budget ϵ .

Gaboardi et. al [23] study the complexity of deciding differential privacy for randomized Boolean circuits. Their results are proved by reduction to majority problems and are incomparable with ours: the only probabilistic choices in [23] are fair coin tosses and e^{ϵ} is taken to be a fixed rational number.

10 Conclusions

We showed that the problem checking differential privacy is in general undecidable, identified an expressive sub-class of programs (DiPWhile) for which the problem is decidable, and presented the results of analyzing many known differential privacy algorithms using our tool DiPC which implements a decision procedure for DiPWhile programs. Advantages of DiPC include the ability to automatically, both prove algorithms to be private for all $\epsilon > 0$, and find counter-examples to demonstrate privacy violations. In addition DiPC can check bounds that are based on concentration inequalities, in particular bounds that use advanced composition theorems. Such bounds are out of reach of most other tools that prove privacy or search for counterexamples.

In the future, it would be interesting to extend this work to handle programs with input/output variables that take values in infinite domains, and parametrized privacy algorithms that work for an unbounded number of input and output variables. Another important problem is developing decision procedures that can prove tight accuracy bounds, and detect violations of accuracy bounds. We also plan to investigate extending the decision procedure to cover algorithms that are currently out of the scope of our decision procedure such as the multiplicative weights and iterative database construction [26, 25], and those involving Gaussian distributions.

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A Semantics of Simple

In this section, we give the semantics of our Simple language. This semantics will be given as a set of computations and a probability space on the set of computations. Recall that we have assumed that in each computation, a reference to a variable is preceded (sometime earlier) by an assignment to the variable.

For the rest of this section, let us fix a Simple program P_{ϵ} and an $\epsilon > 0$. We let $L_{P_{\epsilon}}$ denote the set of labels appearing in P_{ϵ} . The set of Boolean variables, DOM variables (including input/output variables), integer variables and reals variables occurring in P_{ϵ} shall be denoted by $\mathcal{B}_{P_{\epsilon}}$, $\mathcal{X}_{P_{\epsilon}}$, $\mathcal{Z}_{P_{\epsilon}}$ and $\mathcal{R}_{P_{\epsilon}}$ respectively.

In order to define the semantics of P_{ϵ} , we will use an auxiliary function next that given a label, identifies the label of the statement to be executed next. Observe that for most program statements, the next statement to be executed is unique. However, for if and While statements, the next statement depends on the value of a Boolean expression. We will define $next(\ell)$ to be a set of pairs of the form (ℓ', c) , where c is a Boolean condition on the variables of P_{ϵ} , with the understanding that ℓ' is the label of the next statement to be executed if c currently holds. Thus, for a label ℓ , $next(\ell)$ will either be $\{(\ell', true)\}$ or $\{(\ell_1, c), (\ell_2, \neg c)\}$. We do not give a precise definition of $next(\cdot)$, but we will use it when defining the semantics.

States. States of P_{ϵ} will be of the form

 $(\ell, h_{Bool}, h_{\mathsf{DOM}}, h_{\mathbb{Z}}, h_{\mathbb{R}}).$

Informally, $\ell \in L_{\mathsf{P}_{\epsilon}}$ is the label of the statement to be executed, h_{Bool} , h_{DOM} , $h_{\mathbb{Z}}$, and $h_{\mathbb{R}}$ are functions assigning "values" to program variables (of appropriate type). More specifically, we have $h_{Bool} : \mathcal{B}_{P_{\epsilon}} \to \{\mathsf{true}, \mathsf{false}\}, h_{\mathsf{DOM}} : \mathcal{X}_{P_{\epsilon}} \to \mathsf{DOM}, h_{\mathbb{Z}} : \mathcal{Z}_{P_{\epsilon}} \to \mathbb{Z}$ and $h_{\mathbb{R}} : \mathcal{R}_{P_{\epsilon}} \to \mathbb{R}$. We let S denote the set of all states. We define a discrete state ds to be a tuple $(\ell, h_{Bool}, h_{\mathsf{DOM}}, h_{\mathbb{Z}})$ where $\ell, h_{Bool}, h_{\mathsf{DOM}}, h_{\mathbb{Z}}$ are as defined above. Note that a discrete state does not specify values to variables in $\mathcal{R}_{P_{\epsilon}}$. For a state s and an expression e which is a Boolean, real or an integer expression, we let Val(s, e) denote the value obtained by evaluating e in the state s. Note that if e is a boolean expression, Val(s, e) is either True or False. We also define the value of a comparison between two expressions as follows. For a comparison expression $e_1 \sim e_2$, $Val(s, e_1 \sim e_2) = True$ if $Val(s, e_1) \sim Val(s, e_2)$ holds, otherwise $Val(s, e_1 \sim e_2) = False$. The value of a DOM expression e, its

value in state $s = (\ell, h_{Bool}, h_{DOM}, h_{\mathbb{Z}}, h_{\mathbb{R}})$ will be denoted by $h_{DOM}(e)$. For a sequence of DOM expressions DOM $\tilde{e} = e_1, \ldots, e_m, h_{DOM}(\tilde{e})$ will denote the sequence $h_{DOM}(e_1), \ldots, h_{DOM}(e_m)$.

Measurable sets of states. Let $\mathcal{R}_{P_{\epsilon}} = \{r_1, ..., r_t\}$. With each vector $u = (u_1, ..., u_t) \in \mathbb{R}^t$, we associate a unique function $h^u_{\mathbb{R}} : \mathcal{R}_{P_{\epsilon}} \to \mathbb{R}$ such that $h^u_{\mathbb{R}}(r_i) = u_i$ for $1 \leq i \leq t$. Given a discrete state $ds = (\ell, h_{Bool}, h_{\mathsf{DOM}}, h_{\mathbb{Z}})$ and a Borel set $D \subseteq \mathbb{R}^t$, we let $[\![(ds, D)]\!] = \{(\ell, h_{Bool}, h_{\mathsf{DOM}}, h_{\mathbb{Z}}, h^u_{\mathbb{R}}) | u \in D\}$. Now, we define \mathcal{E} , the set of measurable sets of states, to be the σ -algebra generated by the sets of states of the form $[\![(ds, D)]\!]$ where ds is a discrete state and $D \subseteq \mathbb{R}^t$ is a Borel set.

Markov Kernel K_{ϵ} . We give the single step semantics of the program P_{ϵ} as a Markov kernel from the measure space (S, \mathcal{E}) to itself. Formally, $K_{\epsilon} : S \times \mathcal{E} \to \mathbb{R}$, where $K_{\epsilon}(s, C)$ gives the probability that the next state of P_{ϵ} is in Cgiven that its current state is s. We fix the state $s = (\ell, h_{Bool}, h_{DOM}, h_{\mathbb{Z}}, h_{\mathbb{R}})$ and the set $C \in \mathcal{E}$ of states, and define the value of $K_{\epsilon}(s, C)$ based on the following cases.

DOM assignments. Let $next(\ell) = \{(\ell', true)\}$ and let x be the variable being assigned in ℓ . There are two cases to consider. First, consider the case where x is assigned a value of a DOM expression e. In this case, $K_{\epsilon}(s, C) = 1$ if $(\ell', h_{Bool}, h_{DOM}[x \mapsto h_{DOM}(e)], h_{\mathbb{Z}}, h_{\mathbb{R}}) \in C$; otherwise $K_{\epsilon}(s, C) = 0$. The second case is when x is assigned a random value according to $Exp(a\epsilon, F(\tilde{x}), e)$ or choose $(a\epsilon, \tilde{e})$. For $d \in DOM$, let prob(d) be the probability of d based on the distribution; note, that these probabilities will depend on the value of $h_{DOM}(e)$ and $h_{DOM}(\tilde{e})$. Then, $K_{\epsilon}(s, C) = \sum_{d \in D} prob(d)$ where D = $\{d \mid (\ell', h_{Bool}, h_{DOM}[x \mapsto d], h_{\mathbb{Z}}, h_{\mathbb{R}}) \in C\}$. Note that the right hand sum is zero if $D = \emptyset$.

Integer assignments. Let $\mathsf{next}(\ell) = \{(\ell', \mathsf{true})\}\)$ and let z be the variable being assigned in ℓ . Again there are two cases to consider. First, consider the case where z is assigned a value of an integer expression e. In this case, $K_{\epsilon}(s, C) = 1$ if $(\ell', h_{Bool}, h_{\mathsf{DOM}}, h_{\mathbb{Z}}[\mathsf{z} \mapsto Val(s, e)], h_{\mathbb{R}}) \in C$; otherwise $K_{\epsilon}(s, C) = 0$. Next, consider the case when z is assigned a random value according to $\mathsf{DLap}(a\epsilon, e)$. For $j \in \mathbb{Z}$, let $\mathsf{prob}(j)$ be the probability assigned to the integer j by the distribution given by $\mathsf{DLap}(a\epsilon, h_{\mathsf{DOM}}, h_{\mathbb{Z}}[\mathsf{z} \mapsto j], h_{\mathbb{R}}) \in C\}$. Note that the right hand sum is zero if $D = \emptyset$.

Real assignments. Let $next(\ell) = \{(\ell', true)\}$ and let r be the variable being assigned in ℓ . Again there are two cases to consider. First, consider the case where r is assigned a value of a real expression e. In this case, $K_{\epsilon}(s, C) =$ 1 if $(\ell', h_{Bool}, h_{DOM}, h_{\mathbb{Z}}, h_{\mathbb{R}}[r \mapsto Val(s, e)]) \in C$; otherwise $K_{\epsilon}(s, C) = 0$. In the second case, r is assigned a random value according to $Lap(a\epsilon, e)$. In this case, $K_{\epsilon}(s, C) = Prob(D)$ where $D = \{r \in \mathbb{R} \mid (\ell', h_{Bool}, h_{DOM}, h_{\mathbb{Z}}, h_{\mathbb{R}}[r \mapsto$ $r]) \in C\}$ and Prob(D) is the probability given to set D by the distribution $Lap(a\epsilon, h_{DOM}(e))$. Observe that $D \subseteq \mathbb{R}$ is a Borel set.

Boolean assignments. Again let $\mathsf{next}(\ell) = \{(\ell', \mathsf{true})\}\)$ and let **b** be the variable being assigned in ℓ and e the expression being assigned. Now, $K_{\epsilon}(s, C) = 1$ if $(\ell', h_{Bool}[\mathsf{b} \mapsto Val(s, e)], h_{\mathsf{DOM}}, h_{\mathbb{Z}}, h_{\mathbb{R}}) \in C$; otherwise $K_{\epsilon}(s, C) = 0$.

if statement. In this case, $\mathsf{next}(\ell) = \{(\ell_1, c), (\ell_2, \neg c)\}\$ for some Boolean condition c. If either $Val(s, c) = \mathsf{true}$ and $(\ell_1, h_{Bool}, h_{\mathsf{DOM}}, h_{\mathbb{Z}}, h_{\mathbb{R}}) \in C$ or $Val(s, c) = \mathsf{false}$ and $(\ell_2, h_{Bool}, h_{\mathsf{DOM}}, h_{\mathbb{Z}}, h_{\mathbb{R}}) \in C$ then $K_{\epsilon}(s, C) = 1$, otherwise $K_{\epsilon}(s, C) = 0$.

While statement. Again let $next(\ell) = \{(\ell_1, c), (\ell_2, \neg c)\}$. This case is identical to the case of if statement, and so is skipped.

exit statement. In this case, $K_{\epsilon}(s, C) = 1$ if $s \in C$; otherwise $K_{\epsilon}(s, C) = 0$.

Probability Spaces on finite executions. For each i > 0, we define a probability space $\Phi_i = (S^i, \Sigma_i, \phi_i)$ capturing the set of finite executions of length $i S^i$, the class Σ_i of measurable sets of executions of length i and a probability measure ϕ_i , as follows. Let $\vec{C} = (C_1, ..., C_i)$ be a sequence of measurable sets where, for $1 \leq j \leq i$, $C_j \in \mathcal{E}$. For each such \vec{C} , let $Exec(\vec{C}) = \{(s_1, s_2, ..., s_i) \mid s_j \in C_j, 1 \leq j \leq i\}$. The set Σ_i of measurable sets of finite executions of length i, is the σ -algebra generated by the sets of executions $Exec(\vec{C})$ where \vec{C} is a vector of measurable sets as specified above. Essentially, (S^i, Σ_i) is the measurable space obtained by taking the product of (S, \mathcal{E}) , i times. The probability measure ϕ_i is defined by first fixing an initial state and using the Markov kernel K_{ϵ} as follows.

Initial State and initial distribution. For an integrable function g with respect to a measure space (X, Σ, μ) , let $\int_{X_1} g\mu(dx)$ denote the integral of

function g with respect to measure μ over a measurable set $X_1 \in \Sigma$. Let ℓ_{in} be the label of the first statement of P_{ϵ} . Let h_{Bool}^{in} , $h_{\mathbb{Z}}^{in}$, and $h_{\mathbb{R}}^{in}$ be functions such that h_{Bool}^{in} assigns false to every variable in $\mathcal{B}_{P_{\epsilon}}$, and $h_{\mathbb{Z}}^{in}, h_{\mathbb{R}}^{in}$ assign value zero to every variable in $\mathcal{Z}_{P_{\epsilon}}, \mathcal{R}_{P_{\epsilon}}$ respectively. An initial state of P_{ϵ} will be of the form $(\ell_{in}, h_{Bool}^{in}, h_{DOM}^{in}, h_{\mathbb{Z}}^{in}, h_{\mathbb{R}}^{in})$, where h_{DOM}^{in} assigns the given values to input variables and assigns zero to all other variables in $\mathcal{X}_{P_{\epsilon}}$ (recall that all input variables are in $\mathcal{X}_{P_{\epsilon}}$); the values given to the input variables by h_{DOM}^{in} will be the "initial input value". We fix a unique initial state s_{init} . Let ϕ_{init} be a distribution on the measure space (S, \mathcal{E}) such that for any $C' \in \mathcal{E}, \phi_{init}(C') = 1$ if $s_{init} \in C'$; otherwise, $\phi_{init}(C') = 0$. Now, ϕ_i is the unique probability measure defined by the Markov kernel K_{ϵ} with initial distribution ϕ_{init} such that for each sequence of measurable sets $\vec{C} = (C_1, ..., C_i), \phi_i(Exec(\vec{C}))$ is

$$\int_{C_1} \int_{C_2} \cdots \int_{C_i} \mathbf{1} \ K_{\epsilon}(x_{i-1}, \mathrm{d}x_i) \cdots K_{\epsilon}(x_1, \mathrm{d}x_2) \phi_{\mathsf{init}}(\mathrm{d}x_1)$$

where $\mathbf{1}$ is the constant function that takes 1 everywhere. Please see [15] for additional details.

We let $\operatorname{Prob}_{natural}(P_{\epsilon}(\mathbf{in}) = \mathbf{out})$ denote the probability that P_{ϵ} outputs value **out** on the input **in**. We define this probability as follows. Let $\alpha = (s_1, ..., s_i) \in S^i$ be an execution. We say that α is a *required* execution if α is a terminating execution with output **out**, i.e., it satisfies the following two conditions: (i) $s_i = (\ell, h_{Bool}, h_{DOM}, h_{\mathbb{Z}}, h_{\mathbb{R}})$ where ℓ is the label of *ext* statement and valuation of output variables is **out**; (ii) if j < i and $s_j =$ $(\ell', f'_{Bool}, f'_{DOM}, f'_{int}, f'_{\mathbb{R}})$ then ℓ' is not the label of *ext* statement. For each i > 0, let Req_i be the set of all required executions in S^i . It is easy to see that, for each i > 0, $Req_i \in \Sigma_i$ and no execution in Req_i is a prefix of an execution in Req_{i+1} . We define $\operatorname{Prob}_{natural}(P_{\epsilon}(\mathbf{in}) = \mathbf{out}) = \sum_{i>0} \phi_i(Req_i)$.

B Undecidability of checking differential privacy of Simple programs

In this section, we will prove Theorem 5. That is, we will show that both Fixed Parameter Differential Privacy and Differential Privacy are undecidable.

Proof. Recall that a 2-counter Minsky Machine is tuple $\mathcal{M} = (Q, q_s, q_f, \Delta_{inc}^1, \Delta_{jzdec}^2, \Delta_{jzdec}^2, \Delta_{jzdec}^2)$ where

• Q is a finite set of control states.

- $q_s \in Q$ is the initial state.
- $q_f \in Q$ is the final state.
- $\Delta_{inc}^i \subseteq Q \times Q$ is the increment of counter *i* for i = 1, 2.
- $\Delta^i_{jzdec} \subseteq Q \times Q \times Q$ is the conditional jump of counter *i* for i = 1, 2.

 \mathcal{M} is said to be deterministic if from each state q, there is at most one transition out of q. The semantics of \mathcal{M} is defined in terms of a transition system $(Conf, (q_s, 0, 0), \rightarrow)$ where $Conf = Q \times \mathbb{N} \times \mathbb{N}$ is the set of configurations, $(q_s, 0, 0)$ is the initial configuration and \rightarrow is defined as follows:

$$\begin{array}{ll} (q,i,j) \to (q',i+1,j) & \text{if } (q,q') \in \Delta_{inc}^{1}, \\ (q,i,j) \to (q',i,j+1) & \text{if } (q,q') \in \Delta_{inc}^{2}, \\ (q,i,j) \to (q',i,j) & \text{if } i = 0 \text{ and } (q,q',q'') \in \Delta_{jzdec}^{1}, \\ (q,i,j) \to (q'',i-1,j) & \text{if } i \neq 0 \text{ and } (q,q',q'') \in \Delta_{jzdec}^{1}, \\ (q,i,j) \to (q',i,j) & \text{if } j = 0 \text{ and } (q,q',q'') \in \Delta_{jzdec}^{2}, \\ (q,i,j) \to (q'',i,j-1) & \text{if } j \neq 0 \text{ and } (q,q',q'') \in \Delta_{jzdec}^{2}, \\ \end{array}$$

A sequence of configurations s_0, s_1, \ldots, s_k is said to be a computation of \mathcal{M} is $s_0 = (q_s, 0, 0)$ and $s_i \to s_{i+1}$ for $i = 0, 1, \ldots, k-1$. A computation s_0, s_1, \ldots, s_k is said to be a terminating computation of \mathcal{M} if $s_k = (q_f, i, j)$ for some $i, j \in \mathbb{N}$.

We show that given a 2-counter Minsky Machine \mathcal{M} , there is a program $P_{\epsilon}^{\mathcal{M}} \in \mathsf{Simple}$ such that for each $\epsilon > 0$,

- (a) $P_{\epsilon}^{\mathcal{M}}$ has only one input x_{in} and only one output x_{out} taking values in $\mathsf{DOM} = \{0, 1\}.$
- (b) $P_{\epsilon}^{\mathcal{M}}$ terminates with probability 1.
- (c) $P_{\epsilon}^{\mathcal{M}}$ is $(\epsilon, 0)$ -differentially private with respect to the adjacency relation $\Phi = \{(0, 1), (1, 0)\}$ if and only if \mathcal{M} does not halt.

Given a 2-counter Machine $\mathcal{M}, P_{\epsilon}^{\mathcal{M}}$ is constructed as follows. Without loss of generality, let $Q = \{q_1, \ldots, q_m\}$ and let q_1 be the initial state and q_m be the final state. We will model a state in Q using m Boolean variables $\mathbf{b}_1, \ldots, \mathbf{b}_m$. If the current state is q_i then \mathbf{b}_i will be set to true and all other variables will be set to false. The counters will be modeled using real variables as follows. Initially a real variable \mathbf{r}_0 will be sampled from Laplacian distribution. If $\mathbf{r}_0 \leq 0$, we will exit the program. Otherwise, we will initialize two real variables $\mathbf{r}_1, \mathbf{r}_2$ to be $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2$ will model the counters as follows. If the first (second respectively) counter is going to hold natural number i then $\mathbf{r}_1 = (i+1)\mathbf{r}_0$ ($\mathbf{r}_2 = (i+1)\mathbf{r}_0$ respectively). Incrementing

```
Input: x_{in}
 Output: x<sub>out</sub>
\mathsf{x_{out}} \gets 0
\mathsf{r}_0 \leftarrow \mathsf{Lap}(\epsilon, 0)
\mathsf{b}_{\mathsf{test}} \gets \mathsf{r}_0 > 0
if b_{test} then
           \mathsf{r}_{\mathsf{number\_steps}} \leftarrow \mathsf{Lap}(\epsilon, 0)
           \mathsf{r}_{\mathsf{curr\_step}} \gets \mathsf{r}_0
           b_{continue} \gets r_{number\_steps} > r_{curr\_step}
           \mathsf{b}_1 \gets \mathsf{true}
           \mathsf{b}_2 \gets \mathsf{false}
           . . .
           \mathsf{b}_m \gets \mathsf{false}
           \mathsf{r}_1 \gets \mathsf{r}_0
           \mathsf{r}_2 \gets \mathsf{r}_0
           while b_{continue} \ do
                      s_1
                     s_n
                     \mathsf{b}_1 \gets \mathsf{b}_1^{\mathsf{next}}
                      . . .
                      \begin{array}{l} \mathbf{b}_m \leftarrow \mathbf{b}_m^{\mathsf{next}} \\ \mathbf{r}_1 \leftarrow \mathbf{r}_1^{\mathsf{next}} \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2^{\mathsf{next}} \end{array} 
                     \mathsf{r}_{\mathsf{curr\_step}} \gets \mathsf{r}_0 + \mathsf{r}_{\mathsf{curr\_step}}
                 \mathsf{b}_{\mathsf{continue}} \gets \mathsf{r}_{\mathsf{number\_steps}} > \mathsf{r}_{\mathsf{curr\_step}}
           end
           \mathbf{if}~(\mathsf{b}_m \text{ and } \mathsf{EQ}(\mathsf{x}_{\mathbf{in}},1)) then
              x_{out} \leftarrow 1
           \mathbf{end}
 end
 exit
```

Algorithm 3: Program $P_{\epsilon}^{\mathcal{M}}$ simulating 2-counter machine \mathcal{M}

the first counter (second respectively) counter is achieved by adding r_0 to r_1 (r_2 respectively). Decrementing the first counter (second respectively) counter is achieved by sibtracting r_0 from r_1 (r_2 respectively). For encoding the transition relations $\Delta_{inc}^1, \Delta_{inc}^2, \Delta_{jzdec}^1$ and Δ_{jzdec}^2 , we use variables $b_1^{next}, \ldots, b_m^{next}, r_1^{next}, r_2^{next}$ to compute the next configuration as expected. For example, the transition $(q_i, q_j, q_k) \in \Delta_{jzdec}^1$ can be encoded using conditional statements as follows:

$$\begin{array}{l} \mathsf{b}_{i,j,k} \leftarrow \mathsf{r}_1 = \mathsf{r}_0 \\ \text{if } (\mathsf{b}_{i,j,k} \ \text{and} \ \mathsf{b}_i) \\ \text{then } \mathsf{b}_j^{\text{next}} \leftarrow \text{true; } \mathsf{b}_1^{\text{next}} \leftarrow \text{false; } \dots \mathsf{b}_{j-1}^{\text{next}} \leftarrow \text{false; } \\ \mathsf{b}_{j+1}^{\text{next}} \leftarrow \text{false; } \dots; \mathsf{b}_m^{\text{next}} \leftarrow \text{false } \\ \text{else } \mathsf{r}_1^{\text{next}} \leftarrow \mathsf{r}_1 - \mathsf{r}_0; \ \mathsf{b}_k^{\text{next}} \leftarrow \text{true; } \mathsf{b}_1^{\text{next}} \leftarrow \text{false; } \dots \\ \mathsf{b}_{k-1}^{\text{next}} \leftarrow \text{false; } \dots; \ \mathsf{b}_m^{\text{next}} \leftarrow \text{false; } \dots; \ \mathsf{b}_m^{\text{next}} \leftarrow \text{false; } \end{array}$$

Let s_1, s_2, \ldots, s_n be the statements encoding the transition relation. Consider the program $P_{\epsilon}^{\mathcal{M}}$ given in Algorithm 3. The program $P_{\epsilon}^{\mathcal{M}}$ initially samples r_0 from a continuous Laplacian distribution. If the sampled value is ≤ 0 then it outputs 0. Otherwise, it starts simulating \mathcal{M} . In order to make sure that the program terminates, we sample another real variable $\mathsf{r}_{\mathsf{number_steps}}$ and simulate k steps of the program where k is the smallest number such that $k\mathsf{r}_0 > \mathsf{r}_{\mathsf{number_steps}}$.

At the end of the simulation, if the halting state is reached and the input is 1 then it outputs 1. Otherwise, it outputs 0.

Clearly, $P_{\epsilon}^{\mathcal{M}}$ satisfies properties (a) and (b) above. That the program $P_{\epsilon}^{\mathcal{M}}$ has property (c) above follows from the following observations:

- 1. If \mathcal{M} does not halt then $P_{\epsilon}^{\mathcal{M}}$ outputs 0 with probability 1.
- 2. If \mathcal{M} halts then $P_{\epsilon}^{\mathcal{M}}$ outputs 1 with non-zero probability on input 1 and outputs 1 with zero probability on input 0.

This shows that Fixed Parameter Differential Privacy is undecidable. Undecidability of Fixed Parameter Differential Privacy is obtained by taking ϵ_0 to be any constant rational number, say $\frac{1}{2}$.

C DiPWhile encoding of exponential distribution

Example 13. Given $\epsilon > 0$ and offset, let Lap⁺(ϵ , offset) be the continuous distribution whose probability density function (p.d.f.) is given by

$$f_{\epsilon,\mu}(x) = \begin{cases} \epsilon \ e^{-\epsilon(x - \text{offset})} & \text{if } x \ge \text{offset} \\ 0 & \text{otherwise} \end{cases}$$

Observe that the one-sided Laplacian distribution $Lap^+(\epsilon, 0)$ is the standard exponential distribution. Our language is expressive enough to encode one-sided Laplacians as follows. Consider the sequence of statements:

$$\begin{split} &X \leftarrow \mathsf{Lap}(\epsilon, 0); \\ &b \leftarrow X \leq 0; \\ &\text{if } b \, \text{then} \, Y \leftarrow X \, \text{else} \, Y \leftarrow (-1) X \, \text{end}; \\ &Z \leftarrow Y + \text{offset} \end{split}$$

The effect of the sequence of statements is that Z has the one-sided Laplacian distribution $Lap^+(\epsilon, offset)$.

D Formal DTMC Semantics of DiPWhile programs

We define formally $[\![P_{\epsilon}]\!]$, the DTMC semantics of an DiPWhile program P_{ϵ} . Let us recall some key restrictions in DiPWhile programs. The first restriction is that real and integer-valued variables are never assigned within the scope of a while statement. Hence, they are assigned only a bounded number of times, and therefore, without loss of generality, we can assume that they are assigned a value exactly *once*. Second, real valued expressions are never compared against integer valued expressions.

Let us fix some basic notation. Partial functions from A to B will be denoted as $A \hookrightarrow B$. The value of $f : A \hookrightarrow B$ on $a \in A$, will be denoted as f(a). Two partial functions f and g will be equal (denoted $f \simeq g$) if for every element a, either f and g are both undefined, or f(a) = f(b). If $f : A \hookrightarrow B$, $a \in A$ and $b \in B$, then $f[a \mapsto b]$ denotes the partial function that agrees with f on all elements of A except a; on a, $f[a \mapsto b](a) = b$.

In the rest of this section let us fix a DiPWhile program P_{ϵ} . L will denote the set of labels appearing in P_{ϵ} . A valuation *val* for DOM variables is a function that assigns a value in DOM to variables in \mathcal{X} ; we will denote set of all such valuations by V_{DOM} . Given a valuation $val \in V_{\text{DOM}}$ and a real expression e, val(e) denotes the real expression that results from substituting all the DOM variables appearing in e by their value in val. Similarly, for an integer expression, val(e) is the partial evaluation of e with respect to val. Finally, for a comparison $e_1 \sim e_2$ between two expressions e_1 and e_2 , again we will define $val(e_1 \sim e_2)$ to be $val(e_1) \sim val(e_2)$. Let us denote the set of integer expressions, real expressions, and Boolean comparisons, appearing on the right of assignments in P_{ϵ} by P_Z , P_R , and P_B , respectively. Three sets of expressions will be used in defining the semantics, and they are as follows.

$$zExp = \{val(e) \mid val \in V_{DOM}, e \in P_Z\}$$

$$rExp = \{val(e) \mid val \in V_{DOM}, e \in P_R\}$$

$$bExp = \{val(e) \mid val \in V_{DOM}, e \in P_B\}$$

Thus, zExp, rExp, and bExp are partially evaluated expression appearing on the right hand side of assignments in P_{ϵ} . Notice that the sets L, zExp, rExp, and bExp are all finite. Finally, let Const be the set of rational constants appearing as coefficient of ϵ of Laplace and discrete Laplace assignments in P_{ϵ} ; again Const is finite.

In order to define the semantics of P_{ϵ} , we will use an auxiliary function next that given a label, identifies the label of the statement to be executed next. Observe that for most program statements, the next statement to be executed is unique. However, for if and While statements, the next statement depends on the value of a Boolean expression. We will define $next(\ell)$ to be a set of pairs of the form (ℓ', c) with the understanding that ℓ' is the next label if c holds. Thus, for a label ℓ , $next(\ell)$ will either be $\{(\ell', true)\}$ or $\{(\ell_1, c), (\ell_2, \neg c)\}$. We do not give a precise definition of $next(\cdot)$, but we will use it when defining the semantics.

The semantics of P_{ϵ} will given as a finite-state, parametrized DTMC $\llbracket P_{\epsilon} \rrbracket$. To define the parametrized DTMC $\llbracket P_{\epsilon} \rrbracket$, we need to define the states and the transitions.

States. States of $\llbracket P_{\epsilon} \rrbracket$ will be of the form

$$(\ell, f_{Bool}, f_{\mathsf{DOM}}, f_{\mathsf{int}}, f_{\mathsf{real}}, C).$$

Informally, $\ell \in \mathsf{L}$ is the label of the statement to be executed, f_{Bool} , f_{DOM} , f_{int} , and f_{real} are partial functions assigning "values" to program variables (of appropriate type), and C is a collection of inequalities among program variables that hold on the current computational path. Both f_{Bool} and f_{DOM} are valuations for the appropriate set of variables, and so we have $f_{Bool} : \mathcal{B} \hookrightarrow \{\mathsf{true}, \mathsf{false}\}$ and $f_{\mathsf{DOM}} : \mathcal{X} \hookrightarrow \mathsf{DOM}$. For real and integer

variables, instead of tracking exact values, we will track the expressions used in assignments and parameters of (discrete) Laplace mechanisms used in random assignments. Therefore, we have $f_{\text{int}} : \mathcal{Z} \hookrightarrow \mathsf{zExp} \cup (\mathsf{Const} \times \mathsf{DOM})$ and $f_{\mathsf{real}} : \mathcal{R} \hookrightarrow \mathsf{rExp} \cup (\mathsf{Const} \times \mathsf{DOM})$. Finally, $C \subseteq \mathsf{bExp} \cup \{\neg e \mid e \in \mathsf{bExp}\}$. It follows immediately that the set of states of $[P_{\epsilon}]$ is finite.

Well-Formed States. The functions f_* (for $* \in \{Bool, DOM, int, real\}$) assign values to program variables that have been assigned during the computation thus far. Since we assume variables in DiPWhile program are defined before they are used, if a variable z' appears in $f_{int}(z) \in zExp$, then $f_{int}(z')$ must be defined. A similar condition holds for real variables. The comparisons in C are also relationships that must hold on the current path, and so all variables participating in it must be defined. If a state satisfies these consistency properties between f_{int} , f_{real} , and C, we will say it is *well-formed*. All reachable states in $[P_{\epsilon}]$ will be well-formed. So when we define transitions we will assume that the states are well-formed.

Initial States. Let ℓ_{in} be the label of the first statement P_{ϵ} . Let $C^{in} = \emptyset$, and let f_{Bool}^{in} , f_{int}^{in} , and f_{real}^{in} be partial functions with an empty domain. An initial state of $[\![P_{\epsilon}]\!]$ will be of the form $(\ell_{in}, f_{Bool}^{in}, f_{OOM}^{in}, f_{real}^{in}, C^{in})$, where f_{DOM}^{in} is defined only on the input variables; the values given to these variables by f_{DOM}^{in} will be the "initial input value".

We will now define the semantics of transitions in $\llbracket P_{\epsilon} \rrbracket$. Let us fix a state $z = (\ell, f_{Bool}, f_{DOM}, f_{int}, f_{real}, C)$. Transitions out of z will be defined based on the effect of executing the statement labeled ℓ , and so its definition will depend on this statement. We handle each case below.

DOM assignments. Let $next(\ell) = \{(\ell', true)\}$ and let x be the variable being assigned in ℓ . There are two cases to consider. First, consider the case where x is assigned a value for a DOM expression *e*. In this case, $\llbracket P_{\epsilon} \rrbracket$ will transition to

$$(\ell', f_{Bool}, f_{DOM}[\mathsf{x} \mapsto f_{DOM}(e)], f_{int}, f_{real}, C)$$

with probability 1. The second case is when x is assigned a random value according to $\mathsf{Exp}(a\epsilon, F(\tilde{\mathsf{x}}), e)$ or $\mathsf{choose}(a\epsilon, \tilde{e})$. For $d \in \mathsf{DOM}$, let $\mathsf{prob}(d)$ be the probability of d (as a function of ϵ) based on the distribution; note, that these probabilities will depend on the value of $f_{\mathsf{DOM}}(e)$ and $f_{\mathsf{DOM}}(\tilde{e})$. Then, $\llbracket P_{\epsilon} \rrbracket$ will transition to

$$(\ell', f_{Bool}, f_{\mathsf{DOM}}[\mathsf{x} \mapsto d], f_{\mathsf{int}}, f_{\mathsf{real}}, C)$$

with probability prob(d).

Integer assignments. Let $\mathsf{next}(\ell) = \{(\ell', \mathsf{true})\}\)$ and let z be the variable being assigned in ℓ . Again there are two cases to consider. First, consider the case where z is assigned a value for an integer expression e. In this case, $\llbracket P_{\epsilon} \rrbracket$ will transition to

$$(\ell', f_{Bool}, f_{DOM}, f_{int}[z \mapsto f_{DOM}(e)], f_{real}, C)$$

with probability 1. Next, if z is assigned a random value according to $\mathsf{DLap}(a\epsilon, e)$, then $[\![P_{\epsilon}]\!]$ transitions to

$$(\ell', f_{Bool}, f_{DOM}, f_{int}[z \mapsto (a, f_{DOM}(e))], f_{real}, C)$$

with probability 1. Notice that we have a deterministic transition even if the assignment samples from a discrete Laplace. The effect of choosing randomly a value will get accounted for during Boolean assignments.

Real assignments. Let $next(\ell) = \{(\ell', true)\}$ and let r be the variable being assigned in ℓ . First, if z is assigned a value for a real expression e, $\llbracket P_{\epsilon} \rrbracket$ will transition to

$$(\ell', f_{Bool}, f_{DOM}, f_{int}, f_{real}[r \mapsto f_{DOM}(e)], C)$$

with probability 1. If z is assigned a random value according to $Lap(a\epsilon, e)$, then $\llbracket P_{\epsilon} \rrbracket$ transitions to

$$(\ell', f_{Bool}, f_{DOM}, f_{int}, f_{real}[\mathbf{r} \mapsto (a, f_{DOM}(e))], C)$$

with probability 1. Again sampling according to Laplace is modeled deterministically.

Boolean assignments. Again let $next(\ell) = \{(\ell', true)\}$ and let b be the variable being assigned in ℓ . When b is assigned the value of Boolean expression $e, \llbracket P_{\epsilon} \rrbracket$ transitions to

$$(\ell', f_{Bool}[\mathsf{b} \mapsto f_{Bool}(e)], f_{\mathsf{DOM}}, f_{\mathsf{int}}, f_{\mathsf{real}}, C)$$

with probability 1. The interesting case is when **b** is assigned the result of comparing expressions $e_1 \sim e_2$. If the probability of all conditions in *C* holding is 0, then let p_1 be 0. Otherwise, let p_1 denote the probability of $f_{\text{DOM}}(e_1) \sim f_{\text{DOM}}(e_2)$ holding given all conditions in *C* hold; notice that this probability depends on the functions f_{int} and f_{real} that store the parameters to various random sampling steps. Now $[\![P_{\epsilon}]\!]$ will transition to

 $(\ell', f_{Bool}[\mathsf{b} \mapsto \mathsf{true}], f_{\mathsf{DOM}}, f_{\mathsf{int}}, f_{\mathsf{real}}, C \cup \{f_{\mathsf{DOM}}(e_1) \sim f_{\mathsf{DOM}}(e_2)\})$

with probability p_1 , and it will transition to

$$(\ell', f_{Bool}[\mathsf{b} \mapsto \mathsf{false}], f_{\mathsf{DOM}}, f_{\mathsf{int}}, f_{\mathsf{real}}, C \cup \{\neg(f_{\mathsf{DOM}}(e_1) \sim f_{\mathsf{DOM}}(e_2))\})$$

with probability $1 - p_1$. Thus, the effect of the probabilistic sampling steps for integer and real variables gets accounted for when the result of a comparison is assigned to a Boolean variable.

if statement. In this case, $next(\ell) = \{(\ell_1, c), (\ell_2, \neg c)\}$. If $f_{Bool}(c) = true$ then we transition to

$$(\ell_1, f_{Bool}, f_{\mathsf{DOM}}, f_{\mathsf{int}}, f_{\mathsf{real}}, C)$$

with probability 1. On the other hand, if $f_{Bool}(c) = false$ then transition to

$$(\ell_2, f_{Bool}, f_{\mathsf{DOM}}, f_{\mathsf{int}}, f_{\mathsf{real}}, C)$$

with probability 1.

While statement. Again let $next(\ell) = \{(\ell_1, c), (\ell_2, \neg c)\}$. This case is identical to the case of if statement, and so is skipped.

exit statement. In this case we stay in state z with probability 1.

Equivalence of the two semantics. Let in be a valuation over input variables and **out** be a valuation over output variables. We let $\operatorname{Prob}_{DTMC}(P_{\epsilon}(\mathbf{in}) =$ **out**) denote the probability that P_{ϵ} outputs value **out**, on the input **in**, under the DTMC semantics. This probability is defined to be the probability of reaching a state of the form $(\ell, f_{Bool}, f_{\text{DOM}}, f_{\text{int}}, f_{\text{real}}, C)$ where ℓ is the label of an exit statement and f_{DOM} assigns the values given by **out** to output variables, from an initial state in which the values of the input variables is given by **in**, in the DTMC $[\![P_{\epsilon}]\!]$. The following theorem states the equivalence of the natural semantics given in Appendix A to that of the DTMC semantics for DiPWhile programs. **Theorem 14.** For every $\epsilon > 0$ and DiPWhile program P_{ϵ} , and for every pair of evaluations **in**, **out** to the input and output variables respectively, $\mathsf{Prob}_{DTMC}(P_{\epsilon}(\mathbf{in}) = \mathbf{out}) = \mathsf{Prob}_{natural}(P_{\epsilon}(\mathbf{in}) = \mathbf{out}).$

Proof Sketch. Let us fix an $\epsilon > 0$ and a program P_{ϵ} . Then $\llbracket P_{\epsilon} \rrbracket$ can be considered as a (non-paramaterized) DTMC. For any path $\rho = z_1, \ldots, z_i$ in the DTMC $\llbracket P_{\epsilon} \rrbracket$, let $\operatorname{prob}(\rho)$ denote the product of the probabilities of all the transitions in ρ . We call ρ an *initialized* path if it starts with an initial state, and a proper path if $\operatorname{prob}(\rho) > 0$. For any *initialized path* ρ of $\llbracket P_{\epsilon} \rrbracket$, let $\operatorname{prob}_{\mathsf{DOM}}(\rho)$ be the product of all the transitions in ρ that result from random assignments to DOM variables, $\operatorname{prob}_{\mathbb{Z}}(\rho)$ be the product of the probabilities that result from a comparison between integer variables and $\operatorname{prob}_{\mathbb{R}}(\rho)$ be the product of the probabilities that result from a comparison between real variables. It is easy to see that

 $\operatorname{prob}(\rho) = \operatorname{prob}_{\operatorname{DOM}}(\rho) \operatorname{prob}_{\mathbb{R}}(\rho) \operatorname{prob}_{\mathbb{Z}}(\rho).$

We recall some of the notation as defined in Appendix A. Let S be the set of states of P_{ϵ} in the natural semantics. A state $s \in S$ is a tuple $(\ell, h_{Bool}, h_{DOM}, h_{\mathbb{Z}}, h_{\mathbb{R}})$ denoting the label of a statement to be executed, and the values of Boolean, DOM, integer and real variables of P_{ϵ} . A discrete state of P_{ϵ} , ds, is a tuple $(\ell, h_{Bool}, h_{DOM}, h_{\mathbb{Z}})$ specifying the label of the statement and the values of Boolean, DOM and integer variables of P_{ϵ} . For a state $s = (\ell, h_{Bool}, h_{DOM}, h_{\mathbb{Z}}, h_{\mathbb{R}})$, let disc(s) be the discrete state $(\ell, h_{Bool}, h_{DOM}, h_{\mathbb{Z}})$. A discrete state ds is said to be *initial* if $ds = disc(s_{init})$ where s_{init} is the initial state of S.

A discrete execution $\beta = ds_1, \ldots, ds_i$ of P_{ϵ} is a sequence of discrete states. The discrete execution $\beta = ds_1, \ldots, ds_i$ is an *initialized* if ds_1 is the initial discrete state. For a discrete execution β as given above, let $ext(\beta) = \{(s_1, \ldots, s_i) \mid ds_j = disc(s_j), 1 \leq j \leq i\}$. It is easy to see that, for any discrete execution β of length i, $ext(\beta)$ is in Σ_i (see Appendix A) , i.e., is measurable. For a discrete computation β , of length i > 0, let $pr(\beta) = \phi_i(ext(\beta))$ where ϕ_i is the probability function defined on the measure space (S^i, Σ_i) in Appendix A. If $pr(\beta) > 0$ then we call β a proper discrete execution of P_{ϵ} .

Consider an initialized proper discrete execution β of length *i*, as given above, where $ds_j = (\ell_j, h_{\text{DOM}}^j, h_{Bool}^j, h_{\mathbb{Z}}^j)$ for $1 \leq j \leq i$. It can be shown that there exists a unique initialized path $\rho_{\beta} = z_0, \ldots, z_i$ in the DTMC $\llbracket P_{\epsilon} \rrbracket$ corresponding to β such that for each *j*,

1. the state $z_j = (\ell_j, f_{\text{DOM}}^j, f_{Bool}^j, f_{\text{int}}^j, f_{\text{real}}^j, C_j)$ for some appropriate $f_{\text{DOM}}^j, f_{\text{int}}^j, f_{\text{real}}^j$ and C_j , and

2.
$$f_{Bool}^{j}(\mathbf{b}) = h_{Bool}^{j}(\mathbf{b}) (f_{DOM}^{j}(\mathbf{x}) = h_{DOM}^{j}(\mathbf{x}) \text{ respectively}) \text{ whenever } f_{Bool}^{j}(\mathbf{b}) (f_{DOM}^{j}(\mathbf{x}) \text{ respectively}) \text{ is defined.}$$

Let H be the function mapping initialized proper discrete executions of P_{ϵ} to corresponding initialized paths in $[\![P_{\epsilon}]\!]$, as specified above.

For an initialized proper discrete execution β of length i as above, we define a number p_j for each $j \leq i$ as follows. For $ds_j = (\ell_j, h_{\text{DOM}}^j, h_{Bool}^j, h_{\mathbb{Z}}^j)$, let $\widehat{ds_j} = (\ell_j, h_{\text{DOM}}^j, h_{Bool}^j, h_{\mathbb{Z}}^j, h_{\mathbb{R}}^{\text{in}})$ where $h_{\mathbb{R}}^{\text{in}}$ is the function that maps each real variable of P_{ϵ} to 0. Let K_{ϵ} be the Markov kernel as defined in Appendix A. If j > 1 and ℓ_{j-1} is the label of an assignment to an integer variable that samples from a discrete Laplacian variable then $p_j = K_{\epsilon}(\widehat{ds_{j-1}}, \{\widehat{ds_j}\})$, otherwise $p_j = 1$. Let $\operatorname{pr}_{\mathbb{Z}}(\beta) = p_1 \dots p_i$. It can be shown using the definition of measure ϕ_i on Σ_i (See Section A) that

$$\operatorname{pr}(\beta) = \operatorname{prob}_{\operatorname{DOM}}(H(\beta)) \operatorname{prob}_{\mathbb{R}}(H(\beta)) \operatorname{pr}_{\mathbb{Z}}(\beta).$$

Furthermore, $prob(H(\beta)) > 0$ if $pr(\beta) > 0$.

Now consider any initialized proper path $\rho = z_1, \ldots, z_i$ in $[\![P_{\epsilon}]\!]$. From the above observations, it can be shown that $\operatorname{prob}(\rho) = \sum_{u \in H^{-1}(\rho)} \phi_i(\operatorname{ext}(\beta))$. Now, the theorem follows from this observation and the definitions of $\operatorname{Prob}_{DTMC}(P_{\epsilon}(\operatorname{in}) = \operatorname{out})$ and $\operatorname{Prob}_{natural}(P_{\epsilon}(\operatorname{in}) = \operatorname{out})$.

Complexity. Now, we bound the size of the state space of DTMC $[P_{\epsilon}]$ as follows. Let $m = |\mathsf{DOM}| = 2N_{\mathsf{max}} + 1$ and m' be the length of P_{ϵ} . Let n_1, n_2, n_3, n_4 , respectively, be the number DOM variables, boolean variables, integer variables, and real variables occurring in P_{ϵ} . In a state s of M_P , the number of possible values for f_{dom} is $\leq m^{n_1}$, the number of possible values for f_{bool} is $\leq 2^{n_2}$. The number of possible values for f_{int} can be bounded as follows. An integer variable can be assigned a Laplacian distribution whose parameters are pairs of the form $(a\epsilon, e)$ where e is an expression over variables in $U \cap \mathcal{X}$; the number of such pairs is $\leq m_1 m^{n_1}$ where m_1 is the number of values of a in P and m^{n_1} is the bound on the number of values of e. An integer variable can also be assigned a linear expression over integer variables with coefficients that are integer constants or expressions over DOM variables; the number of such linear combinations is $< m_2 m^{n_1}$ where m_2 is the number of such expressions appearing in P. Since, $m_1 + m_2 \leq m'$, we see that number of values that an integer variable can be mapped to is $\leq m'm^{n_1}$. Hence the number of possible values for f_{int} is $\leq (m'm^{n_1})^{n_3}$. By a similar reasoning we observe that the number of possible values for f_{real} is $\leq (m'm^{n_1})^{n_4}$. Now we bound the number of values for C as follows. The only

places where comparisons appear are on the right hand sides of assignments to boolean variables. In each such assignment we have comparisons over linear expressions of integer and real variables ; such comparisons also have integer constants and DOM variables appearing in them. Since the number of integer constants is $\leq m'$ and the number of valuations to DOM variables $\leq m^{n_1}$, we get that the number of possible comparisons is $\leq m'm^{n_1}$. Since C is a subset of such comparisons, the number of possible values for C is $\leq 2^{(m'm^{n_1})}$. Now, the number of states is bounded by the product of possible values to each component of a state, which is seen to be $O(2^n \cdot n^{n_1+n_2+n_3+n_4})$ where $n = m'm^{n_1}$.

E DiPWhile programs are finite, definable, parametrized DTMCs

We show the proof of Theorem 8, namely that for any DiPWhile program P_{ϵ} , $[P_{\epsilon}]$ is a finite, definable, parametrized DTMC.

Proof. From our definition of the DTMC semantics (Appendix D), it follows that $\llbracket P_{\epsilon} \rrbracket$ is a finite parameterized DTMC. We now show that it is definable also. In order to show this, we have to show that the transition probabilities of $\llbracket P_{\epsilon} \rrbracket$ are definable. Observe that, by definition, the transition probabilities of **choose** $(a\epsilon, \tilde{E})$ construct are definable. The other probabilistic transitions arise as a result of comparison between random variables of the same sort or from using the exponential mechanism. These transition probabilities turn out to be from a special class of definable functions. We define this form next.

Definition 15. Let $p(\epsilon) = \sum_{i=1}^{m} a_i \epsilon^{n_i} e^{\epsilon q_i}$ where each a_i is a rational number, n_i is a natural number and q_i is a non-negative rational number. We shall call all such expressions *pseudo-polynomials* in ϵ . Given a real number b > 0 and a pseudo-polynomial $p(\epsilon)$, p(b) is the real number obtained by substituting b for ϵ . The ratio of two pseudo-polynomials in ϵ , $\frac{p_1(\epsilon)}{p_2(\epsilon)}$, shall be called a *pseudo-rational function* in ϵ if $p_2(b) \neq 0$ for all real b > 0. Given a real number b > 0 and a pseudo-rational function $rt(\epsilon) = \frac{p_1(\epsilon)}{p_2(\epsilon)}$, rt(b) is defined to be $\frac{p_1(b)}{p_2(b)}$.

Observe that a pseudo-rational function rt defines a function f_{rt} from the set of strictly positive reals to the set of reals. We will henceforth confuse f_{rt} with rt. Pseudo-rational functions are easily seen to be closed under addition and multiplication.

Corollary 16. Each pseudo-rational function rt is definable in the theory Th_{exp} .

Proof. Let $rt(\epsilon) = \frac{\sum_{i=1}^{m} a_i \epsilon^{n_i} e^{\epsilon q_i}}{\sum_{i=1}^{m'} a_i' \epsilon^{n'_i} e^{\epsilon q'_i}}$. Let N be the least common multiple of all denominators of q_i, q'_i . Let $p_i = q_i N$ and $p'_i = q'_i N$. Let a be the least common multiple of all denominators of a_i, a'_i . Let $b_i = aa_i$ and $b'_i = aa'_i$. It is easy to see that rt is definable by the formula $\phi(x)$:

$$\phi(x) \equiv \forall z. ((x\sum_{i=1}^{m'} b'_i \epsilon^{n'_i} z^{p'_i} = \sum_{i=1}^m b_i \epsilon^{n_i} z^{p_i}) \land (z^N = e^{\epsilon}) \land (z > 0)).$$

Note that in the above formula, z is the Nth root of ϵ .

Now, it follows from our restriction on our scoring functions, namely that they take values in rationals, that the transition probabilities in exponential mechanism are pseudo-rational functions that can be computed.

Let us now consider the case of comparison between random variables. Let $state = (\ell, f_{Bool}, f_{DOM}, f_{int}, f_{real}, C)$ of $\llbracket P_{\epsilon} \rrbracket$ be a state of $\llbracket P_{\epsilon} \rrbracket$. Recall that when we compare random variables in *state*, we add a new linear comparison e to C. Further, in order to compute transition probabilities, we compute the conditional probability that the set of linear comparison $C \cup e$ is true given that C is true. For this, it suffices to show that we can compute the probability that the set of linear comparisons C is true and the probability $C \cup e$ is true. We make the following observations:

- Since every random variable must be defined before it is used, we can simplify C and $C \cup e$ to only refer to program variables that were used in random assignments.
- All our random assignments sample from independent random variables. Since we never compare integer and real random variables, it suffices to compute the probability that a system of linear comparisons over integers with integer coefficients hold and the probability that a system of linear comparisons over reals with rational coefficients hold. We will now show that these probabilities can be computed and are pseudo-rational functions.
- In order to compute the probability that a system of linear comparisons over reals with rational coefficients hold, we only need to consider systems of linear inequalities. Clearly any equality $u_1 = u_2$ can be written as two inequalities, $u_1 \leq u_2$ and $u_2 \leq u_1$. If a comparison

in C is $u_1 \neq u_2$ then we can consider the systems $C_1 = (C \setminus \{u_1 \neq u_2\}) \cup \{u_1 < u_2\}$ and $C_2 = (C \setminus \{u_1 \neq u_2\}) \cup \{u_2 < u_1\}$, compute probabilities of C_1 and C_2 separately and add them up to compute the probability that C holds. Thus, without loss of generality we can assume that C consists of only linear inequalities.

Probability of system of linear inequalities over integers. Let $\overline{Z} = (Z_1, \ldots, Z_n)$ be a discrete random variable taking values in \mathbb{Z}^n . Consider a finite system of linear inequalities C with integer coefficients and with n unknowns Z_1, \ldots, Z_n . A solution of C is a tuple $\overline{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ such that all inequalities in C are satisfied when each $Z_j \in C$ is replaced by b_j . Let $sol(C) \subseteq \mathbb{Z}^n$ denote the set of all solutions of C. The probability that \overline{Z} satisfies C is said to be the probability of the event $E = \{\overline{Z} = \overline{b} \mid \overline{b} \text{ is a solution of } C\}$. We denote this probability by $\mathsf{Prob}(\overline{Z} \models C)$. We have the following:

Lemma 17. Let *C* be a finite system of linear inequalities with integer coefficients and with *n* unknowns Z_1, \ldots, Z_n . Let $Z_j = \mathsf{DLap}(a_j\epsilon, \mu_1), \ldots, Z_n = \mathsf{DLap}(a_n\epsilon, \mu_n)$ be mutually independent discrete Laplacians such that for each $1 \leq j \leq n, a_j$ is a strictly positive rational number and μ_j is an integer. Let $\overline{Z} = (Z_1, \ldots, Z_n)$. There is a pseudo-rational function $rt_{\overline{Z},C}$ in ϵ such that $\mathsf{Prob}(\overline{Z} \models C) = rt_{\overline{Z},C}$. The function $rt_{\overline{Z},C}$ can be computed from $C, (a_1, \mu_1), \ldots, (a_n, \mu_n)$.

Proof. For, each $1 \leq j \leq n$, consider $Y_j = \mathsf{DLap}(a_j\epsilon, 0)$. It is easy to see that Z_j has the same distribution as $Y_j + \mu_j$. Now consider the system of inequalities C' in which each Z_j is replaced by $Y_j + \mu_j$. Let $\overline{Y} = (Y_1, \ldots, Y_n)$. It is easy to see that $\mathsf{Prob}(\overline{Z} \models C) = \mathsf{Prob}(\overline{Y} \models C')$. This observation implies that it suffices to prove the Lemma in the special case that each $\mu_j = 0$. Thus, for the rest of the proof we assume that each $\mu_j = 0$.

Now, consider a set $\mathsf{pos} \subseteq \{1, \ldots, n\}$. Let C_{pos} be the system of inequalities $C \cup \{Z_j \ge 0 \mid j \in \mathsf{pos}\} \cup \{Z_j < 0 \mid j \notin \mathsf{pos}\}$. It is easy to see that the set of solutions of C is the *disjoint* union $\cup_{\{\mathsf{pos}\subseteq 1,\ldots,n\}} C_{\mathsf{pos}}$. Thus, it suffices to the prove that for each $\mathsf{pos} \subseteq \{1,\ldots,n\}$, $\mathsf{Prob}(\overline{Z} \models C_{\mathsf{pos}})$ is a pseudo-rational function that can be computed.

Consider the system of inequalities C'_{pos} obtained from C_{pos} by replacing each Z_j by Y_j for $j \in \text{pos}$ and by $-Y_j$ for $j \notin \text{pos}$. Let $Y = (Y_1, \ldots, Y_n)$. From the fact that Laplacians are symmetric distributions, it follows each Y_j has the same distribution as Z_j . Thus, $\text{Prob}(\overline{Z} \models C_{\text{pos}}) = \text{Prob}(\overline{Y} \models C'_{\text{pos}})$. Observe that the set of solutions of C'_{pos} are a subset of \mathbb{N}^n . Without loss of generality, we can also assume that the terms in each inequality of C'_{pos} are rearranged so that the constant terms in C'_{pos} and the coefficients of the variables Y_j are natural numbers, ie, non-negative integers.

Therefore, C'_{pos} is a system of linear inequalities with natural number coefficients. We are interested in solutions of C'_{pos} over natural numbers. For such system of inequalities, the set of solutions can be written as a *disjoint* union of *simple linear sets* [12]; a set $S \subseteq \mathbb{N}^n$ is said to be *linear* if there are tuples $\overline{b}_0, \overline{p}_1, \ldots, \overline{p}_m \in \mathbb{N}^n$ such that $S = \{\overline{b}_0 + \sum_{i=1}^m k_i \overline{p}_i \mid$ for each i, $k_i \in \mathbb{N}\}$ and *simple* if each $\overline{b} \in S$ has a unique representation as a sum $\overline{b}_0 + \sum_{i=1}^m k_i \overline{p}_i$. \overline{b}_0 is said to be the offset of S and $\overline{p}_1, \ldots, \overline{p}_m$ the periods of S. From the fact that the set of solutions of C'_{pos} can be written as a *disjoint* union of *simple linear sets*, it follows that it suffices to show that $\text{Prob}(\overline{Y} \in S \mid S \text{ is simple linear})$ is a pseudo-rational function in ϵ . In order to show this we need a couple of additional notations.

For two *n*-tuples $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$, $\overline{x} \cdot \overline{y}$ will denote the sum $\sum_{j=1}^n x_j y_j$. Secondly, we will denote the tuple (a_1, \ldots, a_n) by \overline{a} .

Fix a simple semilinear set S. Let \overline{b}_0 be its offset and $\overline{p}_1, \ldots, \overline{p}_m$ its periods. Let $\kappa = \prod_{i=1}^n \frac{1-e^{-a_i\epsilon}}{1+e^{-a_i\epsilon}}$. From the fact that each $\overline{b} \in S$ has a unique representation as a sum $\overline{b}_0 + \sum_{i=1}^m k_i \overline{p}_i$, it follows that

The latter is clearly a pseudo-rational function in ϵ .

Probability of system of linear inequalities over reals. Let $\overline{R} = (R_1, \ldots, R_n)$ be a continuous random variable taking values in \mathbb{R}^n . Consider a finite system of linear inequalities C with rational coefficients and with n unknowns R_1, \ldots, R_n . As in the case of discrete random variables, we can define $sol(C) \subseteq \mathbb{R}^n$, the set of solutions, and $\operatorname{Prob}(\overline{R} \models C)$, the probability that \overline{R} satisfies C. We have the following result.

Lemma 18. Let *C* be a finite system of linear inequalities with rational coefficients and with *n* unknowns R_1, \ldots, R_n . Let $R_1 = \text{Lap}(a_1\epsilon, \mu_1), \ldots, R_n = \text{Lap}(a_n\epsilon, \mu_n)$ be mutually independent Laplacian doistributions such that for each $1 \leq j \leq n$, a_j is a strictly positive rational number and μ_j is a rational number. Let $\overline{R} = (R_1, \ldots, R_n)$. There is a pseudo-rational

function $rt_{\overline{R},C}$ in ϵ such that $\operatorname{Prob}(\overline{R} \models C) = rt_{\overline{R},C}$. The function $rt_{\overline{R},C}$ can be computed from $C, (a_1, \mu_1), \ldots, (a_n, \mu_n)$.

Proof. As in the proof of Lemma 17, it suffices to consider the case when each $\mu_i = 0$ and to show that the probability measure of the set $Sol = sol(C) \cap \{(b_1, \ldots, b_n) \mid b_i \in \mathbb{R}^{>0}\}$ is a computable pseudo-rational function.

Since \overline{R} is continuous, we can also assume that each inequality is of the form \leq . This is because the measure of any set in \mathbb{R}^n that satisfies a linear equation over n unknowns R_1, \ldots, R_n is 0. There are computable finite sets S_1, \ldots, S_m such that (See [8])

- 1. $Sol = S_1 \cup \ldots S_m$,
- 2. The measure of the $S_i \cap S_j$ is 0 for $i \neq j$, and
- 3. Each S_i is a positive repetitive polyhedra. $S \subseteq (\mathbb{R}^{>0})^n$ is said to be a positive repetitive polyhedra if there are constants h_0^-, h_0^+ and functions $h_1^-(x_1), h_1^+(x_1), h_2^-(x_1, x_2), h_2^+(x_1, x_2), \ldots, h_{n-1}^-(x_1, x_2, \ldots, x_{n-1}),$ $h_{n-1}^+(x_1, x_2, \ldots, x_{n-1})$ such that

•
$$S_i = {\{(x_1, \dots, x_n) \mid h_0^- \le x_1 \le h_0^+, \dots, h_{n-1}^-(x_1, \dots, x_{n-1}) \le x_n \le h_{n-1}^+(x_1, \dots, x_{n-1})\}}.$$

- h_0^- is a rational number ≥ 0 .
- h_0^+ is either ∞ or a rational number.
- For each 1 ≤ j ≤ n, h_j⁻ is a linear function in its arguments. In the latter case, h_j⁻ has rational coefficients.
- For each 1 ≤ j ≤ n, h_j⁺ is either ∞ or a linear function in its arguments. h_i⁺ has rational coefficients in the latter case.
- For each $1 \le j \le n$, $h_j^- \ne h_j^+$.

Thanks to conditions (1) and (2) above, it suffices to show that for any positive repetitive polyhedra S, the probability measure of the event $\{\overline{R} = \overline{b} \mid \overline{b} \in S\}$ is a pseudo-rational function.

Fix S and let $h_0^-, h_0^+, h_1^-, h_1^+, \dots, h_{n-1}^-, h_{n-1}^+$ be as above. The measure of the event $\{\overline{R} = \overline{b} \mid \overline{b} \in S\}$ can be computed using the nested integral

$$F = \int_{h_0^-}^{h_0^+} f_{a_1}(x_1) \int_{h_1^-}^{h_1^+} f_{a_2}(x_2) \cdots \int_{h_{n-1}^-}^{h_{n-1}^+} f_{a_n}(x_n) \, dx_n \cdots dx_1$$

where $f_{a_i}(x_i) = \frac{a_i \epsilon}{2} e^{-a_i \epsilon x_i}$ is the pdf of R_i (we always have that $x_i \ge 0$) and the arguments of h_i^+, h_i^- are omitted for readability.

For $1 \leq j \leq n$, let I_j be the nested integral

$$I_j = \int_{h_{j-1}^-}^{h_{j-1}^+} f_{a_j}(x_j) \cdots \int_{h_{n-1}^-}^{h_{n-1}^+} f_{a_n}(x_n) \, dx_n \cdots dx_j.$$

We claim by induction on k = n - j that I_j is a finite sum of terms of the form

$$a\epsilon^m e^{b\epsilon}(x_1^{m_1}e^{\epsilon b_1x_1})\dots(x_{j-1}^{m_{j-1}}e^{\epsilon b_{j-1}x_{j-1}})$$

where $a, b, b_1, \ldots, b_{j-i}$ are rational numbers (including negative numbers), m is an integer, and m_1, \ldots, m_{j-1} are natural numbers. We will assume that the sum is always presented in simplest form, namely, that all cancellations have already taken place in the sum.

Clearly the claim is true when k = 0. Suppose that the claim is true for $k = k_0$. Let $j_0 = n - k_0$. Suppose

$$w = a\epsilon^m e^{b\epsilon} (x_1^{m_1} e^{\epsilon b_1 x_1}) \dots (x_{j_0-1}^{m_{j_0-1}} e^{\epsilon b_{j_0-1} x_{j_0-1}})$$

is a summand in I_{j_0} . Let $k = k_0 + 1$ and $j = n - k = n - k_0 - 1 = j_0 - 1$. Consider the indefinite integral

$$J = \int f_{a_{j_0-1}} w \, dx_{j_0-1} = \int \frac{a_{j_0-1}\epsilon}{2} e^{-a_{j_0-1}\epsilon x_{j_0-1}} w \, dx_{j_0-1} = \frac{a_{j_0-1}}{2} \epsilon^{m+1} e^{b\epsilon} (x_1^{m_1} e^{\epsilon b_1 x_1}) \dots (x_{j_0-2}^{m_{j_0-2}} e^{\epsilon b_{j_0-2} x_{j_0-2}}) \int x_{j_0-1}^{m_{j_0-1}} e^{\epsilon (b_{j_0-1}-a_{j_0-1}) x_{j_0-1}} dx_{j_0-1}$$

Let

$$J' = \int x_{j_0-1}^{m_{j_0-1}} e^{\epsilon(b_{j_0-1}-a_{j_0-1})x_{j_0-1}} dx_{j_0-1}$$

Now, if $b_{j_0-1} - a_{j_0-1} = 0$ then

$$J' = \frac{x_{j_0-1}^{m_{j_0-1}+1}}{m_{j_0-1+1}}.$$

If $b_{j_0-1} - a_{j_0-1} \neq 0$ then by doing a change of variables $t = (b_{j_0-1} - a_{j_0-1})\epsilon x_{j_0-1}$, it is not too hard to show that

$$J' = \sum_{k=0}^{m_{j_0-1}} c_k \epsilon^{t_k} x_{j_0-1}^k e^{\epsilon(b_{j_0-1}-a_{j_0-1})x_{j_0-1}}$$

where c_k is a rational number and t_k an integer for each k.

Thus, the indefinite integeral J is a sum, each of whose terms is of the form

$$a'\epsilon^{m'}e^{b'\epsilon}(x_1^{m'_1}e^{\epsilon b'_1x_1})\dots(x_{j_0-1}^{m'_{j_0-1}}e^{\epsilon b'_{j_0-1}x_{j_0-1}}).$$

If $h_{j_0-2}^-$ and $h_{j_0-2}^+$ are linear functions, we get immediately that $I_j = \int_{h_{j_0-2}}^{h_{j_0-2}^+} f_{a_{j_0-1}} w \, dx_{j_0-1}$ is of the right form. The induction step follows in this case.

If $h_{j_0-2}^+ = \infty$, and each b'_j in a summand of J is strictly negative, then it is also easy to see that the induction step follows. Apriori, it seems that there might be a problem when $b'_j \ge 0$ as in this case, I_j will evaluate to either ∞ or $-\infty$. This, however, will contradict the fact that the nested integral F defines probability of an event (and hence is bounded above by 1). Thus, if $h_{j_0-2}^+ = \infty$ then b_j must be strictly negative.

The claim immediately implies that the measure of the set $Sol = sol(C) \cap \{(b_1, \ldots, b_n) \mid b_i \in \mathbb{R}^{>0}\}$ is a pseudo-rational function. \Box

F Reachability in Parametrized DTMCs

In this section we will prove Lemma 10. Let us first recall how reachability probabilities are computed in (non-parametrized) finite-state DTMCs. Recall that a (non-parametrized) DTMC is a pair (Q, δ) where Q is a finite set of states, and $\delta : Q \times Q \rightarrow [0,1]$ is such that for every $q \in Q$, $\sum_{q' \in Q} \delta(q, q') = 1$. So in a DTMC the transition probabilities are fixed, and are not functions of a parameter. The probability of reaching a set of states $Q' \subseteq Q$ from a state q_0 is computed by solving a more general problem, namely, the problem of computing the probability of reaching Q' from each state $q \in Q$. Let the variable x_q denote the probability of reaching Q' from state q. One simple observation is that if $q \in Q'$ then $x_q = 1$. Second, if Q_0 denotes the set of all states from which Q' is not reachable in the underlying graph (i.e., one where we ignore the probabilities and just have edges for all transitions that are non-zero), then $x_q = 0$ if $q \in Q_0$. Now the set Q_0 can be computed by performing a simple graph search on the underlying graph. For states $q \notin (Q' \cup Q_0)$, we could write x_q as $x_q = \sum_{q' \in Q} \delta(q, q') x_{q'}$. This gives us the following system of linear equations.

$$\begin{array}{ll} x_q = 1 & \text{if } q \in Q' \\ x_q = 0 & \text{if } q \in Q_0 \\ x_q = \sum_{q' \in Q} \delta(q,q') x_{q'} & \text{otherwise} \end{array}$$

The above system of linear equations can be shown to have a unique solution, with the solution giving the probability of reaching Q' from each state q.

Now let us consider a parametrized DTMC $\mathcal{D} = (Z, \Delta)$. Let $\varphi_{zz'}$ be a \mathcal{L}_{exp} formula that defines the function $\Delta(z, z')$. Recall that in the algorithm outlined in the previous paragraph, one crucial step is to compute the set of states that have probability 0 of reaching the target set. This requires knowing the underlying graph of the DTMC, i.e., knowing which transitions have probability 0 and which ones have probability > 0. In a parametrized DTMC this is challenging because the probability of transitions depends on the value of ϵ , and our goal is to compute the reachability probability as a function of ϵ . We will overcome this challenge by "guessing" the underlying graph.

Let $C \subseteq Z \times Z$. We will construct a formula φ_C that will capture the constraints that reachablity probabilities need to satisfy under the assumption that the probability of edges in C is 0, and those outside C is > 0. Based on the assumption that C is exactly the set of 0 probability edges, we can compute the set Z_0^C of states that cannot reach Z'. The formula φ_C will have variables that will have the following intuitive interpretations — $p_{zz'}$ the probability of transitioning from z to z'; x_z the probability of reaching Z' from state z.

$$\varphi_C = \bigwedge_{(z,z')\in C} (p_{zz'}=0) \land \bigwedge_{(z,z')\notin C} (p_{zz'}>0) \land \bigwedge_{z\in Z'} (x_z=1) \land \bigwedge_{z\in Z_0^C} (x_z=0) \land \bigwedge_{z\notin (Z'\cup Z_0^C)} (x_z=\sum_{z'} p_{zz'}x_{z'}).$$

Notice that φ_C is a formula in \mathcal{L}_{exp} . φ_C can be used to construct the formula we want. To construct the formula $\varphi_{z_0Z'}$ that characterizes the probability of reaching Z' from z_0 , we need to account for two things. First, we need to ensure that $p_{zz'}$ is indeed the probability of transitioning from z to z'. Second, we need to account for the fact that we don't know the exact set of edges with probability 0. Based on these observations, we can define $\varphi_{z_0,Z'}$ as follows.

$$\varphi_{z_0 Z'} = [\exists x_z]_{z \neq z_0} [\exists p_{zz'}]_{z, z' \in Z} \bigwedge_{z, z' \in Z} \varphi_{zz'}(\epsilon, p_{zz'}) \land \left(\bigvee_{C \subseteq Z \times Z} \varphi_C\right)$$

In the above definition of $\varphi_{z_0Z'}$ all variables except x_{z_0} (and ϵ) are existentially quantified. Notice, that $\varphi_{z_0Z'}$ is in \mathcal{L}_{exp} provided we pull all the quantifiers to get it in prenex form. Given that Z_0^C can be effectively constructed for any set C, the above formula can also be computed for any parametrized DTMC \mathcal{D} .

Expressions ($b \in \mathcal{B}, x \in \mathcal{X}, d \in \text{DOM}, g \in \mathcal{F}_{Bool}, f \in \mathcal{F}_{DOM}$):

$$B ::= \operatorname{true} |\operatorname{false}| b | \operatorname{not}(B) | B \text{ and } B | B \text{ or } B | g(\tilde{E})$$
$$E ::= d | \mathbf{x} | f(\tilde{E})$$

Basic Program Statements $(a \in \mathbb{Q}^{>0}, \sim \in \{<, >, =, \leq, \geq\}, F$ is a scoring function and choose is a user-defined distribution):

 $\begin{array}{rll} s & ::= & \mathsf{x} \leftarrow E \mid \mathsf{b} \leftarrow B \mid \mathsf{x} \leftarrow \mathsf{Exp}(a\epsilon, F(\tilde{\mathsf{x}}), E) \mid \\ & \mathsf{x} \leftarrow \mathsf{choose}(a\epsilon, \tilde{E}) \mid \mathsf{if} \, B \, \mathsf{then} \, P \, \mathsf{else} \, P \, \mathsf{end} \mid \\ & \mathsf{While} \, B \, \mathsf{do} \, P \, \mathsf{end} \mid \mathsf{exit} \end{array}$

Program Statements ($\ell \in \mathsf{Labels}$)

 $P ::= \ell : s \mid \ell : s; P$

Figure 3: BNF grammar for Finite DiPWhile. DOM is a finite discrete domain. \mathcal{F}_{Bool} , (\mathcal{F}_{DOM} resp) are set of functions that output Boolean values (DOM respectively). \mathcal{B}, \mathcal{X} are the sets of Boolean variables, and DOM variables, respectively. Labels is a set of program labels. For a syntactic class S, \tilde{S} denotes a sequence of elements from S.

G Syntax of Finite DiPWhile programs

The syntax of Finite DiPWhile programs is presented in Figure 3.

G.1 A general semantic class of programs

Our methods imply decidability of checking differential privacy for a large semantic class of programs (which include DiPWhile.) A sufficient condition to ensure the decidability of checking differential privacy is to consider programs with the property that for each input, the probability distribution on the outputs is definable in Th_{exp} :

Definition 19. A parametrized program P_{ϵ} with inputs \mathcal{U} and outputs \mathcal{V} is said to identify a *definable distribution* on \mathcal{V} if for each $\mathbf{in} \in \mathcal{U}$ and $\mathbf{out} \in \mathcal{V}$ the function $\epsilon \mapsto \mathsf{Prob}(P_{\epsilon}(\mathbf{in}) = \mathbf{out})$ is definable in $\mathsf{Th}_{\mathsf{exp}}$.

A parametrized program P_{ϵ} with inputs \mathcal{U} and outputs \mathcal{V} is said to *effectively* identify a definable distribution on \mathcal{V} if there is an algorithm \mathcal{A}

such that for each $\mathbf{in} \in \mathcal{U}$ and $\mathbf{out} \in \mathcal{V}$, \mathcal{A} outputs a formula $\varphi_{\mathbf{in},\mathbf{out}}(\epsilon, x)$ in \mathcal{L}_{exp} that defines the function $\epsilon \mapsto \mathsf{Prob}(P_{\epsilon}(\mathbf{in}) = \mathbf{out})$.

We can conclude by a proof similar to the proof Theorem 11.

Theorem 20. The Fixed Parameter Differential Privacy and Differential Privacy problems are decidable for programs P_{ϵ} that effectively identify a definable distribution, rationals $t \in \mathbb{Q}^{>0}$ and definable functions δ (in the case of the Differential Privacy problem). Furthermore, if P_{ϵ} is not $(t\epsilon, \delta)$ differentially private for some admissible value of ϵ then we can compute a counter-example.

H Detailed Experimental Results

We implemented a simplified version of the algorithm, presented earlier, for proving/disproving differential privacy of DiPWhile programs. Our tool DiPC [3] handles loop-free programs, i.e., acyclic programs. Programs with bounded loops (with constant bounds) can be handled by unrolling loops. The tool takes in an input program P_{ϵ} parametrized by ϵ , and either proves P_{ϵ} to be differentially private for all ϵ or returns a counter-example. The tool can also be used to check differential privacy for a given, fixed ϵ , or to check for $k\epsilon$ -differential privacy for some constant k. The design of the tool will be discussed in detail in Section H.2.

H.1 Examples

We used various examples to measure the effectiveness of our tool. These include SVT [28, 21], Noisy Maximum [18], Noisy Histogram [18] and Randomized Response [20]. Pseudocodes for all variants of these examples that we tried are given in this section for completeness. Though the pseudo-codes don't strictly adhere to the syntax of DiPWhile programs, they can easily be rewritten to fit the syntax.

Sparse Vector Technique (SVT) We looked at six different variants of the Sparse Vector Technique (SVT). Algorithms addressed as SVT1-6, are Algorithms 1-6 in [28], respectively. In these programs, the array q represents the input queries. The array *out* represents the output array, \perp represents False and \top represents True. In all our experiments, we set the threshold T = 0. SVT1 was previously introduced in this paper as Algorithm 1 on page 8. The adjacency relation Φ we used is given by

 $(q_1, q_2) \in \Phi$ if and only if $|q_1[i] - q_2[i]| \leq 1$ for all *i*. While SVT1 and SVT2 are differentially private, the other four variants are not. We will present counter-examples for all four of these variants in Section H.3. The pseudocode for the six variants of SVT are given in Figures 4 and 5.

SVT1) First Instantiation of SVT	(SVT2) Second Instantiation of SVT
----------------------------------	------------------------------------

\mathbf{T}_{1}	Input: $q[1:N]$
Input: $q[1:N]$	Output: $out[1:N]$
Output: $out[1:N]$	$r_{T} \leftarrow lop(-\frac{\epsilon}{T})$
$r_T \leftarrow Lap(\frac{\epsilon}{2\Delta}, T)$	$T_T \leftarrow Lap(\underline{2c\Delta}, T)$
$count \leftarrow 0^{}$	$count \leftarrow 0$
for $i \leftarrow 1$ to N do	for $i \leftarrow 1$ to N do
$r \leftarrow lop(\frac{\epsilon}{2}, a[i])$	$r \leftarrow Lap(\frac{\epsilon}{4c\Delta}, q[i])$
$\mathbf{r} \leftarrow Lap(\underline{q_{c\Delta}}, q[i])$	$b \leftarrow r \ge r_T$
$D \leftarrow r \ge r_T$	if b then
if b then	$out[i] \leftarrow \top$,
$out[i] \leftarrow op$	$r_{T} \leftarrow lap(\frac{\epsilon}{T})$
$count \leftarrow count + 1$	$r_1 \leftarrow Lap(2c\Delta, 1)$
if $count > c$ then	$count \leftarrow count + 1$
l exit	if $count \ge c$ then
end	
	end
	else
$out[i] \leftarrow \bot$	$out[i] \leftarrow \bot$
end	end
end	and
	enu

Figure 4: Sparse Vector Technique Algorithms

Noisy Maximum Noisy maximum algorithms are a differentially private way to compute different statistical measures for a given set of queries. Algorithms addressed as NMax1-4 are Algorithms 5-8, respectively, in [18]. Algorithms NMax1 and NMax2 are mechanisms to compute the index of the query with maximum value after adding a Laplacian (or exponential) noise. Inputs Q_1 and Q_2 are considered adjacent iff $|Q_1[i] - Q_2[i]| \leq 1$ for all *i*. Under this relation, Algorithms NMax1 and NMax2 are both ϵ -differentially private. Algorithms NMax3 and NMax4 are variants to print the maximum value instead of the index. These variants are shown to be not differentially private in Section H.3. The pseudocode for these algorithms can be found in Figure 6. (SVT3) Third Instantiation of SVT



(SVT4) Fourth Instantiation of SVT

```
Input: q[1:N]
Output: out[1:N]
\mathsf{r}_T \leftarrow \mathsf{Lap}(\frac{\epsilon}{4\Delta}, T)
count \leftarrow 0
for i \leftarrow 1 to N do
      \mathsf{r} \leftarrow \mathsf{Lap}(\frac{3\epsilon}{4\Delta}, q[i])
      \mathsf{b} \leftarrow \mathsf{r} \geq \mathsf{r}_T
      if b then
            out[i] \leftarrow \top
             count \leftarrow count + 1
             if count \ge c then
              exit
             end
      else
        out[i] \leftarrow \bot
      end
end
```

(SVT5) Fifth Instantiation of SVT (SVT6) Sixth Instantiation of SVT

Input: $q[1:N]$	Input: $q[1:N]$
Output: $out[1:N]$	Output: $out[1:N]$
$r_T \leftarrow Lap(\frac{\epsilon}{2\Delta}, T)$	$r_T \leftarrow Lap(\frac{\epsilon}{2\Delta}, T)$
for $i \leftarrow 1$ to N do	for $i \leftarrow 1$ to N do
$r \gets q[i]$	$r \leftarrow Lap(\frac{\epsilon}{2\Delta}, q[i])$
$b \gets r \geq r_T$	$b \leftarrow r \ge r_T$
$\mathbf{if} \ b \ \mathbf{then}$	if b then
$out[i] \leftarrow \top$	$out[i] \leftarrow \top$
else	else
$out[i] \leftarrow \bot$	$out[i] \leftarrow \bot$
end	end
end	end

Figure 5: Sparse Vector Technique Algorithms

Histogram Algorithms Histogram algorithms also target computing statistical measures on queries in a differentially private manner. Algorithms referred to as Hist1-2 here are Algorithms 9-10 in [18]. Algorithm Hist1 (NMax1) Correct Noisy Max with Laplacian Noise

Input: q[1:N]**Input:** q[1:N]Output: out Output: out NoisyVector \leftarrow [] NoisyVector \leftarrow [] for $i \leftarrow 1$ to N do for $i \leftarrow 1$ to N do NoisyVector[i] \leftarrow NoisyVector[i] \leftarrow $Lap(\frac{\epsilon}{2}, q[i])$ $Lap^+(\frac{\epsilon}{2}, q[i])$ end end out $\leftarrow \operatorname{argmax}(\operatorname{NoisyVector})$ out $\leftarrow \operatorname{argmax}(\operatorname{NoisyVector})$

(NMax2) Correct Noisy Max with

Exponential Noise

(NMax3) Incorrect Noisy Max with (NMax4) Incorrect Noisy Max with Laplacian Noise Laplacian Noise

Input: $q[1:N]$	Input: $q[1:N]$
Output: out	Output: out
NoisyVector $\leftarrow []$	NoisyVector \leftarrow []
for $i \leftarrow 1$ to N do	for $i \leftarrow 1$ to N do
$NoisyVector[i] \leftarrow$	$NoisyVector[i] \leftarrow$
$Lap(rac{\epsilon}{2},q[i])$	$Lap^+(rac{\epsilon}{2},q[i])$
end	end
$\operatorname{out} \leftarrow$	$\operatorname{out} \leftarrow$
$Disc_{seq}(\max(\operatorname{NoisyVector}))$	$Disc_{seq}(\max(\operatorname{NoisyVector}))$

Figure 6: Noisy Max Algorithms

and Hist2 are variants of noisy maximum, where we return the histogram, instead of the maximum. Under the above adjacency relation where Q_1 and Q_2 are adjacent if $|Q_1[i] - Q_2[i]| \leq 1$ for all *i*, both these variants are not ϵ -differentially private. However, if we consider an alternative definition for the adjacency relation, where Q_1 and Q_2 are adjacent iff $\sum_i \left(|Q_1[i] - Q_2[i]| \right) \leq 1$, then Hist1 is ϵ -differentially private but Hist2 still is not. All experiments listed in Section H.3 for Algorithms NMax1 and NMax2 were run using the second adjacency relation. The pseudocode for these algorithms can be found in Figure 7.



Figure 7: Noisy Histogram Algorithms

Randomized Response All the previous algorithms use the Laplace mechanism. Randomized Response [20], on the other hand, uses discrete probabilities. In this algorithm (henceforth called Rand1), given a set of Boolean input queries, we flip each input query with a probability of $\frac{e^{\epsilon}-1}{2}$ and output the resulting outcome. We also consider a non-private version (called Rand2) where the input query is flipped with probability $\frac{1-\epsilon}{2}$. The pseudocodes can be found in Figure 8.

(Rand1) Differentially Private Randomized Response (Rand2) Non-Differentially Private Randomized Response

 $\begin{array}{c|c} \mathbf{Input:} \ q[1:N] \\ \mathbf{Output:} \ out[1:N] \\ \mathbf{for} \ i \leftarrow 1 \ \mathbf{to} \ N \ \mathbf{do} \\ & \left[\begin{array}{c} \operatorname{out}[i] \leftarrow \\ \left\{ q[i] & \operatorname{with} \operatorname{prob} = \frac{e^{\epsilon}}{1+e^{\epsilon}} \\ \neg q[i] & \operatorname{with} \operatorname{prob} = \frac{1}{1+e^{\epsilon}} \end{array} \right] \\ \mathbf{end} \end{array} \right] \begin{array}{c} \mathbf{for} \ i \leftarrow 1 \ \mathbf{to} \ N \ \mathbf{do} \\ & \left[\begin{array}{c} \operatorname{out}[1:N] \\ \operatorname{out}[1:N] \\ \mathbf{out}[1] \leftarrow \\ \left\{ q[i] & \operatorname{with} \operatorname{prob} = \frac{1}{2} \\ \neg q[i] & \operatorname{with} \operatorname{prob} = \frac{1-\epsilon}{2} \\ \mathbf{end} \end{array} \right] \\ \end{array}$

Figure 8: Randomized Response Algorithms

Sparse Sparse is a variant of SVT that is discussed in [21]. Our reason for considering this example is to demonstrate our tool's ability to handle (ϵ, δ) -differential privacy (see Section H.4). Pseudocode for this algorithm is provided in Section H.4.

H.2 Tool Design

Given a program and an adjacency relation, DiPC outputs true if the program is differentially private and outputs a counter-example if it is not. The tool works in two phases. In the first phase, the tool parses the program, computes symbolic expressions that capture the output distribution, and identify inequalities that must hold for differential privacy. The symbolic expressions for the probability computation, and the logical constraints that must hold, are written in a Wolfram Mathematica®script. In the second phase, Mathematica is run to perform the symbolic computations and check the results.

The computation of the output distribution proceeds in a manner consistent with the decision procedure outlined in the proof of Theorem 11. Recall that the parametrized DTMC semantics, the state tracks constraints that must hold between different real variables. These constraints can be tracked by maintaining a partial order between the variables. One of the engineering challenges we experienced was in the computation of the probability of the partial order holding, given the parameters used during sampling. The "Probability[]" command in Mathematica was very slow and inefficient. Instead we decided to convert the partial order into a set of total orders, and compute the probability of each total order through integration.

For example, to compute the probability of $x_1 < x_2 < x_3 ... < x_n$, where variable x_i has p.d.f D_i , we would first compute the probability $P(x_n > x) = \int_x^{\infty} D_n(y) dy$. We then compute the probabilities $P(x_n > x_{n-1} > x) = \int_x^{\infty} P(x_n > y) D_{n-1}(y) dy$, $P(x_n > x_{n-1} > x_{n-2} > x) = \int_x^{\infty} P(x_n > x_{n-1} > y) D_{n-2}(y) dy$ and so on. Once we have computed $P(x_n > x_{n-1} > ... > x_1 > x)$, we can compute $P(x_n > x_{n-1} > ... > x_1) = Lim_{x \to -\infty} P(x_n > x_{n-1} > ... > x_1 > ... > x_1 > x)$. Additionally, we try to optimize the above process by splitting the partial order into connected components and computed probability for each component. We also deal with constant assignments to real variables by slightly modifying the integration method.

H.3 Experimental Results

We ran all the experiments on an octa-core Intel®Core i7-8550U @ 1.8gHz CPU with 8GB memory. The tool is implemented in C++ and uses Wolfram Mathematica®. As mentioned in Section H.2, the tool works in two phases — in the first phase, a Mathematica script is produced with commands for all the output probability computations and the subsequent inequality checks and in the second phase, the generated script is run on Mathematica. In all the following tables, we refer the times of the Script Generation Phase (i.e. Phase 1) as T1 and that of the Script Validation Phase (i.e. Phase 2) as T2.

Unless stated otherwise, all the experiments were run with the parameters c = 1, $\Delta = 1$ and discretization parameter seq = (-1 < 0 < 1) wherever applicable. The range of input query values was DOM = $\{-1, 0, 1\}$ in all the experiments. The running times in all experiments were averaged over 3 runs of the tool.

Table 3 shows the runtime of our tool for all the listed algorithms with 3 queries. We chose to use 3 queries because counter-examples for most of the programs which were not differentially private could be found with 3 queries; the only exception being SVT3. Majority of the time is taken for running the Mathematica code. We also observed that most of the time spent by Mathematica was in computing the output probability; the time to perform the inequality checks for adjacent inputs was relatively smaller. Consequently, programs which do not use real variables are much faster to run. Results in the table also show that the time taken for disproving Differential Privacy is lower than the time for proving Differential Privacy on average. This is because the tool terminates on finding a counter-example. On the other hand, to prove differential privacy the tool has to check all inequalities.

Table 5 lists the smallest counter-example found for each non differentially private algorithm. Given a program and an adjacency relation, the tool automatically finds an ϵ , the pair of adjacent inputs, and the output value that demonstrate the violation of differential privacy. All four columns in the table were output by the tool. Further, we observe that the counterexamples found were much smaller, in number of queries, compared to those found in [18]. For example, algorithms NMax3 and NMax4 counter-examples need just 3 and 1 queries respectively, compared to the 5 queries required in [18]. Similarly, algorithm SVT5 has a counter-example with just 2 queries, as compared to the 10 queries.

To study the performance of the tool as the number of queries increases,

Algorithm	Runtime $(T1/T2)$	ε-Diff.Pri-vate
SVT1	0s/825s	~
SVT2	0s/768s	✓
SVT3	0s/3816s	✓
SVT4	0s/269s	X
SVT5	0s/2s	X
SVT6	0s/661s	X
NMax1	0s/197s	1
NMax2	0s/59s	1
NMax3	0s/310s	X
NMax4	1s/58s	X
Hist1	0s/1450s	1
Hist2	0s/55s	X
Rand1	0s/0s	X
Rand2	0s/0s	X

Table 3: Runtime for 3 queries for each algorithm searching over adjacency pairs and all $\epsilon_{\dot{c}}0$, with parameters being [c=1, Δ =1, DOM={-1,0,1}, seq = (-1 < 0 < 1)]. For SVT, we also have T=0.

0		<i>c</i>	Runtime $(T1/T2)$	
Ŷ	С	e	Fixed ϵ	General
1	1	1.0	0 s/7 s	0s/16s
1	1	0.5	0s/8s	$0 \mathrm{s} / 16 \mathrm{s}$
2	1	1.0	0s/43s	0s/113s
2	1	0.5	0s/46s	0s/113s
2	2	1.0	0s/95s	0s/155s
2	2	0.5	0s/113s	0s/155s
3	1	1.0	0s/307s	0s/825s
3	1	0.5	0s/265s	0s/825s
3	2	1.0	0s/541s	0s/1202s
3	2	0.5	0s/572s	0s/1202s
4	1	1.0	0s/1772s	0s/4727s
4	1	0.5	0s/1832s	0s/4727s
4	2	1.0	1s/2904s	0s/6715s
4	2	0.5	1s/3295s	0s/6715s

Table 4: Runtimes of SVT1 over different query length and counts, searching over all adjacency pairs and fixed ϵ , with parameters being $[\Delta=1, T=0, \text{DOM}=\{-1,0,1\}].$

Algo	—Q—	Output	Input 1	Input 2	ϵ	Runtime (T1/T2)
SVT3	5	$[\perp \perp \perp \perp 0], seq = (0 < 1)$	[-1 -1 -1 -1 -1]	$[0 \ 0 \ 0 \ 0 \ 0]$	27	18s/5042
SVT4	2	[⊥ ⊤]	[-1 0]	[0 -1]	27/50	0s/81s
SVT5	2	[⊥ ⊤]	[-1 0]	[-1 -1]	27	0s/2s
SVT6	3	$[\perp \perp \top]$	$[-1 \ -1 \ 0]$	$[0 \ 0 \ -1]$	67/92	0s/661s
NMax3	3	-1, seq = $(-1 < 0 < 1)$	[-1 -1 -1]	$[0 \ 0 \ 0]$	27	0s/310s
NMax4	1	0, seq = (-1 < 0 < 1)	[-1]	[0]	27	0s/2s
Hist2	1	[-1], seq = (-1 < 0 < 1)	[-1]	[0]	9/34	0s/3s
Rand2	1	[⊥]	$[\bot]$	[⊤]	9/34	0s/0s

Table 5: Smallest Counter-example found for each non-differentially private algorithm, searching over all adj. pairs and $\epsilon > 0$, with parameters being $[c=1, \Delta=1, DOM=\{-1,0,1\}]$

_Q	с	Runtime (T1/T2)
1	1	0s/16s
2	1	0s/113s
2	2	0s/155s
3	1	0s/825s
3	2	0s/1202s
4	1	0s/4727s
4	2	0s/6715s

	1 Pair	General	
#Oueries	Run-	Run-	ϵ -Diff.
# Queries	time	time	Private
	(T1/T2)	(T1/T2)	
1	0s/15s	0s/25s	1
2	0s/40s	0s/192s	\checkmark
3	0s/100s	0s/1562s	\checkmark
4	0s/199s	1s/10515	s 🗸
5	0s/141s	18s/5042	s 🗶

Table 6: Runtimes of SVT1 over different query length and counts, searching over all adjacency pairs and all $\epsilon_i 0$, with parameters being $[\Delta=1, T=0, \text{DOM}=\{-1,0,1\}]$

Table 7: Runtimes of SVT3 over different query lengths, searching over a single adj. pair ([00...]~[11...]) and all $\epsilon > 0$, with parameters being [c=1, T=0, $\Delta=1$, DOM={-1,0,1}, seq=(0;1)]

we analyzed SVT1 for various number of queries. The running times along with the number of queries and the value for c is shown in Table 6. The table shows that the tool can handle a reasonable number of queries.

In all the experiments so far, the value of ϵ was not fixed. So DiPC had to either prove privacy for all ϵ or find an ϵ where privacy is violated. Many automated tools are designed only to disprove differential privacy for a fixed ϵ . We tried the performance of the tool on SVT1 for a fixed ϵ . The results are reported in Table 4. As can be seen by comparing the numbers in Tables 6 and 4, fixing ϵ makes the problem easier to handle.

Finally, we wanted to explore the scalability of our tool when we checking

differential privacy for a single pair of adjacent inputs. In Table 7, we have the results when a non differentially private algorithm, namely SVT3 was run with a single adjacency pair ($[00...] \sim [11...]$), while varying number of queries. We notice that the running times is significantly lower in this case. Another interesting observation is that the time taken for 5 queries is lower than the time for 4 queries. This is because with 5 queries, the tool successfully finds a counter-example and terminates before checking the remaining inequalities.

H.4 (ϵ, δ) -Differential Privacy

DiPC can also verify (ϵ, δ) -differential privacy. Algorithm 4 (taken from [21]), referred to henceforth as Sparse, was used to evaluate DiPC's performance in this case. This algorithm has been manually proven to be $(\frac{\epsilon}{2}, \delta_{svt})$ -differentially private for any number of queries in [21] by using advanced composition theorems.

When c = 1 and $\delta_{\text{svt}} = e^{-\frac{1}{32}}$, this algorithm is identical to Algorithm SVT1, where parameters c and Δ are replaced by parameter σ . This algorithm is, therefore, ϵ -differentially private. Further, our tool proves that the algorithm is not $\frac{\epsilon}{2}$ -differentially private. Thanks to the advanced composition theorem, we can show that the resulting algorithm is $(\frac{\epsilon}{2}, e^{-\frac{1}{32}})$ differentially private. The tool also shows that for all $\epsilon > 0$, the algorithm is $(\frac{\epsilon}{2}, e^{-2})$ -differentially private for c = 1 for queries of length 3 with $\mathsf{DOM} = \{0, 1\}$ and T = 0 (observe that $e^{-\frac{1}{32}} > e^{-2}$). Additionally, we get a counter-example for $(\frac{\epsilon}{2}, e^{-2.125})$ -differential privacy.

When c = 2 and $\delta_{\text{svt}} = e^{-\frac{1}{64}}$, Sparse differs from SVT1 since in this case we also need to choose r_T again after outputting a \top . The resulting program is $(\frac{\epsilon}{2}, e^{-1/64})$ -differentially private thanks to the advanced composition theorem. DiPC confirms that for queries of length 3, the resulting program is infact $(\frac{\epsilon}{2}, e^{-2})$ -differentially private with DOM = $\{0, 1\}$ and T = 0. Further, DiPC also demonstrates that the resulting program is not $(\frac{\epsilon}{2}, e^{-2.5})$ differentially private.

Here we are able to check the correctness of Sparse automatically, for values of c = 1, 2 and for the above given values of δ_{svt} and for all $\epsilon > 0$. To the best of our knowledge, our approach is the first method to automatically check this. These results are summarized in Table 8.

Input: $q[1:N]$					
Output: $out[1:N]$					
$\sigma \leftarrow \frac{\epsilon}{2\sqrt{32c\ln\frac{1}{\delta_{svt}}}}$					
$r_T \leftarrow Lap(\sigma, T)$	С	$\delta_{\rm svt}$	δ	Runtime $(T1/T2)$	$(\frac{\epsilon}{2}, \delta)$ -Diff. Privacy
for $i \leftarrow 1$ to N do	1	$e^{-\frac{1}{32}}$	0	0s/48s	×
$ \mathbf{r} \leftarrow Lap(\frac{\sigma}{2}, q[i])$	1	$e^{-\frac{1}{32}}$	e^{-3}	0s/142s	×
$b \leftarrow r \ge r_T$	1	$e^{-\frac{1}{32}}$	$e^{-2.125}$	0s/146s	×
if b then	1	$e^{-\frac{1}{32}}$	e^{-2}	0s/161s	1
$out[i] \leftarrow \top,$	2	$e^{-\frac{1}{64}}$	0	0s/72s	×
$\mathbf{r}_T \leftarrow$	2	$e^{-\frac{1}{64}}$	e^{-3}	0s/187s	×
$Lap(\sigma, T)$	2	$e^{-\frac{1}{64}}$	$e^{-2.5}$	0s/182s	×
$count \leftarrow count + 1$	2	$e^{-\frac{1}{64}}$	e^{-2}	0s/288s	1
if $count > c$	-				

Table 8: DiPC result for $(\frac{\epsilon}{2}, \delta)$ -Diff. Privacy of SPARSE (Algorithm 4) with 3 queries, searching over all adj. pairs and $\epsilon > 0$, with parameters being $[T=0, DOM = \{0, 1\}]$

Algorithm 4: Sparse algorithm

 \mathbf{then} exit

 \mathbf{end}

 $\mid out[i] \leftarrow \bot$ end

 \mathbf{else}

 \mathbf{end}