The Only Undoable CRDTs are Counters

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ABSTRACT

In comparing well-known CRDTs representing sets that can grow and shrink, we find caveats. In one, the removal of an element cannot be reliably undone. In another, undesirable states are attainable, such as when an element is present -1 times (and so must be added for the set to become empty). The first lacks a general-purpose undo, while the second acts less like a set and more like a tuple of counters, one per possible element.

Using some group theory, we show that this trade-off is unavoidable: *every undoable CRDT is a tuple of counters*.

1 INTRODUCTION

Conflict-free replicated data types (CRDTs) allow replication of a data structure across multiple machines without risking conflicts between different versions. Even though each machine may concurrently modify its own copy of the data structure, a CRDT guarantees that these concurrent modifications can be merged into a consistent result, upon which the whole network will agree.

Here, we adopt the *operation-based* view of CRDTs [4], in which a CRDT consists of some *state* and some *operations* affecting it, where any two operations that may be performed concurrently must commute. If two machines' local replicas go out of sync by applying different operations concurrently, they can later merge by exchanging logs of applied operations and applying the other machine's operations to their own state. The commutativity condition ensures that both end up in the same final state, despite applying the operations in different orders.

Below, we review several examples of CRDTs for counters and sets. For more details, see Shapiro et al.'s comprehensive survey [3].

1.1 The counter CRDT

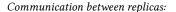
The counter is a simple CRDT, whose state is an integer and whose operations are increment and decrement. These commute, since (n + 1) - 1 = n = (n - 1) + 1. This ensures that once all machines have seen all operations, all will agree on the counter's final value.

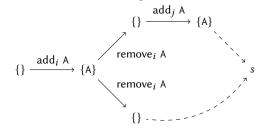
The counter CRDT is *undoable*: After incrementing we may decrement to restore the previous state, and likewise we can undo decrementing by incrementing.

The *modulo-n* counter is a slight variant, where increment wraps around from n - 1 to 0. Like the ordinary counter, all operations on the modulo-*n* counter can be undone.

1.2 The G-Set CRDT

Another simple CRDT is the *grow-only set* or G-Set, whose state is a set of elements and whose operations are add A for each possible element A. Eventual convergence is guaranteed because add A and add B commute.

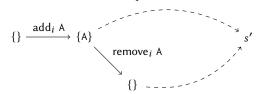




Sequences of operations performed to yield state s:

- $S_1 = add_i A$; remove_i A; add_i A; remove_i A
- $S_2 = \text{add}_i \text{ A}; \text{remove}_i \text{ A}; \text{remove}_i \text{ A}; \text{add}_i \text{ A}$

(a) First replica removes and re-adds Communication between replicas:



Sequences of operations performed to yield state s':

 $S_1 = add_i A$; remove_i A $S_2 = add_i A$; remove_i A

(b) First replica does nothing

Subscripts denote the IDs of add operations (relevant only for OR-Sets)

Figure 1: Example of undoing remove

However, as the name implies, a G-Set can never shrink. There is no remove operation with which to undo an add, and adding one turns out to be tricky. Below, we review several approaches.

1.3 Sets with removal: the OR-Set

In an OR-Set (or *add-wins* set [1]) an element is present if it has been added since it was last removed. We represent this with two G-Sets, *added* and *removed*, each containing pairs of an element and an ID. An element A is deemed present in the set if there is some *i* such that (A, i) is in the added but not the removed set.

The add A operation inserts (A, i) into the added set (with some fresh ID *i*), and the remove A operation inserts (A, j) into the removed set, for each *j* where (A, j) is in the added set.

This means that each remove operation undoes all prior adds. However, undoing a remove is less straightforward. Consider the example in fig. 1a: we start with the empty set and add A to it, at which point two replicas diverge. The first removes and then re-adds A, while the second just removes it. Afterwards, the two replicas merge, yielding state s.

In fig. 1a, S_1 and S_2 describe the sequences of operations performed by the two replicas after merging. The operations commute, so both yield the same final state *s*. In state *s*, both the added and removed sets contain (A, *i*), but only the added set contains (A, *j*). The result is that A is present in the set *s*.

In fig. 1b, instead of removing and re-adding A, the first replica does nothing. Here, A will *not* be present in the final state, as both the added and removed sets contain only (A, i).

We expect that undoing an operation brings us to the same state as if it had never occurred, but this is not the case for OR-Sets. Doing and undoing a remove yields a different result from not removing at all. After removing an element from an OR-Set, there is in general no way to revert to the previous state.

1.4 Sets with removal: the PN-Set

In a PN-Set, an element is present if it has been added more times than it has been removed. The state is an unordered log of operations (add A and remove A), where an element A is deemed present if there are more occurrences of add A than remove A.

On the same examples, the PN-Set gives a different result than the OR-Set. In fig. 1a, in state s, the element A has been added twice and removed twice, and is therefore absent. Similarly, in state s' of fig. 1b, the element A has been added once and removed once, and is therefore absent. Unlike an OR-Set, all PN-Set operations are undoable: add and remove perfectly cancel each other out.

However, the PN-Set allows unexpected extra states. Consider what happens when executing the sequence S_2 from fig. 1a. After performing add A; remove A; remove A, we reach a state in which A is present -1 times: after performing add A, the set will be empty.

This suggests an alternative representation of PN-Sets, as one copy of the counter CRDT for each possible element, where add and remove are implemented as increment and decrement.

1.5 Sets with removal: the T-Set

The extra states of a PN-Set arise because the counters can take values other than 0 and 1. We can eliminate these states by using modulo-2 counters instead of unbounded ones.

However, in a modulo-2 counter increment and decrement are the same operation, so add A and remove A have the same effect, toggling the membership of A. We have eliminated the extra states, but lost the distinction between add and remove.

1.6 A trade-off

In choosing between these CRDTs, we face a trade-off: the OR-Set has intuitive semantics for add and remove, but does not support general undo. The PN-Set and T-Set do support undo, but work more like a tuple of counters than a set, causing side-effects: extra states for PN-Sets and nonstandard semantics for T-Sets.

More sophisticated CRDTs exhibit the same trade-off. For instance, the Logoot-Undo CRDT for collaborative editing [6] allows all operations to be undone and redone, keeping count of how often each operation has been undone. This supports general undo while maintaining commutativity, but like PN-Sets it can be driven to a state where an operation has been performed -1 times, and must be redone to reach the empty state. The generic undo of Yu et al. [7] also keeps undo counters, keeping track of whether an operation has been undone an even or odd number of times (like a T-Set).

The point of this note is that this trade-off is fundamental: *all undoable CRDTs are equivalent to a tuple of counters*.

2 FORMALISING CRDTS

To prove the theorem, we must first formalise undoable CRDTs. We adopt a formulation of operation-based CRDTs close to Shapiro et al.'s CmRDTs, except that we omit some details (e.g. message numbering) that are not relevant here.

A CRDT consists of a set *S* of *abstract states s*, *t*, . . . with a distinguished *initial state s*₀, and a collection *P* of *primitive operations* p, q, \ldots . We assume that *P* is finite, or equivalently that there is some upper bound on the message length needed to communicate a single primitive operation. *S* may be infinite: there may be infinitely many distinct states reachable by sequences of primitive operations.

Each primitive operation $p \in P$ is a partial function from *S* to *S*. That is, not all primitive operations need apply in all states. To reduce parentheses, we write $s \cdot p \cdot q$ instead of q(p(s)). We write $s \cdot p \cdot q$ ok when $s \cdot p \cdot q$ is well-defined: that is, the operation *p* applies in state *s*, and the operation *q* applies in state $s \cdot p$. Note that $s \cdot p \cdot q$ ok implies $s \cdot p$ ok.

For simplicity, we assume that abstract states are neither impossible nor redundant: we assume that distinct members of S represent logically distinct states, and all members of S are reachable by some sequence of primitive operations starting from s_0 . If this isn't true for a concrete implementation, we can choose the abstract states S by discarding unreachable states and picking one representative among groups of logically equivalent states.

The property making states and primitive operations into a CRDT is *commutativity*: any two primitive operations that apply in the same state commute. More formally, the structure is a CRDT if the following axiom is satisfied (Definition 2.6 of Shapiro et al. [4]):

AXIOM 1 (COMMUTATIVITY). If $s \cdot p$ ok and $s \cdot q$ ok, then $s \cdot p \cdot q$ ok, $s \cdot q \cdot p$ ok and $s \cdot p \cdot q = s \cdot q \cdot p$.

Here, we're interested not in plain CRDTs but in *undoable* ones, which also satsify the following:

AXIOM 2 (UNDOABILITY). If $s \cdot p$ ok, then there exists some sequence of primitive operations q_1, \ldots, q_n such that $s \cdot p \cdot q_1 \cdot \ldots \cdot q_n$ is well-defined and equals s.

Usually, a primitive operation p will be undone using just one operation q (so n = 1), but we avoid assuming this.

2.1 From operations to actions

Rather than dealing with individual operations $p, q \in P$, it is more convenient to consider the set P^* of *actions*. An action $a, b \in P^*$ is a finite sequence of primitive operations, which we apply to states using the same notation: if a = pq, then $s \cdot a = s \cdot p \cdot q$. We write ϵ for the empty action (so $s \cdot \epsilon = s$) and *ab* for the concatenation of *a* and *b* (so $s \cdot ab = s \cdot a \cdot b$).

The axioms can be recast in terms of actions (see appendix A):

PROPOSITION 1 (COMMUTATIVITY OF ACTIONS). If $s \cdot a$ ok and $s \cdot b$ ok, then $s \cdot ab$ and $s \cdot ba$ are well-defined and equal.

PROPOSITION 2 (UNDOABILITY OF ACTIONS). If $s \cdot a$ ok, then there exists some action a_s^{-1} such that $s \cdot aa_s^{-1} = s$.

Equivalent CRDTs 2.2

Next, we define what it means for two CRDTs to be equivalent. This is more complicated that merely saying they have the same states and primitive operations, because we want to view the counter CRDT (with increment and decrement operations) as equivalent to a counter CRDT that also exposes an "increment twice" operation.

So, we say that two CRDTs are equivalent if they have the same states and both can implement each other's operations. Formally, a CRDT with states S_1 and primitive operations P_1 is equivalent to one with states S_2 and primitive operations P_2 if there is a oneto-one (invertible) mapping $\phi: S_1 \to S_2$ as well as functions $\psi:$ $P_1 \rightarrow P_2^*$ and $\psi' : P_2 \rightarrow P_1^*$ such that:

- $\phi(s_0) = s'_0$ If $s \cdot p$ ok, then $\phi(s) \cdot \psi(p) = \phi(s \cdot p)$ If $s' \cdot p'$ ok, then $\phi^{-1}(s') \cdot \psi'(p) = \phi^{-1}(s' \cdot p')$

In other words, two equivalent CRDTs are two representations for the same data structure, and we can apply operations to states in either representation. Two machines using equivalent CRDTs can coexist on the same network: as long as they translate their messages back and forth using ψ and ψ' neither will be able to tell that the other is using a different internal representation. For instance, the two representations of PN-Sets in section 1.4 (as unordered logs and as per-element counters) are equivalent.

2.3 The tuple construction

Given two CRDTs A and B, we can combine both into a single CRDT using a straightforward construction. The states of the combined CRDT are pairs (s_A, s_B) of a state of A and a state of B, and all the primitive operations of A and B are primitive operations of the combined CRDT, with the operations of P_A acting on the left half of the state and the operations of P_B acting on the right.

Effectively, the combined CRDT acts as two independent CRDTs, one implementing A and one implementing B. This construction is not limited to just two CRDTs: we may form tuples of n CRDTs in the same way.

This gives us enough ingredients to formally state the theorem:

THEOREM. Every undoable CRDT is equivalent to a tuple of counter and modulo counter CRDTs.

2.4 The group of actions

The proof of this theorem relies on some classical group theory, applied to the group of actions of an undoable CRDT.

First, given any undoable CRDT and a state $s \in S$, we define the relation \equiv_s on actions so that $a \equiv_s b$ whenever $s \cdot a$ ok, $s \cdot b$ ok and $s \cdot a = s \cdot b$. This is a *partial equivalence relation*: it is transitive and symmetric, but not reflexive since $a \equiv_s a$ is not true in general, but only when $s \cdot a$ ok.

Now, given $a_1 \equiv_s b_1$ and $a_2 \equiv_s b_2$, Commutativity tells us that all of a_1, a_2, b_1, b_2 commute with each other (since all apply in state s), and so: $s \cdot a_1 a_2 = s \cdot b_1 a_2 = s \cdot a_2 b_1 = s \cdot b_2 b_1 = s \cdot b_1 b_2$. Therefore:

FACT 1. If $a_1 \equiv_{s} b_1$ and $a_2 \equiv_{s} b_2$, then $a_1 a_2 \equiv_{s} b_1 b_2$

By applying Commutativity with a = b, we learn that any valid action can be done twice (since it may be performed independently by two replicas, which later merge):

FACT 2. If $s \cdot a$ ok, then $s \cdot aa$ ok

Combining this with Undoability, we learn that actions can be undone twice:

FACT 3. If $s \cdot a$ ok, then $(s \cdot a) \cdot a_s^{-1} a_s^{-1}$ ok

But since $s \cdot aa_s^{-1} = s$, this has the surprising consequence that actions can be undone before they are performed:

FACT 4. If $s \cdot a$ ok, then $s \cdot a^{-1}$ ok

Commutativity then tells us that *a* and a_s^{-1} commute:

FACT 5. If $s \cdot a$ ok, then $a_s^{-1}a \equiv_s aa_s^{-1} \equiv_s \epsilon$.

These facts mean that we can form a group $G_s = P^* / \equiv_s$ of the equivalence classes of \equiv_s : concatenation is a binary operation on P^*/\equiv_s thanks to fact 1, and inverses exist thanks to fact 5. In other words, members of the group G_s are denoted by actions that apply to state s, with two actions denoting the same member of the group if they yield the same result when applied to s. Since all members of this group commute, the group is abelian.

2.5 G_s is finitely generated

Just as actions are built out of a finite set *P* of primitive operations, elements of G_s are built out of a finite set P/\equiv_s of generators. To prove this, we first note that any action that can be performed later can be performed now. If $s \cdot ab$ ok, then by fact $5 s \cdot a_s^{-1}$ ok, and so $s \cdot a_s^{-1}ab$ ok by Commutativity, whence:

FACT 6. If $s \cdot ab$ ok, then $s \cdot b$ ok.

Therefore, given any action $a = p_1 p_2 \dots p_n$ such that $s \cdot a$, we have that $s \cdot p_i$ ok for all $1 \le i \le n$: first by noting $s \cdot p_1 p_2 \dots p_i$ ok, and then by applying fact 6. So, each element of G_s can be written as the concatenation of a sequence of primitive operations *p* that apply in state *s*: in other words, G_s is generated by P/\equiv_s .

So, in any undoable CRDT, the actions available from any state have the structure of a *finitely generated abelian group*.

2.6 An old theorem

To show that undoable CRDTs are equivalent to tuples of counters, it's enough that they have isomorphic groups of actions, thanks to the following (proof in appendix):

PROPOSITION 3. If the groups of actions G_{s_0} and $G_{s'_0}$ of two CRDTs are isomorphic, then the CRDTs are equivalent.

The groups of actions of any counter CRDTs is a cyclic group: either \mathbb{Z} , the group of integers with addition (for unbounded counters), or \mathbb{Z}_n , the group of integers with addition modulo *n* (for counters modulo *n*). The group of actions of a tuple of *n* CRDTs is given

by an action for each of the *n* components of the tuple, composed pointwise: this is the *direct sum* of their groups of actions.

Since the group of actions of an undoable CRDT is a finitely generated abelian group, our theorem follows from an old result, the *fundamental theorem of finitely generated abelian groups*:

THEOREM (POINCARÉ 1900; KRONECKER 1870, NOETHER 1926). Every finitely generated abelian group is isomorphic to the direct sum of finitely many cyclic groups.

See e.g. Rotman [2, p.318] for a proof, or Stillwell [5, p.175] for a proof and some history.

3 DISCUSSION

This characterisation of undoable CRDTs has a number of immediate consequences, including:

- All operations are always valid For instance, an undoable CRDT cannot represent a nonnegative counter, in which decrement is available only in nonzero states.
- **Negative states always exist** For any action *a*, there is some state *s* in which applying *a* will bring us back to the initial state s_0 .

In the specific example of set CRDTs, we see that the trade-off described in section 1.6 is unavoidable: in any CRDT with 2^n states representing presence or absence of *n* elements, one of the following must be true:

- Some operations are not undoable (like OR-Set)
- There are an infinite number of extra states, beyond the 2ⁿ states representing membership (like PN-Set)
- All operations must be cyclic, undoing themselves after some number of iterations (like T-Set)

In light of this, designers of distributed data structures must limit themselves to tuples of counters, accept that some operations will not be fully undoable, or use something other than CRDTs.

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REFERENCES

- A. Bieniusa, M. Zawirski, N. Preguiça, M. Shapiro, C. Baquero, V. Balegas, and S. Duarte. An optimized conflict-free replicated set. Research Report RR-8083, INRIA, Oct. 2012. URL https://hal.inria.fr/hal-00738680.
- [2] J. J. Rotman. An introduction to the theory of groups, volume 148. Springer Science & Business Media, 2012.
- [3] M. Shapiro, N. Preguiça, C. Baquero, and M. Zawirski. A comprehensive study of Convergent and Commutative Replicated Data Types. Research Report RR-7506, INRIA, Jan. 2011. URL https://hal.inria.fr/inria-00555588.
- [4] M. Shapiro, N. Preguiça, C. Baquero, and M. Zawirski. Conflict-free replicated data types. In Symposium on Self-Stabilizing Systems, pages 386–400. Springer, 2011.
- [5] J. Stillwell. Classical topology and combinatorial group theory, volume 72. Springer Science & Business Media, 2012.
- [6] S. Weiss, P. Urso, and P. Molli. Logoot-undo: Distributed collaborative editing system on p2p networks. *IEEE Transactions on Parallel and Distributed Systems*, 21(8):1162–1174, 2010.
- [7] W. Yu, V. Elvinger, and C.-L. Ignat. A Generic Undo Support for State-Based CRDTs. In P. Felber, R. Friedman, S. Gilbert, and A. Miller, editors, 23rd International Conference on Principles of Distributed Systems (OPODIS 2019), volume 153 of LIPIcs, pages 14:1–14:17. Schloss Dagstuhl, 2020.

A ADDITIONAL PROOFS

PROOF OF PROPOSITION 1. First, we show that if $s \cdot p$ ok and $s \cdot a$ ok then $pa \equiv_s ap$, by induction on the length of a. If $a = \epsilon$, then the result follows. Otherwise a = qb. By axiom 1, $s \cdot p \cdot q$ and $s \cdot q \cdot p$ are defined and equal. Thus, we have both $(s \cdot q) \cdot p$ ok and $(s \cdot q) \cdot b$ ok, so the inductive hypothesis gives $pb \equiv_{s \cdot q} bp$ or equivalently $qpb \equiv_s qbp$. Gluing these together, we get:

$$pa = pqb \equiv_s qpb \equiv_s qbp = ap$$

Next, we use this fact to prove that if $s \cdot a$ ok and $s \cdot b$ ok then $ab \equiv_s ba$ by similar induction on the length of *b*. If $b = \epsilon$ then the result is again trivial. Otherwise, b = pc and by above, $pa \equiv_s ap$. Thus, we have both $(s \cdot p) \cdot a$ ok and $(s \cdot p) \cdot c$ ok, so the inductive hypothesis gives $ac \equiv_{s \cdot p} ca$ or equivalently $pac \equiv_s pca$, leading to:

$$ab = apc \equiv_s pac \equiv_s pca = ba$$

PROOF OF PROPOSITION 2. Again, we proceed by induction on *a*. If $a = \epsilon$, then $a_s^{-1} = \epsilon$ suffices. Otherwise a = bp, so we choose $a_s^{-1} = q_1q_2 \dots q_nb_s^{-1}$, where q_i are those given by axiom 2 for state $s \cdot b$. Then:

$$s \cdot bpq_1q_2 \dots q_n b_s^{-1} = s \cdot bb_s^{-1} = s \qquad \Box$$

PROOF OF PROPOSITION 3. Given a CRDT with states S_1 and operations P_1 and one with states S_2 and operations P_2 , suppose that an isomorphism ψ exists between G_{s_0} and $G_{s'_0}$. We define the mappings $\phi : S_1 \rightarrow S_2, \phi^{-1} : S_2 \rightarrow S_1$ as follows:

$$\phi(s) = s'_0 \cdot \psi(a) \qquad \text{for some } a \text{ such that } s_0 \cdot a = s$$

$$\phi^{-1}(s') = s_0 \cdot \psi^{-1}(a') \qquad \text{for some } a' \text{ such that } s'_0 \cdot a' = s'$$

Such actions a, a' must exist because all states are reachable in both CRDTs. If several are possible, the choice of a, a' does not matter, since ψ respects \equiv_{s_0} and so must map all such a to equivalent actions.

These functions are inverses:

$$\phi^{-1}(\phi(s)) = \phi^{-1}(s'_0 \cdot \psi(a)) = s_0 \cdot \psi^{-1}(a')$$

where $s'_0 \cdot a' = s'_0 \cdot \psi(a)$
 $s_0 \cdot a = s$

Since $a' \equiv_{s'_0} \psi(a)$,

$$\psi^{-1}(a') \equiv_{s_0} \psi^{-1}(\psi(a)) \equiv_{s_0} a$$

So, $s_0 \cdot \psi^{-1}(a') = s_0 \cdot a = s$. The proof that $\phi(\phi^{-1}(s)) = s$ is identical.

From ψ , we get a mapping $P_1 \rightarrow P_2^*$ (and likewise ψ^{-1} gives a mapping $P_2 \rightarrow P_1^*$). To prove the CRDTs equivalent, we must show that these satisfy the three conditions from section 2.2:

• $\phi(s_0) = s'_0 \cdot \psi(a)$ where $s_0 \cdot a = s_0$. But since $a \equiv_{s_0} \epsilon, \psi(a) \equiv_{s_0} \epsilon$ and so $\phi(s_0) = s'_0$.

Suppose
$$s \cdot p$$
 ok. Then, for some *a* where $s_0 \cdot a = s$

$$\phi(s) \cdot \psi(p) = s_0 \cdot \psi(a) \cdot \psi(p) = s_0 \cdot \psi(ap) = \phi(s \cdot p)$$

As above