

# Optimal Pricing For MHR and $\lambda$ -Regular Distributions\*

YIANNIS GIANNAKOPOULOS and DIOGO POÇAS, TU Munich, Germany  
KEYU ZHU<sup>†</sup>, Georgia Institute of Technology, USA

We study the performance of anonymous posted-price selling mechanisms for a standard Bayesian auction setting, where  $n$  bidders have i.i.d. valuations for a single item. We show that for the natural class of Monotone Hazard Rate (MHR) distributions, offering the same, take-it-or-leave-it price to all bidders can achieve an (asymptotically) optimal revenue. In particular, the approximation ratio is shown to be  $1 + O(\ln \ln n / \ln n)$ , matched by a tight lower bound for the case of exponential distributions. This improves upon the previously best-known upper bound of  $e/(e-1) \approx 1.58$  for the slightly more general class of regular distributions. In the worst case (over  $n$ ), we still show a global upper bound of 1.35. We give a simple, closed-form description of our prices which, interestingly enough, relies only on minimal knowledge of the prior distribution, namely just the expectation of its second-highest order statistic.

Furthermore, we extend our techniques to handle the more general class of  $\lambda$ -regular distributions that interpolate between MHR ( $\lambda = 0$ ) and regular ( $\lambda = 1$ ). Our anonymous pricing rule now results in an asymptotic approximation ratio that ranges smoothly, with respect to  $\lambda$ , from 1 (MHR distributions) to  $e/(e-1)$  (regular distributions). Finally, we explicitly give a class of continuous distributions that provide matching lower bounds, for every  $\lambda$ .

CCS Concepts: • **Theory of computation** → **Computational pricing and auctions**; *Approximation algorithms analysis*; *Algorithmic mechanism design*.

Additional Key Words and Phrases: pricing, optimal auctions, hazard rate distributions, regular distributions,  $\lambda$ -regularity

## 1 INTRODUCTION

In this paper we study a traditional Myersonian auction setting: an auctioneer has an item to sell and he is facing  $n$  potential buyers. Each buyer has a (private) valuation for the item, and these valuations are i.i.d. according to some known continuous probability distribution  $F$ . You can think of this valuation, as modelling the amount of money that the buyer is willing to spend in order to get the item. An auction is a mechanism that receives as input a bid from each buyer, and then decides if the item is going to be sold and to whom, and for what price. Our goal is to design auctions that maximize the seller's expected revenue.

We focus only on truthful auctions, that is, selling mechanisms that give no incentives to the bidders to lie about their true valuation. Such auctions are both conceptually and practically convenient. This restriction is essentially without loss for our revenue maximization objective, due to the Revelation Principle<sup>1</sup>.

In general, such an optimal auction can be rather complicated and even randomized (aka a lottery). However, in his celebrated result, Myerson [44] proved that (under some standard assumptions on

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<sup>1</sup>In this paper we will avoid discussing such subtler issues as implementability and truthfulness, since our goal is to study the performance of specific and very simple pricing mechanisms. The interested reader is pointed to [45] as a good starting point for a deeper investigation of those ideas.

the valuations’ distribution) revenue maximization can be achieved by a very simple deterministic mechanism, namely a second-price auction paired with a reserve value  $r$ . In such an auction, all buyers with bids smaller than  $r$  are ignored and the item is sold to the highest bidder for a price equal to the second-highest bid (or  $r$ , if no other bidder remains). Equivalently, you can think of this as the seller himself taking part in the auction, with a bid equal to  $r$ , and simply running a standard, Vickrey second-price auction; if the auctioneer is the winning bidder, then the item stays with him, that is, it remains unsold. Furthermore, Bulow and Klemperer [12] essentially showed that we can still guarantee a  $1 - \frac{1}{n}$  fraction of this optimal revenue, even if we drop the reserve price  $r$  completely and use just a standard second-price auction.

No matter how simple and powerful the above optimal auction seems, it still requires explicitly soliciting bids from all buyers and using the second-highest as the “critical payment”; this is essentially a centralized solution, that asks for a certain degree of coordination. Arguably, there is an even simpler selling mechanism which, as a matter of fact, is being used extensively in practice, known as *anonymous pricing*: the seller simply decides on a selling price  $p$ , and then the item goes to any buyer that can afford it (breaking ties arbitrarily); that is, we sell the item to any bidder with a valuation greater or equal to  $p$ , for a price of exactly  $p$ .

The question we investigate in this paper, is how well can such an extremely simple selling mechanism perform when compared to an arbitrary, optimal auction. We resolve this in a very positive way proving that, under natural assumptions on the valuation distribution, as the number of buyers grows large, anonymous pricing achieves optimal revenue. More precisely, its approximation ratio is  $1 + O(\ln \ln n / \ln n)$ . Furthermore, we show that in order to get such a near-optimal performance, the seller does not really need to have full knowledge of the bidders’ population; he just needs to know the expectation of the second-highest order statistic of the valuation distribution, that is, (a good estimate of) the *expected* second-highest bid is enough. Finally, we demonstrate how this approximation ratio deteriorates as we gradually relax our distributional assumptions.

## 1.1 Related Work

The seminal reference in auction theory is the work of Myerson [44] who completely characterized the revenue-maximizing auction in single-item settings with bidder valuations drawn from independent (but not necessarily identical) distributions. Under his standard regularity condition (see Section 2.1), this optimal auction has a very simple description when the valuation distributions are identical: it is a second-price auction with a reserve. Furthermore, there is an elegant, closed-form formula that gives the reserve price (see Section 2).

One can achieve good, constant approximations to that optimal revenue by using even simpler auctions, namely *anonymous pricing* mechanisms. These mechanisms offer the same take-it-or-leave-it price to all bidders, and the item is sold to someone who can afford it (breaking ties arbitrarily). An upper bound of  $e/(e-1) \approx 1.58$  on the approximation ratio of anonymous pricing can be shown from the work of Chawla et al. [16]. Blumrosen and Holenstein [11] study the asymptotic performance of pricing when the number of bidders grows large and demonstrate a lower bound on the approximation ratio of  $0.88/0.65 = 1.37$  for anonymous pricing. If we allow for non-continuous distributions that have point-masses, then Dütting et al. [24] provide a matching lower bound of  $e/(e-1)$ . Although the class of MHR distributions (see Section 2.1) is a natural restriction of Myerson’s regularity, that has been extensively studied in optimal auction theory, mechanism design and complexity to derive powerful positive results (see, e.g., [6, 10, 13, 22, 23, 30, 31, 37]), no better bounds are known for anonymous pricing in this class. This is one of our goals in this paper.

Schweizer and Szech [48] propose a quantitative notion of regularity, termed  $\lambda$ -regularity (see Section 2.1), that allows for a smooth interpolation between the general class of regularity à la Myerson and its MHR restriction. This is essentially equivalent to the notion of  $\alpha$ -strong regularity

of Cole and Roughgarden [18] (for  $\alpha = \lambda - 1$ ) and  $\rho$ -concavity of Caplin and Nalebuff [14] (for  $\rho = -\lambda$ ). These parametrizations have proved very useful in developing a fruitful and more “fine-grained” theory of optimal auctions (see, e.g., [17, 26, 42, 43]).

Although not immediately related to our model, an important line of work studies the performance of “simple” auctions, such as pricing and auctions with reserves, for the more general case where bidders’ valuations may be non-identically distributed. In such settings, the elegance of Myerson’s characterization is not in effect any more, and the optimal auction can be rather complicated. Nevertheless, in an influential paper, Hartline and Roughgarden [37] showed that, for regular distributions, a second-price auction with a single anonymous reserve guarantees a 4-approximation to the optimal ratio, and also provided a lower bound of 2. This upper bound was subsequently improved to  $e \approx 2.72$  by Alaei et al. [3], achieved even by the simpler class of anonymous pricing mechanisms. At the same paper, they also provided a lower bound of 2.23 for the approximation ratio of anonymous pricing for non-i.i.d. bidders. This was recently improved to 2.62 [40] and proven to be tight by Jin et al. [39]. For bounds on the approximation ratios between different pricing and reserve mechanisms, under various assumptions on the underlying distributions and the order of the bidders’ arrival, see [3, 16, 24, 40] and [36, Chapter 4]. For anonymous pricing beyond the standard setting of linear utilities see the very recent work of Feng et al. [27].

Finally, we briefly mention that there is a very rich theory about sequential pricing that deals with dynamically arriving buyers and which is inspired by and related to secretary-like online problems and the powerful theory of prophet inequalities. See, e.g., [2, 15, 16, 20, 34, 41, 49]. An intriguing equivalence between pricing and threshold stopping rules was recently established by Correa et al. [21]. In particular, Correa et al. [19] showed an upper bound of 1.34 on the approximation ratio of sequential pricing in the i.i.d. auction model which translates to the same bound for the i.i.d. prophet setting; this resolved a long-standing open question from Hill and Kertz [38]. Similarly, it is not difficult to see that our setting of regular i.i.d. anonymous pricing corresponds to single-threshold rules for the i.i.d. prophet model, for which a tight bound of 1.58 is known [25, 38].

## 1.2 Our Results

In this paper we study the performance of anonymous pricing mechanisms in single-item auction settings with  $n$  bidders that have i.i.d. valuations from the same regular distribution  $F$ . These mechanisms are extremely simple: the seller simply offers the same take-it-or-leave-it price  $p$  to all potential buyers; the item is then sold to a buyer that can meet this price, that is, has a valuation greater than or equal to  $p$ ; the winning bidder pays  $p$  to the seller. Our benchmark is the seller’s expected revenue (with respect to his incomplete, prior knowledge of the buyers’ bids via distribution  $F$ ) and we compare against the maximum revenue achievable by any auction. For our particular model, this optimal auction is a second-price auction with a reserve [44].

Our main result (Section 4.1; see also Fig. 1) is an explicit, closed-form upper bound on the approximation ratio of the revenue of anonymous pricing for MHR distributions. As the number  $n$  of buyers grows large, this ratio tends to the optimal value of 1, at a rate of  $1 + O(\ln \ln n / \ln n)$  (Theorem 4.1). Additionally, we design an upper bound that is fine-tuned to handle also small values of  $n$  (Theorem 4.2), and using this we provide a global, worst-case (with respect to  $n$ ) upper bound of 1.35 on the approximation ratio. Previously, only an upper bound of  $e/(e - 1) \approx 1.58$  was known (for any value of  $n$ ), holding for the entire class of regular distributions.

In Section 4.3 we demonstrate how the aforementioned positive guarantee on the revenue of anonymous pricing can still be (within an exponentially decreasing *additive* constant) achieved even if the seller does not have full knowledge of the prior distribution  $F$  (see Fig. 2). In particular (Theorem 4.6), we give an explicit formula for such a “good” pricing rule that only depends on the expectation of the second-highest order statistic of  $F$ .

To complete the picture, in [Section 4.2](#) we prove that our upper bound analysis is essentially tight, by showing that the exponential distribution provides an (almost) tight gap instance between the revenue of anonymous pricing and that of the optimal auction ([Theorem 4.4](#); see also [Fig. 2](#)).

Finally, in [Section 5](#) we relax our MHR assumption, allowing for  $\lambda$ -regular valuation distributions. This provides a smooth, parametrized generalization of the MHR condition ( $\lambda = 0$ ), all the way to the entire class of standard (Myersonian) regularity ( $\lambda = 1$ ). Extending our ideas from [Section 4](#), we are able to provide upper ([Theorem 5.1](#)) and lower ([Theorem 5.3](#)) bounds on the approximation ratio of anonymous pricing, for all  $\lambda \in [0, 1]$  (see also [Fig. 3](#)). We conclude in [Section 5.1](#), by looking how these bounds behave in the limit as the number of bidders grows arbitrarily large; we derive a tight value for the approximation ratio as a function of  $\lambda$ , which ranges smoothly from optimality for MHR distributions ( $\lambda = 0$ ) to  $e/(e - 1) \approx 1.58$  for the most general case of regular distributions ( $\lambda = 1$ ).

We conclude in [Section 6](#) with some open questions for future work.

*1.2.1 Techniques.* Our upper bound technique differs from related previous approaches [[3](#), [16](#)] in that we do not use the ex-ante relaxation of the revenue-maximization objective. Instead, we deploy explicit upper bounds on the optimal revenue ([Section 3.1](#)) that depend on key parameters of the valuation distribution  $F$ , namely its order statistics and its monopoly reserve. Then, we pair these with a range of critical properties of  $\lambda$ -regular distributions that we develop in [Sections 3.2](#) and [3.3](#). We believe that some of these auxiliary results may be of independent interest, in particular the order statistics tail-bounds of [Lemmas 3.3](#) and [3.6](#) and the reserve-quantile optimal revenue bound of [Lemma 3.4](#) for the special case of MHR distributions.

Our lower bounds are constructed by means of explicitly defining a family of  $\lambda$ -regular, continuous valuation distributions (see [\(19\)](#)), that act as “bad” instances for any  $\lambda$ . We achieve this by generalizing an instance by Dütting et al. [[24](#)] (tailored to the special case of regular distributions) to work for general  $\lambda$ ’s, while at the same time “smoothing it out” to satisfy continuity.

## 2 MODEL AND NOTATION

A seller wants to sell a single item to  $n \geq 2$  bidders. The valuations of the bidders for the item are i.i.d. from a continuous probability distribution supported over an interval  $D_F \subseteq [0, \infty)$ , with cdf  $F$  and pdf  $f$ . Throughout this paper we will assume that  $F$  is  $\lambda$ -regular, for some real parameter  $\lambda \in [0, 1]$  (for formal definitions and discussion, see [Section 2.1](#) right below). For a random variable  $X \sim F$  drawn from  $F$  and  $1 \leq k \leq n$ , we will use  $X_{k:n}$  to denote the  $k$ -th lowest order statistic out of  $n$  i.i.d. draws from  $F$ . That is,  $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ . For completeness and ease of reference, we discuss some useful properties of order statistics in [Appendix A](#).

A pricing mechanism that offers a take-it-or-leave-it price of  $p \in D_F$  to all bidders gives to the seller an expected revenue of

$$\text{PRICE}(F, n, p) \equiv p[1 - F^n(p)],$$

since the probability of no bidder being able to afford price  $p$  is  $F^n(p)$ . We will refer to such a mechanism simply as (*anonymous*) pricing. Thus, the optimal (maximum) revenue achievable via pricing is

$$\text{PRICE}(F, n) \equiv \sup_{p \in D_F} \text{PRICE}(F, n, p).$$

On the other hand, as discussed in the introduction, the optimal revenue attainable by any mechanism may be higher; as a matter of fact, Myerson [[44](#)] showed that it is achieved by a second-price

auction with a reserve equal to the *monopoly reserve*<sup>2</sup>

$$r^* \equiv \operatorname{argmax}_{r \in D_F} r(1 - F(r)) . \quad (1)$$

of the valuation distribution. We denote this optimum revenue by  $\text{MYERSON}(F, n)$ , and it can be shown that

$$\text{MYERSON}(F, n) \equiv \mathbb{E} [\max \{0, \phi(X_{n:n})\}] ,$$

where  $\phi(x) = x - \frac{1-F(x)}{f(x)}$  is the (nondecreasing) virtual valuation function of  $F$  (see [Section 2.1](#)) and  $X_{n:n}$  its maximum order statistic. Keep in mind that, due to the monotonicity of  $\phi$  and the definition of the reserve  $r^*$ , we know that  $\phi(x) \geq 0$  for all  $x \geq r^*$ .

Sometimes it is more convenient to work in quantile space instead of the actual valuation domain. More precisely, the quantile of distribution  $F$  corresponding to a value  $x \in D_F$  is  $q(x) = 1 - F(x)$ . Using this, we can define what is known as the *revenue curve* of distribution  $F$ , by

$$R(q) \equiv F^{-1}(1 - q) \cdot q . \quad (2)$$

In other words, if  $p \in D_F$  is a price and  $q$  is its corresponding quantile, then  $R(q)$  is the expected revenue of selling the item to a single bidder, using a price  $p$ . Thus, the monopoly reserve quantile  $q^*$  that corresponds to the monopoly reserve  $r^*$  defined in (1) is exactly a maximizer of the revenue curve  $R(q)$ . So, for a single bidder ( $n = 1$ ):

$$\text{MYERSON}(F, 1) = \text{PRICE}(F, 1) = \sup_{p \in D_F} p(1 - F(p)) = \sup_{q \in [0, 1]} R(q) = R(q^*) .$$

In general though for more players ( $n \geq 2$ ) this is not the case, and our goal in this paper is exactly to study how well the optimal revenue  $\text{MYERSON}(F, n)$  can be approximated by pricing  $\text{PRICE}(F, n)$ . That is, we want to bound the following approximation ratio:

$$\text{APX}(F, n) \equiv \frac{\text{MYERSON}(F, n)}{\text{PRICE}(F, n)} .$$

Finally, we use  $H_n$  to denote the  $n$ -th harmonic number  $H_n = \sum_{i=1}^n \frac{1}{i}$ ,  $\gamma \approx 0.577$  for the Euler-Mascheroni constant (see also [Lemma B.1](#)) and  $\Gamma, B$  for the standard gamma and beta functions:  $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ ,  $B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$ . Recall that  $\Gamma(n+1) = n!$  for any nonnegative integer  $n$  and  $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$  for all reals  $x, y > 0$  (see, e.g., [46, Chapter 5]).

## 2.1 $\lambda$ -Regular Distributions

Consider a continuous distribution  $F$  supported over an interval  $D_F$  of nonnegative reals, and a real parameter  $\lambda \in [0, 1]$ . We will say that  $F$  is  $\lambda$ -regular if its  $\lambda$ -generalized virtual valuation function

$$\phi_\lambda(x) \equiv \lambda \cdot x - \frac{1 - F(x)}{f(x)}$$

<sup>2</sup>As will become clearer in the following [Section 2.1](#), this quantity is well-defined for all  $\lambda$ -regular distributions with  $\lambda \in [0, 1)$ . For the special case of  $\lambda = 1$ , it might happen that there are multiple maximizers in (1), or even none. To formally deal with such pathological cases, in the former we can take  $r^* = \inf \operatorname{argmax}_{r \in D_F} r(1 - F(r))$ . In the latter we can slightly abuse notation and use  $r^* = \infty$ , which is essentially equivalent to using an arbitrarily large price to approximate within arbitrary accuracy the value of  $\sup_{r \in D_F} r(1 - F(r))$ ; then, the corresponding reserve quantile is  $q^* = 0$  and maximizes the revenue curve in (2) (for a more detailed and rigorous discussion of this issue see, e.g., [28, Appendix 1] and [29, Appendix C]). The aforementioned choices do not affect the validity of any of the results in the rest of our paper.

is monotonically nondecreasing in  $D_F$ . From Schweizer and Szech [48, Proposition 1(iii)], this can be equivalently restated as

$$\begin{cases} [1 - F(x)]^{-\lambda} \text{ is convex,} & \text{if } \lambda \in (0, 1], \\ \ln[1 - F(x)] \text{ is concave,} & \text{if } \lambda = 0. \end{cases}$$

We will use  $\mathcal{D}_\lambda$  to denote the family of all  $\lambda$ -regular distributions. It is not difficult to see that, for any  $0 \leq \lambda \leq \lambda' \leq 1$ , any  $\lambda$ -regular distribution is also  $\lambda'$ -regular. In other words,  $\mathcal{D}_\lambda \subseteq \mathcal{D}_{\lambda'}$ . As a matter of fact, this hierarchy is strict (as demonstrated by the distributions defined in (19)).

For the special case of  $\lambda = 1$ , the above definition recovers exactly the standard notion of regularity à la Myerson [44]. Based on this, for simplicity and consistency, we will feel free in such cases to drop the  $\lambda = 1$  subscripts and refer to the corresponding distributions simply as *regular* and to the function  $\phi_1(x) = \phi(x)$  as the *virtual valuation*. An equivalent way of looking at regularity, is that the revenue curve defined in (2) must be concave.

On the other extreme of the range, for  $\lambda = 0$  we get the definition of *Monotone Hazard Rate (MHR)* distributions.<sup>3</sup> Intuitively, MHR distributions have exponentially decreasing tails. Although they represent the strictest class within the  $\lambda$ -regularity hierarchy, they are still general enough to give rise to a wide family of natural distributions, like the uniform, exponential, normal and gamma.

Given the above discussion,  $\lambda$ -regularity can be seen as a quantitative measure of the “regularity” of the distribution, interpolating smoothly between MHR ( $\lambda = 0$ ) and Myerson-regular ( $\lambda = 1$ ) distributions. It will also be convenient to define, for each  $\lambda \in [0, 1]$ , the worst-case approximation ratio among all  $\lambda$ -regular distributions:

$$\text{APX}(n, \lambda) \equiv \sup_{F \in \mathcal{D}_\lambda} \text{APX}(F, n) = \sup_{F \in \mathcal{D}_\lambda} \frac{\text{MYERSON}(F, n)}{\text{PRICE}(F, n)}.$$

This will be the main quantity of interest throughout our paper.

### 3 PRELIMINARIES

#### 3.1 Bounds on the Optimal Revenue

In this section we collect the bounds on the optimal revenue  $\text{MYERSON}(F, n)$  that we will use for our main positive results in Sections 4.1 and 5 to bound the approximation ratio of pricing. They rely on the regularity of the valuation distribution. The first one is essentially a refinement of the well-known Bulow-Klemperer bound [12] (see, e.g., [36, Corollary 5.3]), and it was proven by Fu et al. [28]:

LEMMA 3.1 (FU ET AL. [28]). *For  $n$  bidders with i.i.d. values from a regular distribution  $F$ ,*

$$\text{MYERSON}(F, n) \leq \mathbb{E}[X_{n-1:n}] + R(q^*)(1 - q^*)^{n-1},$$

where  $X \sim F$  and  $R$  is the revenue curve of  $F$  and  $q^*$  is the quantile corresponding to the monopoly reserve price  $r^*$  of  $F$ ,  $q^* = 1 - F(r^*)$ .

As an immediate consequence, the bound above holds also for the more restricted classes of  $\lambda$ -regular distributions, for any  $\lambda \in [0, 1]$ , and in particular for MHR distributions (see also the discussion in Section 2).

Our second bound on the optimal revenue, designed particularly for MHR distributions, is a new one and might be of independent interest also for future work:

<sup>3</sup>A standard reference for MHR distributions is that from Barlow and Marshall [8]. For an in-depth treatment of the subject, we refer to the book of Barlow and Proschan [9, Chapter 2].

LEMMA 3.2. For every MHR distribution  $F$  with monopoly reserve price  $r^*$  and quantile  $q^* = 1 - F(r^*)$ , and any positive integer  $n$ ,

$$\text{MYERSON}(F, n) \leq r^* \int_0^{q^*} \frac{1 - (1 - z)^n}{z} dz .$$

PROOF. Fix an MHR distribution  $F$ , with monopoly reserve price  $r^*$  and corresponding quantile  $q^*$ . Then, we know that for the virtual valuation (see Section 2) it is

$$\phi(r^*) = r^* - \frac{1 - F(r^*)}{f(r^*)} \geq 0 .$$

Also, from the MHR condition, for any  $x \geq r^*$  it must be that

$$\frac{f(r^*)}{1 - F(r^*)} \leq \frac{f(x)}{1 - F(x)} .$$

Combining the above we get that, for all  $x \geq r^*$ ,

$$1 - F(x) \leq f(x) \cdot \frac{1 - F(r^*)}{f(r^*)} \leq r^* f(x) . \quad (3)$$

Fix also a positive integer  $n$ , and let  $X \sim F$ . Then (see also Appendix A) the maximum order statistic  $X_{n:n}$  is distributed according to  $F^n$ . Observe that (see also Section 2)

$$\frac{\partial \text{PRICE}(F, n, x)}{\partial x} = \frac{\partial x(1 - F^n(x))}{\partial x} = 1 - F^n(x) - nxF^{n-1}(x)f(x)$$

and thus

$$\begin{aligned} \text{MYERSON}(F, n) &= \mathbb{E} [\max \{0, \phi(X_{n:n})\}] \\ &= \int_{r^*}^{\infty} \phi(x) dF^n(x) \\ &= \int_{r^*}^{\infty} \left( x - \frac{1 - F(x)}{f(x)} \right) nF^{n-1}(x)f(x) dx \\ &= \int_{r^*}^{\infty} nxF^{n-1}(x)f(x) - n(1 - F(x))F^{n-1}(x) dx \\ &= \int_{r^*}^{\infty} -\frac{\partial \text{PRICE}(F, n, x)}{\partial x} + 1 - F^n(x) - n(1 - F(x))F^{n-1}(x) dx \\ &= \text{PRICE}(F, n, r^*) + \int_{r^*}^{\infty} 1 + (n - 1)F^n(x) - nF^{n-1}(x) dx . \end{aligned}$$

Also, due to (3), for all  $x \geq r^*$ :

$$\begin{aligned} 1 - F^n(x) - n(1 - F(x))F^{n-1}(x) &= \left[ \frac{1 - F^n(x)}{1 - F(x)} - nF^{n-1}(x) \right] (1 - F(x)) \\ &\leq r^* \left[ \frac{1 - F^n(x)}{1 - F(x)} - nF^{n-1}(x) \right] f(x) . \end{aligned}$$

By performing a change of variable to quantile space, that is setting  $z = 1 - F(x)$  and observing that  $\frac{dz}{dx} = -f(x)$  and  $1 - F(r^*) = q^*$ , we get that

$$\begin{aligned} \int_{r^*}^{\infty} 1 - F^n(x) - n(1 - F(x))F^{n-1}(x) dx &\leq r^* \int_0^{q^*} \frac{1 - (1 - z)^n}{z} - n(1 - z)^{n-1} dz \\ &= -r^* [1 - (1 - q^*)^n] + r^* \int_0^{q^*} \frac{1 - (1 - z)^n}{z} dz \end{aligned}$$

$$= -\text{PRICE}(F, n, r^*) + r^* \int_0^{q^*} \frac{1 - (1 - z)^n}{z} dz .$$

Thus,

$$\text{MYERSON}(F, n) \leq r^* \int_0^{q^*} \frac{1 - (1 - z)^n}{z} dz .$$

□

### 3.2 MHR Distributions

In this section we state some properties of MHR distributions that will play a critical role into deriving our main results in the rest of the paper. [Lemma 3.3](#), in particular, might be of independent interest, since it is providing powerful tail-bounds with respect to the order statistics of the distribution:

**LEMMA 3.3.** *For any continuous MHR random variable  $X$ , integers  $1 \leq k \leq n$  and real  $c \in [0, 1]$ ,*

$$\Pr[X < c \cdot \mathbb{E}[X_{k:n}]] \leq 1 - e^{-c(H_n - H_{n-k})} .$$

**PROOF.** Let  $E$  denote an exponential random variable and also let  $F$  and  $G$  denote the cumulative probability functions of  $X$  and  $X_{k:n}$ , respectively. Then (see also [Appendix A](#))

$$\frac{dG(y)}{dy} = k \binom{n}{k} F^{k-1}(y) (1 - F(y))^{n-k} \frac{dF(y)}{dy} , \quad (4)$$

for almost all  $y \in [0, \infty)$ . To simplify notation, also let  $v = \mathbb{E}[X_{k:n}]$ . Our goal then is to upper-bound  $F(cv)$ , i.e. lower-bound  $1 - F(cv)$ . Since  $F$  is MHR,  $\zeta(c) = \ln(1 - F(cv))$  is a concave function of  $c$  with  $\zeta(0) = 0$ ; thus, assuming  $c \in [0, 1]$  and applying Jensen's inequality,

$$\ln(1 - F(cv)) \geq c \ln(1 - F(v)) \geq c \mathbb{E}[\ln(1 - F(X_{k:n}))] = c \int_0^\infty \ln(1 - F(y)) dG(y) ;$$

the integral can be further simplified using (4) and the changes of variables  $u = F(y)$ ,  $t = -\ln(1 - u)$ ;

$$\begin{aligned} c \int_0^\infty \ln(1 - F(y)) dG(y) &= ck \binom{n}{k} \int_0^\infty \ln(1 - F(y)) F^{k-1}(y) (1 - F(y))^{n-k} dF(y) \\ &= ck \binom{n}{k} \int_0^1 \ln(1 - u) u^{k-1} (1 - u)^{n-k} du \\ &= -ck \binom{n}{k} \int_0^\infty t (1 - e^{-t})^{k-1} (e^{-t})^{n-k} dt \\ &= -c \mathbb{E}[E_{k:n}] \\ &= -c(H_n - H_{n-k}) , \end{aligned}$$

the last equality following from [Appendix A](#). So, applying the exponential function on both sides, finally we get the desired

$$1 - F(cv) \geq e^{-c(H_n - H_{n-k})} .$$

□

The next lemma states some useful bounds on the monopoly reserve of an MHR distribution:

**LEMMA 3.4.** *For any MHR distribution with expectation  $\mu$ , monopoly reserve  $r^*$  and corresponding quantile  $q^*$ :*

- (1)  $q^* \geq 1/e$ ;
- (2)  $\ln(1/q^*) \cdot \mu \leq r^* \leq \frac{\ln(1/q^*)}{1 - q^*} \cdot \mu$ .

PROOF. The first property is by now almost folklore, see e.g. [1, Lemma 1] or [7, Claim B.2]. Alternatively, it can be seen as the limiting case of Lemma 3.7 as  $\lambda \rightarrow 0$ , since  $\lim_{\lambda \rightarrow 0^+} (1-\lambda)^{1/\lambda} = 1/e$ .

For the second property, applying [8, Corollary 3.10] for the first-order moments ( $r = 1$ ), we get that for any quantile  $q = 1 - F(x)$  of our MHR distribution with  $q \geq 1/e$ , it must be that

$$-\ln q \cdot \mu \leq x \leq \left[ \int_0^1 q^y dy \right]^{-1} \cdot \mu = \left( \frac{q-1}{\ln q} \right)^{-1} \mu .$$

By the first property of our lemma, it is valid to use the above inequality with the monopoly reserve quantile  $q^* = 1 - F(r^*)$  and so, by setting  $q \leftarrow q^*$  and  $x \leftarrow r^*$  we have that

$$-\ln(q^*)\mu \leq r^* \leq -\frac{\ln(q^*)}{1-q^*}\mu .$$

□

Finally, the following lemma shows that the high-order statistics of MHR distributions are “well-behaved”, in the sense that they cannot be away from the expectation:

LEMMA 3.5. *For any MHR random variable  $X$  and integer  $n \geq 2$ ,*

$$\mathbb{E}[X_{n-1:n}] \geq \left(1 - \frac{H_n - 1}{n - 1}\right) \cdot \mathbb{E}[X] .$$

PROOF. For convenience denote  $\mu = \mathbb{E}[X]$  and  $\nu = \mathbb{E}[X_{n-1:n}]$ . From Babaiouff et al. [6, Lemma 5.3] we know that, since  $X$  is MHR, its highest-order statistic is upper bounded by

$$\mathbb{E}[X_{n:n}] \leq H_n \cdot \mu .$$

Using this we get that:

$$n \cdot \mu = \mathbb{E} \left[ \sum_{i=1}^n X_{i:n} \right] = \sum_{i=1}^{n-1} \mathbb{E}[X_{i:n}] + \mathbb{E}[X_{n:n}] \leq (n-1)\nu + H_n \cdot \mu ,$$

and thus  $(n-1)\nu \geq (n-H_n)\mu$ , or equivalently,  $\nu \geq \frac{(n-H_n)}{n-1}\mu = \left(1 - \frac{H_n-1}{n-1}\right)\mu$ . □

### 3.3 $\lambda$ -Regular Distributions

The following lemmas are the counterparts of Lemma 3.3 and Lemma 3.4 (Property 1), extending them to  $\lambda$ -regular distributions for  $\lambda > 0$ .

LEMMA 3.6. *Let  $X$  be a  $\lambda$ -regular distribution, for  $\lambda \in (0, 1]$ . Then, for any integer  $1 \leq k \leq n$  and real  $c \in [0, 1]$ ,*

$$\Pr[X \leq c \cdot \mathbb{E}[X_{k:n}]] \leq 1 - \left(1 + c \left( \frac{n!\Gamma(n+1-k-\lambda)}{(n-k)!\Gamma(n+1-\lambda)} - 1 \right)\right)^{-1/\lambda} .$$

PROOF. Let  $F$  and  $G$  be the cumulative density functions of  $X$  and  $X_{k:n}$  respectively. Let also  $\nu = \mathbb{E}[X_{k:n}]$ . As  $F$  is  $\lambda$ -regular, the function  $\zeta(c) = (1 - F(c\nu))^{-\lambda}$  is a convex function on  $c$ , with  $\zeta(0) = 1$ ; thus, assuming  $c \in [0, 1]$ , and applying Jensen’s inequality,

$$\zeta(c) \leq 1 + c(\zeta(1) - 1) = 1 + c((1 - F(\nu))^{-\lambda} - 1) \leq 1 + c(\mathbb{E}[(1 - F(X_{k:n}))^{-\lambda}] - 1) . \quad (5)$$

The rest of the proof follows exactly as in the proof of Lemma 3.3. We use a change of variable to compute the expected value:

$$\mathbb{E}[(1 - F(X_{k:n}))^{-\lambda}] = \int_0^\infty (1 - F(y))^{-\lambda} dG(y)$$

$$\begin{aligned}
&= k \binom{n}{k} \int_0^\infty F(y)^{k-1} (1-F(y))^{n-k-\lambda} dF(y) \\
&= k \binom{n}{k} \int_0^1 u^{k-1} (1-u)^{n-k-\lambda} du \\
&= k \binom{n}{k} \mathbb{B}(k, n+1-k-\lambda) \\
&= \frac{n! \Gamma(n+1-k-\lambda)}{(n-k)! \Gamma(n+1-\lambda)}.
\end{aligned}$$

Plugging this into (5) and rearranging gives us the desired result.  $\square$

LEMMA 3.7 (SCHWEIZER AND SZECH [48]). *For any  $\lambda$ -regular distribution with  $\lambda \in (0, 1]$  and monopoly quantile  $q^*$ ,*

$$q^* \geq (1-\lambda)^{1/\lambda}.$$

#### 4 BOUNDS FOR MHR DISTRIBUTIONS

To facilitate us with stating and proving our bounds for the approximation ratio of pricing, we define the following auxiliary function  $g_n : [0, \infty) \rightarrow [0, \infty)$ , for any positive integer  $n$ ,

$$g_n(c) \equiv c[1 - (1 - e^{-c(H_n-1)})^n] \quad (6)$$

and its (unique) maximizer in  $[0, 1]$  by

$$c_n \equiv \operatorname{argmax}_{c \in [0, 1]} g_n(c). \quad (7)$$

In [Appendix C](#) we prove some properties of  $g_n$  that will be used in the rest of this section.

##### 4.1 Upper Bounds

This section is dedicated to proving the main result of our paper. First ([Theorem 4.1](#)) we show that pricing is indeed asymptotically optimal with respect to revenue and then ([Theorem 4.2](#)) we also provide a more refined upper-bound on the approximation ratio that is fine-tuned to work well for a small number of bidders  $n$ . As we will see in the following [Section 4.2](#), our upper bound analysis of this section is essentially tight (see also [Fig. 2](#)).

THEOREM 4.1. *Using the same take-it-or-leave-it price, to sell an item to  $n$  buyers with i.i.d. valuations from a continuous MHR distribution  $F$ , is asymptotically optimal with respect to revenue. In particular,*

$$\text{APX}(F, n) = 1 + O\left(\frac{\ln \ln n}{\ln n}\right).$$

A plot of the exact values of this upper bound (given by (12) below) can be seen in [Fig. 1](#) (blue).

PROOF. First notice that by using the monopoly reserve price  $r^*$  of  $F$  as a take-it-or-leave-it price to the  $n$  bidders, we get an expected revenue of

$$\text{PRICE}(F, n, r^*) = r^*(1 - F(r^*)^n) = r^*[1 - (1 - q^*)^n] = R(q^*) \frac{1 - (1 - q^*)^n}{q^*}, \quad (8)$$

where  $q^* = 1 - F(r^*)$  is the quantile of the monopoly reserve price, for which we know that  $q^* \geq \frac{1}{e}$  ([Lemma 3.4](#)), and  $R$  denotes the revenue curve (see [Section 2](#)).

Next, for simplicity denote  $v = \mathbb{E}[X_{n-1:n}]$ . For any real  $c \in [0, 1]$ , if we offer a price of  $c \cdot v$  we have

$$\text{PRICE}(F, n, cv) = cv[1 - F(cv)^n] \geq cv \left[1 - \left(1 - e^{-c(H_n-1)}\right)^n\right], \quad (9)$$

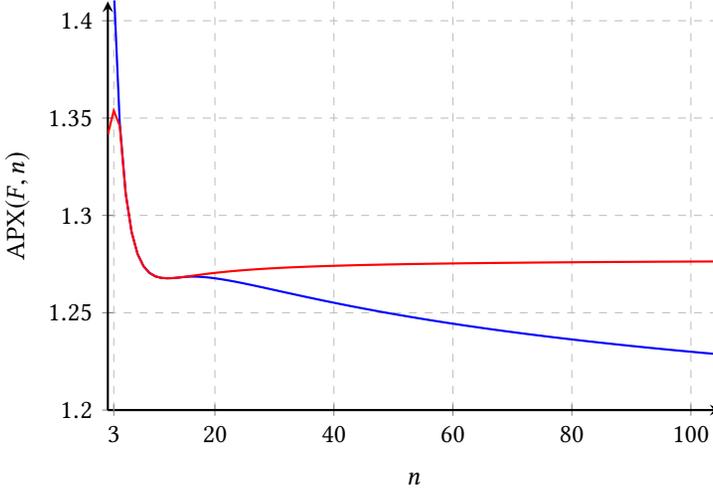


Fig. 1. The upper bounds on the approximation ratio  $\text{APX}(F, n)$  of anonymous pricing for  $n$  i.i.d. bidders with MHR valuations, given by [Theorem 4.1](#) (blue) and [Theorem 4.2](#) (red). The best (smallest) of the two converges to the optimal value of 1 as the number of bidders grows large, at a rate of  $1 + O(\ln \ln n / \ln n)$ . A single, unified plot of this can be seen in [Fig. 2](#) (black), together with a matching lower bound (red). In the worst case ( $n = 3$ ), our upper bound is at most 1.354.

the inequality holding due to [Lemma 3.3](#) (for  $k = n - 1$ ). Optimizing with respect to  $c$  we get that

$$\text{PRICE}(F, n) \geq v \max_{c \in [0, 1]} g_n(c). \quad (10)$$

Using the two lower bounds (8) and (10) on the pricing revenue, in conjunction with the upper bound on the optimal revenue from [Lemma 3.1](#) we can bound the approximation ratio of pricing by

$$\begin{aligned} \text{APX}(F, n) &= \frac{\text{MYERSON}(F, n)}{\text{PRICE}(F, n)} \\ &\leq \frac{v}{v \max_{c \in [0, 1]} g_n(c)} + \frac{R(q^*)(1 - q^*)^{n-1}}{R(q^*) \frac{1 - (1 - q^*)^n}{q^*}} \\ &= \frac{1}{\max_{c \in [0, 1]} g_n(c)} + \frac{q^*(1 - q^*)^{n-1}}{1 - (1 - q^*)^n} \end{aligned} \quad (11)$$

$$\leq \frac{1}{\max_{c \in [0, 1]} g_n(c)} + \frac{(e - 1)^{n-1}}{e^n - (e - 1)^n} \quad (12)$$

$$= 1 + O\left(\frac{\ln \ln n}{\ln n}\right) + O\left(\left(\frac{e}{e - 1}\right)^{-n}\right). \quad (13)$$

[Equation \(12\)](#) holds by observing that function  $x \mapsto \frac{x(1-x)^{n-1}}{1-(1-x)^n}$  is decreasing over  $(0, 1]$ , for any  $n \geq 2$ , and taking into consideration that  $q^* \geq 1/e$ , while for [\(13\)](#) we make use of the asymptotics from [Lemma C.1](#). The upper bound given by [\(12\)](#) is plotted by the blue line in [Fig. 1](#).  $\square$

**THEOREM 4.2.** *The approximation ratio of the revenue obtained by using the same take-it-or-leave-it price, to sell an item to  $n$  buyers with i.i.d. valuations from a continuous MHR distribution  $F$ , is at most*

$$\text{APX}(F, n) \leq \max_{q \in [1/e, 1]} \min \left\{ \frac{1}{1 - (1 - e^{-H_n + 1})^n} + \frac{q(1 - q)^{n-1}}{1 - (1 - q)^n}, \frac{\int_0^q \frac{1 - (1 - z)^n}{z} dz}{1 - (1 - q)^n} \right\}.$$

*In particular, the worst case (maximum) of this quantity is attained at  $n = 3$  and is at most  $\text{APX}(F, 3) \leq 1.354$ .*

A plot of the exact values of this upper bound can be seen in [Fig. 1](#) (red).

**PROOF.** From (11) in the proof of [Theorem 4.1](#) we can get the following upper bound on the approximation ratio, by using (possibly suboptimally)  $c \leftarrow 1$  for the maximization operator:

$$\text{APX}(F, n) \leq \frac{1}{g_n(1)} + \frac{q^*(1 - q^*)^{n-1}}{1 - (1 - q^*)^n} = \frac{1}{1 - (1 - e^{-H_n + 1})^n} + \frac{q^*(1 - q^*)^{n-1}}{1 - (1 - q^*)^n}.$$

On the other hand, using the reserve price of  $F$  as a price and combining the guarantee of (8) with the upper bound on the optimal revenue from [Lemma 3.2](#), gives us

$$\text{APX}(F, n) \leq \frac{r^* \int_0^{q^*} \frac{1 - (1 - z)^n}{z} dz}{R(q^*) \frac{1 - (1 - q^*)^n}{q^*}} = \frac{\int_0^{q^*} \frac{1 - (1 - z)^n}{z} dz}{1 - (1 - q^*)^n},$$

since  $R(q^*) = r^* q^*$ . Recalling that  $q^* \in [1/e, 1]$  and taking the best (i.e., minimum) of the two bounds above, finishes the proof.  $\square$

## 4.2 Lower Bound

The lower bound instance of this section ([Theorem 4.4](#)) shows that our main positive result for the approximation ratio of pricing under MHR distributions in [Theorem 4.1](#) is essentially tight (see also [Fig. 2](#)). It is achieved by an exponential distribution instance. Before proving it, we need the following auxiliary lemma about the maximizers of functions  $g_n$  that we introduced in (6). Its proof can be found in [Appendix C](#).

**LEMMA 4.3.** *For any positive integer  $n$ , function  $g_n$  (defined in (6)) has a unique maximizer. Furthermore, for all  $n \geq 17$ ,*

$$\operatorname{argmax}_{c \geq 0} g_n(c) \leq 1.$$

**THEOREM 4.4.** *For  $n \geq 2$  bidders with exponentially i.i.d. valuations, the approximation ratio of anonymous pricing is at least*

$$\text{APX}(\mathcal{E}, n) \geq \frac{1}{\max_{c \geq 0} g_n(c)},$$

*where function  $g_n$  is defined in (6) and  $\mathcal{E}$  is the exponential distribution. In particular, the upper bound derived in the proof of [Theorem 4.1](#) is tight (up to an exponentially vanishing additive factor).*

A plot of the lower bound given by the theorem above can be seen in [Fig. 2](#) (red).

**PROOF.** Let  $X \sim \mathcal{E}$  be an exponential random variable. Since the revenue of a second-price auction cannot be greater than the optimal one, we have

$$\text{MYERSON}(\mathcal{E}, n) \geq \mathbb{E}[X_{n-1:n}] = H_n - 1,$$

where the equality is taken from [Appendix A](#). Furthermore,

$$\text{PRICE}(\mathcal{E}, n) = \sup_{x \geq 0} x (1 - F_{\mathcal{E}}^n(x)) = \sup_{x \geq 0} x [1 - (1 - e^{-x})^n] = (H_n - 1) \max_{c \geq 0} g_n(c).$$

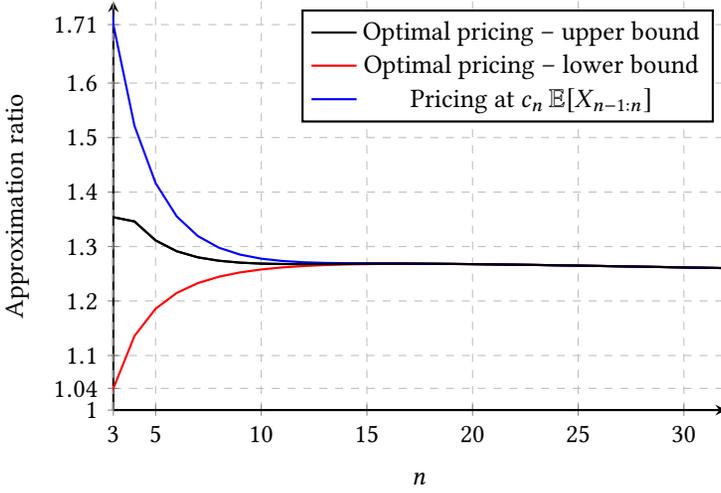


Fig. 2. Bounds on the approximation ratio of anonymous pricing for  $n = 3, \dots, 30$  i.i.d. bidders with MHR valuations: the upper bound on optimal pricing (black) derived in Section 4.1 (see also Fig. 1), the lower bound (red) given by Theorem 4.4, and the upper bound of pricing at the expected value of the second-highest order statistic, scaled down by parameter  $c_n$  (blue), given in Theorem 4.6 of Section 4.3. They are all (asymptotically) optimal, their (additive) difference decreasing exponentially fast. They all converge to the optimal value of 1 at a rate of  $1 + O(\ln \ln n / \ln n)$ .

Putting the above together, we get the desired lower bound on the approximation ratio.

For the tightness, we need to show that our lower bound is within an additive, exponentially decreasing factor of the upper bound given in (12). Since the second term in (12) is at most  $O\left(\left(\frac{e}{e-1}\right)^{-n}\right)$ , it is enough to show that, for a sufficiently large number of bidders  $n$ ,

$$\max_{c \in [0, \infty)} g_n(c) = \max_{c \in [0, 1]} g_n(c).$$

This is exactly what we proved in Lemma 4.3, for any  $n \geq 17$ .  $\square$

### 4.3 Explicit Prices – Knowledge of the Distribution

Our main result from Section 4.1 demonstrates that a seller, facing  $n$  bidders with i.i.d. valuations from an MHR distribution  $F$ , can achieve (asymptotically) optimal revenue by using just an anonymous, take-it-or-leave-it price. Taking a careful look within the proof of Theorem 4.1, we see that this upper bound is derived by comparing the optimal Myersonian revenue (via the bound provided by Lemma 3.1) to that of two different anonymous pricings; namely, first (see (8)) we use the monopoly reserve  $r^*$  of  $F$ , and then (see (9) and (10)) a multiple of the expectation  $v = \mathbb{E}[X_{n-1:n}]$  of the second-highest order statistic of  $F$ , in particular  $c_n \cdot v$  where  $c_n = \operatorname{argmax}_{c \in [0, 1]} g_n(c)$  was defined in (7). Although the latter price requires only the knowledge of  $v = \mathbb{E}[X_{n-1:n}]$ , that is not the case for the former; determining the reserve price  $r^*$  demands, in general, a detailed knowledge of the distribution  $F$ : it is the maximizer of  $r(1 - F(r))$ .

As a result, we would ideally like to provide a more robust solution, that would still provide optimality but depend only on limited information about  $F$ . If we pay even closer attention to the proof of Theorem 4.1, and the derivation of (13) in particular, we will see that the summand of our upper bound that corresponds to the pricing using  $r^*$  is exponentially decreasing, according to  $\left(\frac{e}{e-1}\right)^{-n}$ . Therefore, if we could show that the expected revenue achieved by using an anonymous

price of  $c_n v$  is within a constant factor from that of using an anonymous price of  $r^*$ , then we could deduce that using only price  $c_n v$  yields essentially the same approximation ratio (and in particular, asymptotically optimal revenue). We now proceed to formalize this line of reasoning.

LEMMA 4.5. *For  $n \geq 2$  bidders with i.i.d. valuations from an MHR distribution  $F$  with monopoly reserve  $r^*$  and parameters  $c_n \in [0, 1]$  given by (7),*

$$\text{PRICE}(F, n, c_n \cdot \mathbb{E}[X_{n-1:n}]) \geq (1 - o(1)) \frac{e-1}{e} \cdot \text{PRICE}(F, n, r^*),$$

where  $X \sim F$ .

PROOF. For convenience, denote  $\mu = \mathbb{E}[X]$  and  $v = \mathbb{E}[X_{n-1:n}]$ . By the proof of [Theorem 4.1](#) (see (9) and (10)) we know that by offering an anonymous price of  $c_n \cdot v$  gives us an expected revenue of at least

$$\text{PRICE}(F, n, c_n \cdot v) \geq v \max_{c \in [0,1]} g_n(c) \geq \max_{c \in [0,1]} g_n(c) \frac{n - H_n}{n - 1} \cdot \mu,$$

the second inequality holding due to [Lemma 3.5](#).

On the other hand, from (8) we know that using the reserve price  $r^*$  as an anonymous price to all bidders gives an expected revenue of at most

$$\text{PRICE}(F, n, r^*) = r^* [1 - (1 - q^*)^n] \leq \frac{\ln(1/q^*)}{1 - q^*} [1 - (1 - q^*)^n] \cdot \mu,$$

the inequality holding due to [Lemma 3.4](#).

Putting everything together, we finally get that

$$\begin{aligned} \frac{\text{PRICE}(F, n, r^*)}{\text{PRICE}(F, n, c_n v)} &\leq \frac{\ln(1/q^*)}{1 - q^*} [1 - (1 - q^*)^n] \frac{n - 1}{n - H_n} \frac{1}{\max_{c \in [0,1]} g_n(c)} \\ &\leq \frac{e}{e - 1} \frac{n - 1}{n - H_n} \frac{1}{\max_{c \in [0,1]} g_n(c)} \\ &\leq (1 + o(1)) \frac{e}{e - 1}. \end{aligned} \tag{14}$$

The second inequality holds because  $\frac{\ln(1/q^*)}{1 - q^*} [1 - (1 - q^*)^n] \leq \frac{\ln(1/q^*)}{1 - q^*} \leq \frac{e}{e - 1}$ , since the function  $x \mapsto \frac{\ln(1/x)}{1 - x}$  is decreasing for  $x > 0$  and  $q^* \geq 1/e$  (from Property 1 of [Lemma 3.4](#)). The last inequality is a consequence of [Lemma C.1](#) and the fact that  $H_n \leq \ln(n) + 1$ .  $\square$

As discussed before, [Lemma 4.5](#) shows us that there indeed exists an anonymous price that depends on the knowledge of only the expectation of the second-highest order statistic of the valuation distribution and which, furthermore, guarantees an (asymptotically) optimal revenue. We can even provide a closed-form upper bound for it:

THEOREM 4.6. *Let  $F$  be an MHR distribution and  $X_{n-1:n}$  denote the second-highest out of  $n$  i.i.d. draws from  $F$ . Then, using an anonymous price of  $c_n \cdot \mathbb{E}[X_{n-1:n}]$ , where  $c_n$  is given in (7), to sell an item to  $n \geq 2$  bidders with i.i.d. valuations from  $F$ , guarantees a revenue with approximation ratio of at most*

$$\frac{\text{MYERSON}(F, n)}{\text{PRICE}(F, n, c_n \mathbb{E}[X_{n-1:n}])} \leq \frac{1}{\max_{c \in [0,1]} g_n(c)} \left[ 1 + \frac{1}{e} \frac{n - 1}{n - H_n} \left( \frac{e - 1}{e} \right)^{n-2} \right].$$

A plot of this upper bound can be seen in [Fig. 2](#) (blue).

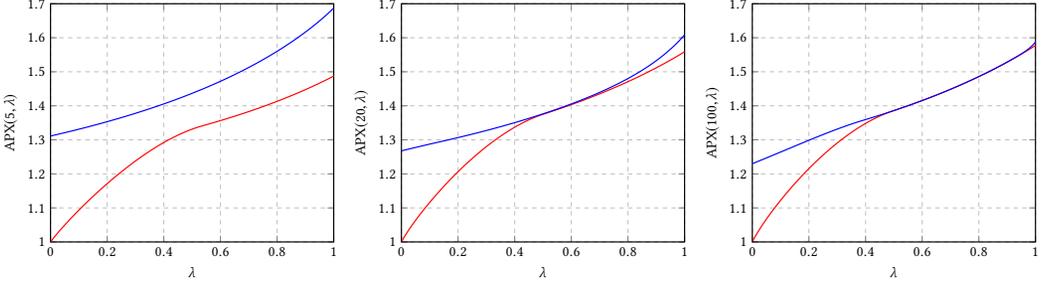


Fig. 3. Upper (Theorem 5.1) and lower (Theorem 5.3) bounds on the approximation ratio of anonymous pricing for  $n = 5$  (left),  $n = 20$  (centre) and  $n = 100$  (right) bidders with i.i.d.  $\lambda$ -regular valuations, as a function of  $\lambda$ .

PROOF. Simulating the proof of the approximation upper bound in Theorem 4.1, but now using (14) to approximate  $\text{PRICE}(F, n, r^*)$  by  $\text{PRICE}(F, n, c_n \mathbb{E}[X_{n-1:n}])$ , the derivation in (11) gives us that

$$\begin{aligned}
 \frac{\text{MYERSON}(F, n)}{\text{PRICE}(F, n, c_n \mathbb{E}[X_{n-1:n}])} &\leq \frac{1}{\max_{c \in [0,1]} g_n(c)} \\
 &+ \frac{\ln(1/q^*)}{1 - q^*} [1 - (1 - q^*)^n] \frac{n-1}{n - H_n} \frac{1}{\max_{c \in [0,1]} g_n(c)} \cdot \frac{q^*(1 - q^*)^{n-1}}{1 - (1 - q^*)^n} \\
 &= \frac{1}{\max_{c \in [0,1]} g_n(c)} \left[ 1 + \frac{n-1}{n - H_n} \ln(1/q^*) q^*(1 - q^*)^{n-2} \right] \\
 &\leq \frac{1}{\max_{c \in [0,1]} g_n(c)} \left[ 1 + \frac{1}{e} \frac{n-1}{n - H_n} \left( \frac{e-1}{e} \right)^{n-2} \right],
 \end{aligned}$$

the last inequality coming from Lemma B.3, together with the fact that  $q^* \in [1/e, 1]$  (see Property 1 of Lemma 3.4).  $\square$

## 5 BOUNDS FOR $\lambda$ -REGULAR DISTRIBUTIONS

In this section we provide the generalized,  $\lambda$ -regular counterparts of our main results for MHR distributions of the previous Section 4. In particular, we provide upper (Theorem 5.1; cf. Theorem 4.1) and lower (Theorem 5.3; cf. Theorem 4.4) bounds on the approximation ratio of anonymous pricing for  $n$  bidders with i.i.d.  $\lambda$ -regular valuations, and also derive an asymptotically tight expression for large  $n$  (Theorem 5.5; cf. Theorem 4.1).

The following quantities will help us to state and prove our results in this section. For an integer  $n \geq 2$  and  $\lambda \in (0, 1]$ , we define the quantities  $\beta_{n,\lambda}, a_{n,\lambda} > 0$  and the function  $g_{n,\lambda} : [0, \infty) \rightarrow [0, \infty)$  as follows:

$$\beta_{n,\lambda} \equiv \frac{n! \Gamma(2 - \lambda)}{\Gamma(n + 1 - \lambda)} = \prod_{k=2}^n \left( 1 - \frac{\lambda}{k} \right)^{-1}, \quad a_{n,\lambda} \equiv \begin{cases} \frac{((1-\lambda)^{1/\lambda} [1 - (1-\lambda)^{1/\lambda}])^{n-1}}{1 - [1 - (1-\lambda)^{1/\lambda}]^n}, & \text{if } \lambda \in (0, 1), \\ \frac{1}{n}, & \text{if } \lambda = 1, \end{cases} \quad (15)$$

$$g_{n,\lambda}(c) \equiv c \left[ 1 - \left[ 1 - [1 + c(\beta_{n,\lambda} - 1)]^{-1/\lambda} \right]^n \right]. \quad (16)$$

The function  $g_n$  introduced in Eq. (6) can be recovered from  $g_{n,\lambda}$  in the limit  $\lambda \rightarrow 0$ . Note also that  $\beta_{n,\lambda}$  corresponds to the fraction in Lemma 3.6 with  $k = n - 1$ . Using Stirling's approximation (see, e.g., Rudin [47, Ch. 8]) we can derive the asymptotics

$$\beta_{n,\lambda} = \frac{n! \Gamma(2 - \lambda)}{\Gamma(n + 1 - \lambda)} \sim \Gamma(2 - \lambda) n^\lambda, \quad (17)$$

valid for fixed  $\lambda$ , and for large  $n$ .

**THEOREM 5.1.** *Let  $F$  be a  $\lambda$ -regular distribution with  $\lambda \in (0, 1]$ , and  $n \geq 2$ . Using the same take-it-or-leave-it price, to sell an item to  $n$  buyers with i.i.d. valuations from  $F$ , achieves an approximation ratio of at most*

$$\text{APX}(F, n) \leq \frac{1}{\sup_{c \in [0, 1]} g_{n, \lambda}(c)} + a_{n, \lambda}$$

with respect to the optimal revenue, where  $a_{n, \lambda}$  and  $g_{n, \lambda}$  are defined as in (15) and (16).

**PROOF.** We follow the same steps as for [Theorem 4.1](#), but use [Lemma 3.6](#) and the quantile bound from [Lemma 3.7](#). Specifically, instead of (9) we bound the revenue from pricing at  $c \cdot v$ , where  $c \in [0, 1]$  and  $v = \mathbb{E}[X_{n-1:n}]$ , as

$$\text{PRICE}(F, n, cv) = cv[1 - F(cv)^n] \geq cv \left[ 1 - \left[ 1 - \left[ 1 + c(\beta_{n, \lambda} - 1) \right]^{-1/\lambda} \right]^n \right]. \quad (18)$$

the inequality holding due to [Lemma 3.6](#) (for  $k = n - 1$ ). Optimizing with respect to  $c$  we get that

$$\text{PRICE}(F, n) \geq v \sup_{c \in [0, 1]} g_{n, \lambda}(c).$$

We can now plug this bound at the derivation in [Theorem 4.1](#) to get

$$\begin{aligned} \text{APX}(F, n) &= \frac{\text{MYERSON}(F, n)}{\text{PRICE}(F, n)} \\ &\leq \frac{v}{v \sup_{c \in [0, 1]} g_{n, \lambda}(c)} + \frac{R(q^*)(1 - q^*)^{n-1}}{R(q^*) \frac{1 - (1 - q^*)^n}{q^*}} \\ &= \frac{1}{\sup_{c \in [0, 1]} g_{n, \lambda}(c)} + \frac{q^*(1 - q^*)^{n-1}}{1 - (1 - q^*)^n} \\ &\leq \frac{1}{\sup_{c \in [0, 1]} g_{n, \lambda}(c)} + a_{n, \lambda}. \end{aligned}$$

For the last step, we use the bound  $q^* \geq (1 - \lambda)^{1/\lambda}$  from [Lemma 3.7](#), together with the definition of  $a_{n, \lambda}$  as in (15).  $\square$

For the lower bounds, we introduce the following family of *rescaled Pareto* distributions  $F_{\lambda, r}$ , with support  $[1, \infty)$ , for  $\lambda \in (0, 1]$  and  $r \in (0, 1]$ :

$$F_{\lambda, r}(x) = 1 - \frac{1}{\left[ 1 + \frac{1}{r}(x - 1) \right]^{1/\lambda}}. \quad (19)$$

These can be seen as lying at the edge of  $\lambda$ -regularity; it is not difficult to see that  $F_{\lambda, r}$  is  $\lambda$ -regular, but *not*  $\lambda'$ -regular for  $\lambda' < \lambda$  (see [Section 2.1](#)). They are inspired by the lower-bound construction of Dütting et al. [[24](#), Appendix C.3], but we generalized them to be  $\lambda$ -regular and also removed the point mass at the right endpoint of the support in order to guarantee continuity. On the other hand, for  $r = 1$ , our distributions reduce to standard Pareto:  $F_{\lambda, 1} = 1 - \frac{1}{x^{1/\lambda}}$ .

We first collect below some basic properties of rescaled Pareto distributions, concerning their expected second-highest order statistic and pricing revenue; their proof can be found in [Appendix C](#).

**LEMMA 5.2.** *For any  $\lambda \in (0, 1]$  and  $r \in (0, 1]$ ,*

- (1)  $\mathbb{E}_{X \sim F_{\lambda, r}}[X_{n-1:n}] = 1 + r(\beta_{n, \lambda} - 1)$ ;
- (2)  $\text{PRICE}(F_{\lambda, r}, n) = \sup_{q \in [0, 1]} \left( 1 + r \left( \frac{1}{q^\lambda} - 1 \right) \right) (1 - (1 - q)^n)$ .

For the sake of exposition we introduce the function  $H_{n,\lambda}(r, q)$ , for  $n \geq 2$ ,  $\lambda \in (0, 1]$ ,  $r \in (0, 1]$ , and  $q = [0, 1]$ ,

$$H_{n,\lambda}(r, q) = \left(1 + r \left(\frac{1}{q^\lambda} - 1\right)\right) (1 - (1 - q)^n). \quad (20)$$

Note that  $H_{n,\lambda}(r, q)$  is continuously defined at  $q = 0$  as the singularity is removable.

**THEOREM 5.3.** *For  $\lambda \in (0, 1]$ ,  $r \in (0, 1]$ , and  $n \geq 2$  bidders with i.i.d. valuations from the rescaled Pareto distribution  $F_{\lambda,r}$  (see (19)), the approximation ratio of anonymous pricing is at least*

$$\text{APX}(F_{\lambda,r}, n) \geq \frac{1 + r (\beta_{n,\lambda} - 1)}{\sup_{q \in [0,1]} H_{n,\lambda}(r, q)}. \quad (21)$$

This implies the following lower bound on the approximation ratio of  $\lambda$ -regular distributions:

$$\text{APX}(n, \lambda) \geq \sup_{r \in [0,1]} \inf_{q \in [0,1]} \frac{1 + r (\beta_{n,\lambda} - 1)}{H_{n,\lambda}(r, q)}. \quad (22)$$

**PROOF.** Using Lemma 5.2 (1), and the fact that the optimal revenue can be lower bounded by the second-highest order statistic, gives

$$\text{MYERSON}(F_{\lambda,r}, n) \geq \mathbb{E}_{X \sim F_{\lambda,r}} [X_{n-1:n}] = 1 + r (\beta_{n,\lambda} - 1).$$

Using Lemma 5.2 (2), and the definition of  $H_{n,\lambda}$  from (20), gives

$$\text{PRICE}(F_{\lambda,r}, n) = \sup_{q \in [0,1]} H_{n,\lambda}(r, q).$$

Taking the ratio between these two quantities proves (21). If we optimize this ratio for  $r \in (0, 1]$ , we obtain

$$\begin{aligned} \text{APX}(n, \lambda) &\geq \sup_{r \in (0,1]} \text{APX}(F_{\lambda,r}, n) \\ &\geq \sup_{r \in (0,1]} \frac{1 + r (\beta_{n,\lambda} - 1)}{\sup_{q \in [0,1]} H_{n,\lambda}(r, q)} \\ &= \sup_{r \in [0,1]} \inf_{q \in [0,1]} \frac{1 + r (\beta_{n,\lambda} - 1)}{H_{n,\lambda}(r, q)}; \end{aligned}$$

note that in the last step, we can allow  $r = 0$  without loss since the right-hand side is well-defined and gives a trivial lower bound of 1.  $\square$

A plot of the bounds given in Theorems 5.1 and 5.3 can be seen in Fig. 3, for different values of  $n$  and  $\lambda$ .

## 5.1 Asymptotic Analysis

By observing the plots in Fig. 3, it seems that the upper and lower bounds are approaching some smooth function of  $\lambda$  as  $n$  grows large; moreover, this function appears to increase from 1 at  $\lambda = 0$  to  $\frac{e}{e-1} \approx 1.58$  at  $\lambda = 1$ , which would recover the known bounds for MHR (this paper, Section 4) and regular (see, e.g., [16, 24]) distributions. In the remainder of this paper we prove this is indeed the case, and characterize the limiting function. In other words, we are interesting in taking the limit (as  $n \rightarrow \infty$ ) of the quantities defined in Theorems 5.1 and 5.3.

For each  $\lambda \in (0, 1]$ , let us define the function  $g_\lambda : [0, \infty) \rightarrow [0, \infty)$  as follows.

$$g_\lambda(c) = c \left(1 - e^{-(c\Gamma(2-\lambda))^{-1/\lambda}}\right). \quad (23)$$

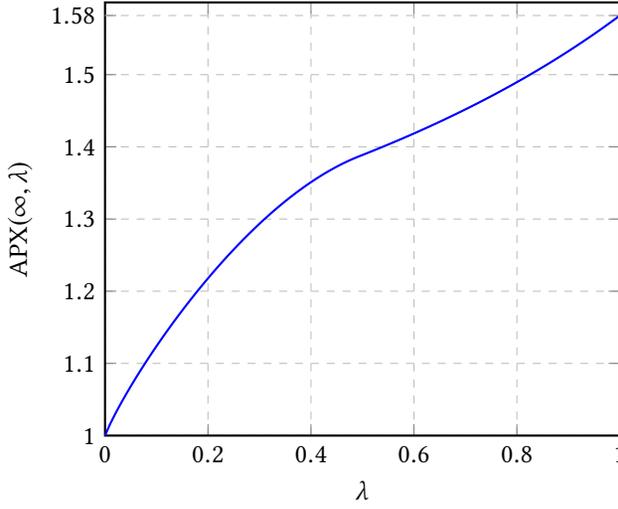


Fig. 4. The asymptotically tight value of the approximation ratio of anonymous pricing, given in [Theorem 5.5](#), as a function of the regularity parameter  $\lambda$  (for  $n \rightarrow \infty$  i.i.d. bidders).

The function  $g_\lambda$  can be obtained from the function  $g_{n,\lambda}$  introduced in [Eq. \(16\)](#) as  $n \rightarrow \infty$ . We also define  $\lambda^* \approx 0.4940$  as the unique positive root over  $(0, 1]$  of the equation (see [Lemma B.4](#))

$$1 - \left(1 + \frac{\Gamma(2-\lambda)^{-1/\lambda}}{\lambda}\right) e^{-\Gamma(2-\lambda)^{-1/\lambda}} = 0. \quad (24)$$

For each  $\lambda \in (0, 1)$ , let also  $\eta(\lambda)$  denote the unique positive solution of the equation  $e^x = 1 + \frac{x}{\lambda}$ , which is related to the function  $\eta(k)$  appearing in Blumrosen and Holenstein [[11](#), Section 3.1.2]. In fact, we must mention here that one can derive a lower bound on the asymptotic approximation ratio from [[11](#)], which coincides with our [Theorem 5.5](#) in the branch  $0 < \lambda \leq \lambda^*$ . Although Blumrosen and Holenstein, in general, study distributions satisfying the von Mises conditions, for their lower bounds they make use of power law distributions of the form  $F(x) = 1 - 1/x^k$ , for  $k > 1$ . These correspond to our rescaled Pareto distribution  $F_{r,\lambda}$  from ([19](#)) with  $r = 1$  and  $\lambda = 1/k$ .

For the main result of this section we shall make use of the following technical lemma whose proof can be found in [Appendix C](#).

**LEMMA 5.4.** *For each  $\lambda \in (0, 1]$ , let  $g_\lambda$  be defined as in ([23](#)) and  $\eta(\lambda)$  be the unique positive solution of the equation  $e^x = 1 + \frac{x}{\lambda}$ . Let also  $\lambda^*$  be the unique root of ([24](#)). The supremum of the function  $g_\lambda$  over  $[0, 1]$  is as follows.*

$$\begin{aligned} \text{If } 0 < \lambda \leq \lambda^*, \text{ then } \quad & \sup_{c \in [0,1]} g_\lambda(c) = \sup_{c \geq 0} g_\lambda(c) = \frac{\eta(\lambda)^{1-\lambda}}{\Gamma(2-\lambda)(\lambda + \eta(\lambda))}. \\ \text{If } \lambda^* \leq \lambda \leq 1, \text{ then } \quad & \sup_{c \in [0,1]} g_\lambda(c) = g_\lambda(1) = 1 - e^{-\Gamma(2-\lambda)^{-1/\lambda}}. \end{aligned}$$

**THEOREM 5.5.** *For  $\lambda \in (0, 1]$ , the approximation ratio of anonymous pricing with arbitrarily many bidders having i.i.d. valuations from a  $\lambda$ -regular distribution, is asymptotically*

$$\lim_{n \rightarrow \infty} \text{APX}(n, \lambda) = \frac{1}{\sup_{0 \leq c \leq 1} g_\lambda(c)} = \begin{cases} \frac{\Gamma(2-\lambda)(\lambda+\eta(\lambda))}{\eta(\lambda)^{1-\lambda}}, & 0 < \lambda \leq \lambda^*, \\ \frac{1}{1-e^{-\Gamma(2-\lambda)^{-1/\lambda}}}, & \lambda^* \leq \lambda \leq 1, \end{cases}$$

where  $g_\lambda$  is defined in (23),  $\lambda^*$  is the unique root of (24),  $\Gamma$  is the gamma function, and  $\eta(\lambda)$  is the unique positive solution of the equation  $e^x = 1 + \frac{x}{\lambda}$ .

A plot of this asymptotic approximation ratio can be seen in Fig. 4.

**PROOF.** The second equality, i.e. the characterization of  $\sup_{c \in [0,1]} g_\lambda(c)$ , comes from Lemma 5.4. The proof of the theorem can be split into three parts; we start by providing an upper bound on the asymptotics; as for the lower bounds, our proof considers the cases  $0 < \lambda \leq \lambda^*$  and  $\lambda^* < \lambda \leq 1$  separately.

To prove an upper bound, we start from the result obtained in Theorem 5.1,

$$\text{APX}(n, \lambda) \leq \frac{1}{\sup_{c \in [0,1]} g_{n,\lambda}(c)} + a_{n,\lambda}.$$

For each  $\lambda \in (0, 1]$  and each  $c \in [0, 1]$ , the Stirling approximation in (17) yields the pointwise convergence  $g_{n,\lambda}(c) \rightarrow g_\lambda(c)$ . Thus, by elementary analysis<sup>4</sup>, we have

$$\liminf_{n \rightarrow \infty} \sup_{0 \leq c \leq 1} g_{n,\lambda}(c) \geq \sup_{0 \leq c \leq 1} g_\lambda(c).$$

As the additive term  $a_{n,\lambda}$  vanishes as  $n \rightarrow \infty$ , we get the desired upper bound,

$$\limsup_{n \rightarrow \infty} \text{APX}(n, \lambda) \leq \frac{1}{\sup_{c \in [0,1]} g_\lambda(c)}. \quad (25)$$

To prove lower bounds, we study the asymptotic behaviour of the result obtained in Theorem 5.3,

$$\text{APX}(n, \lambda) \geq \sup_{r \in [0,1]} \inf_{q \in [0,1]} \frac{1+r(\beta_{n,\lambda}-1)}{H_{n,\lambda}(r,q)}.$$

We begin by setting  $r = 1$ , and applying the change of variables  $c = 1/(\Gamma(2-\lambda)n^\lambda q^\lambda)$ , yielding

$$\begin{aligned} \text{APX}(n, \lambda) &\geq \frac{\beta_{n,\lambda}}{\sup_{0 \leq q \leq 1} \frac{1-(1-q)^n}{q^\lambda}} \\ &= \frac{\beta_{n,\lambda}}{\frac{\Gamma(2-\lambda)n^\lambda}{\sup_{c \geq 1/(\Gamma(2-\lambda)n^\lambda)} c \left[ 1 - \left( 1 - \frac{(c\Gamma(2-\lambda))^{-1/\lambda}}{n} \right)^n \right]}}. \end{aligned}$$

Note that the numerator of the above fraction goes to 1 as  $n \rightarrow \infty$  due to Stirling's approximation. As for the denominator, we need some technical work to ensure that the limit and the supremum operators commute, and to show that it converges to  $\sup_{c \geq 0} g_\lambda(c)$ . This is done in Lemma C.2 in Appendix C. Thus, we get

$$\liminf_{n \rightarrow \infty} \text{APX}(n, \lambda) \geq \frac{1}{\sup_{c \geq 0} g_\lambda(c)}, \quad (26)$$

<sup>4</sup>It is worth mentioning that one could prove, with some technical effort, that the convergence  $g_{n,\lambda}(c) \rightarrow g_\lambda(c)$  is actually uniform on  $c$ , which would allow us to interchange the limit with the supremum and write  $\lim_{n \rightarrow \infty} \sup_{c \in [0,1]} g_{n,\lambda}(c) = \sup_{c \in [0,1]} g_\lambda(c)$ ; however, we will not need this result as the lower bound will be matching, so we omit its proof.

which matches the desired quantity as long as  $0 < \lambda \leq \lambda^*$ , due to [Lemma 5.4](#).

To get tight lower bounds for the case  $\lambda^* < \lambda \leq 1$ , we take an approach very much inspired by that in [\[24\]](#); our starting point is again [Theorem 5.3](#), but now the idea is to carefully choose a value of  $r$  which depends on  $n$ , for which a specific choice of optimal quantile  $q^*$  gives the desired bound.

For each  $n \geq 2$  and  $\lambda \in (0, 1]$ , recall the function  $H_{n,\lambda} : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  from [\(20\)](#). As this function is continuous on a compact set, we can apply Berge’s Maximum Theorem (see, e.g. [\[4, Theorem 17.31\]](#)) to conclude that the correspondence  $r \mapsto \operatorname{argmax}_{q \in [0,1]} H_{n,\lambda}(r, q)$  is non-empty, compact valued and upper hemicontinuous.<sup>5</sup> When  $r = 0$ , we have  $H_{n,\lambda}(0, q) = 1 - (1 - q)^n$ , which has a single peak at  $q = 1$ . Next we consider the case  $r = 1$ , that is,  $H_{n,\lambda}(1, q) = \frac{1}{q^\lambda}(1 - (1 - q)^n)$ . In [Lemma C.3, Appendix C](#), we prove that this function has a single peak, and that for  $\lambda^* < \lambda \leq 1$  and large enough  $n$ , its peak lies to the left of  $\beta_{n,\lambda}^{-1/\lambda}$ . Thus, by hemicontinuity, we can conclude that, for large enough  $n$ , there must exist some value of  $r \in [0, 1]$ , say  $r_n$ , such that  $H_{n,\lambda}(r_n, q)$  is maximized at precisely the quantile  $q_n^* = \beta_{n,\lambda}^{-1/\lambda}$ .<sup>6</sup> We can now plug this into [Theorem 5.3](#), obtaining

$$\begin{aligned} \text{APX}(n, \lambda) &\geq \inf_{0 \leq q \leq 1} \frac{1 + r_n (\beta_{n,\lambda} - 1)}{H_{n,\lambda}(r_n, q)} \\ &= \frac{1 + r_n (\beta_{n,\lambda} - 1)}{(1 + r_n (\beta_{n,\lambda} - 1)) \left(1 - \left(1 - \beta_{n,\lambda}^{-1/\lambda}\right)^n\right)} \\ &= \frac{1}{1 - \left(1 - \beta_{n,\lambda}^{-1/\lambda}\right)^n}; \end{aligned}$$

by taking limits (and with the help of Stirling’s approximation), we get

$$\liminf_{n \rightarrow \infty} \text{APX}(n, \lambda) \geq \frac{1}{1 - e^{-\Gamma(2-\lambda)^{-1/\lambda}}},$$

which matches the desired quantity as long as  $\lambda^* < \lambda \leq 1$ , due to [Lemma 5.4](#). This concludes the proof.  $\square$

## 6 CONCLUSION AND FUTURE DIRECTIONS

In this paper, we studied the performance of anonymous selling mechanisms for a Bayesian setting with a single item and  $n$  bidders with i.i.d. valuations. We completely characterized the asymptotic approximation ratio (with respect to the optimal, Myersonian revenue) for  $\lambda$ -regular distributions with  $\lambda \in [0, 1]$ ,

$$\lim_{n \rightarrow \infty} \sup_{F \in \mathcal{D}_\lambda} \text{APX}(F, n) = \frac{1}{\sup_{0 \leq c \leq 1} g_\lambda(c)}.$$

This quantity increases smoothly (see [Fig. 4](#)) from 1 for MHR distributions to  $\frac{e}{e-1} \approx 1.58$  for regular distributions. For the special case of MHR distributions, we also characterized the rate of convergence and provided explicit, “good” pricing schemes that depend only on the knowledge of the expected second-highest order statistic of the distribution.

A few interesting questions remain. Two quantities of independent interest are

$$\sup_{n \geq 2} \sup_{F \in \mathcal{D}_\lambda} \text{APX}(F, n) \quad \text{and} \quad \sup_{F \in \mathcal{D}_\lambda} \limsup_{n \rightarrow \infty} \text{APX}(F, n).$$

<sup>5</sup>One could, with additional technical work, also prove that  $H_{n,\lambda}(r, q)$  has a single peak (in  $q$ ) for each  $r$ , making this correspondence a continuous function; the rest of the argument would be essentially an application of the intermediate value theorem.

<sup>6</sup>Note that this statement is not merely existential: we can compute  $r_n$  by solving the equation  $\frac{\partial}{\partial q} H_{n,\lambda}(r_n, q_n^*) = 0$ .

The first one corresponds to *global* bounds on the approximation ratio, which require a finer analysis, especially in the “small”  $n$  range. Even for MHR distributions, there is still a gap between the global upper bound of 1.35 (numerically achieved at  $n = 3$  in [Theorem 4.2](#)) and the global lower bound of 1.27 (numerically achieved at  $n = 17$  in [Theorem 4.4](#)).

The second one intuitively captures settings where the seller’s prior knowledge of a large market is assumed to be size-independent. Studying this quantity amounts to asking: do the lower bounds of [Theorem 5.5](#) require constructing a “bad” distribution  $F_n$  for *each*  $n$  separately? In the range closer to MHR, i.e.  $0 < \lambda \leq \lambda^*$ , the answer is “no”: our lower bounds were obtained by setting  $r = 1$  globally, independent of  $n$ . But even for regular distributions, there is still a gap between the size-independent lower bound of 1.40 (achieved by numerically maximizing (26)) and the upper bound of 1.58 (see [Fig. 4](#) at  $\lambda = 1$ ).

Another interesting direction would be to characterize analytically the rate of convergence (with respect to the number of bidders  $n$ ) of the approximation ratio given by [Theorem 5.1](#). In other words, generalize [Theorem 4.1](#) for  $\lambda$ -regular distributions with  $\lambda > 0$ .

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## A ORDER STATISTICS

For the benefit of the reader and for ease of reference, in this section we collect some fundamental facts from probability theory that are used in the paper.

Let  $F$  be (the cumulative function (cdf) of) a continuous probability distribution supported over an interval  $D_F \subseteq [0, \infty)$ . Then  $F$  is an absolutely continuous function and thus almost everywhere differentiable in  $D_F$  with  $F'(x) = f(x)$ , where  $f$  is the density function (pdf) of  $F$ . Let  $X \sim F$  be a random variable drawn from  $F$ . Then, for any  $x \in D_F$ ,  $\Pr[X \leq x] = F(x)$

Let integers  $1 \leq k \leq n$ . Then the pdf  $f_{k:n}$  of  $X_{k:n}$ , i.e. the  $k$ -th (lowest) order statistic out of  $n$  i.i.d. draws from  $F$ , is given by (see [5, Eq. (2.2.2)])

$$f_{k:n}(x) = k \binom{n}{k} F^{k-1}(x) (1 - F(x))^{n-k} f(x), \quad x \in D_F.$$

In particular, the cdf of  $X_{n:n}$  is simply  $F_{n:n}(x) = F^n(x)$ .

The *exponential* distribution has cdf  $F_{\mathcal{E}}(x) = 1 - e^{-x}$  and pdf  $f_{\mathcal{E}}(y) = e^{-y}$ , for  $y \in [0, \infty)$ . In particular, for  $Y \sim \mathcal{E}$ , the expected values of its order statistics are (see, e.g., [5, Eq. (4.6.6)])

$$\mathbb{E}[Y_{k:n}] = H_n - H_{n-k},$$

where  $H_m = \sum_{i=1}^m \frac{1}{i}$  is the  $m$ -th harmonic number.

## B TECHNICAL LEMMAS

LEMMA B.1. *The sequence  $H_n - \ln(n)$  is strictly decreasing for all integers  $n \geq 1$ , and converges (as  $n \rightarrow \infty$ ) to the Euler–Mascheroni constant  $\gamma \approx 0.577$ .*

PROOF. This is a well-known fact from analysis, see e.g. [33, Eq. (6.64)]. □

LEMMA B.2. *For all integers  $n \geq 5$ ,*

$$\ln(n) - \ln(\ln(n)) \leq H_n - 1.$$

PROOF. Since  $\ln n$  is increasing with respect to the positive integer  $n$ , using the value of the constant  $\gamma \approx 0.577$  it is not difficult to numerically check that for any  $n \geq 5$  we have

$$\ln n \geq \ln 5 \approx 1.609 > 1.526 \approx e^{1-\gamma}.$$

As a result, using Lemma B.1, we can see that

$$H_n - 1 \geq \ln(n) + \gamma - 1 = \ln \frac{n}{e^{1-\gamma}} > \ln \frac{n}{\ln n} = \ln n - \ln \ln n.$$

□

LEMMA B.3. For any integer  $n \geq 2$  and any  $x \in [1/e, 1]$ ,

$$\ln(1/x)x(1-x)^{n-2} \leq \frac{1}{e} \left( \frac{e-1}{e} \right)^{n-2}.$$

PROOF. If we define the function  $f : (0, 1] \rightarrow (0, \infty)$  with

$$f_n(x) = \ln \left( \frac{1}{x} \right) x(1-x)^{n-2},$$

then  $f_n(1/e) = \frac{1}{e} \left( \frac{e-1}{e} \right)^{n-2}$ . Thus, it is enough to show that  $f_n$  is monotonically decreasing in  $[1/e, 1]$ . Taking its derivative, we see that indeed

$$\begin{aligned} f'_n(x) &= (1-x)^{n-3} \left[ x - 1 + (1 - (n-1)x) \ln \left( \frac{1}{x} \right) \right] \\ &\leq (1-x)^{n-3} \left[ x - 1 + (1-x) \ln \left( \frac{1}{x} \right) \right] \\ &= -(1-x)^{n-2} \left[ 1 - \ln \left( \frac{1}{x} \right) \right] \\ &\leq 0. \end{aligned}$$

The first inequality is due to the fact that  $n \geq 2$  and  $\ln(1/x) \geq 0$  and the last one due to  $x \geq 1/e$ .  $\square$

LEMMA B.4. The quantity

$$\xi(\lambda) = 1 - \left( 1 + \frac{\Gamma(2-\lambda)^{-1/\lambda}}{\lambda} \right) e^{-\Gamma(2-\lambda)^{-1/\lambda}}$$

has a unique zero  $\lambda^*$  over the interval  $(0, 1]$ ; moreover,  $\xi(\lambda) < 0$  for  $0 < \lambda < \lambda^*$ , and  $\xi(\lambda) > 0$  for  $\lambda^* < \lambda \leq 1$ .

PROOF. For the proof of this result we will use the well-known fact that the gamma function is continuously differentiable over  $[1, 2]$ , and its derivative changes sign from negative to positive at a single point in this interval ([46, Ch. 5]).

Observe that  $\xi$  is continuously differentiable in  $(0, 1]$ , with

$$\xi'(\lambda) = \frac{e^{-\Gamma(2-\lambda)^{-1/\lambda}}}{\lambda^2 \Gamma(2-\lambda)^{1+1/\lambda}} \left( \Gamma(2-\lambda)^{-1/\lambda} + \lambda - 1 \right) \Gamma'(2-\lambda);$$

the first two factors in this expression are strictly positive over  $(0, 1]$ , whereas the third factor changes sign from positive to negative at a unique point, say  $\tilde{\lambda}$ . It follows that  $\xi(\lambda)$  is strictly increasing over  $(0, \tilde{\lambda})$  and strictly decreasing over  $(\tilde{\lambda}, 1]$ . As  $\xi(0^+) = -\infty$  and  $\xi(1) > 0$ , we get the desired result.  $\square$

## C ANALYTIC PROPERTIES OF AUXILIARY FUNCTIONS

LEMMA (LEMMA 4.3). Functions  $g_n$  defined in (6) have a unique point of maximum

$$\xi_n = \operatorname{argmax}_{c \geq 0} g_n(c).$$

Furthermore, for all  $n \geq 17$ ,

$$\xi_n \leq 1.$$

PROOF. Let  $n$  be a positive integer. First we compute the first and second derivatives of  $g_n$ :

$$\begin{aligned} g'_n(c) &= 1 - \left(1 - e^{-c(H_n-1)}\right)^n \left(1 + n \frac{c(H_n-1)}{e^{c(H_n-1)} - 1}\right) \\ g''_n(x) &= \frac{n(H_n-1) \left(1 - e^{-c(H_n-1)}\right)^n \left[e^{c(H_n-1)}(c(H_n-1) - 2) - nc(H_n-1) + 2\right]}{\left(e^{c(H_n-1)} - 1\right)^2}. \end{aligned} \quad (27)$$

There is a unique point  $\tau_n > 0$  on which function  $x \mapsto e^x(x-2) - nx + 2$  changes sign from negative to positive, so the same holds for function  $g''_n$ ; thus,  $g'_n$  is strictly decreasing over  $(0, \tau_n)$  and increasing over  $(\tau_n, \infty)$ . Furthermore, notice that  $\lim_{c \rightarrow 0} g'_n(c) = 1$  and  $\lim_{c \rightarrow \infty} g'_n(c) = 0$ . This means that there has to be a unique point  $\xi_n > 0$  (where  $\xi_n < \tau_n$ ) at which  $g'_n$  changes sign from positive to negative. Therefore,  $\xi_n$  is the unique global maximizer of  $g_n$ .

In order to prove that a positive real  $y$  is above that maximization point, i.e.,  $y \geq \xi_n$ , it is enough to show that function  $g_n$  is decreasing at  $y$ , or equivalently,  $g'_n(y) < 0$ . So, recalling from [Lemma B.1](#) that  $H_n - 1 > \ln(n) + \gamma - 1$ , in order to complete our proof it is enough to show that

$$g'_n\left(\frac{\ln n + \gamma - 1}{H_n - 1}\right) < 0 \quad \text{for all } n \geq 17.$$

We will do this by demonstrating that  $g'_n\left(\frac{\ln n + \gamma - 1}{H_n - 1}\right)$  is a decreasing sequence (with respect to  $n$ ) that gets negative for  $n = 17$ . Using the explicit formula (27) for  $g'_n$ , we can see that

$$g'_n\left(\frac{\ln n + \gamma - 1}{H_n - 1}\right) = 1 - \left(1 - \frac{e^{1-\gamma}}{n}\right)^n \left(1 + \frac{\ln n + \gamma - 1}{e^{\gamma-1} - 1/n}\right)$$

is decreasing, since sequences  $\left(1 - \frac{e^{1-\gamma}}{n}\right)^n$  and  $1 + \frac{\ln n + \gamma - 1}{e^{\gamma-1} - 1/n}$  are both positive and increasing. Finally it is easy to compute (by simple substitution) that for  $n = 17$ ,  $g'_{17}\left(\frac{\ln 17 + \gamma - 1}{H_{17} - 1}\right) \approx -0.019 < 0$ .  $\square$

LEMMA C.1. For the functions  $g_n$  defined in (6),

$$\max_{c \in [0,1]} g_n(c) = 1 - O\left(\frac{\ln \ln n}{\ln n}\right) = 1 - o(1). \quad (28)$$

PROOF. For any integer  $n \geq 5$ , we will lower bound the maximum value of  $g_n(x)$  by evaluating it on the following numbers:

$$\tilde{c}_n = \frac{\ln n - \ln \ln n}{H_n - 1} = \frac{\ln(n/\ln n)}{H_n - 1} \leq 1;$$

the inequality holds due to [Lemma B.2](#). We have that

$$\begin{aligned} g_n(\tilde{c}_n) &= \frac{\ln(n/\ln n)}{H_n - 1} \left[1 - \left(1 - e^{-\ln(n/\ln n)}\right)^n\right] \\ &= \frac{\ln n - \ln \ln n}{H_n - 1} \left[1 - \left(1 - \frac{\ln n}{n}\right)^n\right] \\ &\geq \frac{\ln n - \ln \ln n}{\ln n} \left(1 - \frac{1}{n}\right) \\ &= 1 - O\left(\frac{\ln \ln n}{\ln n}\right), \end{aligned}$$

the inequality being a consequence of the fact that  $(1 - \frac{x}{n})^n \leq e^{-x}$  for any positive real  $x \leq n$  with  $x \leftarrow \ln n$  and also due to  $H_n \leq 1 + \ln n$ .  $\square$

LEMMA (LEMMA 5.2). *For the rescaled Pareto distribution  $F_{\lambda,r}$  given by (19), for  $\lambda \in (0, 1]$  and  $r \in (0, 1]$ , we have the following expressions for its expected second-highest order statistic and optimal anonymous pricing,*

- (1)  $\mathbb{E}_{X \sim F_{\lambda,r}} [X_{n-1:n}] = 1 + r (\beta_{n,\lambda} - 1)$ ;
- (2)  $\text{PRICE}(F_{\lambda,r}, n) = \sup_{0 \leq q \leq 1} \left( 1 + r \left( \frac{1}{q^\lambda} - 1 \right) \right) (1 - (1 - q)^n)$ ,

where  $\beta_{n,\lambda}$  is given by (15).

PROOF. Point 2 follows directly from the change of variables

$$q = 1 - F_{\lambda,r}(x) \quad \Leftrightarrow \quad x = 1 + r \left( \frac{1}{q^\lambda} - 1 \right) = 1 - r + \frac{r}{q^\lambda}.$$

To prove point 1, we use the same change of variables to express the expectation in terms of the gamma function:

$$\begin{aligned} \mathbb{E} [X_{n-1:n}] &= n(n-1) \int_1^\infty x F^{n-2}(x) (1 - F(x)) dF(x) \\ &= n(n-1) \int_0^1 \left( 1 - r + \frac{r}{q^\lambda} \right) q (1 - q)^{n-2} dq \\ &= (1-r)n(n-1)B(2, n-1) + rn(n-1)B(2-\lambda, n-1) \\ &= 1 - r + r \frac{n! \Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} \\ &= 1 + r(\beta_{n,\lambda} - 1). \end{aligned}$$

$\square$

LEMMA (LEMMA 5.4). *For each  $\lambda \in (0, 1]$ , let  $g_\lambda$  be defined as in (23) and  $\eta(\lambda)$  be the unique positive solution of the equation  $e^x = 1 + \frac{x}{\lambda}$ . Let also  $\lambda^*$  be the unique root of (24). The supremum of the function  $g_\lambda$  over  $[0, 1]$  is as follows.*

$$\begin{aligned} \text{If } 0 < \lambda \leq \lambda^*, \text{ then } \quad & \sup_{c \in [0,1]} g_\lambda(c) = \sup_{c \geq 0} g_\lambda(c) = \frac{\eta(\lambda)^{1-\lambda}}{\Gamma(2-\lambda)(\lambda + \eta(\lambda))}. \\ \text{If } \lambda^* \leq \lambda \leq 1, \text{ then } \quad & \sup_{c \in [0,1]} g_\lambda(c) = g_\lambda(1) = 1 - e^{-\Gamma(2-\lambda)^{-1/\lambda}}. \end{aligned}$$

PROOF. Fix  $\lambda \in (0, 1)$ . Observe that  $g_\lambda$  is twice continuously differentiable; with some calculations one can see that

$$\begin{aligned} g'_\lambda(c) &= 1 - e^{-(c\Gamma(2-\lambda))^{-1/\lambda}} \left[ 1 + \frac{(c\Gamma(2-\lambda))^{-1/\lambda}}{\lambda} \right]; \\ g''_\lambda(c) &= \frac{e^{-(c\Gamma(2-\lambda))^{-1/\lambda}}}{\lambda^2 c^{1+2/\lambda} \Gamma(2-\lambda)^{2/\lambda}} \left[ -1 + (1-\lambda)(c\Gamma(2-\lambda))^{1/\lambda} \right]; \end{aligned}$$

one also has the limits  $g_\lambda(0) = 0$ ,  $g_\lambda(\infty) = 0$ ,  $g'_\lambda(0) = 1$ ,  $g'_\lambda(\infty) = 0$ ,  $g''_\lambda(0) = 0$ ,  $g''_\lambda(\infty) = 0$ . Now note that there is a unique point,  $\tau_\lambda = \frac{1}{\Gamma(2-\lambda)(1-\lambda)^\lambda}$ , at which  $g''_\lambda$  changes sign from negative to positive, so that there is a unique point  $\xi_\lambda \leq \tau_\lambda$  at which  $g'_\lambda$  changes sign from positive to negative. Thus  $g_\lambda$  has a unique peak over  $[0, \infty)$  which corresponds to the unique root of its derivative. This peak occurs in the interval  $[0, 1]$  if and only if  $g'_\lambda(1) \leq 0$ ; solving for  $\lambda$ , we get  $\lambda \leq \lambda^*$  as in (24) and

**Lemma B.4.** Thus, if  $\lambda \geq \lambda^*$ , the supremum of  $g_\lambda(c)$  over  $[0, 1]$  is achieved at  $c = 1$ ; this includes the case  $\lambda = 1$  since a similar analysis yields that  $g'_\lambda$  is strictly negative and  $g''_\lambda$  is strictly positive. On the other hand, for  $\lambda \leq \lambda^*$ , by looking at the equation  $g'_\lambda(c) = 0$  and performing the change of variables  $x = (c\Gamma(2 - \lambda))^{-1/\lambda}$  we get  $e^x = 1 + \frac{x}{\lambda}$ , or  $x = \eta(\lambda)$ ; plugging these back into  $g_\lambda(c)$  gives us the desired result.  $\square$

**LEMMA C.2.** For each  $\lambda \in (0, 1]$ , we have the limit

$$\lim_{n \rightarrow \infty} \sup_{c \geq 1/(\Gamma(2-\lambda)n^\lambda)} c \left[ 1 - \left( 1 - \frac{(c\Gamma(2-\lambda))^{-1/\lambda}}{n} \right)^n \right] = \sup_{c \geq 0} g_\lambda(c).$$

**PROOF.** Define the sequence of auxiliary functions  $h_{n,\lambda}$  for  $n \geq 2$ ,

$$h_{n,\lambda}(c) = \begin{cases} c, & c \leq \frac{1}{\Gamma(2-\lambda)n^\lambda}, \\ c \left[ 1 - \left( 1 - \frac{(c\Gamma(2-\lambda))^{-1/\lambda}}{n} \right)^n \right], & c \geq \frac{1}{\Gamma(2-\lambda)n^\lambda}. \end{cases}$$

These functions are continuous, vanish at infinity, and converge pointwise to  $g_\lambda$ , which is also continuous. Also, for any  $\epsilon > 0$ , the restriction of  $h_{n,\lambda}$  to  $[\epsilon, \infty)$  forms an eventually decreasing sequence (with respect to  $n$ ), since  $x \leq n \leq m$  implies  $(1 - x/n)^n \leq (1 - x/m)^m$  ([35, Theorem 35]). Thus, by Dini's Theorem [47, Theorem 7.13], over any interval  $[a, b]$  with  $0 < a < b < \infty$ , the sequence  $h_{n,\lambda}$  converges uniformly to  $g_\lambda$  and we have  $\lim_{n \rightarrow \infty} \sup_{c \in [a, b]} h_{n,\lambda}(c) = \sup_{c \in [a, b]} g_\lambda(c)$ .

To conclude the proof, let  $z_\lambda^*$  be the unique maximizer of  $g_\lambda(c)$  (see also the proof of Lemma 5.4). Take  $0 < \epsilon \leq g_\lambda(z_\lambda^*)$ . Take  $N$  large enough so that  $h_{n,\lambda}$  are decreasing over  $[\epsilon, \infty)$ , for  $n \geq N$ . Take  $\delta$  such that  $h_N(c) \leq \epsilon$  for all  $c \geq \delta$ . Putting all this together, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sup_{c \in [0, \epsilon]} h_{n,\lambda}(c) &\leq \epsilon; \\ \limsup_{n \rightarrow \infty} \sup_{c \in [\epsilon, \delta]} h_{n,\lambda}(c) &= \sup_{c \in [\epsilon, \delta]} g_\lambda(c) \leq g_\lambda(z_\lambda^*); \\ \limsup_{n \rightarrow \infty} \sup_{c \geq \delta} h_{n,\lambda}(c) &\leq \epsilon, \end{aligned}$$

and thus  $\limsup_{n \rightarrow \infty} \sup_{c \geq 0} h_{n,\lambda}(c) \leq \sup_{c \geq 0} g_\lambda(c)$ . Since by elementary analysis we also have  $\liminf_{n \rightarrow \infty} \sup_{c \geq 0} h_{n,\lambda}(c) \geq \sup_{c \geq 0} g_\lambda(c)$ , this concludes the proof.  $\square$

**LEMMA C.3.** For each  $\lambda \in (0, 1]$  and  $n \geq 2$ , the function

$$H_{n,\lambda}(1, q) = \frac{1 - (1 - q)^n}{q^\lambda},$$

defined over  $q \in [0, 1]$ , has a unique maximizer  $\xi_{n,\lambda}$ . Moreover, if in addition  $\lambda^* < \lambda \leq 1$ , then  $\xi_{n,\lambda} \leq \beta_{n,\lambda}^{-1/\lambda}$  for large enough  $n$ . Here  $\lambda^*$  is the unique root of (24) and  $\beta_{n,\lambda}$  is defined as in (15).

**PROOF.** For simplicity we shall assume that  $\lambda < 1$ ; when  $\lambda = 1$  a similar analysis implies that the unique maximizer occurs at  $\xi_{n,\lambda} = 0$ .

The first derivative of  $H_{n,\lambda}$  is

$$\frac{\partial H_{n,\lambda}(1, q)}{\partial q} = \frac{\lambda}{q^{1+\lambda}} \left[ (1 - q)^{n-1} \left[ 1 + q \left( \frac{n}{\lambda} - 1 \right) \right] - 1 \right],$$

which (for  $\lambda < 1$ ) has a positive pole at  $q = 0$  and is negative at  $q = 1$ . Its first factor is always positive, and its second factor can be differentiated:

$$\tilde{H}_{n,\lambda}(1, q) = (1 - q)^{n-1} \left[ 1 + q \left( \frac{n}{\lambda} - 1 \right) \right] - 1;$$

$$\frac{\partial \tilde{H}_{n,\lambda}(1, q)}{\partial q} = \frac{n}{\lambda}(1-q)^{n-2} [(1-\lambda) - q(n-\lambda)] .$$

This function changes sign from positive to negative at a single point,  $\tau_{n,\lambda} = \frac{1-\lambda}{n-\lambda}$ . Thus, both  $\tilde{H}_{n,\lambda}$  and  $\frac{\partial}{\partial q} H_{n,\lambda}$  change sign from positive to negative at a single point  $\xi_{n,\lambda} \geq \tau_{n,\lambda}$ , which enables us to conclude that  $H_{n,\lambda}(1, q)$  has a single peak.

Next, we argue that the unique maximizer of  $H_{n,\lambda}(1, q)$  is smaller than the quantity  $\beta_{n,\lambda}^{-1/\lambda}$ , for large enough  $n$ . In order to do this, observe that

$$\left. \frac{\partial H_{n,\lambda}(1, q)}{\partial q} \right|_{q=\beta_{n,\lambda}^{-1/\lambda}} = \lambda \beta_{n,\lambda}^{1+1/\lambda} \left[ \left(1 - \beta_{n,\lambda}^{-1/\lambda}\right)^{n-1} \left[1 + \beta_{n,\lambda}^{-1/\lambda} \left(\frac{n}{\lambda} - 1\right)\right] - 1 \right] .$$

The first factor of the above expression is strictly positive for all choices of  $n$  and  $0 < \lambda \leq 1$ . The second factor actually has a limit as  $n \rightarrow \infty$ ; it converges to

$$\left(1 + \frac{\Gamma(2-\lambda)^{-1/\lambda}}{\lambda}\right) e^{-\Gamma(2-\lambda)^{-1/\lambda}} - 1 .$$

This quantity is negative precisely when  $\lambda > \lambda^*$ , as proven in [Lemma B.4](#). Thus, as long as  $\lambda > \lambda^*$ , we have that  $\frac{\partial}{\partial q} H_{n,\lambda}(1, \beta_{n,\lambda}^{-1/\lambda})$  is negative for large enough  $n$ , which implies that the unique maximizer of  $H_{n,\lambda}(1, q)$  is to the left of  $\beta_{n,\lambda}^{-1/\lambda}$ .  $\square$