The Computability of LQR and LQG Control

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ABSTRACT

We consider decision problems associated with the linear quadratic regulator (LQR) and linear quadratic Gaussian (LQG) control problems in continuous time. The decision problems ask, given the parameters of a problem and a threshold rational number r, is the optimal cost less than or equal to the threshold r? LQR and LQG are fundamental problems in the theory of linear systems and it is well known that optimal controllers for these problems have a closed-form solution. However, since the closed-form solutions involve transcendental functions, they can only be evaluated numerically. Thus, it is possible that numerical imprecisions prevent answering the decision problem no matter what precision is used for the computations. Indeed, the computability of these natural decision problems has remained open.

We show that the problems are decidable. The decidability is not relative to any given limit on the numerical precision. Our proof uses the Lindemann-Weierstrass theorem from transcendental number theory to show that checking whether an exponential polynomial is less than or equal to a rational threshold is decidable. We show further that (conditional) decidability results for several open problems in linear systems theory can be obtained if one additionally assumes Schanuel's conjecture, a unifying conjecture in transcendental number theory.

CCS CONCEPTS

• **Computing methodologies → Computational control theory**;

KEYWORDS

LQR, LQG, decidability, transcendental numbers

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1 INTRODUCTION

In 1960, Rudolf Kalman published two groundbreaking papers that set the stage of much of modern control theory research [17, 18]. In the first paper [17], he defined the linear-quadratic regulator (LQR) problem, and showed under what circumstances, and how, a (closed-form) feedback controller can be designed for a linear system such that the integral of the square of the tracking error is minimized. The second paper [18] developed a theory of filtering, and the two papers together formed the basis for linear-quadratic-Gaussian (LQG) control.

The LQR and LQG paradigms have been extremely influential in the subsequent development of control theory. However, one aspect that has not been studied is the *computability* of the problem.

Let us be precise. The LQR problem (over finite horizon t_f) asks, given the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

to find a feedback control law such that the following quadratic cost is minimized:

$$x(t_f)^T Sx(t_f) + \int_0^{t_f} \left(x(t)^T Qx(t) + u(t)^T Ru(t) \right) dt.$$

Here, the matrices S, Q are symmetric positive semi-definite, and R is symmetric positive definite.

With every optimization problem, one can associate a natural decision problem. For the LQR problem, the natural *LQR decision problem* is, given a system and a cost function as above, a horizon t_f, and a rational number c, whether there is a controller that achieves a cost less than or equal to c. One can define an analogous decision problem for LQG control. The computability status of these LQR and LQG decision problems has remained open.

It may come as a surprise to the practitioner that the decision problem is open. It is well known, from [17], that the optimal controller has a closed-form solution, obtained through analyzing the associated Ricatti equation. However, the closed-form solution involves transcendental functions and does not immediately imply the decidability of the associated decision problem. In practice, the closed-form is evaluated numerically and the accumulation of numerical errors may make it impossible to answer the decision problem. For example, suppose that the optimal value is exactly c. Then, no matter how precisely we bound the numerical computations, we may never be able to state if the minimum is less than or equal to c or above c.

There is an implicit belief in the community that decision problems for continuous-state, continuous-time problems are not amenable to exact algorithmic analysis, and hence one must formulate the decision problems relative to a numerical precision. For example, Blondel and Tsitsiklis's influential survey on algorithms for control [9] do not discuss continuous-time problems for the most part, and state that "Problems of [continuous-state] type do not admit closed-form or exact algorithmic solutions" and can only be "solved approximately, by discretizing them" [9, pg. 1267]. We show, contrary to this belief, that one can formulate and decide the natural decision problem without requiring a fixed numerical precision.

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In particular, we show how tools from transcendental number theory can provide an algorithmic framework for many decision problems in continuous-time, continuous-space, control theory—including the LQR and LQG decision problems. Our starting point is the study of exponential-polynomials

$$\sum_i a_i e^{b_i},$$

where a_i, b_i are algebraic numbers, that arise as solutions to linear differential equations. We use the Lindemann-Weierstrass theorem from transcendental number theory [24] to characterize properties of exponential-polynomials; in particular, the function $\sum a_i e^{b_i t}$ is identically zero for distinct algebraic numbers b_i if and only if each a_i is zero. A consequence of the result, together with results on computational algebraic number theory, is that it is decidable if an exponential polynomial is greater than or equal to a rational number. We use this decision procedure to show decidability of several important, and open, problems in control, including LQR and LQG control.

While we focus on decidability, using quantitative estimates for the Lindemann-Weierstrass theorem [25], we can provide (somewhat pessimistic) complexity bounds. The complexity bounds are high and we do not have matching lower bounds at this point. We leave the goal of proving optimal complexity results for future work.

For a number of additional questions in continuous-time control, we show *conditional* decision procedures, subject to Schanuel's conjecture, a famous unresolved conjecture in transcendental number theory. Schanuel's conjecture is a far-reaching generalization of Lindemann-Weierstrass theorem and many other results in transcendental number theory [24]. A consequence of Schanuel's conjecture is that the theory of reals with bounded exponentials, sines, and cosines, is decidable [21]. We use this theory to show the conditional decidability of the following problems:

(1) *Controllability of linear time-varying systems*: [26] Given a time-varying linear system
$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
where the matrices $A(t)$ and $B(t)$ have exponential-polynomial entries, check if the system is controllable on an interval $[t_0, t_1]$; and

(2) *State feedback stabilization with guaranteed transient bounds*: [8, 16] given a Hurwitz matrix A, a rational β that is greater than the maximum real parts of the eigenvalues of A, and a rational M, decide if $\|e^{At}\| \leq Me^{\beta t}$ for all $t \geq 0$.

As far as we know, algorithmic techniques for solving these problems (and many others) are open.

Other Related Work. Natural decision problems of other optimization problems are defined analogously to our definition of the LQR and LQG problems, and studied in complexity theory. For example, it is known that linear programming is in polynomial time [19, 20] (and conjectured to be in strongly polynomial time) and integer-programming [10] and quadratic programming are NP-complete [27]. In each case, one can show that the problem satisfies an ordered field property: the optimal value of an instance given by rational numbers is also rational (and in fact, a "small" rational). This is unlike the LQR and LQG problems, where the solution can be transcendental even when the problem is defined using rational numbers only. To the best of our knowledge, whether the natural decision problem for semidefinite programming is in polynomial time remains open; although one can get to within an additive error of ϵ of the optimal in polynomial time (in the size of the problem and $\log(1/\epsilon)$).

A number of results in linear dynamical systems and the analysis of continuous-time Markov chains have used either the Lindemann-Weierstrass theorem or Schanuel's conjecture to show (conditional) decidability results [1, 3, 12, 23]. Moreover, there are deep connections between dynamical systems and problems in Diophantine approximation [2, 15]. While we focus on a few key problems in this paper, we believe many other problems in control of linear systems can be tackled using similar techniques. Also, while Schanuel's conjecture may seem too strong a hammer, related problems in continuous-time linear systems have been open for a long time, and only known to be conditionally decidable under Schanuel's conjecture [5, 23].

Notation. We use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\overline{\mathbb{Q}}$, and $\mathbb{C}$ for the set of natural numbers, integers, rationals, reals, algebraic numbers, and complex numbers. We write $\mathbf{I}_n$ for the identity matrix of size n and e for Euler's constant.

2 PRELIMINARIES

2.1 Algebraic Numbers

A complex number a is *algebraic* if it is the root of a univariate polynomial with integer coefficients. We denote the set of algebraic numbers by $\overline{\mathbb{Q}}$; the notation captures the fact that algebraic numbers form the algebraic closure of $\mathbb{Q}$ over $\mathbb{C}$. A *transcendental* number is a complex number that is not algebraic.

Given complex numbers $a_1, \ldots, a_n$, we say they are *linearly dependent* over $\mathbb{Q}$ if there are $d_1, \ldots, d_n \in \mathbb{Q}$, not all 0 such that $\sum_i d_i a_i = 0$. They are called linearly independent otherwise. Likewise, we say they are *algebraically dependent* over $\mathbb{Q}$ if there is a nonzero polynomial $P \in \mathbb{Q}[x_1, \ldots, x_n]$ such that $P(a_1, \ldots, a_n) = 0$. Otherwise, they are algebraically independent over $\mathbb{Q}$.

The *defining polynomial* of an algebraic number a, written p_a, is the unique univariate polynomial of least degree and whose coefficients do not have common factors, which vanishes at a. The *height* of p is the maximum magnitude of its coefficients. The height and degree of an algebraic number is the height and degree of its defining polynomial, respectively. For a univariate polynomial $p \in \mathbb{Z}[x]$, the bit length of p, written $\|p\|$, is the total length of its list of coefficients encoded in binary. Thus, the degree of p is bounded above by $\|p\|$ and the height by $2^{\|p\|}$.

An algebraic number a can be represented using its defining polynomial, together with rational approximations of its real and imaginary parts that are sufficiently precise to distinguish a from the other roots of p_a. Thus, a representation of a number a consists of $(p_a, c, d, r) \in \mathbb{Z}[x] \times \mathbb{Q}^3$ where a is the unique root of p_a in the circle in $\mathbb{C}$ of radius r centered at $c + d\,i$. For an algebraic number ρ, we write $\|\rho\|$ for the bit length of its representation. Given a real number t and a positive integer m, we say that $q \in \mathbb{Q}$ is an m-bit approximation of t if $|t - q| < 2^{-m}$. It is known that, given a polynomial p, one can compute standard representations of all its roots in time polynomial in $\|p\|$ [13]. Thus, we can compute m-bit approximations for any algebraic number in time polynomial in $\|\rho\|$ and in m.

The following theorem summarizes necessary algorithmic results for computations with algebraic numbers (see [13]).

Theorem 2.1. (1) *[13] Given algebraic numbers a, b, one can compute $a + b$, ab, and check if $a = b$ or $a \geq b$ in time polynomial in $\|a\| + \|b\|$.*
(2) *[11] For any fixed real numbers $0 < a < b$, there is an algorithm which, given integer $p \geq 0$, computes the functions $\exp(x)$, $\sin(x)$, and $\cos(x)$ in time $O(p \log p \log \log p)$, with relative error at most $O(2^{-p})$, uniformly for all $x \in [a, b]$.*
(3) *There is an algorithm which takes as input a real algebraic number ρ and a positive number m, and returns an m-bit approximation of $\exp(\rho)$, $\sin(\rho)$, $\cos(\rho)$ in time polynomial in both $\|\rho\|$ and m.*

2.2 Exponential Polynomials and Matrix Exponentials

Informally, an exponential polynomial is a polynomial function of variables and exponentials of variables. For any polynomial $p(x_1, \ldots, x_{2n})$ in $\overline{\mathbb{Q}}[x_1, \ldots, x_{2n}]$, we associate the *exponential polynomial* $p(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$.

We also consider the ring of finite sums of the form $\sum_{j=1}^{k} \alpha_j e^{\beta_j}$, for algebraic numbers $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$. Note that since algebraic numbers form a field, expressions of the above form are closed under addition and multiplication. Evaluating an exponential polynomial $p(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$ at a point in $\overline{\mathbb{Q}}^n$ yields such a finite sum.

For a matrix $A \in \mathbb{R}^{n \times n}$, we write e^A for the matrix exponential, defined by $e^A := \mathbf{I}_n + A + \frac{1}{2!}A^2 + \ldots$.

Recall that the solution of a linear time invariant system

$$\dot{x}(t) = Ax(t), \quad x \in \mathbb{R}^n, x(0) = x_0 \tag{1}$$

is given by $x(t) = e^{At}x_0$.

Proposition 2.2. *For any matrix $A \in \overline{\mathbb{Q}}^{n \times n}$ and $t \in \overline{\mathbb{Q}}$, each element of the matrix e^{At} is a finite sum of the form $\sum_{j=1}^{k} \alpha_j e^{\beta_j}$ with* algebraic numbers *α_j and β_j, $j = 1, \ldots, k$.*

Proof. Let $A = PJP^{-1}$ be the Jordan canonical form for A, where J is an upper block diagonal matrix

$$J = \begin{pmatrix} J_{\lambda_1} & & & 0 \\ & J_{\lambda_2} & & \\ & & \ddots & \\ 0 & & & J_{\lambda_k} \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A, each J_λ is of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & 0 \\ & & & \ddots & 1 \\ 0 & \ldots & & 0 & \lambda \end{pmatrix},$$

and the size of J_λ is the multiplicity of λ. Then, $e^{At} = Pe^{Jt}P^{-1}$, where

$$e^{Jt} = \begin{pmatrix} e^{J_{\lambda_1} t} & & & 0 \\ & e^{J_{\lambda_2} t} & & \\ & & \ddots & \\ 0 & & & e^{J_{\lambda_k} t} \end{pmatrix}$$

and, for λ of multiplicity $m + 1$,

$$e^{J_\lambda t} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \ldots & \frac{t^m}{m!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \ldots & \frac{t^{m-1}}{(m-1)!}e^{\lambda t} \\ & & & \ddots & te^{\lambda t} \\ 0 & \ldots & & 0 & e^{\lambda t} \end{pmatrix}.$$

Now the eigenvalues of an algebraic matrix, being solutions of the characteristic equation, are algebraic, and each entry of the matrix P (and P^{-1}) is obtained through linear algebra and therefore algebraic. Thus, each entry of e^{At} is an exponential polynomial in t, and for each algebraic number t, is of the form $\sum_j \alpha_j e^{\beta_j}$ for algebraic numbers α_j, β_j. □

In fact, the proof demonstrates that when A is a real-algebraic matrix and x_0 is a real-algebraic vector, each coordinate of the solution vector $x(t)$ of (1) can be expressed as an exponential polynomial function of the parameter t:

$$\sum_{j=1}^{k} P_t(j) e^{\lambda_j t}, \tag{2}$$

where $\lambda_1, \ldots, \lambda_k$ are the distinct (real or complex) eigenvalues of A and each $P_t(j)$ is a polynomial function of t with (possibly complex) algebraic coefficients and with degree one less than the multiplicity of the eigenvalue λ_j.

2.3 Transcendental Number Theory and Decision Problems for Exponential Polynomials

We shall use the following basic result from transcendental number theory (see, e.g., [24]).

Theorem 2.3 (Lindemann-Weierstrass). *If $a_1, \ldots, a_n \in \overline{\mathbb{Q}}$ are linearly independent over $\mathbb{Q}$, then $e^{a_1}, \ldots e^{a_n}$ are algebraically independent over $\mathbb{Q}$.*

The following theorem uses Theorem 2.1 and the Lindemann-Weierstrass theorem to provide a decision procedure.

Theorem 2.4. *Let $p(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$ be an exponential polynomial. For a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \overline{\mathbb{Q}}^n$ and a real algebraic number $r \in \overline{\mathbb{Q}}$, the problem of deciding if $|p(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n})| \geq r$ is decidable.*

Proof. Evaluating the exponential polynomial at α gives a finite sum of exponentials, call it $|\eta|$. Using Theorem 2.1, we can compute this expression to any desired precision in time polynomial in the number of bits in the precision. If the finite sum is degenerate, i.e., every exponential term has coefficient zero, it is an algebraic number and we can decide if the number is greater than or equal to r. Otherwise, we know that the number is not algebraic, using the

Lindemann-Weierstrass theorem, and so $||\eta| - r| > 0$. We compute a sequence of approximations to $||\eta| - r|$, using Theorem 2.1(3) until we can establish $||\eta| - r| > 0$. At this point, computing the sign of $|\eta| - r$ allows us to decide the problem. □

The following corollary is immediate, since the solution of an LTI system is an exponential polynomial.

Corollary 2.5. *Given the system* (1) *with a rational matrix A and a rational vector x_0, a rational time point t_f and a rational vector r, the problem of deciding if $x(t_f) \geq r$ is decidable.*

3 DECISION PROBLEMS

3.1 Linear Quadratic Regulator (LQR)

We follow the formulation of Bertsekas [6] for the finite horizon LQR problem. Consider the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \quad x(0) = x_0, \tag{3}$$

and the quadratic cost over the time interval $[0, t_f]$,

$$J(x_0, t_f) := \\ x(t_f)^T Sx(t_f) + \int_0^{t_f} [x(t)^T Qx(t) + u(t)^T Ru(t)]dt,$$

where S and Q are symmetric positive semidefinite, and R is symmetric positive definite.

Problem 1. *The* LQR decision problem *asks, given a rational time bound t_f, rational matrices A, B, Q, R, S, a rational initial state x_0, and a rational threshold $r > 0$, decide if there is a controller for* (3) *such that $J(x_0, t_f) \leq r$.*

It is well known that the control policy that minimizes the cost $J(x_0, t_f)$ is of the form $u(t) = -R^{-1}B^T K(t)x(t)$, where $K(t)$ is the solution of the matrix Riccati Equation

$$\dot{K}(t) = -K(t)A - A^T K(t) + K(t)BR^{-1}B^T K(t) - Q \tag{4}$$

that is solved backward with the terminal condition $K(t_f) = S$. Moreover, the optimal cost-to-go function is $J^*(t, x) = x^T K(t)x$. This means the minimum cost is $J^*(0, x_0) = x_0^T K(0)x_0$.

It is also well known that $K(t) = \Lambda(t)X^{-1}(t)$, where $\Lambda(t)$ and $X(t)$ satisfy the Hamiltonian differential equation:

$$\frac{d}{dt}\begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix} = H \begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix}, \; H := \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$$

with the final condition $X(t_f) = \mathbf{I}_n$ and $\Lambda(t_f) = S$. Then, $\Lambda(t)$ and $X(t)$ are exponential polynomials of time:

$$\begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix} = e^{H(t-t_f)} \begin{bmatrix} \mathbf{I}_n \\ S \end{bmatrix}.$$

Thus, the entries of $K(t)$ and the optimal cost $J(x_0) = x_0^T K(0)x_0 = x_0^T \Lambda(0)X^{-1}(0)x_0$ are ratios of exponential polynomials.

Theorem 3.1. *The LQR decision problem is decidable.*

Proof: $J(x_0, t_f) \leq r$ becomes an exponential polynomial inequality after multiplying both sides by the determinant of $X(0)$. Then, we use Theorem 2.4. □

We note that one can similarly formulate the decision problem for the *infinite horizon* LQR problem. Since the infinite horizon problem reduces to solving the algebraic Riccati equation, one can reduce the problem to a satisfiability problem in the theory of reals with addition and multiplication. Since that theory is decidable [4], the decidability of the infinite horizon problem follows.

3.2 Linear Quadratic Gaussian (LQG)

A formal treatment of LQG can be found in [14]. Consider the stochastic LTI system

$$\begin{aligned} dx(t) &= [Ax(t) + Bu(t)]dt + Gdw(t), \\ dy(t) &= Cdx(t) + dv(t), \end{aligned} \tag{5}$$

where $w(\cdot)$ and $v(\cdot)$ are Brownian motions, independent from each other and covariance matrices Q and R, respectively. The initial condition is normally distributed $x(0) \sim \mathcal{N}(0, P_0)$. The goal of the Kalman Filter is to find an estimator $\hat{x}(t)$ such that the variance of the estimation is minimized:

$$J(t) = \mathbb{E}\left[(x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t))\right]. \tag{6}$$

It is shown that such an estimator is in fact the expectation of $x(t)$ conditioned on past information

$$\hat{x}(t) := \mathbb{E}\left[x(t) \mid \text{ past } u(t), y(t)\right], \tag{7}$$

and satisfies the following differential equation

$$\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + F(t)[y(t) - C\hat{x}(t)], \tag{8}$$

with $F(t) = P(t)C^T R^{-1}$, and $P(t)$ is the covariance of the estimate satisfying the Riccati equation:

$$\dot{P}(t) = AP(t) + P(t)A^T - P(t)C^T R^{-1} CP(t) + GQG^T,$$

initialized at $P(0) = P_0$. We also know that $J^*(t) = \min J(t) = $ trace $P(t)$.

As in the LQR case, we can write $P(t) = \Lambda(t)X^{-1}(t)$, where $\Lambda(t)$ and $X(t)$ satisfy the Hamiltonian differential equation:

$$\frac{d}{dt}\begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix} = \bar{H} \begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix}, \bar{H} := \begin{bmatrix} -A^T & -C^T R^{-1} C \\ -GQG^T & A \end{bmatrix}$$

with the initial condition $X(0) = \mathbf{I}_n$ and $\Lambda(0) = P_0$. Then, $\Lambda(t)$ and $X(t)$ are exponential polynomials of time:

$$\begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix} = e^{\bar{H}t} \begin{bmatrix} \mathbf{I}_n \\ P_0 \end{bmatrix},$$

and the entries of $P(t)$ and the optimal cost $J^*(t) = $ trace $P(t) = $ trace $(\Lambda(t)X^{-1}(t))$ are ratios of exponential polynomials.

Problem 2. *The* LQG decision problem *asks, given rational matrices A, C, G, Q, R, P_0, rational time bound T, and rational threshold r, whether the least square error of the estimate $J^*(T)$ is less than or equal to r.*

Theorem 3.2. *The LQG decision problem is decidable.*

Proof: $J^*(T) \leq r$ becomes an exponential polynomial inequality after multiplying both sides by the determinant of $X(T)$. Then, we use Theorem 2.4. □

4 SCHANUEL'S CONJECTURE AND FURTHER RESULTS

Let us consider an extension to the LQR decision problem, where we do not set the initial condition but ask if there is an appropriate controller for some initial condition from a set.

Problem 3. *The* LQR with polytopic initial set decision problem *asks, given a rational time bound t_f, rational matrices A, B, Q, R, Q_{t_f}, a bounded polytopic set X_0, and a rational threshold $r > 0$, decide if there is a controller for* (3) *such that $J(t_f, x_0) \leq r$ for some $x_0 \in X_0$.*

Similarly, we can extend the LQG problem so that the covariance of the initial state is not known exactly but is bounded by a polytopic set.

Problem 4. *The* LQG with polytopic initial covariance decision problem *asks, given rational matrices A, C, G, Q, R, a bounded polytopic set P, and rational time bound t_f, and rational threshold r, whether the least square error of the estimate $J^*(t_f)$ is less than or equal to r for $x_0 \sim \mathcal{N}(0, P_0)$ for some $P_0 \in P$.*

Unfortunately, these problems do not fall within the purview of our decision procedure. This is because we existentially quantify over $x_0 \in X_0$ (and $P_0 \in P$) and do not have a fixed x_0 or P_0 with algebraic coordinates.

It turns out that these, and a number of other related problems, can be decided if we assume *Schanuel's conjecture*. Schanuel's conjecture for the complex numbers is a unifying conjecture in transcendental number theory (see, e.g., [24]).

Conjecture 4.1 (**SC**). *Let $a_1, \ldots, a_n$ be complex numbers that are linearly independent over rational numbers $\mathbb{Q}$. Then, among the $2n$ numbers*

$$a_1, \ldots, a_n, e^{a_1}, \ldots, e^{a_n}$$

at least n are algebraically independent.

Schanuel's conjecture is believed to imply many known results in transcendental number theory as well as all reasonable conjectures on the values of the exponential function; it is also believed that it will be a difficult result to prove. Note that the special case where $a_1, \ldots, a_n$ are all algebraic is the Lindemann-Weierstrass theorem.

An important consequence of Schanuel's conjecture is that the theory of reals $(\mathbb{R}, 0, 1, +, \cdot, \leq)$ remains decidable when extended with the exponential and trigonometric functions over bounded domains.

For a function $f : \mathbb{R} \to \mathbb{R}$ and $n \in \mathbb{N}$, define the restriction $f \upharpoonright n$ as

$$f \upharpoonright n(x) = \begin{cases} f(x) & \text{if } x \in [0, n] \\ 0 & \text{otherwise} \end{cases}$$

Let $(\mathbb{R}, 0, 1, +, \cdot, <, \exp \upharpoonright n, \sin \upharpoonright n, \cos \upharpoonright n)$ denote the ordered field of the real numbers $\mathbb{R}$ with addition $+$, multiplication $\cdot$, ordering $<$, as well as the restrictions of the exponential, sine, and cosine functions to $[0, n]$. Let $\mathcal{T}_n[\mathbb{R}, \exp \upharpoonright n, \sin \upharpoonright n, \cos \upharpoonright n]$ denote the first order theory of such structures.

Theorem 4.2 (Macintyre and Wilkie [21, 22]). *Assume **SC**. For any $n \in \mathbb{N}$, the theory $\mathbb{R}_{\mathsf{MW}} := \mathcal{T}_n[\mathbb{R}, \exp \upharpoonright n, \sin \upharpoonright n, \cos \upharpoonright n]$ is decidable.*

Macintyre and Wilkie's result provides a powerful tool to prove conditional decidability results (assuming Schanuel's conjecture) for many problems in linear systems theory. Essentially, we encode the problem into the first order theory of reals extended with restricted exponential, sine, and cosine functions, and invoke the decision procedure.

For example, the LQR with polytopic initial set problem can be encoded as follows. We compute the expression $J(t_f, x_0)$ as in Section 3.1, but treating x_0 as a variable, and ask

$$\exists x_0 \in X_0.\ J(t_f, x_0) \leq r.$$

This is a formula in $\mathbb{R}_{\mathsf{MW}}$ for any polytope $X_0 = \{x_0 \in \mathbb{R}^n \mid Dx_0 \leq E\}$ with algebraic matrices D and E, and can be decided assuming Schanuel's conjecture. A similar encoding shows conditional decidability of the extended LQG problem with polytopic bounds on the initial covariance matrices.

In fact, encoding into the theory of reals with (bounded) exponentials and trigonometric functions enables conditional decidability results for a number of other open questions in control theory. In the following, we mention two such problems.

Controllability of Linear Time-Varying Systems Consider a linear time-varying (LTV) system with state evolution

$$\frac{d}{dt}x(t) = A(t)x(t) + B(t)u(t), \tag{9}$$

where $A(t)$ and $B(t)$ are time-dependent matrices with appropriate dimensions. The system (9) is *controllable* on $[t_0, t_1]$ iff for any states y and z, there is a control input $u(\cdot)$ such that the closed-loop trajectory moves from $x(t_0) = y$ to $x(t_1) = z$. We consider the special case when each element of $A(t)$ and $B(t)$ is an exponential polynomial in t.

Problem 5. *The LTV controllability problem asks, given $A(t)$, $B(t)$, and an interval $[t_0, t_1]$ with rational endpoints, is the system* (9) *controllable on $[t_0, t_1]$?*

One way of characterizing controllability and finding an appropriate control input is through the *controllability Gramian* [26], which involves the integration of the state-transition matrix of the system. A simpler condition for controllability is a rank condition stated next.

Theorem 4.3 (Sontag [26]). *Let $B_0(t) = B(t)$ and define for each $i \geq 0$,*

$$B_{i+1}(t) = A(t)B_i(t) - \frac{d}{dt}B_i(t). \tag{10}$$

The system with analytic matrices $A(t)$ and $B(t)$ is controllable on $[t_0, t_1]$ if and only if $[B_0(\bar{t}), B_1(\bar{t}), \ldots, B_{n-1}(\bar{t})]$ has rank n for some $\bar{t} \in [t_0, t_1]$.

This rank condition is sufficient for controllability according to Proposition 3.5.16 and Corollary 3.5.18 in Sontag [26]. The condition is also necessary according to [26, Exercise 3.5.23]. We show that the characterization of Theorem 4.3 is conditionally decidable when $A(\cdot)$ and $B(\cdot)$ are exponential polynomials. in

Theorem 4.4. *Assume Schanuel's conjecture. Suppose $G : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times m}$ with $m \geq n$ where elements of $G(\cdot)$ are exponential polynomials. Checking if there exists $t \in [t_0, t_1]$ such that $G(t)$ has rank n is decidable.*

Proof: This problem amounts to selecting n columns $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ of $G(\cdot)$ and computing the determinant $\det G_{\mathbf{i}}(t)$ which is again an exponential polynomial. Here, $G_{\mathbf{i}}$ is the matrix obtained by keeping the columns $i_1, \ldots, i_n$ of G. The claim is equivalent to

$$\exists \mathbf{i}.\ \exists t \in [t_0, t_1].\ \det G_{\mathbf{i}}(t) \neq 0.$$

This is a statement in $\mathbb{R}_{\mathrm{MW}}$ and is decidable subject to **SC**. □

The following corollary is immediate, since $A(\cdot)$ and each $B_i(\cdot)$ are exponential polynomials. Note that the quantification over $\mathbf{i}$ is actually a disjunction over the finitely many choices of columns.

Corollary 4.5. *Assume Schanuel's conjecture. The LTV controllability decision problem is decidable when each element of $A(t)$ and $B(t)$ are exponential polynomials in t.*

Exponential Stability An LTI system of the form

$$\frac{d}{dt}x(t) = Ax(t), \quad t \geq 0 \tag{11}$$

is *exponentially stable* if and only if there are constants $M \geq 1$ and $\beta < 0$ such that

$$\|e^{At}\| \leq Me^{\beta t}, \quad t \geq 0. \tag{12}$$

The exponent $\beta < 0$ determines the long-term behavior and the factor $M \geq 1$ bounds the transient behavior. Let $\gamma(A)$ denote the maximum of the real parts of the eigenvalues of A. It is well-known that $\gamma(A) < 0$ implies exponential stability: for any $\beta > \gamma(A)$ there exists a constant $M \geq 1$ such that (12) holds.

Problem 6. *Given A and $\beta > \gamma(A)$, determine the minimal value of $M \geq 1$ satisfying* (12).

This problem is mentioned in [7] as an open problem (see also [16]). The related decision problem is as follows.

Problem 7. *Given A, $\beta > \gamma(A)$, and $M_0 > 1$ decide if there is $M \in [1, M_0]$ such that* (12) *holds.*

The decision version checks if the minimal M belongs to a bounded interval and its decidability is equivalent to the existence of an algorithm that can approximate M with any arbitrary precision using bisection on M_0.

Theorem 4.6. *Assume Schanuel's conjecture. Then, Problem 7 is decidable.*

Proof: We reduce the problem to a sentence in $\mathbb{R}_{\mathrm{MW}}$. The statement of the problem can be written as

$$\exists M \in [1, M_0]\ s.t.\ \forall t \in \mathbb{R}_{\geq 0}.\ \|e^{At}\| \leq Me^{\beta t}.$$

Its negation is

$$\forall M \in [1, M_0].\ \exists t \in \mathbb{R}_{\geq 0}.\ \|e^{At}\| \geq Me^{\beta t}.$$

Due to the monotonicity of the exponential function, the universal quantifier over M can be eliminated:

$$\exists t \in \mathbb{R}_{\geq 0}.\ \|e^{At}\| \geq M_0 e^{\beta t}. \tag{13}$$

Note that $\mathbb{R}_{\mathrm{MW}}$ uses bounded versions of exponentials and trigonometric functions. We first show that t in the statement (13) belongs to a bounded interval. Define $\bar\beta := (\beta+\gamma(A))/2$ and $\bar A := A - \bar\beta \mathbf{I}_n$. Matrix $\bar A$ has all its eigenvalues in the left half-plane, which means the Lyapunov equation $\bar A^T P + P\bar A \leq 0$ has a positive definite solution for P that is computable in the theory of reals. Using [16, Lemma 2.1] we get $\|e^{At}\| \leq \sqrt{\kappa(P)}e^{\bar\beta t}$ for all $t \geq 0$, where $\kappa(P) := \|P\|\|P^{-1}\|$. Then, t in the statement (13) should satisfy the inequality

$$\sqrt{\kappa(P)}e^{\bar\beta t} \geq M_0 e^{\beta t} \quad \Rightarrow \quad t \leq \frac{\ln\kappa(P) - 2\ln M_0}{\beta - \gamma(A)}.$$

Denote this upper bound by τ. Then (13) is equivalent to

$$\begin{aligned} &\exists t \in [0,\tau]\ \exists x_0 \in \mathbb{R}^n \text{ with } \|x_0\| = 1,\\ &\qquad\qquad x_0^T e^{(A-\beta\mathbf{I})^T t} e^{(A-\beta\mathbf{I})t} x_0 \geq M_0^2\\ \Leftrightarrow &\exists t \in [0,\tau]\ \exists x_0 \in [0,1]^n\ \exists y \in \mathbb{R}\ s.t.\\ &\left[\|x_0\|^2 - 1\right]^2 + \left[x_0^T e^{(A-\beta\mathbf{I})^T t} e^{(A-\beta\mathbf{I})t} x_0 - M_0^2 - y^2\right]^2 = 0 \end{aligned}$$

It is not difficult so see that $|y| \leq \sqrt{\kappa(P)}$.

The last equality is an exponential polynomial as a function of t, y, and elements of x_0. □

5 DISCUSSION

We have shown that results from transcendental number theory can be used to show decidability of problems in continuous-time, continuous-state, linear systems theory. While the associated control and optimization problems have been studied extensively, the *computability* status of these basic problems had remained open.

One can ask why a decision problem is interesting, if there is an analytic closed-form solution known, such as in the case of the linear quadratic regular. On top of the natural theoretical interest in the computability status of a fundamental algorithm, we argue that a computability result is interesting for these problems because it helps answer whether numerical computations converge at some finite precision or whether one cannot decide the problem no matter what numerical precision is chosen.

While we focus on decidability issues, one can estimate the *computational complexity* of our decision procedure using quantitative versions of Lindemann-Weierstrass's theorem. A result of Sert [25] shows that, in order to show that an exponential polynomial is distinct from zero, it is sufficient to compute an m-bit approximation where m is bounded by a double exponential function of the input. Unfortunately, we do not know corresponding lower bounds on the computational complexity.

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REFERENCES

[1] Shaull Almagor, Edon Kelmendi, Joël Ouaknine, and James Worrell. Invariants for continuous linear dynamical systems. In *ICALP 2020*, volume 168 of *LIPIcs*, pages 107:1–107:15. Leibniz-Zentrum für Informatik, 2020.

[2] Shaull Almagor, Joël Ouaknine, and James Worrell. The semialgebraic orbit problem. In *36th International Symposium on Theoretical Aspects of Computer Science, STACS 2019, March 13-16, 2019, Berlin, Germany*, volume 126 of *LIPIcs*, pages 6:1–6:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

[3] Adnan Aziz, Kumud Sanwal, Vigyan Singhal, and Robert Brayton. Model-checking continuous-time Markov chains. *ACM Transactions on Computational Logic*, 1(1):162–170, July 2000.

[4] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in Real Algebraic Geometry*. Springer, 2006.

[5] Paul C. Bell, Jean-Charles Delvenne, Raphaël M. Jungers, and Vincent D. Blondel. The continuous Skolem-Pisot problem. *Theoretical Computer Science*, 411(40-42):3625–3634, September 2010.

[6] Dimitri P. Bertsekas. *Dynamic Programming and Optimal Control*. Athena Scientific, 2nd edition, 2000.

[7] Vincent D. Blondel and Alexandre Megretski. *Unsolved Problems in Mathematical Systems and Control Theory*. Princeton University Press, 2004.

[8] Vincent D. Blondel, Eduardo D. Sontag, Mathukumalli Vidyasagar, and Jan C. Willems. *Open Problems in Mathematical Systems and Control Theory*. Communications and control engineering. Springer, 1999.

[9] Vincent D. Blondel and John N. Tsitsiklis. A survey of computational complexity results in systems and control. *Automatica*, 36(9):1249–1274, 2000.

[10] I. Borosh and L.B. Treybig. Bounds on positive integral solutions to linear Diophantine equauons. *Proc. Amer. Math. Soc.*, 55:299–304, 1976.

[11] Richard P. Brent. Fast multiple-precision evaluation of elementary functions. *J. ACM*, 23(2), 1976.

[12] Ventsislav Chonev, Joël Ouaknine, and James Worrell. On the Skolem problem for continuous linear dynamical systems. In *ICALP 2016*, volume 55 of *LIPIcs*, pages 100:1–100:13. Leibniz-Zentrum für Informatik, 2016.

[13] Henri Cohen. *A Course in Computational Algebraic Number Theory*. Springer, 1993.

[14] Mark H.A. Davis. *Linear Estimation and Stochastic Control*. Chapman and Hall, 1977.

[15] Nathanaël Fijalkow, Joël Ouaknine, Amaury Pouly, João Sousa Pinto, and James Worrell. On the decidability of reachability in linear time-invariant systems. In *Proceedings of the 22nd ACM International Conference on Hybrid Systems: Computation and Control, HSCC 2019, Montreal, QC, Canada, April 16-18, 2019*, pages 77–86. ACM, 2019.

[16] Diederich Hinrichsen, Elmar Plischke, and Fabian Wirth. State feedback stabilization with guaranteed transient bounds. *Proceedings of MTNS-2002, Notre Dame, IN, USA*, 2002.

[17] Rudolph E. Kalman. Contributions to the theory of optimal control. *Boletin de la Sociedad Matematica Mexicana*, 5:102–119, 1960.

[18] Rudolph E. Kalman. A new approach to linear filtering and prediction problems. *Journal of Basic Engineering*, 82(1):35–45, 1960.

[19] Narendra Karmarkar. A new polynomial time algorithm for linear programming. *Combinatorica*, 4(4):373–395, 1984.

[20] Leonid G. Khachiyan. A polynomial algorithm in linear programming. *Doklady Akademii Nauk SSSR (English translation: Soviet. Math Dokl. 20: 191âĂŞ194)*, 244:1093–1096, 1979.

[21] Angus Macintyre. Turing meets Schanuel. *Annals of Pure and Applied Logic*, 167(10):901–938, October 2016.

[22] Angus Macintyre and Alex J. Wilkie. On the decidability of the real exponential field. In Piergiorgio Odifreddi, editor, *Kreiseliana. About and Around Georg Kreisel*, pages 441–467. A K Peters, 1996.

[23] Rupak Majumdar, Mahmoud Salamati, and Sadegh Soudjani. On decidability of time-bounded reachability in CTMDPs. In *ICALP 2020*, volume 168 of *LIPIcs*, pages 133:1–133:19. Leibniz-Zentrum für Informatik, 2020.

[24] M. Ram Murty and Purusottam Rath. *Transcendental Numbers*. Springer, 2014.

[25] Alain Sert. Une version effective du théorème de Lindemann-Weierstrass par les déterminants d'interpolation. *Journal of Number Theory*, 76:94–119, 1999.

[26] Eduardo D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer-Verlag, Berlin, Heidelberg, 1998.

[27] Stephen A. Vavasis. Quadratic programming is in NP. *Information Processing Letters*, 36(2):73–77, 1990.