A OMITTED PROOFS IN SECTION 3 AND SECTION 4.2

Proof of Lemma 3.2. The first inequality follows from Fact 3.1, item 1. For the second, let $a_1 < a_2 < \cdots < a_m$ be all the values of $g(x)$ when $x \in \mathcal{V}$ and let a_0 be a negative number arbitrarily close to 0. We have:

$$\begin{aligned}\mathop{\mathbb{E}}_{x\sim\mathcal{D}}\left[(g(x))^2\right] &= \sum_{x\in\mathcal{V}} f(x)\cdot(g(x))^2 \\ &= \sum_{i=1}^{m} a_i^2 \cdot \Pr_{x\sim\mathcal{D}}(g(x)=a_i) \\ &= \sum_{i=1}^{m} a_i^2 \cdot \left(\Pr_{x\sim\mathcal{D}}(g(x)>a_{i-1}) - \Pr_{x\sim\mathcal{D}}(g(x)>a_i)\right) \\ &= a_0^2 + \sum_{i=1}^{m}\left(a_i^2 - a_{i-1}^2\right)\cdot \Pr_{x\sim\mathcal{D}}(g(x)\ge a_i).\end{aligned}$$

Using the identity $x^2 - y^2 = (x+y)(x-y)$, we get:

$$\begin{aligned}\mathop{\mathbb{E}}_{x\sim\mathcal{D}}\left[(g(x))^2\right] &\le a_0^2 + \sum_{i=1}^{m}(a_i - a_{i-1})\cdot 2a_i \cdot \Pr_{x\sim\mathcal{D}}(g(x)\ge a_i) \\ &\le a_0^2 + 2\cdot\left(\max_{x\in\mathcal{V}} g(x)\cdot \Pr_{y\sim\mathcal{D}}(g(y)\ge g(x))\right)\cdot \sum_{i=1}^{m}(a_i - a_{i-1}) \\ &\le a_0^2 + 2\cdot\left(\max_{x\in\mathcal{V}} g(x)\cdot \Pr_{y\sim\mathcal{D}}(g(y)\ge g(x))\right)\cdot \left(\max_{x\in\mathcal{V}} g(x) - a_0\right).\end{aligned}$$

The lemma follows as $a_0 < 0$ was arbitrary. □

Proof of Lemma 3.4. As $x \le x'$, Algorithm 1 did not set the value of $\tilde{\varphi}_j(x')$ after setting the value of $\tilde{\varphi}_j(x)$. Using this and Line 5 of Algorithm 1, we get that it is sufficient to show that the value $a(y^*)$ cannot increase between two consecutive iterations of the While loop. To this end, consider two consecutive iterations and let $x_1, y_1^*, a_1(\cdot)$ and $x_2, y_2^*, a_2(\cdot)$ be the values of the corresponding variables in the first and the second iteration respectively and note that $y_2^* \le x_2 < y_1^* \le x_1$.

By our choice of y_1^* in Line 4 in the first iteration, we have that $a_1(y_1^*) \ge a_1(y_2^*)$. Extending using Line 3, we get:

$$\begin{aligned}a_1(y_1^*) \ge a_1(y_2^*) &= \frac{\sum_{y'\in[y_2^*,x_1]\cap\mathcal{V}_j} f_j(y')\cdot\varphi_j(y')}{\sum_{y'\in[y_2^*,x_1]\cap\mathcal{V}_j} f_j(y')} \\ &= \frac{\sum_{y'\in[y_2^*,x_2]\cap\mathcal{V}_j} f_j(y')}{\sum_{y'\in[y_2^*,x_1]\cap\mathcal{V}_j} f_j(y')}\cdot a_2(y_2^*) + \frac{\sum_{y'\in[y_1^*,x_1]\cap\mathcal{V}_j} f_j(y')}{\sum_{y'\in[y_2^*,x_1]\cap\mathcal{V}_j} f_j(y')}\cdot a_1(y_1^*).\end{aligned}$$

It follows that $a_1(y_1^*) \ge a_2(y_2^*)$, as desired. □

Proof of Lemma 3.5. Let $x_1, y_1^*, a_1(\cdot)$ be the values of the corresponding variables in the iteration when the value of $\tilde{\varphi}_j(x)$ is set. Observe that $y_1^* \le x \le x_1$. If $x = x_1$, we simply have:

$$\tilde{\varphi}_j(x) = a_1(y_1^*) = \frac{\sum_{y'\in[y_1^*,x_1]\cap\mathcal{V}_j} f_j(y')\cdot\varphi_j(y')}{\sum_{y'\in[y_1^*,x_1]\cap\mathcal{V}_j} f_j(y')} \le \frac{\sum_{y'\in[y_1^*,x_1]\cap\mathcal{V}_j} f_j(y')\cdot x}{\sum_{y'\in[y_1^*,x_1]\cap\mathcal{V}_j} f_j(y')} = x,$$

where the penultimate step uses $\varphi(y') \le y' \le x_1 = x$ by Definition 3.3. Otherwise, we have $x < x_1$. Define $x' \in \mathcal{V}_j$ to be the smallest such that $x < x'$ and observe that $x' \le x_1$. By our choice of y_1^* in

Line 4, we have:

$$\begin{aligned}
a_1(y_1^*) &\geq a_1(x') \\
&= \frac{\sum_{y' \in [x', x_1] \cap \mathcal{V}_j} f_j(y') \cdot \varphi_j(y')}{\sum_{y' \in [x', x_1] \cap \mathcal{V}_j} f_j(y')} \\
&= a_1(y_1^*) \cdot \frac{\sum_{y' \in [y_1^*, x_1] \cap \mathcal{V}_j} f_j(y')}{\sum_{y' \in [x', x_1] \cap \mathcal{V}_j} f_j(y')} - \frac{\sum_{y' \in [y_1^*, x] \cap \mathcal{V}_j} f_j(y') \cdot \varphi_j(y')}{\sum_{y' \in [y_1^*, x] \cap \mathcal{V}_j} f_j(y')} \cdot \frac{\sum_{y' \in [y_1^*, x] \cap \mathcal{V}_j} f_j(y')}{\sum_{y' \in [x', x_1] \cap \mathcal{V}_j} f_j(y')}.
\end{aligned}$$

Rearranging, we get:

$$\tilde{\varphi}_j(x) = a_1(y_1^*) \leq \frac{\sum_{y' \in [y_1^*, x] \cap \mathcal{V}_j} f_j(y') \cdot \varphi_j(y')}{\sum_{y' \in [y_1^*, x] \cap \mathcal{V}_j} f_j(y')} \leq \frac{\sum_{y' \in [y_1^*, x] \cap \mathcal{V}_j} f_j(y') \cdot x}{\sum_{y' \in [y_1^*, x] \cap \mathcal{V}_j} f_j(y')} = x,$$

using $\varphi(y') \leq y' \leq x$ by Definition 3.3 in the penultimate step. □

Proof of Lemma 4.6. We omit the subscript $X \sim D^k$ to keep the notation concise. Observe that we have $\Pr(\mathrm{tb}(X) = i) = \frac{1}{k}$ by symmetry and also that:

$$\Pr\Big(\max_{i'} x_{i'} = s\Big) = \Big(\Pr_{x \sim D}(x \leq s)\Big)^k - \Big(\Pr_{x \sim D}(x < s)\Big)^k.$$

This means that it is sufficient to show that:

$$\Pr\Big(\max_{i'} x_{i'} = s \wedge \mathrm{tb}(X) = i\Big) = \frac{1}{k} \cdot \left(\Big(\Pr_{x \sim D}(x \leq s)\Big)^k - \Big(\Pr_{x \sim D}(x < s)\Big)^k\right).$$

We show this by considering all possible values of $\arg\max_{i'} x_{i'}$. We have:

$$\begin{aligned}
\Pr\Big(\max_{i'} x_{i'} = s \wedge \mathrm{tb}(X) = i\Big) &= \sum_{S \ni i} \Pr\Big(\max_{i'} x_{i'} = s \wedge \mathrm{tb}(X) = i \wedge \arg\max_{i'} x_{i'} = S\Big) \\
&= \sum_{S \ni i} \frac{1}{|S|} \cdot \Pr\Big(\max_{i'} x_{i'} = s \wedge \arg\max_{i'} x_{i'} = S\Big). \qquad \text{(Equation 9)}
\end{aligned}$$

We can calculate the term on the right:

$$\begin{aligned}
\Pr\Big(\max_{i'} x_{i'} = s \wedge \mathrm{tb}(X) = i\Big) &= \sum_{S \ni i} \frac{1}{|S|} \cdot \Big(\Pr_{x \sim D}(x = s)\Big)^{|S|} \cdot \Big(\Pr_{x \sim D}(x < s)\Big)^{k-|S|} \\
&= \sum_{k'=1}^{k} \frac{1}{k'} \cdot \binom{k-1}{k'-1} \cdot \Big(\Pr_{x \sim D}(x = s)\Big)^{k'} \cdot \Big(\Pr_{x \sim D}(x < s)\Big)^{k-k'} \\
&= \sum_{k'=1}^{k} \frac{1}{k} \cdot \binom{k}{k'} \cdot \Big(\Pr_{x \sim D}(x = s)\Big)^{k'} \cdot \Big(\Pr_{x \sim D}(x < s)\Big)^{k-k'}.
\end{aligned}$$

The Binomial theorem $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$ then gives:

$$\Pr\Big(\max_{i'} x_{i'} = s \wedge \mathrm{tb}(X) = i\Big) = \frac{1}{k} \cdot \left(\Big(\Pr_{x \sim D}(x \leq s)\Big)^k - \Big(\Pr_{x \sim D}(x < s)\Big)^k\right).$$

□

B SOME LOWER BOUNDS ON $\mathsf{SRev}(\cdot)$

In this section, we analyze the revenue of some auctions that sell the items separately. By definition, the revenue of any such auction is a lower bound for $\mathsf{SRev}(\cdot)$. All lemmas in this section are for a fixed auction setting $(n, m, \mathcal{D})$ (see Section 3).

LEMMA B.1 (VCG WITH RESERVES). *Fix item $j \in [m]$. For all $x \geq 0$, it holds that:*

$$\sum_{v \in \mathcal{V}^n} f^*(v) \cdot x \cdot \mathbb{1}\left(\max(v)|_j \geq x\right) \leq \mathsf{SRev}_j(n).$$

PROOF. Consider the auction that sells item j through a VCG auction with reserve x. Namely, it solicits bits $v_{i,j}$ for item j for each bidder $i \in [m]$ and proceeds as follows: If the highest bid is at least x, then allocate this item to the highest bidder for a price equal to the maximum of x and the second highest bid. Otherwise, the item stays unallocated. Clearly, the auction is truthful and generates revenue at least:

$$\sum_{v \in \mathcal{V}^n} f^*(v) \cdot x \cdot \mathbb{1}\left(\max(v)|_j \geq x\right).$$

Thus, we can upper bound the above quantity by $\mathsf{SRev}_j(n)$ and the lemma follows. □

LEMMA B.2 (SEQUENTIAL POSTED PRICE). *Let non-negative numbers $\{x_{i,j}\}_{i \in [n], j \in [m]}$ be given. It holds that:*

$$\sum_{j=1}^{m} \sum_{v \in \mathcal{V}^n} f^*(v) \cdot \max_{i \in [n]} \left\{x_{i,j} \cdot \mathbb{1}\left(v_{i,j} \geq x_{i,j}\right)\right\} \leq \mathsf{SRev}(n).$$

PROOF. Consider the auction that sells each item $j \in [m]$ separately through the following auction: It goes over all the bidders in decreasing order of $x_{i,j}$, bidder i can either take the item and pay price $x_{i,j}$, in which case the auction terminates, or skip the item, in which case the auction goes to the next bidder. Clearly, the auction is truthful and generates revenue at least:

$$\sum_{j=1}^{m} \sum_{v \in \mathcal{V}^n} f^*(v) \cdot \max_{i \in [n]} \left\{x_{i,j} \cdot \mathbb{1}\left(v_{i,j} \geq x_{i,j}\right)\right\}.$$

Thus, we can upper bound the above quantity by $\mathsf{SRev}(n)$ and the lemma follows. □

LEMMA B.3 (RONEN'S AUCTION [32]). *For all $j \in [m]$ and $x \geq 0$, define $r^*_{\mathsf{Ron},j}(x) = \max_{y > x} y \cdot \Pr_{y' \sim \mathcal{D}_j}(y' \geq y)$. It holds that:*

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{v_{-i} \in \mathcal{V}^{n-1}} f^*(v_{-i}) \cdot r^*_{\mathsf{Ron},j}(\max(v_{-i})|_j) \leq \mathsf{SRev}(n).$$

PROOF. Consider the auction that sells each item $j \in [m]$ separately through the following auction: First, it solicits bids $v_{i,j}$ for item j from each bidder $i \in [n]$. Then, for $i \in [n]$, it sets $y^*_{i,j}(v_{-i})$ to be[12] the maximizer in the definition of $r^*_{\mathsf{Ron},j}(\max(v_{-i})|_j)$, and offers each bidder i to purchase item j at a price of $y^*_{i,j}(v_{-i})$. As $y^*_{i,j}(v_{-i}) > \max(v_{-i})|_j$ by definition, at most one bidder will ever purchase the item and the auction is well defined (Equation 1).

Also, as the price offered to bidder i does not depend on his bid, the auction is also truthful. Thus, its revenue is a lower bound on $\mathsf{SRev}(n)$ and we get:

$$\mathsf{SRev}(n) \geq \sum_{j=1}^{m} \sum_{v \in \mathcal{V}^n} f^*(v) \cdot \sum_{i=1}^{n} y^*_{i,j}(v_{-i}) \cdot \mathbb{1}\left(v_{i,j} \geq y^*_{i,j}(v_{-i})\right)$$

[12] We write $y^*_{i,j}$ as a function of v_{-i} but note that it only depends on the bidders' bids for item j.

$$\geq \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{v_{-i} \in \mathcal{V}^{n-1}} f^*(v_{-i}) \cdot y^*_{i,j}(v_{-i}) \cdot \Pr_{v_i \in \mathcal{V}}\Big(v_{i,j} \geq y^*_{i,j}(v_{-i})\Big)$$

$$\geq \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{v_{-i} \in \mathcal{V}^{n-1}} f^*(v_{-i}) \cdot r^*_{\text{Ron},j}(\max(v_{-i})|_j).$$

□

C PROOFS OF LEMMA 4.4 AND LEMMA 4.5

We show Lemma 4.4 and Lemma 4.5 following the framework of [8]. The first step is common to both the lemmas and shows that $\mathsf{IU}(n'', n')$ is at most $4 \cdot \mathrm{SRev}(n')$ plus an additional term corresponding to the term Core in [8]. This is captured in Lemma C.1. The next step bounds Core in two different ways to show the two lemmas. These can be found in Subsubsection C.2.1 and Subsubsection C.2.2.

C.1 Step 1 – Decomposing $\mathsf{IU}(\cdot)$

Lemma C.1. *For all $n'' \leq n'$, we have:*

$$\mathsf{IU}(n'', n') \leq 4 \cdot \mathrm{SRev}(n') + \textsc{Core},$$

where we define Core as:

$$r^*_{\text{Ron},j}(x) = \max_{y > x} y \cdot \Pr_{y' \sim \mathcal{D}_j}(y' \geq y) \qquad \forall j \in [m].$$

$$r^{(i)}_{\text{Ron}}(w_{-i}) = \sum_{j=1}^{m} r^*_{\text{Ron},j}(\max(w_{-i})|_j) \qquad \forall i \in [n'], w_{-i} \in \mathcal{V}^{n'-1}.$$

$$\mathcal{T}_{i,j}(w_{-i}) = r^{(i)}_{\text{Ron}}(w_{-i}) + \max(w_{-i})|_j \qquad \forall j \in [m], i \in [n'], w_{-i} \in \mathcal{V}^{n'-1}.$$

$$\textsc{Core} = \sum_{j=1}^{m} \sum_{i=1}^{n'} \sum_{w_{-i}} \sum_{\max(w_{-i})|_j \leq v_{i,j} \leq \mathcal{T}_{i,j}(w_{-i})} f^*(w_{-i}) f_j(v_{i,j}) \cdot \big(v_{i,j} - \max(w_{-i})|_j\big).$$

Proof. Fix $n'' \leq n'$. We first get rid of the parameter n'' by showing that $n'' = n'$ is the hardest case for the lemma. We have:

$$\begin{aligned} \mathsf{IU}(n'', n') &= \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n''}} \left[\max_{i \in [n'']} \left\{ v_{i,j} \cdot \big(1 - \mathcal{P}_j(v_i)\big) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathcal{P}_j(v_i) \right\} \right] \\ &\leq \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}} \left[\max_{i \in [n']} \left\{ v_{i,j} \cdot \big(1 - \mathcal{P}_j(v_i)\big) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathcal{P}_j(v_i) \right\} \right] \\ &= \mathsf{IU}(n'). \end{aligned}$$

Henceforth, we focus on upper bounding $\mathsf{IU}(n')$. Note that the term $1 - \mathcal{P}_j(v_i)$ in $\mathsf{IU}(n')$ corresponds to the event that $v_i \notin \mathcal{R}_j(w_{-i})$. By our choice of the regions $\mathcal{R}_j(\cdot)$, whenever this happens, either v_i is less than $\max(w_{-i})|_j$ or there is a $j' \neq j$ such that the utility from j' is at least as much as that from j. To capture these cases we define the sets[13]:

$$\begin{aligned} E_j^{\textsc{Und}}(v_i) &= w_{-i} \in \mathcal{V}^{n'-1} \mid v_{i,j} < \max(w_{-i})|_j \\ E_j^{\textsc{Srp}}(v_i) &= w_{-i} \in \mathcal{V}^{n'-1} \mid \exists j' \neq j : v_{i,j} - \max(w_{-i})|_j \leq v_{i,j'} - \max(w_{-i})|_{j'} \\ E_j^{\mathrm{NF}}(v_i) &= E_j^{\textsc{Srp}}(v_i) \setminus E_j^{\textsc{Und}}(v_i). \end{aligned} \tag{10}$$

As mentioned before, when $v_i \notin \mathcal{R}_j(w_{-i})$, we either have $w_{-i} \in E_j^{\textsc{Und}}(v_i)$ or $w_{-i} \in E_j^{\mathrm{NF}}(v_i)$. Thus, we have the following inequality.

$$1 - \mathcal{P}_j(v_i) \leq \Pr_{w_{-i} \sim \mathcal{D}^{n'-1}}\left(w_{-i} \in E_j^{\textsc{Und}}(v_i)\right) + \Pr_{w_{-i} \sim \mathcal{D}^{n'-1}}\left(w_{-i} \in E_j^{\mathrm{NF}}(v_i)\right). \tag{11}$$

Using Equation 11, we decompose $\mathsf{IU}(n')$ as follows:

$$\begin{aligned} \mathsf{IU}(n') &= \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}}\left[\max_{i \in [n']}\left\{v_{i,j} \cdot \left(1 - \mathcal{P}_j(v_i)\right) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathcal{P}_j(v_i)\right\}\right] \\ &\leq \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}}\left[\max_{i \in [n']}\left\{v_{i,j} \cdot \left(1 - \mathcal{P}_j(v_i)\right)\right\}\right] + \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}}\left[\max_{i \in [n']}\left\{\tilde{\varphi}_j(v_{i,j})^+ \cdot \mathcal{P}_j(v_i)\right\}\right] \\ &\leq \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}}\left[\max_{i \in [n']}\left\{\tilde{\varphi}_j(v_{i,j})^+ \cdot \mathcal{P}_j(v_i)\right\}\right] && (\textsc{Single}) \\ &\quad + \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}}\left[\max_{i \in [n']}\left\{v_{i,j} \cdot \Pr_{w_{-i}}\left(w_{-i} \in E_j^{\textsc{Und}}(v_i)\right)\right\}\right] && (\textsc{Under}) \\ &\quad + \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}}\left[\max_{i \in [n']}\left\{v_{i,j} \cdot \Pr_{w_{-i}}\left(w_{-i} \in E_j^{\mathrm{NF}}(v_i)\right)\right\}\right]. && (\textsc{Non-Favorite}) \end{aligned}$$

We have now split $\mathsf{IU}(n')$ into three terms, Single, Under, and Non-Favorite. We will later show that both Single and Under are at most $\mathsf{SRev}(n')$. As far as the term Non-Favorite goes, we need to decompose it further. We have:

$$\begin{aligned} \textsc{Non-Favorite} &= \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}}\left[\max_{i \in [n']}\left\{\sum_{w_{-i}} f^*(w_{-i}) \cdot v_{i,j} \cdot \mathbb{1}\left(w_{-i} \in E_j^{\mathrm{NF}}(v_i)\right)\right\}\right] \\ &\leq \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}}\left[\max_{i \in [n']}\left\{\sum_{w_{-i}} f^*(w_{-i}) \cdot \left(v_{i,j} - \max(w_{-i})|_j\right) \cdot \mathbb{1}\left(w_{-i} \in E_j^{\mathrm{NF}}(v_i)\right)\right\}\right] \\ &\quad + \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}}\left[\max_{i \in [n']}\left\{\sum_{w_{-i}} f^*(w_{-i}) \cdot \max(w_{-i})|_j \cdot \mathbb{1}\left(w_{-i} \in E_j^{\mathrm{NF}}(v_i)\right)\right\}\right]. \end{aligned}$$

Plugging into the previous decomposition and using the fact that $E_j^{\mathrm{NF}}(v_i)$ and $E_j^{\textsc{Und}}(v_i)$ are disjoint by Equation 10, we get that:

$$\begin{aligned} \mathsf{IU}(n') &\leq \textsc{Single} + \textsc{Under} \\ &\quad + \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}}\left[\max_{i \in [n']}\left\{\sum_{w_{-i}} f^*(w_{-i}) \cdot \max(w_{-i})|_j \cdot \mathbb{1}\left(w_{-i} \notin E_j^{\textsc{Und}}(v_i)\right)\right\}\right] && (\textsc{Over}) \end{aligned}$$

[13]For readers familiar with [8], our naming of these events corresponds to that used in [8], *e.g.*, NF corresponds to Non-Favorite.

$$+\sum_{j=1}^{m}\mathop{\mathbb{E}}_{v\sim\mathcal{D}^{n'}}\left[\sum_{i=1}^{n'}\sum_{w_{-i}}f^*(w_{-i})\cdot\left(v_{i,j}-\max(w_{-i})|_j\right)\cdot\mathbb{1}\left(w_{-i}\in E_j^{\mathrm{NF}}(v_i)\right)\right]. \qquad (\textsc{Surplus})$$

It can now be shown that Over is at most $\mathsf{SRev}(n')$. However, Surplus needs to be decomposed even more before it is analyzable. For this, we first use linearity of expectation to take the expectation over v inside. As the summand corresponding to i only depends on v_i, we get:

$$\textsc{Surplus}=\sum_{j=1}^{m}\sum_{i=1}^{n'}\sum_{w_{-i}}\mathop{\mathbb{E}}_{v_i}\left[f^*(w_{-i})\cdot\left(v_{i,j}-\max(w_{-i})|_j\right)\cdot\mathbb{1}\left(w_{-i}\in E_j^{\mathrm{NF}}(v_i)\right)\right].$$

Writing the expectation is a sum and noting that $w_{-i}\in E_j^{\mathrm{NF}}(v_i)$ only happens when $v_{i,j}\geq\max(w_{-i})|_j$ and $w_{-i}\in E_j^{\textsc{Srp}}(v_i)$ by Equation 10, we get that:

$$\textsc{Surplus}\leq\sum_{j=1}^{m}\sum_{i=1}^{n'}\sum_{w_{-i}}\sum_{v_{i,j}\geq\max(w_{-i})|_j}f^*(w_{-i})f_j(v_{i,j})\cdot\left(v_{i,j}-\max(w_{-i})|_j\right)\cdot\Pr_{v_{i,-j}}\left(w_{-i}\in E_j^{\textsc{Srp}}(v_i)\right).$$

To continue, we define, for all $j\in[m]$, the function $r^*_{\mathsf{Ron},j}(x)=\max_{y>x}y\cdot\Pr_{y'\sim\mathcal{D}_j}(y'\geq y)$. This definition is identical to that in Lemma B.3 and is closely connected to the payment of the highest bidder in Ronen's auction for item j when the second highest bid is x [32]. We also define, for all i,w_{-i}, the quantity $r^{(i)}_{\mathsf{Ron}}(w_{-i})=\sum_{j=1}^{m}r^*_{\mathsf{Ron},j}(\max(w_{-i})|_j)$ and, for all $j\in[m]$, the quantity $\mathcal{T}_{i,j}(w_{-i})=r^{(i)}_{\mathsf{Ron}}(w_{-i})+\max(w_{-i})|_j$. Using $v_{i,-j}$ to denote the tuple $(v_{i,1},\cdots,v_{i,j-1},v_{i,j+1},\cdots v_{i,m})$, we continue decomposing Surplus as:

$$\textsc{Surplus}\leq\sum_{j=1}^{m}\sum_{i=1}^{n'}\sum_{w_{-i}}\sum_{v_{i,j}>\mathcal{T}_{i,j}(w_{-i})}f^*(w_{-i})f_j(v_{i,j})\cdot\left(v_{i,j}-\max(w_{-i})|_j\right)\cdot\Pr_{v_{i,-j}}\left(w_{-i}\in E_j^{\textsc{Srp}}(v_i)\right) \qquad (\textsc{Tail})$$

$$+\sum_{j=1}^{m}\sum_{i=1}^{n'}\sum_{w_{-i}}\sum_{\max(w_{-i})|_j\leq v_{i,j}\leq\mathcal{T}_{i,j}(w_{-i})}f^*(w_{-i})f_j(v_{i,j})\cdot\left(v_{i,j}-\max(w_{-i})|_j\right). \qquad (\textsc{Core})$$

We call the first term above Tail and the second term as Core. We shall show that Tail is at most $\mathsf{SRev}(n')$ while Core can be bounded as a function of $\mathsf{BVCG}(n')$ and $\mathsf{SRev}(n')$. First, we state our final decomposition for $\mathsf{IU}(n')$:

$$\mathsf{IU}(n')\leq\textsc{Single}+\textsc{Under}+\textsc{Over}+\textsc{Tail}+\textsc{Core}. \qquad (12)$$

To finish the proof of Lemma C.1, we now show that each of the first four terms above is bounded by $\mathsf{SRev}(n')$.

Bounding Single. If the term Single did not have the factor $\mathcal{P}_j(v_i)$ inside, it will just be maximum (over all auctions) value of (Myerson's) ironed virtual welfare, and we could use Proposition 3.6 to finish the proof. As adding the factor $\mathcal{P}_j(v_i)$ can only decrease the value of Single, we derive:

$$\textsc{Single}\leq\sum_{j=1}^{m}\mathop{\mathbb{E}}_{v\sim\mathcal{D}^{n'}}\left[\max_{i\in[n']}\left\{\tilde{\varphi}_j(v_{i,j})^+\right\}\right]\leq\mathsf{SRev}(n'). \qquad (\text{Proposition 3.6})$$

Bounding Under. Roughly speaking, the term $v_{i,j}$ contributes to Under only if it is not the highest amongst n' bids. As the fact that $v_{i,j}$ is not the highest amongst n' bids implies that it is also not the highest amongst $n' + 1$ bids, we get:

$$\begin{aligned}
\textsc{Under} &\leq \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}} \left[\max_{i \in [n']} \left\{ \sum_{w_{-i}} f^*(w_{-i}) \cdot v_{i,j} \cdot \mathbb{1}\left(v_{i,j} < \max(w_{-i})|_j\right) \right\} \right] && \text{(Equation 10)} \\
&\leq \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}} \left[\max_{i \in [n']} \left\{ \sum_{w} f^*(w) \cdot v_{i,j} \cdot \mathbb{1}\left(v_{i,j} \leq \max(w)|_j\right) \right\} \right].
\end{aligned}$$

Now, consider each term w inside the max as the bids of n' bidders. In this interpretation (as formalized in Lemma B.1), the term inside the max is at most the revenue generated by a VCG auction where the reserve for item j is $v_{i,j}$. Using Lemma B.1, we get:

$$\textsc{Under} \leq \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}} \left[\max_{i \in [n']} \left\{ \mathsf{SRev}_j(n') \right\} \right] = \sum_{j=1}^{m} \mathsf{SRev}_j(n') = \mathsf{SRev}(n').$$

Bounding Over. We first manipulate Over so that w_{-i} can be moved outside the max. Using Equation 10, we have:

$$\begin{aligned}
\textsc{Over} &\leq \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}} \left[\max_{i \in [n']} \left\{ \sum_{w} f^*(w) \cdot \max(w_{-i})|_j \cdot \mathbb{1}\left(v_{i,j} \geq \max(w_{-i})|_j\right) \right\} \right] \\
&\leq \sum_{w} f^*(w) \cdot \sum_{j=1}^{m} \mathop{\mathbb{E}}_{v \sim \mathcal{D}^{n'}} \left[\max_{i \in [n']} \left\{ \max(w_{-i})|_j \cdot \mathbb{1}\left(v_{i,j} \geq \max(w_{-i})|_j\right) \right\} \right].
\end{aligned}$$

We now analyze the term corresponding to each w separately. For each w, consider a sequential posted price auction that sells each item separately. When selling item j, the auction visits the bidders in non-increasing order of $\max(w_{-i})|_j$ and offers them the item at price $\max(w_{-i})|_j$. The revenue generated by this auction is at least term corresponding to w above. Lemma B.2 formalizes this and gives:

$$\textsc{Over} \leq \sum_{w} f^*(w) \cdot \mathsf{SRev}(n') = \mathsf{SRev}(n').$$

Bounding Tail. At a high level, the term Tail is large only when bidder i gets high utility from item j but there exists an item $j' \neq j$ that gives even higher utility. This should be unlikely. More formally, by Equation 10 and a union bound, we have:

$$\Pr_{v_{i,-j}} \left(w_{-i} \in E_j^{\textsc{Srp}}(v_i) \right) \leq \sum_{j' \neq j} \Pr_{v_{i,j'}} \left(v_{i,j} - \max(w_{-i})|_j \leq v_{i,j'} - \max(w_{-i})|_{j'} \right).$$

As Tail only sums over $v_{i,j} > \mathcal{T}_{i,j}(w_{-i}) \geq \max(w_{-i})|_j$, the definition of $r^*_{\mathsf{Ron},j'}(x)$ allows us to further bound this by:

$$\begin{aligned}
\Pr_{v_{i,-j}} \left(w_{-i} \in E_j^{\textsc{Srp}}(v_i) \right) &\leq \sum_{j' \neq j} \frac{r^*_{\mathsf{Ron},j'}(\max(w_{-i})|_{j'})}{v_{i,j} - \max(w_{-i})|_j} \\
&\leq \frac{r^{(i)}_{\mathsf{Ron}}(w_{-i})}{v_{i,j} - \max(w_{-i})|_j}.
\end{aligned} \tag{13}$$

Plugging Equation 13 into the term $\textsc{Tail}$, we have:

$$\begin{aligned}\textsc{Tail} &\leq \sum_{j=1}^{m}\sum_{i=1}^{n'}\sum_{w_{-i}}\sum_{v_{i,j}>\mathcal{T}_{i,j}(w_{-i})} f^*(w_{-i})f_j(v_{i,j})\cdot r_{\mathsf{Ron}}^{(i)}(w_{-i}) \\ &\leq \sum_{j=1}^{m}\sum_{i=1}^{n'}\sum_{w_{-i}} f^*(w_{-i})\cdot r_{\mathsf{Ron}}^{(i)}(w_{-i})\cdot \Pr_{v_{i,j}}\big(v_{i,j}>\mathcal{T}_{i,j}(w_{-i})\big).\end{aligned} \tag{14}$$

Now, we claim that $r_{\mathsf{Ron}}^{(i)}(w_{-i})\cdot\Pr_{v_{i,j}}\big(v_{i,j}>\mathcal{T}_{i,j}(w_{-i})\big)\leq r^*_{\mathsf{Ron},j}(\max(w_{-i})|_j)$. In the case $\Pr_{v_{i,j}}\big(v_{i,j}>\mathcal{T}_{i,j}(w_{-i})\big)=0$, this holds trivially. Otherwise, there exists $x\in\mathcal{V}_j$ be the smallest such that $x>\mathcal{T}_{i,j}(w_{-i})=r_{\mathsf{Ron}}^{(i)}(w_{-i})+\max(w_{-i})|_j$ and we get:

$$r_{\mathsf{Ron}}^{(i)}(w_{-i})\cdot\Pr_{v_{i,j}}\big(v_{i,j}>\mathcal{T}_{i,j}(w_{-i})\big)\leq x\cdot\Pr_{v_{i,j}}\big(v_{i,j}\geq x\big)\leq r^*_{\mathsf{Ron},j}(\max(w_{-i})|_j).$$

We continue Equation 14 as:

$$\textsc{Tail}\leq\sum_{j=1}^{m}\sum_{i=1}^{n'}\sum_{w_{-i}} f^*(w_{-i})\cdot r^*_{\mathsf{Ron},j}(\max(w_{-i})|_j).$$

The last expression is closely related to the revenue of a Ronen's auction [32] that sells the items separately, and is captured in Lemma B.3. Using Lemma B.3, we conclude:

$$\textsc{Tail}\leq\mathsf{SRev}(n').$$

This conclude s the proof of Lemma C.1. □

C.2 Step 2 – Bounding $\textsc{Core}$

The next (and final) step in the proof of Lemma 4.4 and Lemma 4.5 is to upper bound the term $\textsc{Core}$ that was left unanalyzed in Lemma C.1. To this end, we first recall some definitions made in Subsection C.1. Recall that, for all $j\in[m]$, $r^*_{\mathsf{Ron},j}(x)=\max_{y>x} y\cdot\Pr_{y'\sim\mathcal{D}_j}(y'\geq y)$ roughly (but not exactly) corresponds to the payment of the highest bidder in a Ronen's auction when the second highest bid is x. We also defined, for all i, w_{-i} $r_{\mathsf{Ron}}^{(i)}(w_{-i})=\sum_{j=1}^{m} r^*_{\mathsf{Ron},j}(\max(w_{-i})|_j)$ and for all $j\in[m]$, $\mathcal{T}_{i,j}(w_{-i})=r_{\mathsf{Ron}}^{(i)}(w_{-i})+\max(w_{-i})|_j$. The term $\textsc{Core}$ equals:

$$\textsc{Core}=\sum_{j=1}^{m}\sum_{i=1}^{n'}\sum_{w_{-i}}\sum_{\max(w_{-i})|_j\leq v_{i,j}\leq\mathcal{T}_{i,j}(w_{-i})} f^*(w_{-i})f_j(v_{i,j})\cdot\big(v_{i,j}-\max(w_{-i})|_j\big).$$

Observe that the term $\big(v_{i,j}-\max(w_{-i})|_j\big)$ in the above equation is closely related to the utility that bidder with valuation v_i gets from item j in a VCG auction when the bids of the other bidders are w_{-i}. To capture this, we define the notation:

$$\begin{aligned}\mathsf{Util}_{i,j,w_{-i}}(v_{i,j}) &= \max\big(v_{i,j}-\max(w_{-i})|_j,0\big)\\ \widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j}) &= \mathsf{Util}_{i,j,w_{-i}}(v_{i,j})\cdot\mathbb{1}\Big(\mathsf{Util}_{i,j,w_{-i}}(v_{i,j})\leq r_{\mathsf{Ron}}^{(i)}(w_{-i})\Big).\end{aligned}$$

These will primarily be used in the following form:

$$\mathsf{U}_{i,w_{-i}}(v_i)=\sum_{j=1}^{m}\mathsf{Util}_{i,j,w_{-i}}(v_{i,j})\qquad\text{and}\qquad\hat{\mathsf{U}}_{i,w_{-i}}(v_i)=\sum_{j=1}^{m}\widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j}).\tag{15}$$

Using this notation, Core satsifies:

$$\textsc{Core} = \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot \mathop{\mathbb{E}}_{v_i}\left[\hat{\mathsf{U}}_{i,w_{-i}}(v_i)\right]. \tag{16}$$

Observe that, written this way, Core is closely related to the random variable $\hat{\mathsf{U}}_{i,w_{-i}}(v_i)$. It is in this form that we upper bound Core in Subsubsection C.2.1 and Subsubsection C.2.2. But first, let us show using Lemma 3.2 that the variance of $\hat{\mathsf{U}}_{i,w_{-i}}(v_i)$ is small.

Lemma C.2. *It holds for all $i \in [n']$ and all w_{-i} that:*

$$\mathsf{Var}_{v_i \sim \mathcal{D}}\left(\hat{\mathsf{U}}_{i,w_{-i}}(v_i)\right) \leq 2 \cdot \left(r^{(i)}_{\mathsf{Ron}}(w_{-i})\right)^2.$$

Proof. Recall that $\mathcal{D} = \bigtimes_{j=1}^{m} \mathcal{D}_j$ is such that all the items are independent. Using the fact that variance is linear when over independent random variables (Fact 3.1, item 3) and Equation 15, we get:

$$\mathsf{Var}_{v_i \sim \mathcal{D}}\left(\hat{\mathsf{U}}_{i,w_{-i}}(v_i)\right) = \sum_{j=1}^{m} \mathsf{Var}_{v_{i,j} \sim \mathcal{D}_j}\left(\widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j})\right). \tag{17}$$

Our goal now is to bound each term using Lemma 3.2. To this end, note that $\widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j})$ is always at most $r^{(i)}_{\mathsf{Ron}}(w_{-i})$ and thus, we can conclude that $\max_{v_{i,j}} \widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j}) \leq r^{(i)}_{\mathsf{Ron}}(w_{-i})$. Moreover, we have for all $v_{i,j}$ that

$$\begin{aligned}
&\widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j}) \cdot \Pr_{v'_{i,j} \sim \mathcal{D}_j}\left(\widehat{\mathsf{Util}}_{i,j,w_{-i}}(v'_{i,j}) \geq \widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j})\right) \\
&\qquad \leq \widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j}) \cdot \Pr_{v'_{i,j} \sim \mathcal{D}_j}\left(v'_{i,j} \geq \max(w_{-i})|_j + \widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j})\right).
\end{aligned}$$

Now, if $\widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j}) = 0$, then, the right hand side is 0 and consequently, is at most $r^*_{\mathsf{Ron},j}(\max(w_{-i})|_j)$. We show that the latter holds even when $\widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j}) > 0$. Indeed, we have:

$$\begin{aligned}
&\widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j}) \cdot \Pr_{v'_{i,j} \sim \mathcal{D}_j}\left(v'_{i,j} \geq \max(w_{-i})|_j + \widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j})\right) \\
&\leq \left(\widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j}) + \max(w_{-i})|_j\right) \cdot \Pr_{v'_{i,j} \sim \mathcal{D}_j}\left(v'_{i,j} \geq \max(w_{-i})|_j + \widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j})\right) \\
&\leq r^*_{\mathsf{Ron},j}(\max(w_{-i})|_j). && \text{(Definition of } r^*_{\mathsf{Ron},j}(\cdot))
\end{aligned}$$

Thus, we can conclude that:

$$\max_{v_{i,j}} \widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j}) \cdot \Pr_{v'_{i,j} \sim \mathcal{D}_j}\left(\widehat{\mathsf{Util}}_{i,j,w_{-i}}(v'_{i,j}) \geq \widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j})\right) \leq r^*_{\mathsf{Ron},j}(\max(w_{-i})|_j).$$

Plugging this and $\max_{v_{i,j}} \widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j}) \leq r^{(i)}_{\mathsf{Ron}}(w_{-i})$ into Lemma 3.2, we get:

$$\mathsf{Var}_{v_{i,j} \sim \mathcal{D}_j}\left(\widehat{\mathsf{Util}}_{i,j,w_{-i}}(v_{i,j})\right) \leq 2 \cdot r^*_{\mathsf{Ron},j}(\max(w_{-i})|_j) \cdot r^{(i)}_{\mathsf{Ron}}(w_{-i}).$$

Plugging into Equation 17, we get:

$$\mathsf{Var}_{v_i \sim \mathcal{D}}\left(\hat{\mathsf{U}}_{i,w_{-i}}(v_i)\right) \leq 2 \cdot r^{(i)}_{\mathsf{Ron}}(w_{-i}) \cdot \sum_{j=1}^{m} r^*_{\mathsf{Ron},j}(\max(w_{-i})|_j) \leq 2 \cdot \left(r^{(i)}_{\mathsf{Ron}}(w_{-i})\right)^2.$$

□

C.2.1 Bounding Core *for Lemma 4.4.* In this section, we finish our proof of Lemma 4.4 by upper bounding the right hand side of Equation 16 by the revenue of a BVCG auction (and $\mathrm{SRev}(n')$). Specifically, we shall consider a BVCG auction with n' bidders, where the fee charged for player i, when the types of the other bidders are w_{-i} is:

$$\mathsf{Fee}_{i,w_{-i}} = \max\left(\mathop{\mathbb{E}}_{v_i}\left[\hat{\mathsf{U}}_{i,w_{-i}}(v_i)\right] - 2 \cdot r_{\mathsf{Ron}}^{(i)}(w_{-i}), 0\right).$$

The following lemma shows that most bidders will agree to pay this extra fee, and thus, expectation of the total fee is at most $2 \cdot \mathrm{BVCG}(n')$.

Lemma C.3. *It holds that:*

$$\sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot \mathsf{Fee}_{i,w_{-i}} \leq 2 \cdot \mathrm{BVCG}(n').$$

Proof. Consider the BVCG auction defined by $\mathsf{Fee}_{i,w_{-i}}$. That is, consider the auction where the auctioneer first asks all bidders $i \in [n']$ for their bids w_i and runs a VCG auction based on these bids. If bidder $i \in [n']$ is not allocated any items in the VCG auction, he departs without paying anything. Otherwise, he gets all the items allocated to him in the VCG auction if and only if he agrees to pay an amount equal to $\mathsf{Fee}_{i,w_{-i}}$ in addition to the prices charged by the VCG auction.

This auction is truthful as we ensure that $\mathsf{Fee}_{i,w_{-i}} \geq 0$. Moreover, if bidder i does not pay at least $\mathsf{Fee}_{i,w_{-i}}$, we must have that his utility from the VCG auction is (strictly) smaller that $\mathsf{Fee}_{i,w_{-i}}$. Thus, we get the following lower bound on $\mathrm{BVCG}(n')$.

$$\begin{aligned}
\mathrm{BVCG}(n') &\geq \sum_{i=1}^{n'} \sum_{w \in \mathcal{V}^{n'}} f^*(w) \cdot \mathsf{Fee}_{i,w_{-i}} \cdot \mathbb{1}\left(\mathsf{Fee}_{i,w_{-i}} \leq \mathsf{U}_{i,w_{-i}}(w_i)\right) \\
&\geq \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot \mathsf{Fee}_{i,w_{-i}} \cdot \Pr_{w_i}\left(\mathsf{Fee}_{i,w_{-i}} \leq \mathsf{U}_{i,w_{-i}}(w_i)\right) \\
&\geq \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot \mathsf{Fee}_{i,w_{-i}} \cdot \Pr_{w_i}\left(\mathsf{Fee}_{i,w_{-i}} \leq \hat{\mathsf{U}}_{i,w_{-i}}(w_i)\right),
\end{aligned}$$

where the last step is because U upper bounds $\hat{\mathsf{U}}$. The next step is to lower bound the probability on the right hand side. We do this using Chebyshev's inequality (Fact 3.1, item 2) and use the variance bound in Lemma C.2. We have:

$$\Pr_{w_i}\left(\hat{\mathsf{U}}_{i,w_{-i}}(w_i) < \mathsf{Fee}_{i,w_{-i}}\right) \leq \frac{1}{2}.$$

Plugging in, we have:

$$\mathrm{BVCG}(n') \geq \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot \mathsf{Fee}_{i,w_{-i}} \cdot \frac{1}{2}.$$

and the lemma follows. □

We now present our proof of Lemma 4.4.

Proof of Lemma 4.4. From Equation 16 and the definition of $\mathsf{Fee}_{i,w_{-i}}$, we have:

$$\mathrm{Core} \leq \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot \left(\mathsf{Fee}_{i,w_{-i}} + 2 \cdot r_{\mathsf{Ron}}^{(i)}(w_{-i})\right)$$

$$\leq \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot \mathsf{Fee}_{i,w_{-i}} + 2 \cdot \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot r^{(i)}_{\mathsf{Ron}}(w_{-i})$$
$$\leq \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot \mathsf{Fee}_{i,w_{-i}} + 2 \cdot \sum_{j=1}^{m} \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot r^{*}_{\mathsf{Ron},j}(\max(w_{-i})|_j).$$

These two terms can be bounded by Lemma C.3 and Lemma B.3 respectively yielding $\textsc{Core} \leq 2 \cdot \mathsf{BVCG}(n') + 2 \cdot \mathsf{SRev}(n')$. Plugging into Lemma C.1, we get:

$$\mathsf{IU}(n'', n') \leq 4 \cdot \mathsf{SRev}(n') + \textsc{Core} \leq 2 \cdot \mathsf{BVCG}(n') + 6 \cdot \mathsf{SRev}(n').$$

□

C.2.2 Bounding Core *for Lemma 4.5.* Now, we finish our proof of Lemma 4.5 by upper bounding the right hand side of Equation 16 by the revenue of a *prior-independent* BVCG auction. The auction defined in Subsubsection C.2.1 was not prior independent as to compute the fees charged to the bidders required knowledge of the distribution $\mathcal{D}$. Our main idea follows [23], we construct an auction with $n'+1$ bidders, and treat the last bidder as 'special'. This special bidder does not receive any items or pay anything, but his bids allow us to get a good enough estimate of the distribution $\mathcal{D}$.

We shall reserve s to denote the bid of the special bidder and $w \in \mathcal{V}^n$ will denote the bids of the other bidders. For $i \in [n']$, the notation w_i will (as before) denote the bid of player i, while w_{-i} will denote the bids of all the other players *excluding the special player*. This time the fee for player $i \in [n']$ is defined as (recall Equation 15):

$$\mathsf{Fee}_{i,w_{-i},s} = \mathsf{U}_{i,w_{-i}}(s). \tag{18}$$

Importantly, this is determined by the bids of the bidders and is independent of $\mathcal{D}$. We also define, for all i and w_{-i}, the set $\mathcal{N}_{i,w_{-i}}$ as follows:

$$\mathcal{N}_{i,w_{-i}} = \left\{ v \in \mathcal{V} \;\middle|\; \hat{\mathsf{U}}_{i,w_{-i}}(v) \geq \frac{1}{2} \cdot \mathop{\mathbb{E}}_{v' \sim \mathcal{D}}\left[\hat{\mathsf{U}}_{i,w_{-i}}(v')\right] \right\}. \tag{19}$$

We now show a prior-independent analogue of Lemma C.3.

Lemma C.4. *It holds that:*

$$\sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot \Pr_{v' \sim \mathcal{D}}\left(v' \in \mathcal{N}_{i,w_{-i}}\right)^2 \cdot \mathop{\mathbb{E}}_{v' \sim \mathcal{D}}\left[\hat{\mathsf{U}}_{i,w_{-i}}(v')\right] \leq 4 \cdot \mathsf{PI\text{-}BVCG}(n'+1).$$

Proof. We follow the proof approach in Lemma C.3 but this time use the fees defined in Equation 18 as they are prior-independent. More specifically, we consider the auction that first receives the bids w and s of the non-special and special players respectively and runs a VCG auction based on w. Thus, the special bidder never receives or pays anything. If bidder $i \in [n']$ is not allocated any items in the VCG auction, he departs without paying anything. Otherwise, he gets all the items allocated to him in the VCG auction if and only if he agrees to pay an amount equal to $\mathsf{Fee}_{i,w_{-i},s}$ in addition to the prices charged by the VCG auction.

This auction is truthful as we ensure that $\mathsf{Fee}_{i,w_{-i},s} \geq 0$. Moreover, if bidder i does not pay at least the amount $\mathsf{Fee}_{i,w_{-i},s}$, we must have that his utility from the VCG auction is (strictly) smaller than $\mathsf{Fee}_{i,w_{-i},s}$. From Equation 18, we get the following lower bound on the revenue of this auction:

$$\mathsf{PI\text{-}BVCG}(n'+1) \geq \sum_{i=1}^{n'} \sum_{s \in \mathcal{V}} \sum_{w \in \mathcal{V}^{n'}} f(s) f^*(w) \cdot \mathsf{U}_{i,w_{-i}}(s) \cdot \mathbb{1}\left(\mathsf{U}_{i,w_{-i}}(s) \leq \mathsf{U}_{i,w_{-i}}(w_i)\right)$$

$$\geq \sum_{i=1}^{n'} \sum_{w_{-i}} \sum_{s, w_i \in \mathcal{N}_{i,w_{-i}}} f(s) f(w_i) f^*(w_{-i}) \cdot \mathsf{U}_{i,w_{-i}}(s) \cdot \mathbb{1}\big(\mathsf{U}_{i,w_{-i}}(s) \leq \mathsf{U}_{i,w_{-i}}(w_i)\big).$$

As U upper bounds $\hat{\mathsf{U}}$ and we only consider $s \in \mathcal{N}_{i,w_{-i}}$, we have $\mathsf{U}_{i,w_{-i}}(s) \geq \hat{\mathsf{U}}_{i,w_{-i}}(s) \geq \frac{1}{2} \cdot \mathbb{E}_{v' \sim \mathcal{D}}\big[\hat{\mathsf{U}}_{i,w_{-i}}(v')\big]$. Plugging in, we have:

$$\begin{aligned}
&\mathsf{PI\text{-}BVCG}(n'+1) \\
&\quad \geq \frac{1}{2} \cdot \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot \mathop{\mathbb{E}}_{v' \sim \mathcal{D}}\big[\hat{\mathsf{U}}_{i,w_{-i}}(v')\big] \cdot \sum_{s, w_i \in \mathcal{N}_{i,w_{-i}}} f(s) f(w_i) \cdot \mathbb{1}\big(\mathsf{U}_{i,w_{-i}}(s) \leq \mathsf{U}_{i,w_{-i}}(w_i)\big).
\end{aligned}$$

By symmetry, we conclude that:

$$\mathsf{PI\text{-}BVCG}(n'+1) \geq \frac{1}{4} \cdot \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot \Pr_{v' \sim \mathcal{D}}\big(v' \in \mathcal{N}_{i,w_{-i}}\big)^2 \cdot \mathop{\mathbb{E}}_{v' \sim \mathcal{D}}\big[\hat{\mathsf{U}}_{i,w_{-i}}(v')\big].$$

□

We now present our proof of Lemma 4.5.

Proof of Lemma 4.5. Call a pair i, w_{-i} "high" if

$$\mathop{\mathbb{E}}_{v' \sim \mathcal{D}}\big[\hat{\mathsf{U}}_{i,w_{-i}}(v')\big] \geq 6 \cdot r^{(i)}_{\mathsf{Ron}}(w_{-i}), \tag{20}$$

and call it "low" otherwise. Using Chebyshev's inequality (Fact 3.1, item 2) and the variance bound in Lemma C.2, we have for all high (i, w_{-i}) that:

$$1 - \Pr_{v' \sim \mathcal{D}}\big(v' \in \mathcal{N}_{i,w_{-i}}\big) \leq \frac{4 \cdot \mathsf{Var}_{v_i \sim \mathcal{D}}\Big(\hat{\mathsf{U}}_{i,w_{-i}}(v_i)\Big)}{\Big(\mathbb{E}_{v' \sim \mathcal{D}}\big[\hat{\mathsf{U}}_{i,w_{-i}}(v')\big]^2\Big)} \leq \frac{2}{9}.$$

Thus, if the pair (i, w_{-i}) is high, we get that $\Pr_{v' \sim \mathcal{D}}\big(v' \in \mathcal{N}_{i,w_{-i}}\big)$ is at least $\frac{7}{9}$. We now bound Core from Equation 16 and finish the proof. We have:

$$\begin{aligned}
\textsc{Core} &\leq \sum_{\text{high } (i,w_{-i})} f^*(w_{-i}) \cdot \mathop{\mathbb{E}}_{v' \sim \mathcal{D}}\big[\hat{\mathsf{U}}_{i,w_{-i}}(v')\big] + \sum_{\text{low } (i,w_{-i})} f^*(w_{-i}) \cdot \mathop{\mathbb{E}}_{v' \sim \mathcal{D}}\big[\hat{\mathsf{U}}_{i,w_{-i}}(v')\big] \\
&\leq \frac{81}{49} \cdot \sum_{\text{high } (i,w_{-i})} f^*(w_{-i}) \cdot \Pr_{v' \sim \mathcal{D}}\big(v' \in \mathcal{N}_{i,w_{-i}}\big)^2 \cdot \mathop{\mathbb{E}}_{v' \sim \mathcal{D}}\big[\hat{\mathsf{U}}_{i,w_{-i}}(v')\big] \\
&\qquad + 6 \cdot \sum_{\text{low } (i,w_{-i})} f^*(w_{-i}) \cdot r^{(i)}_{\mathsf{Ron}}(w_{-i}),
\end{aligned}$$

where, for high (i, w_{-i}), we plug in $\Pr_{v' \sim \mathcal{D}}\big(v' \in \mathcal{N}_{i,w_{-i}}\big) \geq \frac{7}{9}$, while for low (i, w_{-i}), we use Equation 20. This gives:

$$\textsc{Core} \leq \frac{81}{49} \cdot \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot \Pr_{v' \sim \mathcal{D}}\big(v' \in \mathcal{N}_{i,w_{-i}}\big)^2 \cdot \mathop{\mathbb{E}}_{v' \sim \mathcal{D}}\big[\hat{\mathsf{U}}_{i,w_{-i}}(v')\big] + 6 \cdot \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot r^{(i)}_{\mathsf{Ron}}(w_{-i}).$$

Using Lemma C.4 and using the definition of $r^{(i)}_{\mathsf{Ron}}(w_{-i})$, we get:

$$\textsc{Core} \leq 7 \cdot \mathsf{PI\text{-}BVCG}(n'+1) + 6 \cdot \sum_{j=1}^{m} \sum_{i=1}^{n'} \sum_{w_{-i}} f^*(w_{-i}) \cdot r^*_{\mathsf{Ron},j}(\max(w_{-i})|_j).$$

Using Lemma B.3 on the second term, we have $\textsc{Core} \leq 7 \cdot \text{PI-BVCG}(n'+1) + 6 \cdot \text{SRev}(n')$. Plugging into Lemma C.1, we get $\text{IU}(n'', n') \leq 4 \cdot \text{SRev}(n') + \textsc{Core} \leq 7 \cdot \text{PI-BVCG}(n'+1) + 10 \cdot \text{SRev}(n')$. As we assumed that all the items are regular, we have from Proposition 3.8 that $\text{SRev}(n') \leq \text{VCG}(n'+1) \leq \text{PI-BVCG}(n'+1)$. This yields:

$$\text{IU}(n'', n') \leq 17 \cdot \text{PI-BVCG}(n'+1).$$

□

D PROOF OF COROLLARY 27 OF [8]

This section recalls the proof of Corollary 27 from [8] as Lemma D.1. Our presentation is different from [8] as we do not need their ideas in full generality.

Lemma D.1. *Let $(n, m, \mathcal{D})$ be an auction setting as in Subsection 3.2. Let $n' > 0$ and suppose that for all $i \in [n]$, valuations $w_{-i} \in \mathcal{V}^{n'-1}$ are given. For all $(\overline{\pi}, \overline{p})$ that correspond to a truthful auction $\mathcal{A}$, we have that:*

$$\text{Rev}(\mathcal{A}, n) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{E}_{v_i}\left[\overline{\pi}_{i,j}(v_i) \cdot \left(v_{i,j} \cdot \mathbb{1}\left(v_i \notin \mathcal{R}_j^{(n')}(w_{-i})\right) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathbb{1}\left(v_i \in \mathcal{R}_j^{(n')}(w_{-i})\right)\right)\right].$$

Proof. We start with some notation. We use $v_\varnothing$ to denote a dummy valuation for the bidders and adopt the convention $\overline{\pi}_{i,j}(v_\varnothing) = \overline{p}_i(v_\varnothing) = 0$ for all $i \in [n]$, $j \in [m]$. Suppose that non-negative numbers $\Lambda = \left\{\lambda_i(v_i, v_i')\right\}_{i \in [n], v_i \in \mathcal{V}, v_i' \in \mathcal{V} \cup \{v_\varnothing\}}$ are given that satisfy for all $i \in [n]$ and $v_i \in \mathcal{V}$ that:

$$f(v_i) - \sum_{v_i' \in \mathcal{V} \cup \{v_\varnothing\}} \lambda_i(v_i, v_i') + \sum_{v_i' \in \mathcal{V}} \lambda_i(v_i', v_i) = 0. \tag{21}$$

As $(\overline{\pi}, \overline{p})$ correspond to a truthful auction $\mathcal{A}$, we have from Equation 4 that $\text{Rev}(\mathcal{A}, n) = \sum_{i=1}^{n} \mathbb{E}_{v_i \sim \mathcal{D}}\left[\overline{p}_i(v_i)\right]$. Continuing using the non-negativity of $\lambda_i(v_i, v_i')$ and Equation 3, we have:

$$\begin{aligned}\text{Rev}(\mathcal{A}, n) \leq &\sum_{i=1}^{n} \sum_{v_i \in \mathcal{V}} f(v_i) \cdot \overline{p}_i(v_i) \\ &+ \sum_{i=1}^{n} \sum_{v_i \in \mathcal{V}} \sum_{v_i' \in \mathcal{V} \cup \{v_\varnothing\}} \lambda_i(v_i, v_i') \cdot \left(\sum_{j=1}^{m} \left(\overline{\pi}_{i,j}(v_i) - \overline{\pi}_{i,j}(v_i')\right) \cdot v_{i,j} - \left(\overline{p}_i(v_i) - \overline{p}_i(v_i')\right)\right).\end{aligned}$$

This can be rearranged to:

$$\begin{aligned}\text{Rev}(\mathcal{A}, n) \leq &\sum_{i=1}^{n} \sum_{v_i \in \mathcal{V}} \left(f(v_i) - \sum_{v_i' \in \mathcal{V} \cup \{v_\varnothing\}} \lambda_i(v_i, v_i') + \sum_{v_i' \in \mathcal{V}} \lambda_i(v_i', v_i)\right) \cdot \overline{p}_i(v_i) \\ &+ \sum_{i=1}^{n} \sum_{v_i \in \mathcal{V}} \sum_{j=1}^{m} \left(\sum_{v_i' \in \mathcal{V} \cup \{v_\varnothing\}} \lambda_i(v_i, v_i') \cdot v_{i,j} - \sum_{v_i' \in \mathcal{V}} \lambda_i(v_i', v_i) \cdot v_{i,j}'\right) \cdot \overline{\pi}_{i,j}(v_i).\end{aligned}$$

Plugging in Equation 21, we get:

$$\text{Rev}(\mathcal{A}, n) \leq \sum_{i=1}^{n} \sum_{v_i \in \mathcal{V}} \sum_{j=1}^{m} \left(f(v_i) \cdot v_{i,j} - \sum_{v_i' \in \mathcal{V}} \lambda_i(v_i', v_i) \cdot \left(v_{i,j}' - v_{i,j}\right)\right) \cdot \overline{\pi}_{i,j}(v_i).$$

Rearranging again, and denoting by $\Phi^{\Lambda}_{i,j}(v_i) = v_{i,j} - \frac{1}{f(v_i)} \cdot \sum_{v'_i \in \mathcal{V}} \lambda_i(v'_i, v_i) \cdot \left(v'_{i,j} - v_{i,j}\right)$, we get:

$$\mathrm{Rev}(\mathcal{A}, n) \leq \sum_{i=1}^{n} \sum_{v_i \in \mathcal{V}} \sum_{j=1}^{m} f(v_i) \cdot \overline{\pi}_{i,j}(v_i) \cdot \Phi^{\Lambda}_{i,j}(v_i). \tag{22}$$

Observe that Equation 22 holds for any Λ that is non-negative and satisfies Equation 21. In order to show Lemma D.1, we construct a suitable Λ and apply Equation 22. This is done by defining Λ' and Λ^* as below and setting $\Lambda = \Lambda' + \Lambda^*$.

Defining Λ'. We start with some notation. For $j \in [m]$, let $f_{-j}(\cdot)$ denote the probability mass function of the distribution $\bigtimes_{j' \neq j} \mathcal{D}_{j'}$. Also, for $j \in [m]$ and $v_i \in \mathcal{V}$, let $\mathrm{dec}_j(v_i)$ be defined to be $v_\varnothing$ if $v_{i,j} = \min \mathcal{V}_j$. Otherwise define $\mathrm{dec}_{j,k}(v_i) = v_{i,k}$ for all $k \neq j$ and $\mathrm{dec}_{j,j}(v_i) = \max_{x \in \mathcal{V}_j, x < v_{i,j}} x$. Recall the definition of the regions $\left\{\mathcal{R}^{(n')}_j(w_{-i})\right\}_{j \in \{0\} \cup [m]}$ from Equation 5 and for all $i \in [n], v_i \in \mathcal{V}, v'_i \in \mathcal{V} \cup \{v_\varnothing\}$, define the numbers:

$$\lambda'_i(v_i, v'_i) = \begin{cases} f(v_i), & \text{if } v_i \in \mathcal{R}^{(n')}_0(w_{-i}) \text{ and } v'_i = v_\varnothing \\ \Pr_{y \sim \mathcal{D}_j}\left(y \geq v_{i,j}\right) \cdot f_{-j}(v_{i,-j}), & \text{if } \exists j \in [m] : v_i, v'_i \in \mathcal{R}^{(n')}_j(w_{-i}) \wedge v'_i = \mathrm{dec}_j(v_i) \\ \Pr_{y \sim \mathcal{D}_j}\left(y \geq v_{i,j}\right) \cdot f_{-j}(v_{i,-j}), & \text{if } \exists j \in [m] : v_i \in \mathcal{R}^{(n')}_j(w_{-i}) \wedge v'_i = v_\varnothing \\ & \quad \text{and } \mathrm{dec}_j(v_i) \notin \mathcal{R}^{(n')}_j(w_{-i}) \\ 0, & \text{otherwise} \end{cases}.$$

Defining Λ^*. For all $i \in [n]$, we define $\lambda^*_i(\cdot)$ using the procedure described in Algorithm 2. In Line 6 of Algorithm 2, when we say we invoke Algorithm 1 restricted to values at least x, we mean that Line 4 of Algorithm 1 would only include values that are at least x in the arg max and Line 6 of Algorithm 1 will abort as soon as $y^* = x$ (instead of when $y^* = \min(\mathcal{V}_j)$). Algorithm 1 guarantees that the output $\varphi^{v_{i,-j}}_j(\cdot)$ produced in this manner is a lower bound of $\tilde{\varphi}_j(\cdot)$, and therefore, also a lower bound of $\tilde{\varphi}_j(\cdot)^+$, for all values at least x. Moreover it satisfies, for all $y \geq x$ and with equality when $y = x$, that:

$$\sum_{y' \geq y \in \mathcal{V}_j} f_j(y') \cdot \varphi^{v_{i,-j}}_j(y') \geq \sum_{y' \geq y \in \mathcal{V}_j} f_j(y') \cdot \varphi_j(y'). \tag{23}$$

We now finish the proof of Lemma D.1. Having defined Λ' and $\Lambda^\star$, we first observe that they are both non-negative (Λ^* is non-negative due to Equation 23). Moreover, observe that setting $\Lambda = \Lambda' + \Lambda^*$ satisfies Equation 21. Plugging into Equation 22, we get:

$$\mathrm{Rev}(\mathcal{A}, n) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{E}_{v_i}\left[\overline{\pi}_{i,j}(v_i) \cdot \Phi^{\Lambda}_{i,j}(v_i)\right].$$

Where, using Equation 23 and Definition 3.3, the value $\Phi^{\Lambda}_{i,j}(v_i)$ can be simplified to:

$$\Phi^{\Lambda}_{i,j}(v_i) = v_{i,j} \cdot \mathbb{1}\left(v_i \notin \mathcal{R}^{(n')}_j(w_{-i})\right) + \varphi^{v_{i,-j}}_j(v_{i,j}) \cdot \mathbb{1}\left(v_i \in \mathcal{R}^{(n')}_j(w_{-i})\right).$$

Plugging in and using the fact that $\varphi^{v_{i,-j}}_j(\cdot) \leq \tilde{\varphi}_j(\cdot)^+$, we get:

$$\mathrm{Rev}(\mathcal{A}, n) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{E}_{v_i}\left[\overline{\pi}_{i,j}(v_i) \cdot \left(v_{i,j} \cdot \mathbb{1}\left(v_i \notin \mathcal{R}^{(n')}_j(w_{-i})\right) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathbb{1}\left(v_i \in \mathcal{R}^{(n')}_j(w_{-i})\right)\right)\right].$$

□

Algorithm 2 Computing $\lambda_i^*(\cdot)$ for $i \in [n]$.

Set $\lambda_i^*(v_i, v_i') = 0$ for all $v_i \in \mathcal{V}$ and $v_i' \in \mathcal{V} \cup \{v_\varnothing\}$.
for $j \in [m]$ **do**
  for $v_{i,-j} \in \mathcal{V}_{-j}$ **do**
    $S \leftarrow \min\left\{x \in \mathcal{V}_j \mid (x, v_{i,-j}) \in \mathcal{R}_j^{(n')}(w_{-i})\right\}$. If $S = \emptyset$, continue to next iteration.
    $x^* \leftarrow \min(S)$.
    $\varphi_j^{v_{i,-j}}(\cdot) \leftarrow$ the output of Algorithm 1 when restricted to values at least x^*.
    for $x \in \mathcal{V}_j$ such that $x^* \le x < \max \mathcal{V}_j$ **do**
      $x' \leftarrow$ smallest element $> x$ in $\mathcal{V}_j$.
      Set both $\lambda_i^*((x, v_{i,-j}), (x', v_{i,-j}))$ and $\lambda_i^*((x', v_{i,-j}), (x, v_{i,-j}))$ to
$$\frac{f_{-j}(v_{i,-j})}{x' - x} \cdot \sum_{x'' > x \in \mathcal{V}_j} f_j(x'')\left(\varphi_j^{v_{i,-j}}(x'') - \varphi_j(x'')\right).$$
    end for
  end for
end for