Climbing LP Algorithms

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Abstract

NP (search) problems allow easy correctness tests for solutions. *Climbing* algorithms allow also easy assessment of how close to yielding the correct answer is the configuration at any stage of their run. This offers a great flexibility, as how sensible is any deviation from the standard procedures can be instantly assessed.

An example is the ***Dual Matrix Algorithm (DMA)*** for linear programming, variations of which were considered by A.Y.Levin in 1965 and by Yamnitsky and myself in 1982. It has little sensitivity to numerical errors and to the number of inequalities. It offers substantial flexibility and, thus, potential for further developments.

1 Introduction

A ***Climbing*** algorithm $A(x)$ is supplied with an easily computable ***valuation*** $V(s)$. A runs by iterating a ***step*** transformation $T(s)$ starting from $s \leftarrow x$ and ending with output $y \leftarrow s$ once the required valuation $V(s) = R(x)$ is reached. The efficiency of the algorithm reflects the effort required to compute T, V and the progress in V toward $R(x)$ each iteration of T assures.

The following account is an example of such algorithm. It adds algebraic details to previously published geometric versions. Some of these details exploit the flexibility provided by its climbing nature to help with worst case-performance by deviating in appropriate cases from the standard procedure. The main point is to emphasize the flexibility assured by this climbing nature as an example that may be useful to follow for some other algorithms and problems.

1.1 The Idea

Notations. $\overset{\text{df}}{=}$ indicates definitions. $\det(C)$ denotes the determinant of a matrix C; the Euclidean norm of a vector b is $|b|$. For clarity row vectors may be underlined: $\underline{1} = (1, \ldots, 1)$, column overlined: $\overline{\mathbf{0}} = (0, \ldots, 0)$. $\mathbf{I}$ is the identity matrix, $\underline{e}_k, \overline{e}_k$ are its k-th row and column. We are to solve a system of inequalities $Ax > \overline{\mathbf{0}}$ for a rational vector x, given an $m \times n$ integer matrix A. Our inequalities are $a_k x > 0$ for $a_k \overset{\text{df}}{=} \underline{e}_k A$, linearly independent for $k \le n$. Let n and all entries of A be $< l$ bits long, so each a_k has $< L = nl$ bits. By Hahn-Banach Theorem, the system $Ax > \overline{\mathbf{0}}$ is inconsistent iff $bA = \underline{0}$ for some vector $\underline{b} \ge \underline{0} \ne b$. The same holds if $|bA| < b\overline{1}/4^L$.

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The DMA searches for b in the form dB, where matrix B has no negative entries, $C \overset{\text{df}}{=} BA = V^{-1}$, $d \overset{\text{df}}{=} uV = \underline{1}D > \underline{0}$ for a diagonal D, $u \overset{\text{df}}{=} \sum_{k \le n} a_k$. We must grow $b\overline{1}$ to $>4^L$. This growth is hard to keep monotone, so a lower bound $\log(n! \det(DC)) < n \log(b\overline{1}) + 3L$ is grown instead. It is $-\log$ of the **volume of the simplex** $\lhd^B$ with faces $Cx = \overline{\mathbf{0}}$, $ux = 1$, vertices $VD^{-1}\overline{e}_k$, $\overline{\mathbf{0}}$, and center $\mathbf{v} = VD^{-1}\overline{1}/(n+1)$. The original simplex $\lhd^o$ starts with $B^o_{k,k} = 1$, $B^o_{k,k' \ne k} = 0$.

It turns out that by incrementing a single entry of B one can always increase $\ln\det(DC)$ by $> 1/2n^2$, as long as $x = \mathbf{v}$ fails the $Ax > \overline{\mathbf{0}}$ requirement. This provides an $O(n^3 L)$ steps algorithm. Each step takes $O(n^2)$ arithmetic operations, on $O(L)$-bit numbers and one call of a procedure which points an inequality $a_k x > 0$ violated by a given solution candidate x. This call is the only operation that may depend on the number m of inequalities, which could even be an infinite family with an oracle providing the violated a_k.

1.2 Some Comparisons

The above bound has n times more steps than Ellipsoid Method (EM). However, the EM is much more demanding with respect to the precision with which the numbers are to be kept. The simplex $\lhd^B$ cannot possibly fail to include all solutions of $Ax > \overline{\mathbf{0}}$, $ux < 1$, *whatever* B with no negative entries is taken. In contrast, the faithful transformation of ellipsoids in the EM is the only guarantee that they include all solutions.

Also, for $m = O(n)$ several Karmarkar-type algorithms have lower polynomial complexity bounds. Yet, they work in the dual space and their bounds are in terms of the number m of inequalities, while the above DMA bound is in terms of n. For DMA, m may even be infinite, e.g., forming a ball instead of a polyhedron. Then dual-space complexity bounds break down, while the DMA complexity is not affected (as long as a simple procedure finds a violated inequality for any candidate x).

To assure fast progress, numbers are kept with $O(L)$ digits. This bound cannot be improved since some consistent systems have no solutions with shorter entries. Yet, this or any other precision is not actually *required* by DMA. Any rounding (or, indeed, any other deviation from the procedure) can be made any time as long as $\log\det(DC)$ keeps growing, which is immediately observable. This leaves DMA open to a great variety of further developments. In contrast, an inappropriate rounding in the EM, can yield ellipsoids which, while still shrinking fast, lose any relation to the set of solutions and produce a false negative output.

1.3 A Historical Background

The bound $\det(DC)$ is inversely proportional to the volume of $\triangleleft^B$, which parallels the EM. Interestingly, in history this parallel went in the opposite direction: The simplexes enclosing the solutions were first used by A.Y.Levin in 1965 [1] and the EM was developed by Nemirovsky and Yudin in 1976 [2] as their easier to analyze variation.

The A.Y.Levin's algorithm starts with a large simplex guaranteed to contain all solutions. Its center of gravity is checked, and, if it fails some inequality, the corresponding hyperplane cuts out a "half" of the simplex. The process repeats with the resulting polyhedron. Each cut decreases the volume by a constant factor and so, after some number $q(n)$ of steps the remaining body can be re-enclosed in a new smaller simplex. Only a weak upper bound $q(n) < n\log n$ was proven by A.Y.Levin; it did not preclude the simplex from turning into a too complex to manipulate in polynomial time polyhedron.

Nemirovsky and Yudin replaced simplexes with ellipsoids and made $q(n) = 1$. Both they and A.Y.Levin used real numbers and looked for approximate solutions with a given accuracy. Khachian in 1979 [3] modified the EM for rationals and exact solutions. Yamnitsky and myself in 1982 [4] proved $q(n) = 1$ for the original A.Y.Levin's simplex splitting method. Below, an algebraic version of that geometric algorithm and some implementation improvements are considered and analyzed.

2 Main Algorithm and Analysis

Let $u \overset{\mathrm{df}}{=} \sum_{k\le n} a_k$, $C \overset{\mathrm{df}}{=} BA{=}V^{-1}$, $d \overset{\mathrm{df}}{=} uV$, $d_k \overset{\mathrm{df}}{=} d\overline{e}_k$, $c_k \overset{\mathrm{df}}{=} \underline{e}_k C$, $v_k \overset{\mathrm{df}}{=} V\overline{e}_k/d_k$.
For some i,j,s let $a \overset{\mathrm{df}}{=} a_i$, $v \overset{\mathrm{df}}{=} v_j$, $t \overset{\mathrm{df}}{=} (s^2{-}1)av$. Then $C' \overset{\mathrm{df}}{=} (B{+}\overline{e}_j\underline{e}_i/td_j)A{=}(V')^{-1}$, $d' \overset{\mathrm{df}}{=} uV'$, $\delta_k \overset{\mathrm{df}}{=} d'\overline{e}_k/d_k$.

With $\sigma \overset{\mathrm{df}}{=} \mathbf{I}{+}\overline{e}_j aV/td_j$, $C'{=}\sigma C$, $\det(\sigma){=}(1{+}av/t) = 1{+}1/(s^2{-}1)$, $V{=}V'\sigma$, $d{=}d'\sigma$. Thus, $d_k{=}d'\sigma\overline{e}_k = d_k\delta_k{+}\delta_j av_k d_k/t$ and $1{=}\delta_k{+}\delta_j av_k/t{=}\delta_k{+}\frac{\delta_j av_k}{(s^2-1)av}$.
Taking $k{=}j$, $1{=}\delta_j(1{+}1/(s^2{-}1))$, $\delta_j{=}(s^2{-}1)/s^2 = 1{-}1/s^2 = 1/\det(\sigma)$, $\delta_k{=}1{-}\frac{av_k}{s^2 av}$. Our gain is $\ln\lambda$ for $\lambda \overset{\mathrm{df}}{=} \prod_k \delta_k \det(C')/\det(C) = \prod_k \delta_k \det(\sigma){=}\prod_{k\ne j}\delta_k$.

Now $\mathbf{v} \overset{\mathrm{df}}{=} \sum_k v_k$ take i,j with $a_i\mathbf{v} \le 0$, $a_iv_j{=}\max_k a_iv_k$. Then $\delta_k \ge \delta_j$, $\sum_k \delta_k = n - \frac{a\mathbf{v}}{s^2av} \ge n$, and $\prod_{k\ne j}\delta_k \ge \delta_j^{(n-2)}(\delta_j{+}n(1{-}\delta_j)) = (1{-}1/s^2)^{n-1}(1{+}n/(s^2{-}1))$.

So, $\ln\lambda \ge (n{-}1)\ln(1{-}1/s^2){+}\ln(1{+}n/(s^2{-}1))$. For $s = n{-}1$ and $f(s) \overset{\mathrm{df}}{=} s\ln(1{+}1/s)$ this is $f(s){-}f(s{-}1) > 1/2n^2$.

This $> 1/2n^2$ gain holds if s is accurate to $O(l)$ digits, so t can be rounded to $O(l)$ significant digits, too.

3 Some Improvements

Inverting matrices may take cubic time, but when a matrix with a known inverse is moderately modified, Sherman-Morrison formula gives its inverse in $O(n^2)$ steps. In our case the inverse of C' is $V' \overset{\mathrm{df}}{=} V{-}\frac{vaV}{avs^2}$. Finally, the following **occasional deviations from the standard step** help the worst-case performance and also illustrate the potential allowed by the flexibility of the algorithm.

Digits. The nodes v_k^o of the starting simplex $\triangleleft^o$ lie in a 4^L ball. Rounding t, DB to $O(L)$ digits preserves the $> 1/2n^2$ gain in steps with $\max_k \log|v_k| < 4L$. Yet, at some steps a longest edge (v_i, v_j) of $\triangleleft^B$ may grow up to $2^{O(nL)}$ long. But there would be only $O(n)$ of such steps, since they allow large gains in $\ln\det(DC)$ as follows. Let $\underline{w} \overset{\mathrm{df}}{=} (v_j{-}v_i)^T$, $M \overset{\mathrm{df}}{=} \max_k wv_k^o$, $m \overset{\mathrm{df}}{=} \max_{k',k} w(v_k^o{-}v_{k'}^o)$, $t' \overset{\mathrm{df}}{=} (wv_j{-}M)/m$, $t \overset{\mathrm{df}}{=} \max\{0, t'{-}1\}/(|w|^2 d_i)$, $h_{i,j} \overset{\mathrm{df}}{=} (Mu - w)t$. $\triangleleft^B$ has area p of its projection along w and volume $p|w|/n$. Its slice of height $m/|w|{=}O(4^L)$ cut by $h_{i,j}x{\ge}0$, $h_{j,i}x{\ge}0$ encloses $\triangleleft^o$. Note that $h_{i,j} = bA$ for $b \overset{\mathrm{df}}{=} h_{i,j}V^oB^o = t(M\underline{1}{-}wV^o)B^o \ge \underline{0}$. The new simplex replaces faces c_i, c_j in $\triangleleft^B$ with $c_i{+}h_{i,j}$, $c_j{+}h_{j,i}$. It is up to 3 times wider and higher than the above slice. So its volume is $O(4^L 3^n p)$. This gains $> \log|w| - 3L$ in $\log\det DC$.

Maintaining Sparsity. Given a row $\underline{b}$ of B let $S \overset{\mathrm{df}}{=} \{i : b_i{\ne}0\}$. When $|S|$ exceeds $2n$, we can (in $O(n^3)$ arithmetic operations) simplify B to get $|S|{\le}n$ without changing C. We find a set F of $|S|{-}n$ linearly independent vectors $\underline{f}$ such that $\{i : f_i{\ne}0\} \subset S$ and $fA{=}\underline{0}$.

Then repeat the following. Use an $f{\in}F$ and $i{\in}S$ with maximal $\frac{|f_i|}{b_i}$ to annul b_i via $b \leftarrow b - \frac{b_i}{f_i}f$. This preserves bA, keeps $b{\ge}\underline{0}$, and shrinks S. As S looses an entry i we use an $f{\in}F$ to annul the i-th component in all other vectors in F and drop f from F.

References

[1] A.Yu. Levin. 1965. On an Algorithm for the Minimization of Convex Functions. *Soviet Math., Doklady,* 6:286-290.

[2] D.B. Yudin, A.S. Nemirovsky. 1976. Informational Complexity and Effective Methods for Solving Convex Extremum Problems. *Economica i Mat. Metody,* 12(2):128-142; transl. *MatEcon* 13:3-25.

[3] L.G. Khachian. 1979. A Polynomial Algorithm for Linear Programming. *Soviet Math., Doklady,* 20(1):191-194.

[4] B. Yamnitsky, L.A. Levin. 1982. An Old Linear Programming Algorithm Runs in Polynomial Time. FOCS-82, pp: 327-328