Multiple Source Replacement Path Problem

Manoj Gupta, IIT Gandhinagar, gmanoj@iitgn.ac.in Rahul Jain[†] , Goldman Sachs, Bangalore rahul.e.jain@gs.com Nitiksha Modi, IIT Gandhinagar, nitiksha.modi@iitgn.ac.in

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Abstract

One of the classical line of work in graph algorithms has been the Replacement Path Problem: given a graph G, s and t, find shortest paths from s to t avoiding each edge e on the shortest path from s to t. These paths are called replacement paths in literature. For an undirected and unweighted graph, (Malik, Mittal, and Gupta, Operation Research Letters, 1989) and (Hershberger and Suri, FOCS 2001) designed an algorithm that solves the replacement path problem in $\tilde{O}(m+n)$ time¹. It is natural to ask whether we can generalize the replacement path problem: can we find all replacement paths from a source s to all vertices in G? This problem is called the Single Source Replacement Path Problem.

Recently (Chechik and Cohen, SODA 2019) designed a randomized combinatorial algorithm that solves the Single Source Replacement Path Problem in $\tilde{O}(m\sqrt{n}+n^2)$ time. One of the questions left unanswered by their work is the case when there are many sources, not one. When there are n sources, the combinatorial algorithm of (Bernstein and Karger, STOC 2009) can be used to find all pair replacement path in $\tilde{O}(mn+n^3)$ time. However, there is no result known for any general σ . Thus, the problem we study is defined as follows: given a set of σ sources, we want to find the replacement path from these sources to all vertices in G. We give a randomized combinatorial algorithm for this problem that takes $\tilde{O}(m\sqrt{n\sigma}+\sigma n^2)$ time. This result generalizes both results known for this problem. Our algorithm is much different and arguably simpler than (Chechik and Cohen, SODA 2019). Like them, we show a matching conditional lower bound using the Boolean Matrix Multiplication conjecture.

1 Introduction

One of the classical line of work in graph algorithms is the replacement path problem. The general setting for this problem is as follows: we are given a graph G and two vertices s and t. We want to output the length of all shortest paths avoiding edges on the st path. Note that we just want to output the length of these paths – not the path itself. These paths are called replacement paths in the literature.

Replacement paths were first investigated due to their relation with auction theory, where they were used to compute the Vickrey pricing of edges owned by selfish agents [20, 23]. One can also generalize the replacement path problem and ask to output k –not just one – replacement paths [29, 3, 16]. This is also called the k-replacement path problem.

The replacement path problem has been extensively studied. There are algorithms [21, 22, 20] that compute replacement paths in an undirected and unweighted graph in $\tilde{O}(m+n)$ time. For directed, unweighted graphs, Roditty and Zwick [29] designed an algorithm that finds all replacement paths in

[†] The work was done when the author was a student at IIT Gandhinagar

 $^{{}^{1}\}tilde{O}$ notation hides polylog *n* factor.

 $O(m\sqrt{n})$ time. For the k-shortest path problem, Roditty [27] presented an algorithm with an approximation ratio $^{3}/_{2}$, and the running time $O(k(m\sqrt{n}+n^{3/2}\log n))$. Bernstein [3] improved the above result to get an approximation factor of $(1+\epsilon)$ and running time $O(km/\epsilon)$. The same paper also gives an improved algorithm for the approximate st replacement path problem.

Let us now generalize the replacement path problem, where we want to find replacement paths from *s* to every other vertex in the graph. We define this problem formally.

Single Source Replacement Path (SSRP): Given a graph G and a source s, design an algorithm that can find the length of shortest paths from s to every vertex $t \in V$ avoiding each edge on st path.

Grandoni and Williams [17] were the first to study the SSRP problem in a directed graph. For a directed graph having weights in the range [1, M], they designed an algorithm for the SSRP problem that takes $O(Mn^{\omega})$ time (where ω is the matrix multiplication exponent [15, 31]). They also presented results when there are negative weights on the edges. Recently, Chechik and Cohen [8] designed an algorithm that solves the single source replacement path problem in $\tilde{O}(m\sqrt{n}+n^2)$ time in an undirected graph. They also show that this time is nearly optimal using the conditional lower bound of Boolean matrix multiplication. Other related work includes [13, 14, 28, 30, 32].

We now generalize the SSRP problem when there are multiple sources, we call this the Multiple Source Replacement Path Problem.

Multiple Source Replacement Path Problem (MSRP): Given a graph G and a set of sources \mathcal{S} , design an algorithm that finds length of all replacement paths from s to t where $s \in \mathcal{S}$ ($|\mathcal{S}| = \sigma$) and $t \in V$.

To the best of our knowledge, there is only one work that designs a combinatorial algorithm to solve the above problems. If there are n sources, then the work of Bernstein and Karger [4] can be used to find the replacement path for any pair of vertices in $\tilde{O}(mn+n^3)$ time. In their work [4], Bernstein and Karger built a distance oracle of size $\tilde{O}(n^2)$ that can answer the following query: Query(x, y, e): find the length of the replacement path from x to y avoiding e where x, $y \in V$. They designed an algorithm that builds this oracle in $\tilde{O}(mn)$ time and answers queries in O(1) time. Once this oracle is built, one can just query this oracle and find all replacement paths between any pair of vertices. This takes $O(n^3)$ time. Thus, when there are n sources, the MSRP problem can be solved in $\tilde{O}(mn+n^3)$ time.

The main open question left behind by these two works is the case when there are $|\mathcal{S}| = \sigma$ sources. In this paper, we solve this question by showing the following theorem:

Theorem 1. There is a randomized combinatorial algorithm that solves the MSRP problem in $\tilde{O}(m\sqrt{n\sigma} + \sigma n^2)$ time².

The reader can see that there are two combinatorial results for the above problem. If $\sigma=1$, then we have the result of Chechik and Cohen [8]. And when $\sigma=n$, we have the result of Bernstein and Karger [4]. Our result generalizes both these results. Additionally, we extend the conditional lower first presented in [8] by giving a combinatorial reduction from Boolean Matrix Multiplication (BMM) to MSRP problem.

Theorem 2. For a combinatorial algorithm MSRP(n,m) with runtime of T(n,m), there is a combinatorial algorithm for BMM(n,m) problem with runtime of $O(\sqrt{\frac{n}{\sigma}}T(O(n),O(m)))$.

1.1 Related Work

Bernstein and Karger [4] solved the MSRP problem when $\sigma = n$. As mentioned previously, their aim was to build a single edge fault tolerant distance oracle of size $\tilde{O}(n^2)$ with a query time of O(1). Demestrescu et al. [11] were the first to design this distance oracle of size $\tilde{O}(n^2)$ and query time O(1). However, they

²Note that the second term in the running time is needed as there may be $\Omega(\sigma n^2)$ terms to output.

did not specify the running time of their algorithm. Bernstein and Karger [4] answered this question. Other related work in this area are [5, 10, 9, 12]

The fault-tolerant distance oracle itself can be generalized when there is a single source or σ many sources. Bilo et al. [6] designed a distance oracle of size $\tilde{O}(\sigma^{1/2}n^{3/2})$ that can answer single fault queries in $O(\sqrt{n\sigma})$ time. Gupta and Singh [19] reduced the query time of this oracle to $\tilde{O}(1)$.

Other related problems include the fault tolerant subgraph problem. The aim of this problem is to find a subgraph of G such that the shortest path from $s \in \mathcal{S}$ is preserved in the subgraph after any edge deletion. Parter and Peleg [26] designed an algorithm to compute single fault tolerant subgraph with $O(n^{3/2})$ edges. They also showed that their result can be easily extended to multiple sources with $O(\sigma^{1/2}n^{3/2})$ edges. This result was later extended to dual fault by Parter [16] with $O(n^{5/3})$ edges. Gupta and Khan [18] extended the above result to multiple sources with $O(\sigma^{1/3}n^{5/3})$ edges. All the above results are optimal due to a result by Parter [25] which states that a multiple source k fault tolerant subgraph requires $O(\sigma^{\frac{1}{k+1}}n^{2-\frac{1}{k+1}})$ edges. There is only one positive result known for a general σ , Bodwin et al. [7] showed the existence of a k fault tolerant subgraph of size $O(k\sigma^{1/2^k}n^{2-1/2^k})$.

2 Previous Approach: Chechik and Cohen [8]

Before we dive into our approach, let us look at the previous approach to the problem. To this end, let us first define few terms.

- 1. Let *st* denote the shortest path from *s* to *t* in *G*.
- 2. Let *P* be the shortest replacement path from *s* to *t* avoiding an edge *e* on the *st* path. SUFFIX(*P*) denotes the suffix of *P* from the point it leaves the original *st* path.
- 3. *Landmark vertices*: Let \mathcal{L} be a set obtained by sampling each vertex in G with a probability of $\frac{1}{\sqrt{n}}$.

Figure 1: SUFFIX(P) starts with the blue path from a. It merges back to st path at c and continues till t.

One can show that \mathcal{L} contains $\tilde{O}(\sqrt{n})$ vertices with high probability. We can also show that if $\operatorname{Suffix}(P)$ contains $\tilde{O}(\sqrt{n})$ vertices, then with a high probability, $\operatorname{Suffix}(P)$ contains a landmark vertex.

We now describe the result of Chechik and Cohen [8] in detail. In

[8], the authors solved the SSRP problem using the following important observation from [1]: "For any replacement path from s to t avoiding e, there exists a vertex u such that the replacement path can be broken into two shortest paths in G (1) su path and (2) ut path". Note that the result is non-trivial as these two paths are shortest paths in the original graph – that is none of these paths pass through e in G. If we are able to find the vertex u, then we can easily find the replacement path. Unfortunately, finding u is not easy (in the running time we want to achieve).

We now try to explain how Chechik and Cohen [8] overcome this problem using landmark vertices. Chechik and Cohen [8] showed the following: if SUFFIX(P) is sufficiently long, say $\tilde{O}(\sqrt{n})$, then it contains two landmark vertices u and v such that P can be broken into three parts (1) su path (2) uv path and (3) vt path. Again all these paths are shortest paths in G and do not pass through the edge to be avoided. Since the number of landmark vertices is only $\tilde{O}(\sqrt{n})$, finding u and v becomes slightly easy. Once we have the above result, we can use the following simple algorithm (See Algorithm 1) to find the shortest path from s to t avoiding e (assuming that SUFFIX(P) is sufficiently long).

The reader can check that the time taken by Algorithm 1 for a fixed (t, e) pair is $\tilde{O}(n)$ (assuming that checking if su, uv and vt contain e takes O(1) time). Since there are $O(n^2)$ such pairs, the algorithm takes

Algorithm 1: Algorithm for finding a replacement path P from s to t avoiding e assuming that Suffix(P) is long

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1 d(s,t,e) \leftarrow \infty; // d(s,t,e) will be equal to |P| at the end of the algorithm.

2 for each pair u,v \in \mathcal{L} do

3 | if su,uv and vt avoids e then

4 | d(s,t,e) \leftarrow \min\{d(s,t,e),|su|+|uv|+|vt|\};

5 return d(s,t,e)
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 $\tilde{O}(n^3)$ time. However, remember that Chechik and Cohen [8] solve the SSRP problem in $\tilde{O}(m\sqrt{n}+n^2)$ time. Thus, we need to reduce this running time from $\tilde{O}(n^3)$ to $\tilde{O}(n^2)$. The above simple algorithm is the heart of their paper ("the double pivot case") and reducing the time to $\tilde{O}(n^2)$ require some more modification to the simple algorithm – we do not describe this modification as it is technically heavy and orthogonal to the approach used by us. We refer the interested reader to [8] for details.

3 Our Approach

We now describe our approach for the case when there is only one source, that is $\sigma = 1$. In between our explanation, we will also point out the differences between [8] and our result.

We will use the classical result of [21, 20, 22], that can find the replacement path from s to t in $\tilde{O}(m+n)$ time. An inefficient algorithm will then be to run this algorithm for each s, t pair where $t \in V$, giving a running time of $\tilde{O}(mn)$. Since we want a better running time, we restrict the use of the result of [21, 20, 22] to only landmark vertices \mathcal{L} . Thus, we find the replacement path from s to vertices in \mathcal{L} . Since there are $\tilde{O}(\sqrt{n})$ landmark vertices, the total time taken is $\tilde{O}(m\sqrt{n}+n\sqrt{n})$. We now use the set \mathcal{L} to find replacement paths for all other vertices. We first note the first difference between [8] and our result, the use of landmark vertices is completely different from that in [8]. We use the classical result to find the replacement path between s to all landmark vertices, unlike [8] where the properties of landmark vertices are used (in Algorithm 1).

We now describe our strategy to find a replacement path P from s to t avoiding e if Suffix(P) is sufficiently long. We claim the following results (which we show using Lemma 9): If Suffix(P) is of length $\tilde{O}(\sqrt{n})$, then with a high probability a landmark vertex, say v, will lie on Suffix(P) such that vt path does not contain e. Note that this result is similar to the one used by Chechik and Cohen [8] – here we are arguing about vt path only. Thus, our algorithm to find replacement path (whose suffix is sufficiently long) is as follows:

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Algorithm 2: Algorithm for finding a replacement path P from s to t avoiding e assuming that SUFFIX(P) is long

1 d(s,t,e) \leftarrow \infty; // d(s,t,e) will be equal to |P| at the end of the algorithm.

2 for each \ v \in \mathcal{L} do

3 | if vt avoids e then

4 | Let p be the length of the replacement path from s to v avoiding e;

5 | d(s,t,e) \leftarrow \min\{d(s,t,e),p+|vt|\};

6 return d(s,t,e)
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In the above algorithm, we use the fact that we have already found replacement paths from s to all the landmark vertices. Even though we know that some landmark vertex lies on SUFFIX(P), we do not

know which one. Thus, we have to scan all of \mathcal{L} to find the vertex which lies in SUFFIX(P). This process itself takes $\tilde{O}(\sqrt{n})$ time. This implies that it would take $\tilde{O}(n\sqrt{n})$ time to find all replacement paths of t, which in turns implies an $\tilde{O}(n^2\sqrt{n})$ running time. Since we cannot afford such a running time, we use the following scaling trick. This scaling trick is the second difference between [8] and our paper. This trick greatly simplifies the algorithm as well as the analysis in our paper.

We look at an edge e at a distance of $[2^k, 2^{k+1}]$ from t on st path (assume that $2^k \ge \sqrt{n} \log n$). One can argue that the suffix of the replacement path, say P, avoiding e will have length $\ge 2^k$. This is because e itself is at a distance $\ge 2^k$ from t. Since $|\operatorname{SUFFIX}(P)| > \tilde{O}(\sqrt{n})$, we know that one of our landmark vertex, say v, lies on $\operatorname{SUFFIX}(P)$. If we can find v then we can use Algorithm 2 to find the length of P. At the same time, we don't have enough time to look at all the landmark vertices to find v. Here comes our main idea. Since $|\operatorname{SUFFIX}(P)| \ge 2^k$, we can choose a smaller set of landmark vertices \mathcal{L}_k of size $\tilde{O}(\frac{n}{2^k})$ (a set where each vertex is sampled with probability $\frac{1}{2^k}$). We show (using Lemma 9) that there exists a landmark vertex $v \in \mathcal{L}_k$ such that v lies on $\operatorname{SUFFIX}(P)$ and vt path avoids e. Since, we have already found all replacement paths for vertices in \mathcal{L}_k (using [21, 20, 22]), we can use \mathcal{L}_k (instead of \mathcal{L}) to find replacement paths to t when the edges are in the range $[2^k, 2^{k+1}]$.

Thus, we use the same algorithm in Algorithm 2, but use the landmark set \mathcal{L}_k for edges that are at a distance of $[2^k, 2^{k+1}]$ from t. As the value of k increases the number of vertices in \mathcal{L}_k decreases. The end effect is that we take the same amount of time to process edges in the range $[2^k, 2^{k+1}], [2^{k+1}, 2^{k+2}], [2^{k+2}, 2^{k+3}], \dots, [n/2]$ from t. In general, the reader will see that it will take $\tilde{O}(n)$ time to process all the edges in any given range. This would imply a running time of $\tilde{O}(n)$ for finding all replacement paths of t whose suffix is long. This approach reduces the running time from $\tilde{O}(m\sqrt{n}+n^2\sqrt{n})$ to $\tilde{O}(m\sqrt{n}+n^2)$.

Note that we cannot use the above approach for edges which are near to t in st path (that is, those edges whose distance from t is $\leq \sqrt{n} \log n$) or those replacement paths that have short suffixes. It turns out that dealing with these replacement paths is relatively easy. Even Chechik and Cohen [8] have a simple approach for these paths. In our paper, we design another algorithm to deal with these replacement paths. This completes the description of our algorithm for the single source case.

The above approach gives us a simple algorithm for finding replacement paths from a single source to all vertices. We then extend our result to multiple sources. Ideally, we would have liked to use the result of [21, 20, 22] to find all replacement paths from all sources to vertices in \mathcal{L} . However, this does not give us the required running time. To overcome this barrier, we show that we can adapt the result of Bernstein and Karger [4] to find all replacement paths between all sources and vertices in \mathcal{L} (in the required running time). This completes the overview of our approach.

4 Notation

We use the following notation throughout the paper:

- Unless stated otherwise, uv will denote the shortest path between the vertex u and v in the graph G. |uv| denotes the length of this shortest path.
- Let e = (u, v) be an edge on st path such that u is closer to s than v. We will abuse notation and use se to denote su path and et to denote vt path.
- $st \diamond e$ is the shortest path from s to t avoiding e. $|st \diamond e|$ is the length of this shortest path.
- uv + vy denotes the concatenation of two paths, one ending at v and other starting at v.
- Let *P* be a path from *s* to *t*, not necessarily the shortest path. A sub-path uv on *P* will be denoted by P[u, v].

- The shortest path tree of a vertex $v \in V$ is denoted by \mathcal{T}_v . The shortest path tree can be built by performing Breadth First Search (BFS) algorithm from v.
- d(s,t,e): In our algorithm, we want to find the shortest replacement path from each source to each vertex. d(s,t,e) is the length of the replacement path (from s to t avoiding e) calculated by our algorithm. We will normally initialize $d(s,t,e)=\infty$ and prove that at the end of the algorithm d(s,t,e)=|st| e. Also, d(s,t) will denote the length of the shortest st path. Formally, $d(s,t)=d(s,t,\emptyset)=|st|$.
- The term *with a high probability* means with a probability $\geq 1 \frac{1}{n^c}$ where $c \geq 1$.

5 Preliminaries

We first sample a set of random vertices which we call as landmark vertices.

Definition 3. (Landmark vertices) Let \mathcal{L}_k be a set of vertices sampled randomly from G with a probability of $\frac{4}{2^k}\sqrt{\frac{\sigma}{n}}$ where $0 \le k \le \log \sqrt{n\sigma}$. Let $\mathcal{L} = \bigcup_{k=0}^{\log \sqrt{n\sigma}} \mathcal{L}_k$. Along with these vertices, \mathcal{L} also contains all source nodes.

The following lemma bound the number of vertices in \mathcal{L} .

Lemma 4. The size of \mathcal{L}_k is $\tilde{O}(\frac{\sqrt{n\sigma}}{2^k})$ with a very high probability. Thus, the size of \mathcal{L} is $\tilde{O}(\sqrt{n\sigma})$ with a very high probability.

Proof. Let X_k be a random variable denoting the size of \mathcal{L}_k . The expected size of X_k is, $\mathbb{E}[X_k] = \frac{4\sqrt{n\sigma}}{2^k}$. Using Chernoff's bound, we know that $P[X_k \geq (1+\delta)\mathbb{E}[X_k]] \leq e^{-\frac{\delta \mathbb{E}[X]}{3}}$ where $\delta \geq 1$. Putting $\delta = \log n$, we get $P[X_k \geq (1+\log n)\frac{4\sqrt{n\sigma}}{2^k}] \leq e^{-\frac{4\log n\sqrt{n\sigma}}{3\times 2^k}}$. Since 2^k can at most be $\sqrt{n\sigma}$, we get $P[X_k \geq (1+\log n)\frac{4\sqrt{n\sigma}}{2^k}] \leq e^{-\frac{4\log n}{3}} = n^{-4/3}$. Using union bound, the probability that the size of \mathcal{L} is $\geq \tilde{O}(\sqrt{n\sigma})$ is $\leq (\log \sqrt{n\sigma}) \times n^{-4/3} \leq n^{-1}$ (where the last ineqality is true for a high enough value of n).

Let us first build some elementary data-structure that will help us in our algorithm. Using Breadth First Search (BFS) algorithm, we can find the shortest path from s to all other vertices in G in O(m+n) time. We will assume that at the end of BFS algorithm, we will find the distance from s to every other vertex in G, that is d(s,v)=|sv|. Also, assume that we obtain the shortest path tree of s, that is \mathcal{T}_s , as the output of BFS algorithm. We store d(s,v) in a hash-table for efficient retrieval. To this end, we use the following data structure:

Lemma 5. (Pagh and Rodler [24]) There exists a randomized hash-table with constant look-up time in the worst case and constant insertion time in expectation.

For each landmark vertex r, we find the shortest path from r to every other vertex in G. This can again be done using BFS algorithm and the total running time is $\tilde{O}((m+n)\sqrt{n\sigma})$ as the number of landmark vertices is $\tilde{O}(\sqrt{n\sigma})$. We store the length of the shortest path from r to every other vertex v, that is d(r,v), in a hash-table.

Using the result of [21, 20, 22], we know that all replacement paths from s to a landmark vertex r can be found in $\tilde{O}(m+n)$ time. If there is only one source, that is $\sigma=1$, then we can use this result to find all replacement paths between the single source s and all landmark vertices. For a single source case, since the number of landmark vertices is $\tilde{O}(\sqrt{n})$, the time taken to find all replacement paths is $\tilde{O}((m+n)\sqrt{n})$.

However, when there are many sources, the above strategy gives a running time of $\tilde{O}((m+n)\sigma\sqrt{n\sigma})$ where the second and third multiplicand represent the number of sources and the number landmark vertices. We cannot afford such a huge running time. So, we adapt the result of Bernstein and Karger [4] and show that it can be used to find all replacement paths from all sources to all landmark vertices in $\tilde{O}(m\sqrt{n\sigma}+\sigma n^2)$ time. Given this result (which we will show in Section 8), for each $s\in \mathcal{S}$ and $r\in \mathcal{L}$, we store d(s,r,e) in a hash-table for each $e\in sr$ path.

Lastly, we used the following classical result to compute least common ancestors quickly:

Lemma 6. (See [2] and its references) Given any tree \mathcal{T}_{v} (on n vertices) rooted at v, we can build a data-structure of size O(n) in O(n) time which can find least common ancestor i.e. LCA(x, y) where $x, y \in \mathcal{T}_{v}$ in O(1) time.

Remember that we want to find all replacement paths from s to t for each $s \in \mathcal{S}$ and $t \in V$. Fix a $s \in \mathcal{S}$ and $t \in V$. We partition the edges on the st path into two sets, far and near.

- (*k*-Far Edges) Edges which are at a distance $[2^{k+1}\sqrt{\frac{n}{\sigma}}\log n, 2^{k+2}\sqrt{\frac{n}{\sigma}}\log n]$ away from t on st path (where $0 \le k \le \log \sqrt{n\sigma}$).
- (Near Edges) Edges which are at a distance $< 2\sqrt{\frac{n}{\sigma}}\log n$ away from t on st path.

6 Far Edges

Fix a source $s \in \mathcal{S}$ and $t \in V$. Assume that we are trying to find a replacement path for a k-far edge e on st path. Since the replacement path avoids e, it has to diverge from the st path before e. We now again look at the suffix of a replacement path and describe its properties.

Definition 7. Let P be the shortest replacement path from s to t avoiding an edge e on the st path. Then, SUFFIX(P) denotes the suffix of P from the point it leaves the original st path.

Since P avoids e on st path, it has to diverge from this path before edge e. SUFFIX(P) is the sub path of P that starts from this diverging vertex. We now make an important observation about the length of SUFFIX(P).

Observation 8. If P is the shortest replacement path from s to t avoiding a k-far edge e on st path, then $|SUFFIX(P)| > 2^{k+1} \sqrt{\frac{n}{\sigma}} \log n$.

The above observation holds because of the following simple argument: as SUFFIX(P) start before e on st path, its length should be \geq length of et path. We now claim that there exists a vertex of \mathcal{L}_k on SUFFIX(P). This can be shown easily using elementary probability.

Lemma 9. Let \mathscr{P} be the set of all replacement paths from $s \in \mathscr{S}$ to $t \in V$ that avoid a far edge. Given any path $P \in \mathscr{P}$ such that P avoids a k-far edge e on st path, with a high probability there exists a vertex $r \in \mathscr{L}_k$ on SUFFIX(P) such that the distance of r to t on SUFFIX(P) is $\leq 2^k \sqrt{\frac{n}{\sigma}} \log n$.

Proof. Fix a $P \in \mathscr{P}$. Since P avoids a k-far edge on st path, by Observation 8, $|\operatorname{Suffix}(P)| > 2^{k+1} \sqrt{\frac{n}{\sigma}} \log n$. Let X_P be the event that there does not exist a vertex of \mathscr{L}_k at a distance $\leq 2^k \sqrt{\frac{n}{\sigma}} \log n$ from t on $\operatorname{Suffix}(P)$. Then the probability that X_P occurs is $P[X_P] = (1 - \frac{4}{2^k} \sqrt{\frac{\sigma}{n}})^{2^k} \sqrt{\frac{n}{\sigma}} \log n \leq \frac{1}{n^4}$. Note that the size of \mathscr{P} is $\leq n^2 \sigma \leq n^3$. Thus, the probability that X_P occurs for any $P \in \mathscr{P}$ is $P[\cup_{P \in \mathscr{P}} X_P] \leq \frac{1}{n}$.

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Algorithm 3: Algorithm for finding a replacement path for a k-far edge e

1 d(s,t,e) \leftarrow \infty;
2 for each r \in \mathcal{L}_k do

3 \int \mathbf{if} \ d(r,t) \leq 2^k \sqrt{\frac{n}{\sigma}} \log n then

4 \int d(s,t,e) \leftarrow \min\{d(s,t,e),d(s,r,e)+d(r,t)\};
5 return d(s,t,e)
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We now use a simple algorithm (See Algorithm 3) to find the shortest replacement path from s to t avoiding a k-far edge e.

Let P be the replacement path from s to t avoiding a k-far edge e. Using Lemma 9, we know that there exists a landmark vertex $r \in \text{SUFFIX}(P)$ ($r \in \mathcal{L}_k$) such that the distance of r to t on $\text{SUFFIX}(P) \le 2^k \sqrt{\frac{n}{\sigma}} \log n$. Thus, the shortest path from r to t, that is rt, has length $\le 2^k \sqrt{\frac{n}{\sigma}} \log n$. We first claim that this path cannot pass through e. This is due to the fact that e is a k-far edge and the shortest path from e to t is $\ge 2^{k+1} \sqrt{\frac{n}{\sigma}} \log n$. Given such a $r \in \mathcal{L}_k$, finding the replacement path becomes easy, it is d(s,r,e) + d(r,t). We have already calculated both these terms in the preprocessing phase.

However, our algorithm does not know this particular r before-hand. So, it tries all the vertices in \mathcal{L}_k and finds the required r. The running time of the above algorithm for a fixed t and a k-far edge is $\tilde{O}(\frac{\sqrt{n\sigma}}{2^k})$. Since there can be at most $2^k\sqrt{\frac{n}{\sigma}}\log n$ k-far edges on st path, the total time taken to find the replacement path for all k-far edges for a fixed t is $\tilde{O}(n)$. Since $k \leq \log \sqrt{n\sigma}$, the total time taken to find replacement path for all far edges in st path is $\tilde{O}(n)$. Thus, we can find replacement path for each far edge in st path for each $s \in \mathcal{S}$ and $t \in V$ in $\tilde{O}(\sigma n^2)$ time.

7 Near Edges

There can be two types of replacement path that avoids a near edge e on a st path where $s \in \mathcal{S}$ and $t \in V$.

- 1. (Small replacement path) $|st \diamond e| \le |se| + 2\sqrt{\frac{n}{\sigma}} \log n$
- 2. (Large replacement path) $|st \diamond e| > |se| + 2\sqrt{\frac{n}{\sigma}} \log n$

We say that first set have small replacement paths avoiding a near edge, while the second set of paths have large replacement paths avoiding a near edge.

7.1 Small Replacement Paths avoiding a near edge

In this section, we will find all small replacement paths from s to t that avoid a near edge. To this end, we will make an auxiliary graph G_s . This graph will encode the shortest path from s to other vertices $t \in V$ avoiding near edges on st path. After making this graph, we will run Dijkstra's algorithm on it. At the end of this section, we will show that the output of Dijkstra's algorithm will give us all small replacement paths.

Construction of the auxiliary graph: The graph G_s contains a single source node [s]. For each $t \in V$, there is a node [t] in G_s . For each near edge $e \in st$ path, there is a node [t,e] in G_s . We will now add edges in this graph. There is an edge from [s] to [v] with weight |sv| for each $v \in V$. There is an edge from [v] to [t,e] of weight 1 if e does not lie in sv path and v is a neighbor of t. For each [v,e], there is an edge from [v,e] to [t,e] with weight 1 if t is a neighbor of v.

Size of the auxiliary graph: Let us first find the number of vertices in G_s . For each $t \in V$, there is a node [t] and [t,e] where e is a near edge on st path. Thus, for each $t \in V$, we add $\tilde{O}(\sqrt{\frac{n}{\sigma}})$ vertices in G_s . Thus, the total number of vertices in G_s is $\tilde{O}(n\sqrt{\frac{n}{\sigma}})$. Let us now calculate the number of edges in G_s . There may be an edge from [s] to every other node in G_s . For each $v \in V$, there may be an edge from [v] to [t,e] where t is a neighbor of v. But there are only $\tilde{O}(\sqrt{\frac{n}{\sigma}})$ vertices of type $[t,\cdot]$. This implies that the total number of edges of [v] is $\tilde{O}(\deg(v)\sqrt{\frac{n}{\sigma}})$. Similarly, there are at most $\tilde{O}(\deg(v))$ edges out of node [v,e]. This implies that the total number of edges in G_s is $\tilde{O}(n\sqrt{\frac{n}{\sigma}}+\sum_{v\in V}\deg(v)\sqrt{\frac{n}{\sigma}}+\sum_{[v,e]\in G_s}\deg(v))=\tilde{O}(m\sqrt{\frac{n}{\sigma}}+n\sqrt{\frac{n}{\sigma}})$.

Time taken to construct the auxiliary graph: Let us now try to find the time taken to construct

Time taken to construct the auxiliary graph: Let us now try to find the time taken to construct the graph G_s . We can find all near edges on st path in $\tilde{O}(\sqrt{\frac{n}{\sigma}})$ time using \mathscr{T}_s . Thus, creating the nodes in the graph takes $\tilde{O}(n\sqrt{\frac{n}{\sigma}})$ time. Let us now find the time taken to add an edge in the graph. For each [v], we need to add an edge from [v] to [t,e] if $e \notin sv$ path and t is a neighbor of v. We can check if e lies in sv path by using LCA query in \mathscr{T}_s . Thus, adding the edge takes O(1) time. Similarly adding an edge out of [v,e] also takes O(1) time. Thus, the time taken to make G_s is proportional to the number of vertices and edges in G_s .

Time to run Dijkstra's algorithm in the auxiliary graph: We now run Dijkstra's algorithm in G_s . Let w[t,e] be the weight of the path from [s] to [t,e] returned by Dijkstra's algorithm. We then set $d(s,t,e) \leftarrow \min\{d(s,t,e),w[t,e]\}$. The time taken to run Dijkstra's algorithm in G_s is $\tilde{O}(m\sqrt{\frac{n}{\sigma}}+n\sqrt{\frac{n}{\sigma}})$. Thus, the total time taken to construct all σ auxiliary graphs and run Dijkstra's algorithm in them is $\tilde{O}(m\sqrt{n\sigma}+\sigma n^2)$ time.

Proof of Correctness: We are now ready to prove the correctness of our algorithm. To this end, we show the following:

Lemma 10. Fix $a \ t \in V$ and $s \in \mathcal{S}$. Let P be a replacement path avoiding a near edge e on st path. If $|P| \le |se| + 2\sqrt{\frac{n}{\sigma}}\log n$, Then our algorithm sets d(s,t,e) to |P|.

Proof. We will prove the statement using induction on edge length of the replacement path. That is, we will prove the statement for all the replacement paths of edge length 0, then edge length 1 and so on. The base case is trivial, there is only one replacement path of edge length 0, that is $d(s,s,\emptyset) = 0$. Using the induction hypothesis, let us assume that the statement is true for all replacement paths of edge length i-1. Let us assume that P contains i edges and satisfies the condition of the lemma. And the last edge on this path is (v,t). There are three cases:

1. e does not lie on sv path.

If e does not lie on sv path, then d(s, v, e) = d(s, v). But even in the auxiliary graph G_s , there is a path from $[s] \to [v] \to [t, e]$. The weight of this path is |sv| + 1. Thus, even our algorithm will set d(s, t, e) = |sv| + 1.

2. *e* lies on *sv* path and *e* is a far edge on *sv* path.

We will show that this case cannot arise. If e is a far edge on sv path, then $|sv| = |se| + |ev| > |se| + 2\sqrt{\frac{n}{\sigma}}\log n$. Thus the replacement path from s to v avoiding e, that is $P\setminus (v,t)$, has weight $> |se| + 2\sqrt{\frac{n}{\sigma}}\log n$. This implies $|P| > |se| + 2\sqrt{\frac{n}{\sigma}}\log n$. This contradicts the statement of the lemma, namely $|P| \le |se| + 2\sqrt{\frac{n}{\sigma}}\log n$.

3. e lies on sv path, e is a near edge on sv path and $P \setminus (v, t)$ is a large replacement path. Even in this case, the weight of $P \setminus (v, t)$ is $> |se| + 2\sqrt{\frac{n}{\sigma}} \log n$. So similar to above, this case cannot arise.

4. *e* lies on sv path, *e* is a near edge on sv path and $P \setminus (v, t)$ is a small replacement path.

The replacement path from s to v avoiding e is $P \setminus (v, t)$. This path has i - 1 edges. Using induction hypothesis, we have set d(s, v, e) correctly. Thus, Dijkstra's algorithm will set d(s, t, e) = d(s, v, e) + |vt|.

7.2 Large Replacement Paths avoiding a near edge

Let *P* be a replacement path from *s* to *t* avoiding *e* such that $|st \diamond e| > |se| + 2\sqrt{\frac{n}{\sigma}}\log n$. We will first prove a simple observation:

Lemma 11. Let P be a replacement path from s to t avoiding a near edge e such that $|P| > |se| + 2\sqrt{\frac{n}{\sigma}} \log n$. Then $|\text{Suffix}(P)| > 2\sqrt{\frac{n}{\sigma}} \log n$.

Proof. Remember that the suffix of P will start from a vertex before e on st path. Let this vertex be z. Then $|sz| \leq |se|$. Also, $|P| = |sz| + |\operatorname{SUFFIX}(P)|$. But $|P| > |se| + 2\sqrt{\frac{n}{\sigma}}\log n$. This implies that $|sz| + |\operatorname{SUFFIX}(P)| > |se| + 2\sqrt{\frac{n}{\sigma}}\log n$. Since $|sz| \leq |se|$, it follows that $|\operatorname{SUFFIX}(P)| > 2\sqrt{\frac{n}{\sigma}}\log n$.

Since $\mathrm{Suffix}(P) > 2\sqrt{\frac{n}{\sigma}}\log n$, with a high probability, there exists a landmark vertex $r \in \mathcal{L}_0$ such that the distance of r to t on $\mathrm{Suffix}(P)$ is $\leq \sqrt{\frac{n}{\sigma}}\log n$. The proof for this is similar to Lemma 9. We state this lemma without proof.

Lemma 12. Let \mathscr{P} be the set of all large replacement paths from $s \in \mathscr{S}$ to $t \in V$ that avoid a near edge. Given any path $P \in \mathscr{P}$ such that P avoids a near edge e on st path, with a high probability there exists a vertex $r \in \mathscr{L}_0$ on SUFFIX(P) such that the distance of r to t on SUFFIX(P) is $\leq \sqrt{\frac{n}{\sigma}} \log n$.

Now, we will find the r stated in the above lemma. Once we find this r, we can calculate d(s,t,e) as follows: d(s,t,e) = d(s,r,e) + d(r,t,e). We would have liked to write d(r,t) instead of d(r,t,e) as we have not calculated d(r,t,e) beforehand. In the following lemma, we will show that $e \notin rt$, implying that d(r,t,e) = d(r,t).

Lemma 13. Let P be the shortest replacement path from s to t avoiding a near edge e on st path such that $|P| > |se| + 2\sqrt{\frac{n}{\sigma}}\log n$. Then there exists a landmark vertex r on SUFFIX(P) such that $st \diamond e = sr \diamond e + rt$ and $e \notin rt$.

Proof. By Lemma 11, $|\operatorname{Suffix}(P)| > 2\sqrt{\frac{n}{\sigma}}\log n$. Using Lemma 12, there exists a landmark vertex $r \in \mathcal{L}_0$ such that the distance of r to t on $\operatorname{Suffix}(P)$ is $\leq \sqrt{\frac{n}{\sigma}}\log n$. Thus, $st \diamond e = sr \diamond e + rt \diamond e$. Also we claim that $|rt \diamond e| \leq \sqrt{\frac{n}{\sigma}}\log n$, since the distance from r to t on $\operatorname{Suffix}(P)$ is $\leq \sqrt{\frac{n}{\sigma}}\log n$. We will now show that $rt \diamond e = rt$, that is e does not lie on rt path. Assume for contradiction that e lies on the rt path.

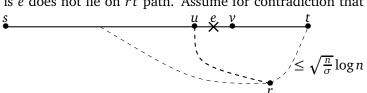


Figure 2: Alternate path avoiding e, $|P'| = su + ur + rt \diamond e$

Let e = (u, v). Since e lies on both st and rt path, there exists a path su and ru. Since $rt \diamond e$ path

itself is of length $\leq \sqrt{\frac{n}{\sigma}} \log n$, $|ru| \leq \sqrt{\frac{n}{\sigma}} \log n$. Consider the following replacement path avoiding e, $P' = su + ur + rt \diamond e$. Thus, $|P'| \leq |se| + 2\sqrt{\frac{n}{\sigma}} \log n$. This contradicts the condition of the lemma, which says that the shortest replacement path avoiding e, that is $|P| > |se| + 2\sqrt{\frac{n}{\sigma}} \log n$.

Thus, we need to process only those $r \in \mathcal{L}$ such that $e \notin rt$. Our algorithm, thus, is very simple.

```
Algorithm 4: Algorithm for finding large replacement paths avoiding a near edge

1 foreach r \in \mathcal{L}_0 do

2 | if e does not lie on rt path then

3 | d(s,t,e) \leftarrow \min\{d(s,r,e) + d(r,t), d(s,t,e)\};
```

The IF condition in the above algorithm can be checked in O(1) time using LCA queries in \mathscr{T}_r . Also, d(s,r,e) and d(r,t) are calculated during pre-processing and can be queried in O(1) time using the hash-table. Thus, the running time of the above algorithm for a fixed e and t is $\tilde{O}(\sqrt{n\sigma})$. Since there are at most $\tilde{O}(\sqrt{n\sigma})$ near edges in st path, finding all replacement paths takes $\tilde{O}(n)$ time. Thus, finding all large replacement paths for all vertices and for all sources takes $\tilde{O}(\sigma n^2)$ time.

For the SSRP problem, our algorithm is now complete. We can use [21, 20, 22] to find all replacement paths from s to vertices in $\mathscr L$ in $\tilde O(m\sqrt n)$ time. The reader can check that the total running time taken by our algorithm in Section 6 and 7 is $\tilde O(m\sqrt {n\sigma} + \sigma n^2)$ which is $\tilde O(m\sqrt {n} + n^2)$ when $\sigma = 1$. Thus, we claim the following theorem:

Theorem 14. There is a randomized algorithm that solves the SSRP problem in $\tilde{O}(m\sqrt{n}+n^2)$ time.

In the rest of the paper, we will generalize the result to multiple sources.

8 Generalizing to Multiple Sources

When there is one source s, we can find the replacement path from s to all vertices in \mathcal{L} using the algorithm of [21, 20, 22]. However, we cannot use this algorithm when there are multiple sources as it leads to the running time of $\tilde{O}((m+n)\sigma\sqrt{n\sigma})$. In this section, we describe a procedure that will find all replacement paths from $s \in \mathcal{L}$ to $r \in \mathcal{L}$ in $\tilde{O}(m\sqrt{n\sigma} + \sigma n^2)$ time. Some lemmas, definitions, and results in this section can be seen as the generalization of the result by Bernstein and Karger [4].

We sample another set of vertices which we call as centers (to differentiate them from landmark vertices). Let \mathscr{C}_k be the set of centers sampled with the probability $\frac{4}{2^k}\sqrt{\frac{\sigma}{n}}$ where $0 \le k \le \log \sqrt{n\sigma}$. Thus (similar to Lemma 4), with a high probability, $|\mathscr{C}_k| = \tilde{O}\left(\frac{\sqrt{n\sigma}}{2^k}\right)$. A center is said to have priority k if it lies in \mathscr{C}_k . Additionally, we add all vertices of \mathscr{S} in \mathscr{C}_0 . Like landmark vertices (similar to Lemma 4), the total number of centers is $\tilde{O}(\sqrt{n\sigma})$. We run BFS algorithm from each center c and find the shortest path tree \mathscr{T}_c . This takes $\tilde{O}(m\sqrt{n\sigma})$ time.

Fix a source $s \in \mathcal{S}$ and a vertex $r \in \mathcal{L}$. We can go over the path from a source s to r to find a center with the highest priority. We then move from s to this highest priority center, finding a list of centers with priority in ascending order. Let c_1 be the first center in the sr path. Then c_2 be the next center with a higher priority than c_1 . This continues till we reach the highest center on the path sr. Then, we find the list of centers in descending order of priority. There are at most $O(\log n)$ centers thus found. Since we are just walking on the path sr in this procedure, the time taken is the size of the path which is O(n). Since there are $\tilde{O}(\sigma\sqrt{n\sigma})$ pairs of possible s and r, the total time taken to find the list of centers is $\tilde{O}(n\sigma\sqrt{n\sigma}) = \tilde{O}(\sigma n^2)$ time. These centers naturally form an interval in the sr path, which we define next:

Definition 15. (Interval on a sr path) Let sr be a path such that $s \in \mathcal{S}$ and $r \in \mathcal{L}$. Assume that we find the centers $c_1, c_2, ..., c_\ell$ on this path, then we say that the path can be divided into intervals $sc_1, c_1c_2, ..., c_\ell r$.

Note that we have to find the replacement path from a source s to a landmark vertex r avoiding an edge e. To this end, we first find the pair of centers c_1 and c_2 between which e lies in sr path. The replacement path can be of the following three types:

- It passes through c_1 .
- It passes through c_2 .
- It avoids the interval c_1c_2 .

The above observation is named path cover lemma in [4].

Lemma 16. (Path Cover Lemma) Given a source s and a landmark vertex r, for any edge e on the sr path, let c_1c_2 be the centers between which e lies in sr path. Then

$$sr \diamond e = \min \begin{cases} sc_1 + c_1r \diamond e, \\ sc_2 \diamond e + c_2r, \\ sr \diamond [c_1c_2] \end{cases}$$
 where the last distance represents the shortest path from s to r avoiding the interval c_1c_2 .

The non-trivial part of the first and the second term in $sr \diamond e$ is $c_1 r \diamond e$ and $sc_2 \diamond e$. In the first term, we want to find a replacement path from a center to a landmark vertex and in the second term we want to find a replacement path from a source to a center.

As in [4], we club together first two terms of the path cover lemma.

Definition 17. (MTC, Minimum through centers) Given any source s and a vertex $r \in \mathcal{L}$. For any edge e on the sr path, let c_1c_2 be the centers between which e lies in sr path. Then

$$MTC(s, r, e) = \min \begin{cases} sc_1 + c_1 r \diamond e, \\ sc_2 \diamond e + c_2 r \end{cases}$$

Thus, MTC, minimum through centers, defines first two terms in the path cover lemma. We first calculate the MTC term. To calculate the MTC term, we have to find a replacement path from a center to a landmark vertex and find a replacement path from a source to a center.

Let us do the second part first.

8.1 Finding the replacement path from a source to a center

The second term in the MTC mandates us to find the replacement path from each source s to each center c. However, we need not find this replacement path for each edge on sc path. We need find the replacement path only for those edges that lie in the interval ending at c on the sc path. To formalize this, let us show the following lemma:

Lemma 18. Let xy be an interval in x path. Assume that the priority of x is x and the priority of y is greater than priority of x. Then $|xy| = \tilde{O}\left(2^k \sqrt{\frac{n}{\sigma}}\right)$.

Proof. We will show that a path P from x to s or r on the sr path of length $\geq 2^{k+1} \sqrt{\frac{n}{\alpha}} \log n$ must have a vertex in \mathscr{C}_{k+1} . This will imply that $|xy| = \tilde{O}(2^k \sqrt{\frac{n}{\sigma}})$.

Consider the subpath P[x,r]. The probability that none of the first $2^{k+1}\sqrt{\frac{n}{\sigma}}$ vertices in P[x,r] have priority k+1 is $(1-\frac{4}{2^k}\sqrt{\frac{\sigma}{n}})^{2^{k+1}}\sqrt{\frac{\pi}{\sigma}}\log n \leq \frac{1}{n^8}$. Since there are polynomial numbers of intervals and the centers are chosen independently of landmark vertices, the statement of the lemma is true for every possible interval with high probability using the union bound.

If c has priority k, then by the above Lemma 18, we just need to find the replacement path for the first $\tilde{O}(2^k\sqrt{\frac{n}{\sigma}})$ edges on cs path. This is because we are sure that any interval ending at c will contain $\tilde{O}(2^k\sqrt{\frac{n}{\sigma}})$ edges if c has priority k. In the ensuing discussion, we will assume that we are finding the replacement path from c to s for the first $\tilde{O}(2^k\sqrt{\frac{n}{\sigma}})$ edges on cs path.

If *P* is a small replacement path from *s* to *c* avoiding a near edge, then we have already found it in Section 7.1. Thus, our aim will be to find following replacement paths.

- 1. e is a far edge on sc path.
- 2. e is a near edge but $|P| > |se| + 2\sqrt{\frac{n}{\sigma}}\log n$.

We now show an important result which binds these replacement paths.

Lemma 19. Let P be a replacement path from s to $c \in \mathscr{C}$ avoiding e. (a) If e is an i-far edge on sc path, then there exists a vertex $c' \in \mathscr{C}_i$ on SUFFIX(P) such that $|c'c| \leq 2^i \sqrt{\frac{n}{\sigma}} \log n$ and $e \notin c'c$ (b) If e is a near edge and $|P| > |se| + 2\sqrt{\frac{n}{\sigma}} \log n$, then there exists a vertex $c' \in \mathscr{C}_0$ on SUFFIX(P) such that $|c'c| \leq \sqrt{\frac{n}{\sigma}} \log n$ and $e \notin c'c$.

- *Proof.* (a) Let us first consider the case when e is an i-far edge. We can now apply Lemma 9. Even though Lemma 9 is proved for landmark vertices, one can see that it can be used even using centers as sampling probability of both these sets are same. Thus, there exists a $c' \in \mathscr{C}_i$ in Suffix(P) such that $|c'c| \le 2^i \sqrt{\frac{\pi}{\alpha}} \log n$. Since e is an i-far edge, e cannot lie in c'c path.
 - (b) Let us now consider the case when e is a near edge and $|P| > |se| + 2\sqrt{\frac{n}{\sigma}}\log n$. By Lemma 12, there exists a $c' \in \text{SUFFIX}(P)$ such that $|c'c| \le \sqrt{\frac{n}{\sigma}}\log n$ and by Lemma 13, $e \notin c'c$. Again note that Lemma 12 and 13 is proved for landmark vertices, one can see that it can be used even for centers in \mathscr{C}_0 .

We will make an auxiliary graph G_s that will find all the required replacement paths from s to each center. This graph will encode replacement path from s to every center. After making this graph, we will run Dijkstra's algorithm on it. At the end of this section, we will show that the output of Dijkstra's algorithm will give us all required replacement paths.

Construction of the auxiliary graph: The graph G_s contains a single source node [s]. For each center c, we add a node [c] in G_s . For each $c \in \mathcal{C}_k$, we will add an $\tilde{O}(2^k\sqrt{\frac{n}{\sigma}})$ nodes in G_s . These are $[c,e_1],[c,e_2],\dots,[c,e_{\ell 2^k}\sqrt{\frac{n}{\sigma}}\log n}]$ representing the first $\tilde{O}(2^k\sqrt{\frac{n}{\sigma}})$ edges on cs path (where ℓ is a suitably chosen high constant). Let us now add edges in G_s . For each [c,e], if e is a near edge on sc path having small replacement path, then we would have already found it in Section 7.1. If w[c,e] is the weight of this small replacement path, then we add an edge from [s] to [c,e] with weight w[c,e]. Else for [c,e], the replacement path P satisfies the condition of Lemma 19. To account for these replacement paths, we add an edge from [c'] to [c,e] if $e \notin sc'$ and $e \notin c'c$. Further, we add an edge from [c',e] (if it exists) to [c,e] if $e \notin c'c$.

Size of the auxiliary graph: We now calculate the number of vertices in G_s . For each center of priority k, there are $\tilde{O}(2^k\sqrt{\frac{n}{\sigma}})$ nodes in G_s . Thus, the total number of nodes is $\sum_{k=1}^{\log n} \frac{\sqrt{n\sigma}}{2^k} 2^k \sqrt{\frac{n}{\sigma}} = \tilde{O}(n)$. Let us now count the number of edges in G_s . In the worst case, there can be an edge between any two nodes in G_s . Thus, the number of edges in G_s is $\tilde{O}(n^2)$.

Time taken to construct the auxiliary graph: For each center c of priority k, we add $\tilde{O}(2^k \sqrt{\frac{n}{\sigma}})$ vertices in G_s . This can be found by moving from c to s in the tree \mathscr{T}_s . For all centers, this takes $\sum_{k=1}^{\log n} \frac{\sqrt{n\sigma}}{2^k} 2^k \sqrt{\frac{n}{\sigma}} = \tilde{O}(n)$ time. Thus, each vertex can be added in O(1) time in G_s . We will see that we

can check if an edge can be added between any pair of node in G_s in O(1) time. To add an edge from [c'] to [c,e], we need to check if $e \notin sc'$ and $e \notin c'c$. This can be done using LCA queries in \mathcal{T}_s and \mathcal{T}_c . Similarly, an edge from [c',e] to [c,e] can be added in O(1) time. Thus, the time taken to construct G_s is $\tilde{O}(n^2)$ in the worst case.

Time taken to run Dijkstra's algorithm in the auxiliary graph: We now run Dijkstra's algorithm in G_s . For each $[c,e] \in G_s$, we set d(c,s,e) to be the weight of the path from [s] to [c,e] as returned by Dijkstra's algorithm. Since, there are $\tilde{O}(n)$ vertices and $\tilde{O}(n^2)$ edges in G_s , the total time to run Dijkstra's algorithm in G_s is $\tilde{O}(n^2)$. Since we run the above algorithm for σ sources, the total time taken by the above procedure is $\tilde{O}(\sigma n^2)$.

Proof of Correctness: We now prove the correctness of the above procedure.

Lemma 20. Let c be a center of priority k. Let P be a replacement path from $s \in \mathcal{S}$ to $c \in \mathcal{C}$ avoiding e such that e is one of first $\ell 2^k \sqrt{\frac{n}{\sigma}} \log n$ edges in cs path (where $\ell \geq 2$ is a suitably chosen constant). Then Dijkstra's algorithm in G_s will set d(c,s,e) to |P|.

Proof. We will prove the lemma by induction on the length of the path. Since s is also a center in \mathcal{C}_0 , the base case is that our algorithm finds a path of length 0 from s to itself. Assume that there is a replacement path P from s to c avoiding e that satisfies the statement of the lemma. If P is a small replacement path for a near edge, then we have already added an edge from [s] to [c,e] with an appropriate weight. So, assume that P satisfies the statement of the Lemma 19. By Lemma 19, if e is an i-far edge, then there exists a vertex $c' \in \mathcal{C}_i$ in Suffix(P) such that $|c'c| < 2^i \sqrt{\frac{n}{\sigma}} \log n$ and $e \notin c'c$. We will first show that a node corresponding to c' always exists in G_s . We claim that either sc' does not pass through e or e is one of the first e0 from e1 given and e2 given and e3 given and e4 given and e5 given and e6 given and e6 given and e7 given and e8 given and e8 given and e9 given are given as a small replacement and e9 given and e

Similarly, if e is a near edge then there exists a vertex $c' \in \mathcal{C}_0$ such that $e \notin c'c$. One can argue as above that either e does not lie in sc' or [c', e] vertex exists in G_s .

Thus, there are following cases:

1. $e \notin sc'$

In this case, we have P = sc' + c'c. In G_s , we have an edge from [s] to [c'] with weight |sc'| and [c'] to [c,e] with weight |c'c| if c'c does not pass through e. Thus, the Dijkstra's algorithm will be able to find this path.

2. $e \in sc'$

We have already shown that [c',e] will exist in G_s . P[s,c'] is a replacement path whose edge length is strictly less than |P|. By induction hypothesis, we assume that Dijkstra's algorithm finds P[s,c'] correctly. Thus, it sets d(c,s,e) to |P| correctly.

Till now, we have found all replacement path that will be used in the second term of MTC. We will now try to find replacement paths that will be used in the first term of MTC.

8.2 Finding the replacement path from a center to a landmark vertex

Now, we calculate the first term of MTC. Let P be a replacement path from a source s to a landmark vertex r that is passing through the center c. To calculate the first term in MTC(s, r, e), we should calculate

 $|sc| + |cr \diamond e|$ if e lies in the interval starting with c on sr path. This implies that the center c lies in sr path and e lies in the interval starting with c in cr path.

If c has priority k, then by Lemma 18, on any sr path, we are sure to find a center of priority k+1 at a distance of $\tilde{O}(2^k\sqrt{\frac{n}{\sigma}})$ from c. Thus, we just need to find the replacement path for edges till a distance of $\tilde{O}(2^k\sqrt{\frac{n}{\sigma}})$ from c.

Before moving ahead, let us first make an important observation. We need to find a replacement path from c to r avoiding e only if there exists a replacement path from some source s to r avoiding e that passes through c. Otherwise, there is no need to even find a replacement path from c to r avoiding e.

Remember that a replacement path from a source to r avoiding e and passing through c can be of three types.

- 1. Small replacement path that avoids a near edge.
- 2. Large replacement path that avoids a near edge.
- 3. Replacement path that avoids a far edge.

Let us look at the first set of replacement paths as we have already found these paths in Section 7.1.

8.2.1 Small replacement paths avoiding a near edge

For each $r \in \mathcal{L}$, we have already found the replacement path from each source to r. We can enumerate all the edges on each of these replacement paths too. Remember that the algorithm in Section 7.1 only finds the length of the small replacement path, not the replacement path itself. However, we can use Dijkstra's algorithm to find the actual path too. The time taken to enumerate a path is equal to the length of the path. Since, there are σ sources, $\tilde{O}(\sqrt{n\sigma})$ vertices in \mathcal{L} , and only $\tilde{O}(\sqrt{\frac{n}{\sigma}})$ near edges, we need to enumerate $\tilde{O}(\sigma\sqrt{n\sigma}\sqrt{\frac{n}{\sigma}}) = \tilde{O}(\sigma n)$ paths. Since there can be n edges on each path, the total time to enumerate all the paths is $\tilde{O}(\sigma n^2)$.

We can pre-process each enumerated replacement path to find whether a vertex lies on it – this can be done using LCA queries. For a landmark vertex r and a center c, we can check if there exists a small replacement path from a source to r passing through c and avoiding a near edge e. To this end, we will first check all enumeration that represents a replacement path from a source to r avoiding e. There are σ such enumerations. If e lies in any of the enumerations, then there is a replacement path avoiding a near edge of small length passing through e. Thus, we can find a small replacement path from e to e avoiding a near edge e in e0(e0) time. Once again we reiterate, that we will find this replacement path only if there is a replacement path from a source to e1 avoiding e2 that passes through e3. If there is no such path, then there is no need to find a small replacement path from e3 to e4 avoiding a near edge e5.

Given a (c, r, e), the time taken to find a small replacement path is $O(\sigma)$. Since there are $\tilde{O}(\sqrt{n\sigma})$ centers, $\tilde{O}(\sqrt{n\sigma})$ landmark vertices and $\tilde{O}(\sqrt{\frac{n}{\sigma}})$ near edges, the total time taken to find all small replacement paths for all possible (c, r, e) tuples is $\tilde{O}(\sqrt{n\sigma}\sqrt{n\sigma}\sqrt{n\sigma}\sqrt{\frac{n}{\sigma}}\sigma) = \tilde{O}(\sigma n^2)$. We store the length of all small replacement paths in d(c, r, e) (where $c \in \mathcal{C}$ and $e \in \mathcal{L}$ and $e \in \mathcal{L}$ and $e \in \mathcal{L}$ and $e \in \mathcal{L}$ are reduced in a hash-table for efficient retrieval.

8.2.2 Other replacement paths

Once we have dealt with small replacement paths, two other types of replacement paths are left. Remember that this replacement path P is from a source s to a landmark vertex r that is passing through the center c. Also, e lies in cr path. And our aim is to find the length of the sub-path P[c, r].

The replacement path *P* can be of two types:

- P avoids a far edge e on sr path and passes through c.
 If e is a far edge on sr path then it is also a far edge on cr path.
- 2. e is a near edge but $|P| > |se| + 2\sqrt{\frac{n}{\sigma}}\log n$. Since e is a near edge for sr path, it is also a near edge in cr path. Thus, even the subpath P[c,r] satisfies, $|P[c,r]| > |ce| + 2\sqrt{\frac{n}{\sigma}}\log n$.

Thus, we need to find a replacement path P[c, r] such that:

- 1. P[c, r] avoids a far edge e on cr path.
- 2. *e* is a near edge but $|P[c,r]| > |ce| + 2\sqrt{\frac{n}{\sigma}}\log n$.

Lemma 21. Let c lie on sr path such that $e \in cr$. Let P be a replacement path from s to r avoiding e and passing through c. Let P[c,r] be the corresponding replacement path from c to r avoiding e such that (a) e is a far edge on cr path or (b) e is a near edge and $|P[c,r]| > |ce| + 2\sqrt{\frac{n}{\sigma}}\log n$. Then (1) $|SUFFIX(P[c,r])| > 2\sqrt{\frac{n}{\sigma}}\log n$ and (2) There exists a vertex $r' \in \mathcal{L}$ in SUFFIX(P[c,r]) such that $e \notin r'r$.

- *Proof.* 1. If e is a far edge, P[c,r] has to diverge from cr before e. Thus, SUFFIX(P[c,r]) will be $> 2\sqrt{\frac{n}{\sigma}}\log n$. And by Lemma 13, even for a near edge, if $|P[c,r]| > |ce| + 2\sqrt{\frac{n}{\sigma}}\log n$, then $|SUFFIX(P[c,r])| > 2\sqrt{\frac{n}{\sigma}}\log n$.
 - 2. Let us first consider the case when e is a far edge. By Lemma 9, there exists a r' in SUFFIX(P[c,r]) such that $|r'r| \le \sqrt{\frac{n}{\sigma}} \log n$. Since e is a far edge, e cannot lie in r'r path. Let us now consider the case when e is a near edge nd $|P[c,r]| > |ce| + 2\sqrt{\frac{n}{\sigma}} \log n$. By Lemma 12 and 13, there exists a $r' \in \text{SUFFIX}(P[c,r])$ such that $e \notin r'r$.

With all armoury at hand, we are now ready to find the required replacement paths from a center to all landmark vertices. Fix a center c with priority k. We will now find the replacement path from c to each landmark vertex. Also remember that we want to find replacement path avoiding all edges at a distance $\tilde{O}(\sqrt{\frac{n}{\sigma}}2^k)$ from c. We will now create an auxiliary graph that will help us in finding all the required replacement paths.

Construction of the auxiliary graph: We make an auxiliary directed weighted graph G_c with a single source node [c]. In this graph, for each landmark vertex r, we will have at most $\tilde{O}(2^k\sqrt{\frac{n}{\sigma}})$ nodes, $[r,e_1],[r,e_2],\ldots,[r,e_{\ell 2^k}\sqrt{\frac{n}{\sigma}}\log n]$ (where $\ell \geq 2$ is a suitably chosen constant). The second term in the tuple represents the first $\tilde{O}(2^k\sqrt{\frac{n}{\sigma}})$ edges on the cr path. Also, there is a node [r] for each landmark vertex r. We now add edges in G_c . There is an edge from [c] to [r] (where $r \in \mathcal{L}$) of weight |cr|. For each [r,e], there are three types of incoming edges to it.

- 1. If there exists a small replacement path from a source to r avoiding the near edge e passing through c, then we have already found it in Section 8.2.1. Let the weight of this path be w[c, r, e]. We add an edge from [c] to [r, e] with the weight w[c, r, e].
- 2. Edge from [r'] (where $r' \in \mathcal{L}$) to [r,e] of weight |r'r| if cr' path does not pass through e and r'r path does not pass through e.
- 3. For each $r' \in \mathcal{L}$, if [r', e] exists, then there is an edge from [r', e] to [r, e] of weight |rr'| if r'r path does not pass through e.

This completes the description of G_c .

Size of the auxiliary graph: Let us first count the number of vertices in G_c . For each $r \in \mathcal{L}$, there is a node [r] in G_c . Let us now count the number of nodes of type [r,e] in G_c . Since there are $\tilde{O}(\sqrt{n\sigma})$ vertices in \mathcal{L} and at most $\tilde{O}(2^k\sqrt{\frac{n}{\sigma}})$ nodes of tupe [c,e], there are at most $\tilde{O}(\sqrt{n\sigma}\sqrt{\frac{n}{\sigma}}2^k)=\tilde{O}(n2^k)$ vertices in the auxiliary graph. Let us now count the number of edges in G_c . There may be an edge from [c] to every other vertices in the graph. Each vertex of type [r'] can have an edge to all other vertices in the graph. Thus the number of such edges is $\tilde{O}(n2^k\sqrt{n\sigma})$. Each vertex of type [r',e] can have edge to at most $\tilde{O}(\sqrt{n\sigma})$ other edges of type [r,e]. There are at most $\tilde{O}(n2^k\sqrt{n\sigma})$ such edges. Thus, in total there are $\tilde{O}(n2^k\sqrt{n\sigma})$ edges G_c .

Time taken to construct the auxiliary graph: For each landmark vertex r, we add at most $\tilde{O}(2^k \sqrt{\frac{n}{\sigma}})$ vertices in G_c . This can be done by moving up from the vertex c in \mathcal{T}_r . Thus, adding a vertex in G_c takes O(1) time. We will see that even adding an edge in G_c takes O(1) time. To add an edge from [c] to [r,e], we need to check if there is a small replacement path from c to r avoiding e. We have already found this path in Section 8.2.1 and can be retrived in O(1) time from the hash-table. Similarly, to add an edge from [r'] to [r,e], we can check if $e \in cr'$ (using LCA query in \mathcal{T}_c) and $e \in r'r$ (using LCA query in $\mathcal{T}_{r'}$) in O(1) time. Similarly, a edge from node [r',e] to [r.e] can be added in O(1) time. The reader can check that the time taken to construct this auxiliary graph is proportional to its size.

Time taken to run Dijkstra's algorithm in the auxiliary graph: We now run Dijkstra's algorithm in G_c . For each $[r,e] \in G_c$, we set d(c,r,e) to be the weight of the path from [c] to [r,e] as returned by Dijkstra's algorithm. We now calculate the time taken to run Dijkstra's algorithm. The time taken to run Dijkstra's algorithm in G_c is $\tilde{O}(n2^k\sqrt{n\sigma})$. Since there are $\frac{\sqrt{n\sigma}}{2^k}$ centers of priority k and at most $\log n$ such priorities, the total time taken to run Dijkstra's algorithm in all auxiliary graphs is $\tilde{O}(\sum_{k=0}^{\log n} \frac{\sqrt{n\sigma}}{2^k} n \sqrt{n\sigma} 2^k) = \tilde{O}(\sigma n^2).$ **Proof of Correctness:** We now show that the shortest path calculated above gives us the replace-

ment path from c to all landmark vertices.

Lemma 22. Let c be a center of prioroty k. Let P be a replacement path from s to r avoiding e and passing through c such that e is one of the first $\tilde{O}(2^k \sqrt{\frac{n}{\sigma}})$ edges on cr path. Then Dijkstra's algorithm in G_c gives the replacement path P[c,r] avoiding e.

Proof. We will prove the lemma by induction on the length of the path P[c, r]. The base case is that our algorithm finds a path of length 0 from c to itself (for the base case we will assume that c is also a landmark vertex). Assume that there is a replacement path P from s to r avoiding e that passes through c. If P is a small replacement path for a near edge, then we have already added an edge from [c] to [r,e]with an appropriate weight. So, assume that P satisfies the statement of the Lemma 21. By Lemma 21, there exists a vertex $r' \in \mathcal{L}$ such that $e \notin r'r$. We now claim that either $e \notin cr'$ or e is one of the first $\tilde{O}(2^k \sqrt{\frac{n}{\sigma}})$ edges in cr' path. This is because if $e \in cr'$ and $e \in cr$, then the subpath ce is same for both of them.

There are following cases:

1. $e \notin cr'$

In this case, we have P[c, r] = cr' + r'r. In G_c , we have an edge from [c] to [r'] with weight |cr'|and [r'] to [r,e] with weight |r'r| if r'r does not pass through e. Thus, the Dijkstra's algorithm will be able to find P[c, r].

2. $e \in cr'$

In this case, the replacement path from c to r is $cr' \diamond e + r'r$. We have already shown that [c, r']exists in G_c . We have added an edge from [r', e] to [r, e] with weight |rr'|. By induction hypothesis, we assume that Dijkstra's algorithm correctly calculates $cr' \diamond e$. Thus, it will correctly calculate $cr \diamond e$ too.

Given the result in this section (Section 8.2) and the result in Section 8.1, we can now calculate the first two terms in the path cover lemma (See Lemma 16). In the ensuing discussion, we will be calculating a replacement path from a source to a landmark vertex that avoids an entire interval.

8.3 Replacement path avoiding an interval

In this section, we will find the replacement path from s to r avoiding the interval that contains edge e. To this end, we use the concept of bottleneck vertex (adapted as bottleneck edge for our purpose) introduced in [4]. Let us first define a few terms which will be used in this section.

Definition 23. Let [s, r, i] denote the i-th interval on the sr path where $s \in \mathcal{S}$ and $r \in \mathcal{L}$. A bottleneck edge is the hardest edge on this interval to avoid. Formally, the bottleneck edge $\mathcal{B}[s, r, i] := \max_{e \in [s, r, i]} \{sr \diamond e\}$.

Thus, the path cover lemma looks as follows for an edge that lies in the i-th interval of sr path.

Lemma 24. If e lies on the i-th interval on the sr path, then
$$sr \diamond e = \min \begin{cases} \mathsf{MTC}(s,r,e), \\ sr \diamond \mathscr{B}[s,r,i] \end{cases}$$

Proof. If $|sr \diamond e|$ avoids the *i*-th interval then it avoids $\mathscr{B}[s,r,i]$ too. Thus, $|sr \diamond e| \geq |sr \diamond \mathscr{B}[s,r,i]|$. But $\mathscr{B}[s,r,i]$ is the bottleneck edge of the *i*-th interval, so $|sr \diamond e| \leq |sr \diamond \mathscr{B}[s,r,i]|$. This implies that $|sr \diamond e| = |sr \diamond \mathscr{B}[s,r,i]|$. If $|sr \diamond e|$ does not avoid the *i*-th interval, then $|sr \diamond e| = \mathsf{MTC}(s,r,e)$. This proves the statement of the lemma.

Thus, two things are left now: find the bottleneck edge for each interval and then find the replacement path avoiding the bottleneck edge.

8.3.1 Finding Bottleneck edge in each interval in sr path

We now show how to find a bottleneck edge in the i^{th} interval of the path sr. We first observe that the bottleneck edge will have the highest MTC value among all edges in the i-th interval. This is true as by Lemma 24, the last term for each edge in the interval is the same. So, to find a bottleneck edge, we should look at the edge in the interval which maximizes the first two terms, that is the MTC value.

To find the bottleneck edge of the i-th interval, we just need to go over each edge in the i^{th} interval and check the MTC value (whose constituents we have already calculated). This takes O(n) time for all intervals on the sr path. Since there are σ sources and $\tilde{O}(\sqrt{n\sigma})$ vertices in \mathcal{L} , the total time taken to find bottleneck edges is $\tilde{O}(n \sigma \sqrt{n\sigma}) = \tilde{O}(\sigma n^2)$.

8.3.2 Finding the replacement path avoiding the bottleneck edge in sr path

Let us now find the replacement path P avoiding the bottleneck edge of the i-th interval on the sr path. Let $e \leftarrow \mathcal{B}[s,r,i]$, that is e is the bottleneck edge of the i-th interval in sr path. If P is a small replacement path avoiding a near edge on sr path, then we would have already found it in Section 7.1. Thus, our focus will be to find P when:

- 1. P avoids a far bottleneck edge e.
- 2. *P* avoids a near bottleneck edge *e* but $P > |se| + 2\sqrt{\frac{n}{\sigma}}\log n$.

By Lemma 21, there exists a r' in SUFFIX(P) such that $e \notin r'r$. We will now crucially use this property to make another auxiliary graph G_s .

Construction of the auxiliary graph: This graph G_s contains a source vertex [s]. There is a vertex [r] for each $r \in \mathcal{L}$. For a bottleneck edge of interval i in sr path, there is a vertex [s,r,i] where $i \leq \log n$. We now find the edges in G_s . There is an edge from [s] to [r] with weight |sr|. If $\mathcal{B}[s,r,i]$ happen to be a near edge whose replacement path has small weight, then we add an edge from [s] to [s,r,i] with appropriate weight (see Section 7.1). Else, there can be three types of edges to [s,r,i].

- 1. Edge from [s] to [s,r,i] of weight MTC($s,r,\mathcal{B}[s,r,i]$). This edge represents the first two terms in the path cover lemma for the bottleneck edge $\mathcal{B}[s,r,i]$ (we have already calculated these in Section 8.1 and 8.2).
- 2. For each $r' \in \mathcal{L}$, there is an edge from [s] to [s,r,i] with weight MTC $(s,r',\mathcal{B}[s,r,i]) + |r'r|$ if $\mathcal{B}[s,r,i]$ does not lie in r'r.
- 3. If $\mathscr{B}[s,r,i]$ lies in the j^{th} interval on the sr' path, then there is an edge from [s,r',j] to [s,r,i] with weight |r'r| if $\mathscr{B}[s,r,i]$ does not lie in r'r path.

This completes the construction of G_s .

Size of the auxiliary graph: The number of nodes of type [r] in G_s is $\tilde{O}(\sqrt{n\sigma})$. The number of nodes in G_s of type [s,r,i] is $\tilde{O}(\sqrt{n\sigma})$ since there are $\tilde{O}(\sqrt{n\sigma})$ landmark vertices and $\log n$ interval in any sr path. We now find the number of edges in G_s . For each vertex [s,r,i], there are at most $\tilde{O}(\sqrt{n\sigma})$ from the source [s] (due to point (2) in above enumeration). Also, there are at most $\tilde{O}(\sqrt{n\sigma})$ edges from other vertices in \mathcal{L} (due to point (3) in above enumeration). Thus, the total number of edges in G_s is $\tilde{O}(\sqrt{n\sigma}\sqrt{n\sigma}) = \tilde{O}(n^2)$.

Time taken to construct the auxiliary graph: For each source, we have already found the bottleneck edge of each interval in sr path in Section 8.3.1(where r is a landmark vertex). Thus, adding vertices in G_s takes O(1) time. If there is a small replacement path from [s] to [s,r,i], then we have already found it in Section 7.1 and can be added in O(1) time. We add an edge from [s] to [s,r,i] with MTC $(s,r,\mathcal{B}[s,r,i])$. Again, we have calculated the MTC term in Section 8.1 and 8.2. So, we can add this edge in O(1) time. For each $r' \in \mathcal{L}$, we add an edge from [s] to [s,r,i] if $e \notin r'r$. Again, this edge can be added in O(1) time. The hardest part is point (3) in the above enumeration. For each interval [s,r',j], we first need to check if $\mathcal{B}[s,r,i]$ in the j-th interval in sr' path. This can be done by first finding if $\mathcal{B}[s,r,i]$ lies in sr' path – by doing LCA queries in \mathcal{T}_s . If $\mathcal{B}[s,r,i]$ lies in sr' path, then we can calculate the distance of $\mathcal{B}[s,r,i]$ relative to s and r'. This can be done easily as we have already stored distances from s to all other vertices in the graph in $d(s.\cdot)$ (in the pre-processing phase). Thus, all edges in G_s can be added in O(1) time. Thus, the time taken to construct G_s is equal to the worst case size of G_s , that is $\tilde{O}(n^2)$.

Time taken to run Dijkstra's algorithm in the auxiliary graph: We run Dijkstra's algorithm in G_s to find the shortest replacement path for each bottleneck edge. We set $d(s, r, \mathcal{B}[s, r, i])$ to the weight of the shortest path from [s] to [s, r, i] as returned by Dijkstra's algorithm in G_s . The time taken by Dijkstra's algorithm in G_s is $\tilde{O}(n^2)$. Since there are σ such graphs, the total time taken is $\tilde{O}(\sigma n^2)$.

Proof of Correctness: We now prove the correctness of the above algorithm.

Lemma 25. Let P be the shortest path from $s \in \mathcal{S}$ to $r \in \mathcal{L}$ avoiding the bottleneck edge in the i-th interval of sr path. Then Dijkstra's algorithm in G_s correctly finds P.

Proof. We will prove using induction on the edge length of the replacement paths. Since a source is also in \mathcal{L} , Dijkstra's algorithm correctly finds the replacement path of length 0 (which is our base case). Let us assume that the number of edges in P is k. By induction hypothesis, Dijkstra's algorithm has correctly found replacement path from s to any $r' \in \mathcal{L}$ avoiding a bottleneck edge on sr' path whose edge length is < k. If P is a small replacement path avoiding a near bottleneck edge $\mathcal{B}[s,r,i]$ in sr path, then we would have already found it in Section 7.1 and put an edge from [s] to [s,r,i] of appropriate weight in G_s . So, assume that P satisfies the statement of the Lemma 19. (Remember that we prove this lemma for centers, but the reader can check that the same lemma holds for landmark vertices as both the sets have same sampling probability). By Lemma 19, there exists a vertex $r' \in \mathcal{L}$ such that $\mathcal{B}[s,r,i] \notin r'r$.

There are following cases

1. $\mathscr{B}[s,r,i] \notin sr'$

In this case, we have P = sr' + r'r. In G_s , we have an edge from [s] to [r'] with weight |sr'| and [r'] to $[s,r,\mathcal{B}[s,r,i]]$ with weight |r'r|. Thus, the Dijkstra's algorithm will be able to find this path.

2. $\mathscr{B}[s,r,i] \in sr'$

Let us assume that $\mathcal{B}[s,r,i]$ lies on the *j*-th interval in sr' path. Then,

$$d(s,r,\mathcal{B}[s,r,i]) = \min \begin{cases} \operatorname{MTC}(s,r',\mathcal{B}[s,r,i]) + |r'r| \\ sr' \diamond \mathcal{B}[s,r',j] + |r'r| \end{cases}.$$

Here we are just expanding the path cover lemma for the tuple $(s, r', \mathcal{B}[s, r, i])$. For the first case, we have added an edge from [s] to [s, r, i] with weight MTC $(s, r', \mathcal{B}[s, r, i])$.

And for the second term, since $sr' \diamond \mathcal{B}[s,r',j]$ has edge length strictly less than k, using induction hypothesis, we can assume that we have set $d(s,r',\mathcal{B}[s,r,j]) = |sr' \diamond \mathcal{B}[s,r',j]|$. Thus, Dijkstra's algorithm will find P correctly.

Thus, we claim the main theorem of the paper:

Theorem 26. There is a randomized combinatorial algorithm that solves the MSRP problem in $\tilde{O}(m\sqrt{n\sigma} + \sigma n^2)$.

9 Conditional Lower Bounds

We now try to prove a conditional lower bound of $\Omega(m\sqrt{n\sigma})$ for MSRP problem with σ sources in undirected and unweighted graphs by giving a combinatorial reduction from Boolean Matrix Multiplication (BMM) to MSRP. This can be seen as a simple extension of the lower bound obtained in [8].

Let BMM(n, m) be a combinatorial algorithm for multiplying two matrices A and B both of size $n \times n$ such that the total number of 1's in both A and B is m. A combinatorial algorithm does not use any matrix multiplication. The conditional lower bound relies on the conjecture for combinatorial BMM, that there does not exist any truly subcubic algorithm for it.

Conjecture 27. In the Word RAM model with words of $O(\log n)$ bits, any combinatorial algorithm for multiplying two Boolean matrices A and B of size $n \times n$ with a total number of m 1's in them requires $(mn)^{1-o(1)}$ time in expectation to compute.

Let Msrp(n, m) denote our multiple source replacement path algorithm for unweighted graph with n vertices, m edges and σ sources. We will now reduce Bmm(n, m) to Msrp(n, m)

Theorem 28. For a combinatorial algorithm MSRP(n, m) with runtime of T(n, m), there is a combinatorial algorithm for BMM(n, m) problem with runtime of $O(\sqrt{\frac{n}{\sigma}}T(O(n), O(m)))$.

Proof. Consider three matrices A,B and C such that $C=A\times B$. Next we show how to compute C using our MSRP(n,m) algorithm. To this end, we create $\sqrt{\frac{n}{\sigma}}$ graphs $\{G_1,G_2,\ldots,G_{\sqrt{\frac{n}{\sigma}}}\}$ each containing 3 sets of vertices $V_a=\{a(1),a(2),\ldots,a(n)\},V_B=\{b(1),b(2),\ldots,b(n)\}$ and $V_c=\{c(1),c(2),\ldots,c(n)\}$. Each graph will have O(n) vertices and O(m) edges. In each graph G_i , we will use MSRP algorithm to find all values of rows $C[(i-1)\times\sqrt{n\sigma}+j]$ for all $1\leq i\leq\sqrt{\frac{n}{\sigma}},1\leq j\leq\sqrt{n\sigma}$.

Let us give the construction of G_i . For all $1 \le x, y \le n$, we add an edge between a(x) and b(y) if A[x][y] = 1. Similarly we add an edge between b(x) and c(y) if B[x][y] = 1. We add additional vertices in graph $v(1), v(2), \ldots, v(\sqrt{n\sigma})$. We create σ paths such that $P_j = v((j-1)\sqrt{\frac{n}{\sigma}}+1), v((j-1)\sqrt{\frac{n}{\sigma}}+2), \ldots, v(j\sqrt{\frac{n}{\sigma}})$, where $1 \le j \le \sigma$ and each vertex $v(j\sqrt{\frac{n}{\sigma}})$ is a source vertex.

After that, we connect v(j) to $a((i-1)\sqrt{n\sigma}+j)$ by a path of $2((j-1) \mod \sqrt{\frac{n}{\sigma}})+1$ additional vertices, where $1 \le j \le \sqrt{n\sigma}$. Thus, the distance of v(1) from a(1) is 1, distance of v(2) from a(2) is 3 and so on. The distance is again reset at $j=\sqrt{\frac{n}{\sigma}}+1$, the distance of $v(\sqrt{\frac{n}{\sigma}}+1)$ from $a(\sqrt{\frac{n}{\sigma}}+1)$ is 1. This graph construction will give all values of row $C[(i-1)\sqrt{n\sigma}+j]$, for all $1 \le j \le \sqrt{n\sigma}$ by running MSRP algorithm on G_i .

Let us consider first graph G_1 and its first source $v(\sqrt{\frac{n}{\sigma}})$. If the shortest path from $v(\sqrt{\frac{n}{\sigma}})$ to c_ℓ for all $1 \le \ell \le n$ is of length $\sqrt{\frac{n}{\sigma}} + 3$ then we set $C[1][\ell] = 1$ (in this case path will be $v(\sqrt{\frac{n}{\sigma}}), v(\sqrt{\frac{n}{\sigma}} - 1), \dots, v(1), a_1, b_{\ell'}, c_\ell)$), otherwise it is 0. For edge failure e(v(1), v(2)), if there is path length is $\sqrt{\frac{n}{\sigma}} + 5$ then $C[2][\ell] = 1$ (in this case path will be $v(\sqrt{\frac{n}{\sigma}}), v(\sqrt{\frac{n}{\sigma}} - 1), \dots, v(2), \dots, a_2, b_{\ell'}, c_\ell)$ in $G_i/e(v(1), v(2))$), otherwise 0. Similarly, one can see that we can find all values of rows $C[1], C[2], \dots, C[\sqrt{n\sigma}]$ by executing MSRP algorithm in G_1 . Thus, by running MSRP algorithm on $G_1, G_2, \dots, G_{\sqrt{\frac{n}{\sigma}}}$, we can find all rows of C.

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