Separability Problems in Creative Telescoping*

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ABSTRACT

For given multivariate functions specified by algebraic, differential or difference equations, the separability problem is to decide whether they satisfy linear differential or difference equations in one variable. In this paper, we will explain how separability problems arise naturally in creative telescoping and present some criteria for testing the separability for several classes of special functions, including rational functions, hyperexponential functions, hypergeometric terms, and algebraic functions.

CCS CONCEPTS

• **Computing methodologies** → **Algebraic algorithms**.

KEYWORDS

Creative telescoping, Separable functions, Separation of variables, Zeilberger's algorithm

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1 INTRODUCTION

The method of separation of variables has been used widely in solving differential equations [23]. In order to solve the one-dimensional heat equation

$$\frac{\partial y}{\partial t} - c\frac{\partial^2 y}{\partial x^2} = 0, \text{ where } c \in \mathbb{C},$$

together with the boundary conditions $y(t, 0) = y(t, L) = 0$, where $L \neq 0$, one can try to find a nonzero solution of the form

$$y = u(t)v(x),$$

and then substitute this form into the equation to get

$$\frac{\frac{\partial u(t)}{\partial t}}{u} = c\frac{\frac{\partial^2 v(x)}{\partial x^2}}{v}.$$

Since both sides only depend on one variable, there exits some constant $\lambda \in \mathbb{C}$ such that

$$\frac{\partial u}{\partial t} - \lambda u = 0 \quad \text{and} \quad c\frac{\partial^2 v}{\partial x^2} - \lambda v = 0.$$

Note that the above two equations are also satisfied by $y = u(t)v(x)$, which are linear differential equations in only one variable. After solving these special equations by taking the boundary conditions into account, a special solution of the heat equation can be given as

$$y(t, x) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 ct}{L^2}\right), \tag{1.1}$$

where $d_n \in \mathbb{C}$ are coefficients determined by the initial conditions. Motivated by this example, one can ask the following natural question.

Problem 1.1 (Separability Problem). *Given a multivariate function specified by certain equations (e.g. algebraic, differential or difference equations), decide whether this function satisfies linear differential or difference equations in one of the arising variables.*

To make the problem more tractable, we will consider some special classes of functions, such as rational functions, algebraic functions, hyperexponential functions and hypergeometric terms. Our main contributions are separability criteria for these classes of functions, especially the algebraic case is new. We will also show the close connection between the separability problem and Zeilberger's method of creative telescoping [29, 30].

The remainder of this paper is organized as follows. We specify the separability problem and the existence problem of telescopers precisely in Section 2 together with the definition of orders and (local) dispersions of rational functions. After this, we explain how the separability problems arise naturally in creative telescoping for rational functions in Section 3, hyperexponential functions and hypergeometric terms in Section 4, and for algebraic functions in Section 5. Separability criteria will be given for these classes of special functions. We then conclude our paper with some comments on the separability problem on D-finite functions and P-recursive sequences.

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2 PRELIMINARIES

Let $\mathbb{F}$ be a field of characteristic zero and let $\mathbb{E} = \mathbb{F}(t, \mathbf{x})$ be the field of rational functions in t and $\mathbf{x} = (x_1, \ldots, x_m)$ over $\mathbb{F}$. Let δ_t, δ_{x_i} be the usual partial derivations $\partial/\partial_t, \partial/\partial_x$ with $x \in \{x_1, \ldots, x_m\}$, respectively. The shift operators σ_t and σ_{x_i} on $\mathbb{E}$ are defined as the $\mathbb{F}$-automorphisms such that for any $f \in \mathbb{E}$, $\sigma_t(f(t, \mathbf{x})) = f(t+1, \mathbf{x})$ and

$$\sigma_{x_i}(f(t, \mathbf{x})) = f(t, x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_m).$$

The ring of linear functional operators in t and $\mathbf{x}$ over $\mathbb{E}$ is denoted by $\mathbb{E}\langle\partial_t, \partial_\mathbf{x}\rangle$, where $\partial_\mathbf{x} = (\partial_{x_1}, \ldots, \partial_{x_m})$ and ∂_v with $v \in \{t, \mathbf{x}\}$ is either the derivation D_v such that $D_v f = f D_v + \delta_v(f)$ or the shift operator S_v such that $S_v f = \sigma_v(f) S_v$ for any $f \in \mathbb{E}$, and ∂_t and ∂_{x_i} commute. For $v \in \{t, \mathbf{x}\}$, we let Δ_v denote the difference operator $S_v - \mathbf{1}$, where $\mathbf{1}$ stands for the identity map on $\mathbb{E}$. Abusing notation, we let δ_v and σ_v denote arbitrary extensions of δ_v and σ_v to derivation and $\overline{\mathbb{F}}$-automorphism of $\overline{\mathbb{E}}$, the algebraic closure of $\mathbb{E}$. The functions we consider will be in certain differential or difference extension of $\mathbb{E}$, which is also an $\mathbb{E}\langle\partial_t, \partial_\mathbf{x}\rangle$-module via the action defined by simply interpreting D_v, S_v by δ_v, σ_v, respectively, for $v \in \{t, \mathbf{x}\}$. The ring $\mathbb{F}(t)\langle\partial_t\rangle$ is a subring of $\mathbb{E}\langle\partial_t, \partial_\mathbf{x}\rangle$ that is also a left Euclidean domain. Efficient algorithms for basic operations in $\mathbb{F}(t)\langle\partial_t\rangle$, such as computing the least common left multiple (LCLM) of operators, have been developed in [6, 10].

Definition 2.1 (Separable functions). *Let $\mathfrak{M}$ be an $\mathbb{E}\langle\partial_t, \partial_\mathbf{x}\rangle$-module and $f \in \mathfrak{M}$. We say that $f(t, \mathbf{x})$ is ∂_t-separable if there exists a nonzero $L \in \mathbb{F}(t)\langle\partial_t\rangle$ such that $L(f) = 0$.*

As an example, the special solution (1.1) of the one-dimensional heat equation is both D_t-separable and D_x-separable. We should mention that the notion of separable functions in Definition 2.1 and that of separable polynomials and extensions in Galois theory [20, Chapter V] are not connected. The problem of separating variables in bivariate polynomial ideals in [11] is somehow also irrelevant to the separability problem studied in this paper. Note that ∂_t-separable functions are just the D-finite functions in the differential case and the P-recursive sequences in the shift case, which are both introduced in [28]. By the closure properties of D-finite functions and P-recursive sequences, we have the same closure properties for ∂_t-separable functions.

Proposition 2.2. *Let $\mathfrak{M}$ be an $\mathbb{E}\langle\partial_t, \partial_\mathbf{x}\rangle$-module. If $f, g \in \mathfrak{M}$ are ∂_t-separable, so are $f + g$, $f \cdot g$, and $a \cdot f$ for all $a \in \mathbb{F}(t)$.*

We will focus on the separability problem on functions in an $\mathbb{E}\langle\partial_t, \partial_\mathbf{x}\rangle$-module.

Definition 2.3 (Creative telescoping). *Let $\mathfrak{M}$ be an $\mathbb{E}\langle\partial_t, \partial_\mathbf{x}\rangle$-module and $f \in \mathfrak{M}$. A nonzero operator $L \in \mathbb{F}(t)\langle\partial_t\rangle$ is called a* telescoper *of type $(\partial_t, \partial_\mathbf{x})$ for f if there exist $Q_1, \ldots, Q_m \in \mathbb{E}\langle\partial_t, \partial_\mathbf{x}\rangle$ such that*

$$L(t, \partial_t)(f) = \partial_{x_1}(Q_1(f)) + \cdots + \partial_{x_m}(Q_m(f)), \qquad (2.1)$$

where $\partial_t \in \{D_t, S_t\}$ and $\partial_{x_i} \in \{D_{x_i}, \Delta_{x_i}\}$.

The central problem in the Wilf-Zeilberger theory of automatic proving of special-function identities is related to the existence and the computation of telescopers for special functions. In the next sections, we will show that this central problem of creative telescoping is closely connected to the separability problem on the corresponding class of special functions.

Let $V = (V_1, \ldots, V_s)$ be any set partition of the variables $\mathbf{v} = \{t, x_1, \ldots, x_m\}$. A rational function $f \in \mathbb{F}(t, \mathbf{x})$ is called a *split* function with respect to the partition V if $f = f_1 \cdots f_s$ with $f_i \in \mathbb{F}(V_i)$ and a *semi-split* function with respect to V if there are split functions $g_j \in \mathbb{F}(t, \mathbf{x})$ such that $f = g_1 + \cdots + g_n$. By definition, we have $f = p/q$ with $p, q \in \mathbb{F}[t, \mathbf{x}]$ and $\gcd(p, q) = 1$ is semi-split with respect to the partition V if and only if the denominator q is a split polynomial with respect to the partition V. Split rational functions will be used to describe the separability of given functions.

Let $\mathbb{K} = \mathbb{F}(\mathbf{x})$ and $p \in \mathbb{K}[t]$ be an irreducible polynomial in t. For any $f \in \mathbb{K}(t)$, we can write $f = p^m a/b$, where $m \in \mathbb{Z}, a, b \in \mathbb{K}[t]$ with $\gcd(a, b) = 1$ and $p \nmid ab$. The integer m is called the *order* of f at p, denoted by $\nu_p(f)$. Conventionally, we set $\nu_p(0) = +\infty$. We collect some basic properties of valuations as follows and refer to [9, Chapter 4] for their proofs.

Proposition 2.4. *Let $f, g \in \mathbb{K}(t)$ and $p \in \mathbb{K}[t]$ be an irreducible polynomial. Then,*

(i) $\nu_p(fg) = \nu_p(f) + \nu_p(g)$.
(ii) $\nu_p(f + g) \geq \min\{\nu_p(f), \nu_p(g)\}$ and equality holds if $\nu_p(f) \neq \nu_p(g)$.
(iii) If $\nu_p(f) \neq 0$, then $\nu_p(D_t(f)) = \nu_p(f) - 1$. In particular, for any $i \in \mathbb{N}$, $\nu_p(D_t^i(f)) = \nu_p(f) - i$ if $\nu_p(f) < 0$.

The dispersion introduced by Abramov in [1] can be viewed as a shift analogue of the order. For any polynomial $u \in \mathbb{K}[t]$ with $\deg_t(u) \geq 1$, the *dispersion* of u, denoted by $\mathrm{dis}(u)$, is defined as $\max\{k \in \mathbb{N} \mid \gcd(u, \sigma_t^k(u)) \neq 1\}$, which is the maximal integer root-distance $|\alpha - \beta|$ with α, β being roots of u in $\bar{K}$. Define $\mathrm{dis}(u) = 0$ if $u \in K \setminus \{0\}$ and $\mathrm{dis}(0) = +\infty$. For a rational function $f = a/b \in \mathbb{K}(t)$ with $a, b \in \mathbb{K}[t]$ and $\gcd(a, b) = 1$, define $\mathrm{dis}(f) = \mathrm{dis}(b)$. For later use, we introduce a local version of Abramov's dispersion. Let $p \in \mathbb{K}[t]$ be an irreducible polynomial. If $\sigma_t^i(p) \mid u$ for some $i \in \mathbb{Z}$, the *local dispersion* of u at p, denoted by $\mathrm{dis}_p(u)$, is defined as the maximal integer distance $|i - j|$ with $i, j \in \mathbb{Z}$ satisfying $\sigma_t^i(p) \mid u$ and $\sigma_t^j(p) \mid u$; otherwise we define $\mathrm{dis}_p(u) = 0$. Conventionally, we set $\mathrm{dis}_p(0) = +\infty$. For a rational function $f = a/b \in \mathbb{K}(t)$ with $a, b \in \mathbb{K}[t]$ and $\gcd(a, b) = 1$, we also define $\mathrm{dis}_p(f) = \mathrm{dis}_p(b)$. By definition, we have

$$\mathrm{dis}(u) = \max\{\mathrm{dis}_p(u) \mid p \text{ is an irreducible factor of } u\}.$$

The set $\{\sigma_t^i(p) \mid i \in \mathbb{Z}\}$ is called the σ_t-orbit at p, denoted by $[p]_{\sigma_t}$. Note that $\mathrm{dis}_p(u) = \mathrm{dis}_q(u)$ if $q \in [p]_{\sigma_t}$. So we can define the local dispersion of a rational function f at a σ_t-orbit at p, denoted by $\mathrm{dis}_{[p]_{\sigma_t}}(f)$.

Example 2.5. *Let $u = x(x+1)(x-5)(x^2+1)(x^2+4x+5) \in \mathbb{Q}[x]$. Then we have $\mathrm{dis}_x(u) = 6$ and $\mathrm{dis}_{x^2+1}(u) = 2$. Abramov's dispersion of u is then equal to 6.*

We now show how the local dispersions change under the action of linear recurrence operators, which was first proved for Abramov's dispersion in [1, 2] and [25, Section 3.1].

Lemma 2.6. *Let $f = a/b \in \mathbb{K}(t)$ with $a, b \in \mathbb{K}[t]$ and $\gcd(a, b) = 1$ and let $p \in \mathbb{K}[t]$ be an irreducible factor of b. Let $L = \sum_{i=0}^{\rho} \ell_i S_t^i \in \mathbb{K}[t]\langle S_t\rangle$ be such that $\ell_\rho \ell_0 \neq 0$ and $\sigma_t^i(p)$ does not divide $\ell_\rho \ell_0$ for any*

$i \in \mathbb{Z}$. Then $dis_p(L(f)) = dis_p(f) + \rho$. In particular, $dis_p(\Delta_t(f)) = dis_p(f) + 1$.

Proof. Let $d = \mathrm{dis}_p(b)$. Without loss of generality, we may assume that $p \mid b$ but $\sigma_t^i(p) \nmid b$ for any $i < 0$. Since $\gcd(a, b) = 1$ and σ_t is a $\mathbb{K}$-automorphism of $\mathbb{K}[t]$, we have $\gcd(\sigma_t^i(a), \sigma_t^i(b)) = 1$ for any $i \in \mathbb{Z}$. Applying L to f yields

$$L(f) = \sum_{i=0}^{\rho} \ell_i \sigma_t^i \left(\frac{a}{b}\right) = \frac{\sum_{i=0}^{\rho} \ell_i \sigma_t^i(a) u_i}{u},$$

where $u = b\sigma_t(b) \cdots \sigma_t^{\rho}(b)$ and $u_i = u/\sigma_t^i(b)$. Write $L(f) = A/B$ with $A, B \in \mathbb{K}[t]$ and $\gcd(A, B) = 1$. Then $B \mid u$ and $\mathrm{dis}_p(L(f)) = \mathrm{dis}_p(B)$ by definition. Since $\sigma_t^i(p) \nmid \ell_0$ and $\sigma_t^i(p) \nmid \ell_\rho$ for any $i \in \mathbb{Z}$, we have both p and $\sigma_t^{d+\rho}(p)$ do not divide the sum $\sum_{i=0}^{\rho} \ell_i \sigma_t^i(a) u_i$, but they divide u. So $p \mid B$ and $\sigma_t^{d+\rho}(p) \mid B$, which implies that $\mathrm{dis}_p(B) \geq d + \rho$. Since $B \mid u$, we have $\mathrm{dis}_p(B) \leq \mathrm{dis}_p(u) = d + \rho$. Therefore, $\mathrm{dis}_p(L(f)) = d + \rho$. ∎

3 THE RATIONAL CASE

We first explain how the existence problem of telescopers for rational functions is naturally connected to the separability problem on this class of functions. Let $f(t, x)$ be a bivariate rational function in $\mathbb{F}(t, x)$. By the Ostrogradsky-Hermite reduction [19, 24], we can decompose f into the form

$$f = D_x(g) + \frac{a}{b},$$

where $g \in \mathbb{F}(t, x)$ and $a, b \in \mathbb{F}(t)[x]$ with $\gcd(a, b) = 1$, $\deg_x(a) < \deg_x(b)$ and b being squarefree in x over $\mathbb{F}(t)$. Moreover, $f = D_x(h)$ for some $h \in \mathbb{F}(t, x)$ if and only if $a = 0$. Then f has a telescoper of type (S_t, D_x) if and only if a/b does. Applying a nonzero operator $L = \sum_{i=0}^{\rho} \ell_i S_t^i \in \mathbb{F}(t)\langle S_t \rangle$ to a/b yields

$$L\left(\frac{a}{b}\right) = \sum_{i=0}^{\rho} \ell_i(t) \sigma_t^i \left(\frac{a}{b}\right) = \sum_{i=0}^{\rho} \frac{\ell_i(t) a(t+i, x)}{b(t+i, x)} = \frac{p}{q},$$

where $p, q \in \mathbb{F}[t, x]$ with $\gcd(p, q) = 1$. Since the shift operator S_t is an $\mathbb{F}(x)$-automorphism and preserves the degrees in t and x, we have $b(t + i, x)$ is squarefree in x over $\mathbb{F}(t)$ for any $i \in \mathbb{N}$ and $\deg_x(a(t + i, x)) < \deg_x(b(t + i, x))$. So $\deg_x(p) < \deg_x(q)$ and q is also squarefree in x over $\mathbb{F}(t)$. This implies the operator L is a telescoper of type (S_t, D_x) for a/b, i.e., $L(a/b) = D_x(g)$ for some $g \in \mathbb{F}(t, x)$ if and only if $p = 0$, i.e., $L(a/b) = 0$. Therefore, we conclude that f has a telescoper of type (S_t, D_x) if and only if a/b is S_t-separable.

We can also consider telescopers of type (D_t, S_x). By Abramov's reduction [3, 4], we can decompose $f \in \mathbb{F}(t, x)$ into the form

$$f = \Delta_x(g) + \frac{a}{b},$$

where $g \in \mathbb{F}(t, x)$ and $a, b \in \mathbb{F}(t)[x]$ with $\gcd(a, b) = 1$, $\deg_x(a) < \deg_x(b)$ and b being shift-free in x over $\mathbb{F}(t)$, i.e., $\gcd(b, \sigma_x^i(b)) = 1$ for all nonzero $i \in \mathbb{Z}$. Applying a nonzero operator $L = \sum_{i=0}^{\rho} \ell_i(t) D_t^i$ in $\mathbb{F}(t)\langle D_t \rangle$ to a/b yields

$$L\left(\frac{a}{b}\right) = \sum_{i=0}^{\rho} \ell_i \delta_t^i \left(\frac{a}{b}\right) = \sum_{i=0}^{\rho} \frac{\ell_i(t) a_i}{b^{i+1}} = \frac{p}{q},$$

where $a_i, p, q \in \mathbb{F}[t, x]$ with $\gcd(p, q) = 1$ and $\deg_x(a_i) < (i + 1)\deg_x(b)$. Since b is shift-free in x, so is b^i for any $i \in \mathbb{N}$. Note that any factor of a shift-free polynomial is still shift-free. So q is shift-free and $\deg_x(p) < \deg_x(q)$. This implies that the operator L is a telescoper of type (D_t, S_x) for a/b, i.e., $L(a/b) = \Delta_x(g)$ for some $g \in \mathbb{F}(t, x)$ if and only if $p = 0$, i.e., $L(a/b) = 0$. Then we also have that f has a telescoper of type (D_t, S_x) if and only a/b is D_t-separable.

The next theorem characterizes all possible separable rational functions in terms of semi-split rational functions, which was implicitly used in [16, Theorem 4.6].

Theorem 3.1. *A rational function $f \in \mathbb{F}(t, \mathbf{x})$ is ∂_t-separable if and only if f is semi-split in t and $\mathbf{x}$.*

Proof. Assume that f is semi-split in t and $\mathbf{x}$. Then $f = a_1 b_1 + \cdots + a_n b_n$, where $a_i \in \mathbb{F}(t)$ and $b_i \in \mathbb{F}(\mathbf{x})$ for all i with $1 \leq i \leq n$. Since each $a_i b_i$ is annihilated by the operator $L_i := \partial_t - \partial_t(a_i)/a_i \in \mathbb{F}(t)\langle \partial_t \rangle$, the rational function f is annihilated by $\mathrm{LCLM}(L_1, \ldots, L_n)$. So f is ∂_t-separable.

For the necessity we assume that $f = a/b$ with $a, b \in \mathbb{F}[t, \mathbf{x}]$ and $\gcd(a, b) = 1$ is ∂_t-separable, i.e., there exists a nonzero operator $L = \sum_{i=0}^{\rho} \ell_i \partial_t^i \in \mathbb{F}(t)\langle \partial_t \rangle$ with $\ell_\rho \neq 0$ such that $L(f) = 0$. It suffices to show that the denominator b is split with respect to t and $\mathbf{x}$. Suppose for the sake of contradiction that b is not split. Then b has at least one irreducible factor p such that p is not split. Now we proceed by a case distinction according to the type of ∂_t. In the case when $\partial_t = D_t$, we have $\nu_p(\ell_i D_t^i(f)) = \nu_p(f) - i$ for each i with $\ell_i \neq 0$, since $\nu_p(f) < 0$ and $\nu_p(\ell_i) = 0$, which implies further that $\nu_p(L(f)) = \nu_p(f) - \rho$ by Proposition 2.4. But $\nu_p(L(f)) = \nu_p(0) = +\infty$, which leads to an contradiction. In the case when $\partial_t = S_t$, we may always assume that $\ell_i \in \mathbb{F}[t]$ and $\ell_0 \neq 0$ since σ_t is an $\mathbb{F}(\mathbf{x})$-automorphism of $\mathbb{F}(t, \mathbf{x})$. Since ℓ_0 and ℓ_ρ are free of x, we have $\sigma_t^i(p) \nmid \ell_0 \ell_\rho$ for any $i \in \mathbb{Z}$. By Lemma 2.6, we get $\mathrm{dis}_p(L(f)) = \mathrm{dis}_p(f) + \rho < \infty$, which contradicts with $\mathrm{dis}_p(L(f)) = \mathrm{dis}_p(0) = +\infty$. ∎

Remark 3.2. *With the above theorem, we can detect easily the ∂_t-separability of rational functions by the computation of contents and derivatives of multivariate polynomials in t.*

4 THE HYPEREXPONENTIAL AND HYPERGEOMETRIC CASES

The separability problem on hyperexponential functions and hypergeometric terms was first studied in [21], which was later connected to the existence of parallel telescopers for hyperexponential functions [14]. We motivate this problem by revisiting Zeilberger's algorithm which computes telescopers for hypergeometric terms (see [26, Chapter 6]).

Let $H(t, x)$ be a nonzero hypergeometric term over $\mathbb{F}(t, x)$, i.e., both $\sigma_t(H)/H$ and $\sigma_x(H)/H$ are in $\mathbb{F}(t, x)$. If telescopers of type (S_t, S_x) exist for H, Zeilberger's algorithm starts from an ansatz: for fixed $\rho \in \mathbb{N}$, set $L = \sum_{i=0}^{\rho} \ell_i S_t^i \in \mathbb{F}(t)\langle S_t \rangle$ with the ℓ_i's being undetermined coefficients. Applying L to H yields

$$T := L(H) = \sum_{i=0}^{\rho} \ell_i \sigma_t^i(H) = \sum_{i=0}^{\rho} \ell_i a_i H = \frac{\sum_{i=0}^{\rho} \ell_i P_i}{Q} H,$$

where $a_i = \sigma_t^i(H)/H = P_i/Q \in \mathbb{F}(t,x)$ with $P_i, Q \in \mathbb{F}[t,x]$. The second step of Zeilberger's algorithm is computing the Gosper form of $L(H)$ that gives

$$\frac{\sigma_x(L(H))}{L(H)} = \frac{\sigma_x\left(\sum_{i=0}^{\rho} \ell_i P_i\right)}{\sum_{i=0}^{\rho} \ell_i P_i} \frac{\sigma_x(p)}{p} \frac{q}{r},$$

where $(p,q,r) \in \mathbb{F}(t)[x]^3$ is a Gosper form of the rational function $Q\sigma_x(H)/(\sigma_x(Q)H)$ satisfying that $\gcd(q, \sigma_x^i(r)) = 1$ for all $i \in \mathbb{N}$. The last step is finding $\ell_0, \ldots, \ell_\rho \in \mathbb{F}(t)$, not all zero, such that the equation

$$\left(\sum_{i=0}^{\rho} \ell_i P_i\right) p = q\sigma_x(z) - \sigma_x^{-1}(r)z.$$

has a polynomial solution in $\mathbb{F}(t)[x]$. If so, then $L = \sum_{i=0}^{\rho} \ell_i S_t^i$ is a telescoper for H. It may happen that the final choice of the ℓ_i's satisfies that $\sum_{i=0}^{\rho} \ell_i P_i = 0$. This means division by zero may happen in the second step. To avoid this, we should first detect whether $L(H) = 0$ for some $L \in \mathbb{F}(t)\langle S_t\rangle$, i.e., the separability problem on hypergeometric terms. In this special situation, Zeilberger's algorithm still works and returns $z = 0$.

The following theorem characterizes all possible separable hyperexponential functions and hypergeometric terms, whose proof was given in [21, Lemma 4] or in [14, Proposition 10].

Theorem 4.1. *Let $\mathfrak{M}$ be an $\mathbb{E}\langle \partial_t, \partial_{\mathbf{x}}\rangle$-module and let $H \in \mathfrak{M}$ be such that*

$$\partial_t(H) = aH \text{ and } \partial_{x_i}(H) = b_i H \text{ with } a, b_i \in \mathbb{F}(t, \mathbf{x}).$$

Then the following holds,

(*i*) *Hyperexponential case: H is D_t-separable if and only if there exist $p \in \mathbb{F}(\mathbf{x})[t]$ and $r \in \mathbb{F}(t)$ such that*

$$a = \frac{\delta_t(p)}{p} + r.$$

(*ii*) *Hypergeometric case: H is S_t-separable if and only if there exist $p \in \mathbb{F}(\mathbf{x})[t]$ and $r \in \mathbb{F}(t)$ such that*

$$a = \frac{\sigma_t(p)}{p} \cdot r.$$

Remark 4.2. *The above form for $\partial_t(H)/H$ can be detected by algorithms for computing the Gosper form and its differential analogue in [7, 18].*

5 THE ALGEBRAIC CASE

In this section, we solve the separability problem on algebraic functions. We first explain the connection between this problem and the following existence problem of telescopers for rational functions in three variables.

Problem 5.1. *Given $f \in \mathbb{F}(t,x,y)$, decide whether there exists a nonzero operator $L \in \mathbb{F}(t)\langle D_t\rangle$ such that $L(f) = \Delta_x(g) + D_y(h)$ for some $g, h \in \mathbb{F}(t,x,y)$.*

By applying first the Ostrogradsky-Hermite reduction in y and then Abramov's reduction in x to $f \in \mathbb{F}(t,x,y)$, we get

$$f = \Delta_x(u) + D_y(v) + r \text{ with } r = \sum_{i=1}^{I} \frac{\alpha_i}{y - \beta_i}$$

where $u, v, r \in \mathbb{F}(t,x,y)$, $\alpha_i, \beta_i \in \overline{\mathbb{F}(t,x)}$ and the β_i's are in distinct σ_x-orbits. Then f has a telescoper of type (D_t, S_x, D_y) if and only if r does. By Theorem 4.21 in [13] or Theorem 4.43 in [12], we have that r has a telescoper of type (D_t, S_x, D_y) if and only if for each i with $1 \le i \le I$, either α_i is D_t-separable in $\overline{\mathbb{F}(t,x)}$ or $\beta_i \in \overline{\mathbb{F}(t)}$ and $\alpha_i \in \mathbb{F}(t,x)(\beta_i)$ has a telescoper of type (D_t, S_x). The existence problem of telescopers of type (D_t, S_x) in $\mathbb{F}(t,x)(\beta)$ with $\beta \in \overline{\mathbb{F}(t)}$ has been solved in [16]. To completely solve Problem 5.1, it remains to solve the following separability problem.

Problem 5.2. *Given an algebraic function $f(t, \mathbf{x})$ over $\mathbb{F}(t, \mathbf{x})$, decide whether $f(t,\mathbf{x})$ is D_t-separable.*

We assume that $\mathbb{F}$ is an algebraically closed and computable subfield of $\mathbb{C}$ in the remaining part of this section.

5.1 A descent theorem

We first recall some basic notions and results from the theory of algebraic functions of one variable [17]. Let k be a field of characteristic zero and $k(x,y)$ be an algebraic function field of one variable over k, i.e., the transcendence degree of $k(x,y)$ over k is one. This means there exists a nonzero polynomial $f \in k[X,Y]$ such that $f(x,y) = 0$. The field of constants of $k(x,y)$ is defined as the set of elements of $k(x,y)$ which are algebraic over k. A subring R of $k(x,y)$ is called a *valuation ring* if $k \subset R \subsetneqq k(x,y)$ and for any $a \in k(x,y)$ nonzero, either $a \in R$ or $a^{-1} \in R$. Any valuation ring R of $k(x,y)$ is a local ring, whose unique maximal ideal $\mathfrak{p}$ is called a *place* of $k(x,y)$ and the quotient field $R/\mathfrak{p}$ is called the *residue field* of the place $\mathfrak{p}$, denoted by $\Sigma_{\mathfrak{p}}$.

Lemma 5.3. *Let $k(x,y)$ and $f \in k[X,Y]$ be as above. Assume that $(\bar{x}, \bar{y}) \in k^2$ satisfies that $f(\bar{x},\bar{y}) = 0$ and $\frac{\partial f}{\partial Y}(\bar{x},\bar{y}) \neq 0$. Then there is a unique place $\mathfrak{p}$ of $k(x,y)$ containing $x - \bar{x}$ and $y - \bar{y}$. Furthermore, the residue field $\Sigma_{\mathfrak{p}}$ of $\mathfrak{p}$ is isomorphic to k and k is the field of constants of $k(x,y)$.*

Proof. By Corollary 2 of [17, page 8], there is a place of $k(x,y)$ containing $x - \bar{x}$ and $y - \bar{y}$, say $\mathfrak{p}$. Let $\mathfrak{a}$ be the discrete valuation ring (DVR) with respect to $\mathfrak{p}$. It is easy to see that the ring $k[x,y]$ is contained in $\mathfrak{a}$. Let $\mathfrak{m}$ be the ideal in $k[x,y]$ generated by $x - \bar{x}$ and $y - \bar{y}$. Then $\mathfrak{m}$ is a maximal ideal. Denote by R the localization of $k[x,y]$ at $\mathfrak{m}$ and we still use $\mathfrak{m}$ to denote the unique maximal ideal of R. Rewriting $f(X,Y)$ as a polynomial in $X - \bar{x}$, $Y - \bar{y}$ yields that

$$\left(\frac{\partial f}{\partial Y}(\bar{x},\bar{y}) + (Y - \bar{y})A\right)(Y - \bar{y}) + (X - \bar{x})B$$

for some $A, B \in k[X - \bar{x}, Y - \bar{y}]$. Since $\frac{\partial f}{\partial Y}(\bar{x},\bar{y}) \neq 0$, one has that $\frac{\partial f}{\partial Y}(\bar{x},\bar{y}) + (y - \bar{y})A(x - \bar{x}, y - \bar{y})$ is invertible in R and so $y - \bar{y} \in (x - \bar{x})R$. It implies that R is a regular local ring, i.e., a DVR. Therefore $R = \mathfrak{a}$, since $R \subset \mathfrak{a}$. This concludes that $\mathfrak{p}$ is unique.

We have that $\Sigma_{\mathfrak{p}} = R/\mathfrak{m} = k[x,y]/\mathfrak{m} \cong k$. Since the field of constants of $k(x,y)$ is a subfield of $\Sigma_{\mathfrak{p}}$ under the natural homomorphism, it coincides with k. ∎

Remark 5.4. *Let $k(x,y)$ and $(\bar{x},\bar{y})$ be as in Lemma 5.3. The above proof implies that $k(x,y)$ can be embedded into the field of formal Laurent series $k((x - \bar{x}))$.*

Theorem 5.5. *Let $\mathbb{F} \subseteq k \subseteq \mathbb{C}$ be fields with $\mathbb{F}$ being algebraically closed. Let $f(t,Y)$ be an irreducible polynomial in $k[t,Y]$. Let $k(t,y)$ be the quotient field of $k[t,Y]/\langle f\rangle$. Assume that*

(1) *the places of $k(t)$ that ramify in $k(t,y)$ are defined over $\mathbb{F}$, i.e., their uniformizing parameters can be chosen to be $1/t$ or $t-c$ with $c \in \mathbb{F}$.*
(2) *there exists a solution (a,α) of the system*

$$\begin{aligned} f(a,\alpha) &= 0, \\ \frac{\partial f}{\partial Y}(a,\alpha) &\neq 0, \end{aligned}$$

where $a \in \mathbb{F}$ and $\alpha \in k$.

Then there exists $\beta \in \overline{\mathbb{F}(t)}$ such that $k(t,y) = k(t,\beta)$.

Proof. Since (a,α) is a simple point of $f(t,Y)=0$ in k^2, by [27], $f(t,Y)$ is absolutely irreducible over k. This implies that f is irreducible over $\mathbb{C}$, i.e., $\mathbb{C}[t,Y]/\langle f\rangle$ is an integral domain. Let $\mathbb{C}(t,y)$ be the quotient field of $\mathbb{C}[t,Y]/\langle f\rangle$. Then $k(t,y)$ can be considered as a subfield of $\mathbb{C}(t,y)$ under the natural homomorphism. From Theorem 3 in [17, page 92], none of places of $\mathbb{C}(t,y)$ is ramified with respect to $k(t,y)$. Therefore the condition 1 holds for $\mathbb{C}(t,y)$. Proposition 2.1 in [22, page 10] states that there is $\beta \in \overline{\mathbb{F}(t)}$ such that $\mathbb{C}(t,y) = \mathbb{C}(t,\beta)$. Now there are $g_0(t), \cdots, g_{n-1}(t) \in \mathbb{C}(t)$ such that

$$\beta = \sum_{i=0}^{n-1} g_i(t) y^i, \tag{5.1}$$

where $n = [\mathbb{C}(t,y):\mathbb{C}(t)]$. For each i, let $g_i = q_i/q$ with $q_i, q \in \mathbb{C}[t]$ and let $s = \max_i\{\deg_t q_i, \deg_t q\}$. Equation (5.1) implies that $q\beta = \sum_{i=0}^{n-1} q_i y^i$ and therefore the set

$$\left\{t^j\beta,\ t^j y^i\right\}_{j=0,\ldots s,\, i=0,\ldots n-1}$$

is linearly dependent over $\mathbb{C}$. This set lies in $k(t,y,\beta)$ and, since it is linearly dependent over D_t-constants in a larger differential field, it is linearly dependent over D_t-constants in $k(t,y,\beta)$. Denote by $\tilde{k}$ the set of D_t-constants of $k(t,y,\beta)$. If $\tilde{k} = k$, then $\beta \in k(t,y)$, which will conclude the proposition. Therefore it suffices to prove that $\tilde{k} = k$. It is easy to verify that $\tilde{k}$ coincides with the field of constants of $k(t,y,\beta)$. In the following, we will show that the field of constants of $k(t,y,\beta)$ is equal to k.

From Remark 5.4, $k(t,y)$ and $\mathbb{C}(t,y)$ can be embedded into $k((t-a))$ and $\mathbb{C}((t-a))$ respectively. We will consider them as the subfields of $k((t-a))$ and $\mathbb{C}((t-a))$ respectively. Since $\beta \in \mathbb{C}(t,y) \cap \overline{\mathbb{F}(t)}$, $\mathbb{F}$ is algebraically closed and $a \in \mathbb{F}$, $\beta \in \mathbb{F}((t-a))$. Therefore, $k(t,y,\beta) \subseteq k((t-a))$. Since k is algebraically closed in $k((t-a))$, the field of constants of $k(t,y,\beta)$ is equal to k. This completes the proof. ∎

5.2 Separability criteria

Throughout this section, assume that $\mathbb{F} \subseteq \mathbb{C}$ with $\mathbb{F}$ being algebraically closed. Let $P = \sum_{i=0}^n A_i Y^i \in \mathbb{F}(t,\mathbf{x})[Y]$ be the minimal polynomial of $y \in \overline{\mathbb{F}(t,\mathbf{x})}$. We can always pick $(a,\alpha) \in \mathbb{F} \times \overline{\mathbb{F}(\mathbf{x})}$ such that

$$A_n(\mathbf{x},a) \neq 0,\ P(\mathbf{x},a,\alpha) = 0 \text{ and } \frac{\partial P}{\partial Y}(\mathbf{x},a,\alpha) \neq 0. \tag{5.2}$$

Let $K = \mathbb{F}(\mathbf{x},\alpha)$ and $\ell = [K(t,y):K(t)]$.

Lemma 5.1. *Asume that $z \in \overline{\mathbb{F}(t,\mathbf{x})}$ satisfies $P(z)=0$. Then y is D_t-separable if and only if z is also D_t-separable.*

Proof. Note that z and y are conjugated over $\mathbb{F}(t,\mathbf{x})$. By Theorem 3.2.4 in [9], any field automorphism of the splitting field of P commutes with the derivation D_t. So for any $L \in \mathbb{F}(t)\langle D_t\rangle$, $L(z)=0$ if and only if $L(y)=0$. ∎

The above lemma implies that to detect if there is a nonzero $L \in \mathbb{F}(t)\langle D_t\rangle$ such that $L(y)=0$, it suffices to detect if there exists such operator for z. In the following, we will characterize all possible D_t-separable algebraic functions. Let us firstly prove that the function field over $\mathbb{F}(\mathbf{x},\alpha,t)$ generated by a D_t-separable algebraic function satisfies the condition 1 of Theorem 5.5.

Proposition 5.6. *Let K be as above. If y is D_t-separable then $K(t,y)$ satisfies the condition 1 of Theorem 5.5.*

Proof. Assume that y is D_t-separable, i.e., there exists a nonzero $L \in \mathbb{F}(t)\langle D_t\rangle$ such that $L(y)=0$. Let $\mathfrak{p}$ be a place of $K(t)$ and $\mathfrak{q}$ a place of $K(t,y)$ that is ramified with respect to $\mathfrak{p}$. Suppose that p and q are uniformizing parameters of $\mathfrak{p}$ and $\mathfrak{q}$ respectively, and e is the corresponding ramification index. Then $p = aq^e$ for some invertible a in the DVR with respect to $\mathfrak{q}$. If $p = c/t$ for some $c \in K$ then there is nothing to prove. In the following, assume that p is an irreducible polynomial in $K[t]$. Let $\wp$ be a place of $\mathbb{C}(t,y)$ lying above $\mathfrak{q}$. Then by Theorem 3 in [17, page 92]), $\wp$ is not ramified with respect to $\mathfrak{q}$ and so q is a uniformizing parameter of $\wp$. Since $p \in \wp$, the uniformizing parameter of $\wp \cap \mathbb{C}(t)$ can be selected as a factor of p, say $t-c$ for some $c \in \mathbb{C}$. It is easy to see that $p/(t-c)$ is an invertible element in the DVR with respect to $\wp$. It implies that $t-c = \bar{a}q^e$ for some invertible element $\bar{a}$ and thus $K(t,y)$ can be embedded into $\mathbb{C}((t-c)^{1/e})$. Therefore $y \in \mathbb{C}((t-c)^{1/e}))$ and c is a singular point of L. Note that the singular points of L lie in the algebraically closed field $\mathbb{F}$. So $c \in \mathbb{F}$ and then $p = b(t-c)$ for some $b \in K$. In other words, $t-c$ is a uniformizing parameter of $\mathfrak{p}$. Hence $K(t,y)$ satisfies the condition 1 of Theorem 5.5. ∎

By Proposition 5.6 and Theorem 5.5, if y is D_t-separable, then there is $\beta \in \overline{\mathbb{F}(t)}$ such that $K(t,y) = K(t,\beta)$. We now characterize separable algebraic functions as follows.

Proposition 5.7. *Let $P = \sum_{i=0}^n A_i Y^i \in \mathbb{F}[t,\mathbf{x}][Y]$ with $A_n \neq 0$ be the minimal polynomial of $y \in \overline{\mathbb{F}(t,\mathbf{x})}$. Suppose that y is D_t-separable. Let $K = \mathbb{F}(\mathbf{x},\alpha)$ with $\alpha \in \overline{\mathbb{F}(\mathbf{x})}$ be as in (5.2) and $\beta \in \overline{\mathbb{F}(t)}$ be such that $K(t,y) = K(t,\beta)$. Then*

(1) *$A_n(\mathbf{x},t)$ is split, i.e., $A_n(\mathbf{x},t) = a(\mathbf{x})b(t)$, where $a(\mathbf{x}) \in \mathbb{F}[\mathbf{x}]$, $b(t) \in \mathbb{F}[t]$, and*
(2) *there are $a_i(t) \in K[t]$ such that*

$$y = \frac{1}{b(t)q(t)} \sum_{i=0}^{\ell-1} a_i(t)\beta^i, \tag{5.3}$$

where b is as in (1), $\ell = [K(t,y):K(t)]$ and $q(t)$ is the discriminant of the base $\{1,\beta,\cdots,\beta^{\ell-1}\}$.

Proof. Let $r_i = A_i/A_n = p_i/q_i \in \mathbb{F}(t,\mathbf{x})$ with $0 \le i \le n$, $p_i, q_i \in \mathbb{F}[t,\mathbf{x}]$ and $\gcd(p_i,q_i)=1$. Since y is D_t-separable, so are all of the conjugate roots of $P(Y)=0$. By Vieta's formulas, the

r_i's are polynomials of these roots, which therefore are also D_t-separable by Proposition 2.2. By Theorem 3.1, q_i is split for all i with $0 \le i \le n$. Since A_n is the LCM of the q_i's, we have $A_n(\mathbf{x}, t)$ is also split.

Let S be the integral closure of $K[t]$ in $K(t, y)$. Then $A_n(\mathbf{x}, t)y \in S$. Since $\{1, \beta, \cdots, \beta^{\ell-1}\}$ is a base of $K(t, y)$ over $K(t)$, one has that

$$A_n(\mathbf{x}, t)y = \frac{1}{q(t)} \sum_{i=0}^{\ell-1} g_i(t)\beta^i,$$

where $g_i(t) \in K[t]$. Setting $a_i(t) = g_i(t)/a(\mathbf{x})$, we obtain the required expression for y. ∎

Recall that $K = \mathbb{F}(\mathbf{x}, \alpha)$ and $\ell = [K(t, y) : K(t)]$. Since the derivative of y is also in $K(t, y)$ for any $i \in \mathbb{N}$, we have that $Y = (1, y, y^2, \cdots, y^{\ell-1})^t$ satisfies a linear differential system of the form

$$Y' = AY, \quad \text{where } A \in \mathrm{Mat}_\ell(K(t)). \tag{5.4}$$

We will call (5.4) the associated differential equation of y over $K(t)$. The following proposition will allow us to design an algorithm for testing the separability of algebraic functions.

Proposition 5.8. *Let y and K be as above. Assume that (5.4) is the associated differential equation of y over $K(t)$. Then y is D_t-separable if and only if there is an invertible matrix G with entries in $K[t]$ such that*

$$G^{-1}G' - G^{-1}AG \in \mathrm{Mat}_\ell(\mathbb{F}(t)).$$

Furthermore, if y is D_t-separable then

$$G^{-1}G' - G^{-1}AG = \frac{(b^{\ell-1}q)'}{b^{\ell-1}q} - B$$

where b, q, ℓ are as in (5.3) and B is the associated differential equation of β over $\mathbb{F}(t)$ with β being as in (5.3).

Proof. Assume that there exists a nonzero $L \in \mathbb{F}(t)\langle D_t\rangle$ such that $L(y) = 0$. Then by Proposition 5.7, y has the form (5.3). Let E be the Galois closure of $K(t, \beta)$ over $K(t)$. Let $\beta_1 = \beta, \beta_2, \cdots, \beta_\ell$ be the conjugates of β and $\sigma_i \in \mathrm{Gal}(E/K(t))$ such that $\sigma_i(\beta) = \beta_i$. Then $\sigma_1(y), \cdots, \sigma_\ell(y)$ are all zeroes of $P(\mathbf{x}, t, y)$. We will denote the Vandermonde matrix generated by $\sigma_1(y), \cdots, \sigma_\ell(y)$ by $U(y)$ and the one generated by $\beta_1, \cdots, \beta_\ell$ by $U(\beta)$. Then $U(y)$ is a fundamental matrix of the system (5.4) and $U(\beta)$ is a fundamental matrix of a system $Y' = BY$ with $B \in \mathrm{Mat}_\ell(\mathbb{F}(t))$. Using the argument similar to that in the proof of Proposition 5.8, we have that for all j with $1 \le j \le \ell - 1$,

$$y^j = \frac{1}{b(t)^j q(t)} \sum_{i=0}^{\ell-1} a_{i,j}(t)\beta^i, \tag{5.5}$$

where $a_{i,j}(t) \in K[t]$ and $b(t), q(t)$ are as in Proposition 5.7. Applying σ_l to both sides of the equalities (5.5) implies that

$$\sigma_l(y)^j = \frac{1}{b(t)^j q(t)} \sum_{i=0}^{\ell-1} a_{i,j}(t)\beta_l^i, \tag{5.6}$$

where $j = 1, \cdots, \ell - 1, l = 1, \cdots, \ell$. Let $\tilde{a}_{i,j} = a_{i,j} b^{\ell-1-j}$ and

$$G = \begin{pmatrix} b(t)^{\ell-1}q(t) & 0 & \cdots & 0 \\ \tilde{a}_{0,1}(t) & \tilde{a}_{1,1}(t) & \cdots & \tilde{a}_{\ell-1,1}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{a}_{0,\ell-1}(t) & \tilde{a}_{1,\ell-1}(t) & \cdots & \tilde{a}_{\ell-1,\ell-1}(t) \end{pmatrix}$$

that is an element in $\mathrm{Mat}_\ell(K[t])$. Then the equations (5.6) can be rewritten as $U(y) = (GU(\beta))/(b(t)^{\ell-1}q(t))$. Hence G is invertible and an easy calculation yields that

$$\begin{aligned} U(\beta)' &= (b^{\ell-1}qG^{-1}U(y))' \\ &= \left((b^{\ell-1}q)' - b^{\ell-1}qG^{-1}G' + b^{\ell-1}qG^{-1}AG\right) G^{-1}U(y) \\ &= BU(\beta) = b^{\ell-1}qBG^{-1}U(y). \end{aligned}$$

This implies that

$$G^{-1}AG - G^{-1}G' = B - \frac{(b^{\ell-1}q)'}{b^{\ell-1}q} \in \mathrm{Mat}_\ell(\mathbb{F}(t)).$$

Now we prove the converse. Assume that there is an invertible matrix $G \in \mathrm{Mat}_\ell(K[t])$ such that

$$\tilde{B} = G^{-1}AG - G^{-1}G' \in \mathrm{Mat}_\ell(\mathbb{F}(t)).$$

Then $U(y) = GF$, where F is a fundamental matrix of $Y' = \tilde{B}Y$ with entries in some differential extension field of $K(t)$. Obviously, the entries of both G and F are annihilated by nonzero operators in $\mathbb{F}(t)\langle D_t\rangle$ and thus so are the sum of products of entries of G and F, in particular, so is y. ∎

Remark 5.9. *Once β is computed, one can obtain the linear differential equations $Y' = BY$ satisfied by $U(\beta)$.*

5.3 An algorithm for testing separability

All arguments presented in this section are summarized in Algorithm 5.11 which decides whether a given algebraic function $y \in \overline{\mathbb{F}(t, \mathbf{x})}$ is D_t-separable or not. For the sake of simplicity, we may take $\mathbb{F} = \bar{\mathbb{Q}}$, the field of all algebraic numbers over $\mathbb{Q}$. Let $P = \sum_{i=0}^n A_i Y^i \in \mathbb{F}[t, \mathbf{x}][Y]$ be the minimal polynomial of y. We may assume that A_n is split, otherwise we can conclude with Proposition 5.7 that y is not D_t-separable. Under this assumption, y is D_t-separable if and only if $A_n y$ is D_t-separable. Therefore without loss of generality, we may assume that

$$P(\mathbf{x}, t, Y) = Y^n + A_{n-1}(\mathbf{x}, t)Y^{n-1} + \cdots + A_0(\mathbf{x}, t), \tag{5.7}$$

where $A_i \in \mathbb{F}[\mathbf{x}, t]$. Let $(a, \alpha) \in \mathbb{F} \times \overline{\mathbb{F}(\mathbf{x})}$ satisfy

$$P(\mathbf{x}, a, \alpha) = 0, \ \frac{\partial P}{\partial Y}(\mathbf{x}, a, \alpha) \neq 0, \tag{5.8}$$

and let $K = \mathbb{F}(\mathbf{x}, \alpha)$. Then $P(\mathbf{x}, t, Y)$ may be factorized into a product of irreducible polynomials in $K[t, Y]$. There is a unique factor of $P(\mathbf{x}, t, Y)$ in $K[t, Y]$ vanishing at (a, α), denoted by $\bar{P}(\mathbf{x}, \alpha, t, Y)$. Let $K(t, y)$ be the quotient field of $K[t, Y]/\langle \bar{P}(\mathbf{x}, \alpha, t, Y)\rangle$. Now suppose that y is D_t-separable. Then Proposition 5.6 and Theorem 5.5 imply that there is $\beta \in \overline{\mathbb{F}(t)}$ such that $K(t, y) = K(t, \beta)$. We shall show how to find such β.

Let $R = \mathbb{F}(t)[\mathbf{x}]$ and S the integral closure of R in $K(t, y)$. Then $\alpha, y \in S$. Suppose that

$$\bar{P}(\mathbf{x}, \alpha, t, Y) = B_\ell Y^\ell + B_{\ell-1}Y^{\ell-1} + \cdots + B_0, \tag{5.9}$$

where $B_\ell \in \mathbb{F}[\mathbf{x}], B_i \in \mathbb{F}[\mathbf{x}, \alpha, t]$ with $i = 0, \cdots, \ell - 1$. Note that

$$\begin{aligned}[K(t,y):\mathbb{F}(\mathbf{x},t)] &= [K(t,y):K(t)][K(t):\mathbb{F}(\mathbf{x},t)] \\ &= [K(t,y):K(t)][K:\mathbb{F}(\mathbf{x})] = \ell[K:\mathbb{F}(\mathbf{x})].\end{aligned}$$

The set

$$\left\{\alpha^i y^j \,|\, i = 0, \cdots, [K:\mathbb{F}(\mathbf{x})] - 1, j = 0, \cdots, \ell - 1\right\}$$

is a base of $K(t, y)$ over $\mathbb{F}(\mathbf{x}, t)$. Let $D(\mathbf{x}, t)$ be the discriminant of the above base and let $F(\mathbf{x}, Y)$ be an irreducible polynomial in $\mathbb{F}[\mathbf{x}, Y]$ such that $F(\mathbf{x}, \alpha) = 0$. Then we have

Lemma 5.10. *Let $(\mathbf{c}, b) \in \mathbb{F}^{m+1}$ be such that $F(\mathbf{c}, b) = 0$ and $D(\mathbf{c}, t)B_\ell(\mathbf{c}) \neq 0$. Then $\bar{P}(\mathbf{c}, b, t, Y)$ is irreducible in $\mathbb{F}[t, Y]$ and for any root $Y = \gamma$ of $\bar{P}(\mathbf{c}, b, t, Y) = 0$, we have that $K(t, y)$ is isomorphic over $K(t)$ to $K(t, \gamma)$.*

Proof. Let $\beta \in K(t, y)$ be such that $K(t, y) = K(t, \beta)$. Since β is algebraic over $\mathbb{F}(t)$ we have that β is integral over $R = \mathbb{F}(t)[\mathbf{x}]$. Therefore we may write

$$\beta = \frac{1}{D(\mathbf{x}, t)} \sum b_{i,j} \alpha^i y^j,$$

where the $b_{i,j} \in R$. Let $(\mathbf{c}, b)$ satisfy the hypothesis of the lemma and consider the ideal

$$\mathfrak{p} = \langle x_1 - c_1, \ldots, x_m - c_m, \alpha - b\rangle \lhd R[\alpha].$$

Note that $\mathfrak{p}$ is a maximal ideal. The Going Up Theorem implies that there is a maximal ideal $\mathfrak{q} \lhd S$ such that $\mathfrak{q} \cap R[\alpha] = \mathfrak{p}$. In particular, $D(\mathbf{x}, t) \notin \mathfrak{q}$. There is a natural map $\phi : S \to S/\mathfrak{q}$. We will let M denote the field $S/\mathfrak{q}$. The element $\gamma = \phi(y)$ is a root of $\bar{P}(\mathbf{c}, b, t, \gamma) = 0$. Since the minimal polynomial $Q(t, Y)$ of β lies in $\mathbb{F}[t, Y]$, it remains unchanged when we apply ϕ to its coefficients. Therefore $\phi(\beta)$ satisfies $Q(t, \phi(\beta)) = 0$. In particular, the degree of $\phi(\beta)$ over $\mathbb{F}(t)$ is equal to ℓ, the degree of $K(t, \beta)$ over $K(t)$. Since

$$\phi(\beta) = \frac{1}{D(\mathbf{c}, t)} \sum \phi(b_{i,j})\phi(\alpha)^i \gamma^j$$

we have that $\phi(\beta) \in \mathbb{F}(t)(\gamma)$. Note that $\bar{P}(\mathbf{c}, b, t, Y) \neq 0$. The element γ satisfies $\bar{P}(\mathbf{c}, b, t, \gamma) = 0$ and so it has degree at most ℓ over $\mathbb{F}(t)$. Since $\phi(\beta) \in \mathbb{F}(t, \gamma)$, we have that

$$\begin{aligned}\ell \geq [\mathbb{F}(t,\gamma):\mathbb{F}(t)] \geq [\mathbb{F}(t,\phi(\beta)):\mathbb{F}(t)] &= [K(t,\beta):K(t)] \\ &= [K(t,y):K(t)] = \ell\end{aligned}$$

and so $[\mathbb{F}(t, \gamma) : \mathbb{F}(t)] = \ell$. Therefore $\bar{P}(\mathbf{c}, b, t, Y)$ is irreducible. Furthermore $\mathbb{F}(t, \beta)$ is isomorphic over $\mathbb{F}(t)$ to $\mathbb{F}(t, \phi(\beta)) = \mathbb{F}(t, \gamma)$. This implies that $K(t, y)$ is isomorphic over $K(t)$ to $K(t, \gamma)$. ∎

Let $\bar{P}(\mathbf{x}, \alpha, t, Y)$ be as above. Lemma 5.10 implies that if y is D_t-separable then one can compute $(\mathbf{c}, b) \in \mathbb{F}^{m+1}$ such that $\bar{P}(\mathbf{c}, b, t, Y)$ is irreducible over $\mathbb{F}(t)$. Furthermore, β can be taken to be a zero of $\bar{P}(\mathbf{c}, b, t, Y)$. To see this, let φ be the isomorphism map over $K(t)$ from $K(t, \gamma)$ to $K(t, y)$ where γ is a zero of $\bar{P}(\mathbf{c}, b, t, Y) = 0$ in $\overline{\mathbb{F}(t)}$. Set $\beta = \varphi(\gamma)$. Then one has that $K(t, \beta) = K(t, y)$ and $\bar{P}(\mathbf{c}, b, t, \beta) = 0$. From $\bar{P}(\mathbf{c}, b, t, Y)$, we can construct the associated differential equation of β over $\mathbb{F}(t)$. Denote this associated differential equation by $Y' = BY$ with $B \in \mathrm{Mat}_\ell(\mathbb{F}(t))$. The proof of Proposition 5.8 implies that if y is D_t-separable then there is an invertible matrix G with entries in $K[t]$ such that

$$G' = AG - G\left(B - \frac{q'(t)}{q(t)}\right),$$

where $q(t)$ is the discriminant of $\{1, \beta, \cdots, \beta^{\ell-1}\}$ and $Y' = AY$ is the associated differential equation of y over $K(t)$. Here the polynomial $b(t)$ in (5.5) disappears because we assume that P is monic in Y. Note that G is a polynomial solution of the linear differential equation $Y' = AY - Y(B - q(t)'/q(t))$, which can be computed by algorithms developed in [5, 8].

We summarize the above results as the following algorithm.

Algorithm 5.11. *Input: An irreducible polynomial*

$$P(t, \mathbf{x}, Y) = A_n Y^n + A_{n-1} Y^{n-1} + \cdots + A_0 \in \mathbb{F}[t, \mathbf{x}, Y].$$

Output: "Yes" if y is D_t-separable, otherwise "No", where $y \in \overline{k(\mathbf{x}, t)}$ is a root of $P(Y) = 0$.

(1) *If A_n is not split, then y is not D_t-separable and return "No".*
(2) *Transform $P(\mathbf{x}, t, Y)$ into a monic polynomial by replacing Y by Y/A_n and clear the denominators.*
(3) *Compute β:*
(3.a) *Find $(a, \alpha) \in \mathbb{F} \times \overline{\mathbb{F}(\mathbf{x})}$ satisfying the conditions (5.8).*
(3.b) *Decompose P into a product of irreducible polynomials over $\mathbb{F}(\mathbf{x}, \alpha)$. Let $\bar{P}(\mathbf{x}, \alpha, t, Y)$ be the irreducible factor satisfying that $\bar{P}(\mathbf{x}, \alpha, a, \alpha) = 0$.*
(3.c) *Compute $D(\mathbf{x}, t)$, the discriminant of the base $\{\alpha^i \bar{y}^j\}$, where $\bar{y}$ is a zero of $\bar{P}(\mathbf{x}, \alpha, t, Y)$ in $\overline{\mathbb{F}(t, \mathbf{x})}$.*
(3.d) *Compute a point $(\mathbf{c}, b) \in \mathbb{F}^{m+1}$ such that*

$$D(\mathbf{c}, t)B_\ell(\mathbf{c}) \neq 0 \text{ and } F(\mathbf{c}, b) = 0,$$

where F is the minimal polynomial of α over $\mathbb{F}(\mathbf{x})$ and $B_\ell(\mathbf{x})$ is the leading coefficient of $\bar{P}(\mathbf{x}, \alpha, t, Y)$.
(3.e) *Let β be a zero of $\bar{P}(\mathbf{c}, b, t, Y) = 0$ in $\overline{\mathbb{F}(t)}$.*
(4) *Compute G:*
(4.a) *Compute $q(t)$, the discriminant of the base $\{\beta^j | j = 0, \cdots, \ell - 1\}$ and compute the associated differential equations of y and β, which are denoted by $Y' = AY$ and $Y' = BY$ respectively.*
(4.b) *By algorithms developed in [5, 8], compute a base of polynomial solutions of $Z' = AZ - Z(B - q(t)'/q(t))$, where $Z = (z_{ij})$ with indeterminate entries, say $\{Q_1, \cdots, Q_s\}$.*
(4.c) *Compute $C = \det(z_1 Q_1 + \cdots + z_s Q_s)$ with $z_1, \cdots, z_s$ being indeterminates. If $C = 0$ then return "No", otherwise return "Yes".*

We now show an example to illustrate the main steps of the above algorithm.

Example 5.12. *Let $\mathbb{E} = \bar{\mathbb{Q}}(t, x)$ and y be the algebraic function over $\mathbb{E}$ defined by*

$$P(x, t, Y) := Y^2 - 2(xt + 1)Y + (xt + 1)^2 - t.$$

We are going to decide whether y is D_t-separable or not. We will follow the above algorithm step by step. Since $P(x, t, Y)$ is monic in Y. We begin with the third step, i.e., computing β.

(3) *Compute $\beta = \sqrt{t} + 1$:*

(3.a) *Set* $(a, \alpha) = (1, x)$. *One sees that* $P(x, 1, x) = 0$ *and*
$$\frac{\partial P}{\partial Y}(x, 1, x) = -2 \neq 0.$$
So $\bar{\mathbb{Q}}(x, \alpha) = \bar{\mathbb{Q}}(x)$.

(3.b) *Since* $P(x, t, Y)$ *is irreducible over* $\bar{\mathbb{Q}}(x)$, *we take* $\bar{P}(x, \alpha, t, Y)$ *to be* $P(x, t, Y)$.

(3.c) *Set* $D(x, t) = 4t$, *which is the discriminant of the base* $\{1, \bar{y}\}$ *with* $P(x, t, \bar{y}) = 0$.

(3.d) *One sees that* $B_2(x) = 1$ *and* $F = z - x$. *So the point* $(0, 0)$ *satisfies* $D(0, t)B_2(0) \neq 0$ *and* $F(0, 0) = 0$.

(3.e) *Set* $\beta = \sqrt{t} + 1$ *which is a zero of* $P(0, t, Y) = Y^2 - 2Y + 1 - t$.

(4) *Compute G:*

(4.a) *Set* $q(t) = 4t$, *which is the discriminant of the base* $\{1, \beta\}$, *and set*
$$A = \begin{pmatrix} 0 & 0 \\ \frac{x}{2} - \frac{1}{2t} & \frac{1}{2t} \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2t} & \frac{1}{2t} \end{pmatrix}.$$
Then $Y' = AY$ *and* $Y' = BY$ *are the associated differential equations of* y *and* β *respectively.*

(4.b) *Set* $Z = (z_{ij})_{1 \leq i, j \leq 2}$, *and compute a base of the polynomial solutions of the system* $Z' = AZ - Z(B - 1/t)$. *One has that*
$$\left\{ Q_1 := \begin{pmatrix} t & 0 \\ xt^2 + t & 0 \end{pmatrix}, \ Q_2 := \begin{pmatrix} 0 & 0 \\ -t & t \end{pmatrix} \right\}$$
is a required base.

(4.c) *One has that* $\det(z_1 Q_1 + z_2 Q_2) = z_1 z_2 t^2 \neq 0$. *So* y *is* D_t*-separable.*

6 CONCLUSION AND FUTURE WORK

We present a connection between the separability problems and the existence problems in creative telescoping. Separability criteria are given for rational functions, hyperexponential functions, hypergeometric terms and algebraic functions. Some results in the algebraic case have been generalized to the case of D-finite functions whose annihilating operators of minimal order are completely reducible in [15]. Moreover in [15], for a D-finite function y, a nonzero operator $L \in \mathbb{F}(t)\langle D_t \rangle$ such that $L(y) = 0$ is also computed if it exists. The existence problems of telescopers for rational functions in three variables are now completely settled by combining the results in [12] with the separability criteria in this paper.

In terms of future research, the first natural direction is to design efficient algorithms for computing minimal annihilators of separable functions. The second direction is to solve the separability problem for P-recursive sequences, which may have applications in solving the general termination problem of Zeilberger's algorithms beyond the hypergeometric case. We can also try to develop more symbolic computational tools for the method of separation of variables for partial differential equations as in [23].

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