# Solving sparse polynomial systems using Gröbner bases and resultants 

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#### Abstract

Solving systems of polynomial equations is a central problem in nonlinear and computational algebra. Since Buchberger's algorithm for computing Gröbner bases in the 60s, there has been a lot of progress in this domain. Moreover, these equations have been employed to model and solve problems from diverse disciplines such as biology, cryptography, and robotics. Currently, we have a good understanding of how to solve generic systems from a theoretical and algorithmic point of view. However, polynomial equations encountered in practice are usually structured, and so many properties and results about generic systems do not apply to them. For this reason, a common trend in the last decades has been to develop mathematical and algorithmic frameworks to exploit specific structures of systems of polynomials.

Arguably, the most common structure is sparsity; that is, the polynomials of the systems only involve a few monomials. Since Bernstein, Khovanskii, and Kushnirenko's work on the expected number of solutions of sparse systems, toric geometry has been the default mathematical framework to employ sparsity. In particular, it is the crux of the matter behind the extension of classical tools to systems, such as resultant computations, homotopy continuation methods, and most recently, Gröbner bases. In this work, we will review these classical tools, their extensions, and recent progress in exploiting sparsity for solving polynomial systems.


This manuscript complements its homonymous tutorial presented at the conference ISSAC 2022.

## CCS CONCEPTS

- Computing methodologies $\rightarrow$ Symbolic and algebraic algorithms; Hybrid symbolic-numeric methods; Symbolic calculus algorithms; Equation and inequality solving algorithms.


## KEYWORDS

Sparse polynomials, Gröbner bases, resultants, solving polynomial systems

## 1 INTRODUCTION

Systems of polynomial equations give us one of the simplest and general ways of dealing with non-linear objects. They are expressive enough to encode algebraic varieties and effective enough to allow us to compute with them. They generalize two ubiquitous kinds of equations appearing in Mathematics, linear equations and univariate polynomials. Solving polynomial equations, that is, to
find exact or approximate solutions for the equations of the systems, is one of the central problems in this setting and their applications span several domains in science and engineering [17, 33].

There are several tools to solve polynomial systems, e.g., geometric resolutions [64, 67, 68], Gröbner bases [56, 60][75, Chp. 2], homotopy continuation [18, 70, 86], normal form algorithms [2, 79, 88, 97], resultants [24, 49][34, Chp. 3], subdivisions [36, 89, 99], subresultants [43, 65, 95] and triangular decompositions [28, 101]. The aforementioned is a non-exhaustive list of methods and references; the interested reader can find the latest developments on many of these methods in the proceedings of the previous editions of ISSAC. Some nice introductions to the theory of solving systems of polynomial equations can be found in [34, 42, 75]. In this manuscript, and in its associated tutorial, we focus on Gröbner bases, resultants, and normal forms algorithms.

In practice, the polynomial systems that we encounter are structured, e.g., sparse polynomials in biology [62] and statistics [44], symmetric in cryptography [52], and determinantal in optimization [71]. However, the general-purpose strategies to solve polynomial systems does not exploit this structure. For this reason, in the last years, an important trend in polynomial system solving is to improve these techniques for specific structures. In this text, we will focus on the sparsity of the inputs. Besides sparsity, other kinds of structures include, e.g., symmetry [4, 54], determinantal [52, 77], black box evaluations [19], and degenerations [21].

### 1.1 Sparsity

Sparsity is arguably the most common structure appearing in practice. Before giving the definition of sparsity, let us introduce some notation.

Notation. Given $\alpha \in \mathbb{Z}^{n}$, we define $x^{\alpha}:=\prod_{i=1}^{n} x_{i}^{\alpha_{i}}$. The monomials $\left\{x^{\alpha}: \alpha \in \mathbb{Z}^{n}\right\}$ generate the $\mathbb{C}$-algebra of Laurent polynomials $\mathbb{C}\left[x_{1}^{ \pm}, \ldots, x_{n}^{ \pm}\right]$. We will write our polynomials in $\mathbb{C}\left[x_{1}^{ \pm}, \ldots, x_{n}^{ \pm}\right]$as a sum of terms, i.e., $\sum_{\alpha \in \mathbb{N}^{n}} c_{\alpha} x^{\alpha}$, where $c_{\alpha} \in \mathbb{C}$ is the coefficient of the monomial $x^{\alpha}$ and there are finite $\alpha \in \mathbb{Z}^{n}$ such that $c_{\alpha} \neq 0$. The support of a given polynomial $f$ is the finite set of exponents with non-zero coefficients, i.e.,

$$
\operatorname{Supp}\left(\sum_{\alpha \in \mathbb{N}^{n}} c_{\alpha} x^{\alpha}\right):=\left\{\alpha \in \mathbb{Z}^{n}: c_{\alpha} \neq 0\right\}
$$

We abuse notation and we also identify $\operatorname{Supp}(f)$ with a set of the monomials with exponents in $\operatorname{Supp}(f)$. In this text, we will only work with rational polytopes, i.e., its vertices are integer points. Given two polytopes $P_{1}$ and $P_{2}$, we define its Minkowski sum $P_{1}+$ $P_{2}$ as their point-wise addition, i.e., $P_{1}+P_{2}:=\left\{\alpha+\beta: \alpha \in P_{1}, \beta \in\right.$
$\left.P_{2}\right\}$. For each $i \in \mathbb{N}$, let $i P:=P+(i-1) P$, and $0 P=\{0\}$. We denote by $\Delta_{n}$ the $n$-dimensional standard simplex, i.e., the convex hull of $\left\{0, e_{1}, \ldots, e_{n}\right\} \subset \mathbb{R}^{n}$, where $\left\{e_{1}, \ldots, e_{n}\right\}$ is the canonical basis of $\mathbb{R}^{n}$. Intuitively, a polynomial $f \in \mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ is sparse when Supp $(f)$ is small, e.g., $\operatorname{Supp}(f)$ has less elements than $\operatorname{Supp}(g)$, where $g \in$ $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ is a generic polynomial of the same degree as $f$. However, in this text we will use a related, but different, notion of sparsity given by the Newton polytope of the polynomial.

Definition 1.1 (Newton polytope). Given a polynomial $f$, its Newton polytope $N P(f) \subset \mathbb{R}^{n}$ is the convex hull of $\operatorname{Supp}(f)$ over $\mathbb{R}^{n}$.

Observe that the Newton polytope of a generic polynomial in $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ of degree $d$ is $d \Delta_{n}$, i.e., the $d$-dilation of the $n$-dimensional standard simplex.

Example 1.2 (Generalized eigenvalue problem). For most generalized eigenvalue problems in $\mathbb{C}^{2 \times 2}$, we can represent them as a system of polynomial equations in two variables $\lambda, w$, where ( $1, \lambda$ ) denotes the generalized eigenvalue and $v:=(1, w)$, the eigenvector. For example, for the pencil $(A, B), A:=\left(\begin{array}{ll}1 & 2 \\ 3 & 4\end{array}\right), B:=\left(\begin{array}{cc}3 & 4 \\ -2 & -4\end{array}\right)$, we obtain the polynomial system $f_{1}=f_{2}=0$,

$$
(A+\lambda B) v=0 \Longleftrightarrow\left\{\begin{array}{l}
f_{1}:=1+3 \lambda+2 w+4 \lambda w=0 \\
f_{2}:=3-2 \lambda+4 w-4 \lambda w=0
\end{array} .\right.
$$

The polynomials have degree 2 and their support are $\operatorname{Supp}\left(f_{1}\right)=$ $\operatorname{Supp}\left(f_{2}\right)=\{1, \lambda, w, \lambda w\}$. The polynomials are sparse; written in terms of monomials of degree at most 2 , we have that

$$
f_{1}=1+3 \lambda+0 \lambda^{2}+2 w+0 w^{2}+4 \lambda w .
$$

The gray square in the following picture is their Newton polytope.


When we talk about sparsity, we look at Newton polytopes and not at the supports because of the next theorem, attributed to Bernstein [12], Khovanskii [73], and Kushnirenko [76].

Theorem 1.3 (BBK bound). Fix a polynomial system ( $f_{1}, \ldots, f_{n}$ ) with a finite number of solutions over $\left(\mathbb{C}^{*}\right)^{n}$, where $\mathbb{C}^{*}:=\mathbb{C} \backslash\{0\}$. Then, this number of solutions is upper bounded by the mixed volume of $\operatorname{NP}\left(f_{1}\right), \ldots, \operatorname{NP}\left(f_{n}\right)$, i.e., $\operatorname{MV}\left(\operatorname{NP}\left(f_{1}\right), \ldots, \operatorname{NP}\left(f_{n}\right)\right)$, where
$\operatorname{MV}\left(P_{1}, \ldots, P_{n}\right):=(-1)^{n}+\sum_{k=1}^{n}(-1)^{n-k} \sum_{\substack{I \subset\{1, \ldots, n\} \\ \# I=k}} \#\left(\left(P_{I_{1}}+\cdots+P_{I_{k}}\right) \cap \mathbb{Z}^{n}\right)$
Moreover, consider finite subsets $A_{1}, \ldots, A_{n} \subset \mathbb{Z}^{n}$ and a system of generic polynomials $\left(f_{1}, \ldots, f_{n}\right)$ supported on $A_{1}, \ldots, A_{n}$, i.e., each $f_{i}=\sum_{\alpha \in A_{i}} c_{i, \alpha} x^{\alpha}$, where $\left(c_{i, \alpha}\right)_{i \in\{1, \ldots, n\}, \alpha \in A_{i}}$ is a generic vector. Then, the aforementioned bound for this generic system is an equality.

For a proof and extensions of this theorem, see [85]. The generic conditions of the previous theorem can be relaxed [15, 26, 29].

Example 1.4 (Cont. Ex. 1.2). Without considering the sparsity of the input, Bézout's theorem tells us that the system should have (at most) $2^{2}$ solutions. However, it has only two, each corresponding to a different eigenspace. This number agrees with the mixed volume of the Newton polytopes defining the equations.

When working with sparse polynomials, we usually consider two class of systems. We say that a sparse polynomial system $\left(f_{1}, \ldots, f_{r}\right)$ is unmixed when there is an integer polytope $P \subset \mathbb{R}^{n}$ and integers $d_{1}, \ldots, d_{r}$ such that $\operatorname{NP}\left(f_{i}\right)=d_{i} P$, i.e., each Newton polytope is a dilation of a common integer polytope. Otherwise, the system is mixed.

In this manuscript, we will discuss Gröbner bases and resultants in the context of sparse polynomials. We will employ the Newton polytopes of the input polynomials to speed up the computations and derive complexity bounds for our algorithms depending on the sparsity pattern of the inputs. These techniques rely on toric geometry. Toric geometry was also employed to solve sparse polynomial systems using other techniques such as homotopy continuation [ $50,70,82$ ] or geometric resolutions [68].

Remark 1.5 (Other notions of sparsity). As the BKK bound shows, from a complex point of view, the solutions of the sparse systems are "determined" by the Newton polytopes of the input polynomials and not by their supports. However, from a real point of view, the support matters. In real algebraic geometry, a polynomial whose support is small is called a fewnomial. Khovanskii showed that the number of positive real solutions of a fewnomial system is upper bounded by a quantity singly exponential in the size of the supports of the fewnomials [74]. Determining the number of real solutions of a fewnomial system is an active area of research. For example, the bounds were improved for general [16] and particular systems [14]. Moreover, probabilistically, stronger bounds hold [20, 72].

A different kind of sparsity that has been studied is the chordal structure. This structure only cares about the variables appearing in each polynomial, regardless of the specific monomials. There are dedicated algorithms to compute Gröbner bases [30] and triangular sets [87] that exploit this notion.

### 1.2 Multiplication maps

An intermediate tool used to solve polynomial systems using the Gröbner bases and resultants is multiplication maps. To define them, let $f:=\left\{f_{1}, \ldots, f_{r}\right\} \subset \mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ be an affine polynomial system with a finite number of solutions $\delta$ over $\mathbb{C}^{n}$, counting multiplicities. The quotient ring $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right] /\langle f\rangle$ is a finite dimensional $\mathbb{C}$-vector space whose dimension is $\delta$ [34, Chp. 4.2], so we will fix a basis $b_{1}, \ldots, b_{\delta}$ to identify $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right] /\langle f\rangle$ with $\mathbb{C}^{\delta}$. Given $g \in \mathbb{C}\left[x_{1}, \ldots, x_{n}\right] /\langle f\rangle$, the multiplication map of $g$ is the morphism

$$
\begin{aligned}
M_{g}: \mathbb{C}\left[x_{1}, \ldots, x_{n}\right] /\langle f\rangle & \rightarrow \mathbb{C}\left[x_{1}, \ldots, x_{n}\right] /\langle f\rangle \\
h & \mapsto M_{g}(h):=g h .
\end{aligned}
$$

Using the basis $\left\{b_{1}, \ldots, b_{\delta}\right\}$, we think about $M_{g}$ as a matrix, which we call the multiplication matrix. Multiplication matrices are ubiquitous in algorithms for solving polynomial systems:

- On one hand, we can approximate the solutions by computing the eigenvalues of these maps. More precisely, the eigenvalue theorem [79] states that the eigenvalues of $M_{g}$ correspond to the evaluations ${ }^{1}$ of $g$ at the $\delta$ solutions of the

[^0]system $\left(f_{1}, \ldots, f_{r}\right)$. See [32] for more details on this theorem and its history. Moreover, for each solution $p$ of $f$, the vector given by the evaluations $\left(b_{i}(p)\right)_{i \leq \delta}$ belongs to the eigenspace of $M_{g}$ of eigenvalue $g(p)$ [2].

- On the other hand, multiplication maps are a standard tool to compute Gröbner bases for zero-dimension ideals. The algorithm FGLM $[56,57]$ allows us to derive, from the multiplication matrices, Gröbner bases of $\langle f\rangle$ with respect to any monomial order, e.g., lexicographical, or to find a rational univariate representation of the solutions [92].
To compute multiplication matrices, we use a normal form, which is, roughly speaking, a homomorphism $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right] \rightarrow \mathbb{C}^{\delta}$ whose kernel is $\langle f\rangle$. We can obtain such normal forms using, for example, Gröbner bases [75, Chp. 2], border bases [88][42, Chp. 4], or Sylvester formulas for the resultant [2, 97, 98]. We can compute these matrices by performing $d^{O(n)}$ arithmetic operations, where $d$ is the maximal among the degrees of the polynomials in $f$ [78].

The aforementioned strategies do not employ the sparsity of the inputs to speed up the computations (at least directly ${ }^{2}$ ). Moreover, their complexity bounds are independent of this structure and to achieve them, we must destroy the sparsity of the inputs, i.e., to perform random linear change of coordinates or to consider random combinations of the input polynomials. In the following sections, we discuss alternative approaches to avoid these issues.

## 2 THE SPARSE RESULTANT

In this section, we will assume that the reader is familiar with the classical (projective) resultant. We refer to [34, Chp. 3] and [22] for an introduction.

The sparse resultant is a generalization of the classical resultant that allow us to decide if a sparse affine polynomial system in $n$ variables given by $n+1$ polynomials has a common solution in $\left(\mathbb{C}^{*}\right)^{n}$. It is one of the first available tools to solve sparse polynomial systems and a subject of study since the 90 's.

Given finite sets $A_{0}, \ldots, A_{n} \subset \mathbb{Z}^{n}$, we can parameterize each different polynomial system supported in $A:=\left(A_{0}, \ldots, A_{n}\right)$ with points ${ }^{3}$ in $\mathbb{P}^{A}:=\mathbb{P}^{\# A_{0}-1} \times \cdots \times \mathbb{P}^{\# A_{n}-1}$,
$\left(\boldsymbol{c}_{0}, \ldots, \boldsymbol{c}_{\boldsymbol{n}}\right) \in \mathbb{P}^{A_{0}} \times \cdots \times \mathbb{P}^{A_{n}} \mapsto\left(\sum_{\alpha \in A_{0}} c_{0, \alpha} x^{\alpha}, \ldots, \sum_{\alpha \in A_{n}} c_{n, \alpha} x^{\alpha}\right)$.
We use the incidence variety $\Omega$ to characterize all the sparse systems of polynomials supported on $A_{0}, \ldots, A_{n}$ with solutions in $\left(\mathbb{C}^{*}\right)^{n}$.

$$
\Omega:=\left\{\left(p,\left(F_{0}, \ldots, F_{n}\right)\right) \in\left(\mathbb{C}^{*}\right)^{n} \times \mathbb{P}^{A}:(\forall i) F_{i}(p)=0\right\}
$$

If we consider the projection map $\pi:\left(\mathbb{C}^{*}\right)^{n} \times \mathbb{P}^{A} \rightarrow \mathbb{P}^{A}, \pi(\Omega) \subset$ $\mathbb{P}^{\boldsymbol{A}}$ determines the systems with solutions at $\left(\mathbb{C}^{*}\right)^{n}$. However, as $\pi(\Omega)$ is not an algebraic variety, we consider its algebraic closure $\overline{\pi(\Omega)}$ in $\mathbb{P}^{A}$. Under combinatorial assumptions on $\boldsymbol{A}$ [93, Cor. 1.1], $\overline{\pi(\Omega)}$ has co-dimension 1 on $\mathbb{P}^{A}$ and it is an irreducible hypersurface defined by a multihomogeneous polynomial $\operatorname{Res}_{A}$ called

[^1]the sparse resultant of $A^{4}$. For each $i$, the sparse resultant $\operatorname{Res}_{A}$ is homogeneous with respect to the coefficients of the polynomial $F_{i}$ supported in $A_{i}$ and its degree is $\operatorname{MV}\left(P_{0}, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{n}\right)$, where $P_{i}$ is the convex hull of $A_{i}$. The sparse resultant is a generalization of the classical (projective) resultant. More precisely, if we have that $A_{i}=d_{i} \Delta_{n} \cap \mathbb{Z}^{n}$, for some $d_{i} \in \mathbb{N}$, for each $i$, the sparse and classical resultant agree. The interested reader can find more properties of the sparse resultant in, e.g., [34, Chp. 7],[63, Chp. 8], [40, 93].

### 2.1 Vanishing of the sparse resultant

As the classical resultant is a special case of the sparse resultant, it is easy to verify that having solutions on the torus $\left(\mathbb{C}^{*}\right)^{n}$ is a necessary but not sufficient condition for the vanishing of the sparse resultant. To understand when this vanishes, let us recall when the classical (projective) resultant vanishes. Given an affine polynomial $f \in \mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$, we define its classical homogenization, $f^{h} \in$ $\mathbb{C}\left[x_{0}, x_{1}, \ldots, x_{n}\right]$, as the polynomial $f^{h}=x_{0}^{\operatorname{deg}(f)} f\left(\frac{x_{1}}{x_{0}}, \ldots, \frac{x_{n}}{x_{0}}\right)$. Given an affine polynomial system $f_{0}, \ldots, f_{n} \in \mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$, the classical resultant of $\left(f_{0}, \ldots, f_{n}\right)$ will vanish if and only if the homogenization of the system, i.e. $f_{0}^{h}, \ldots, f_{n}^{h} \in \mathbb{C}\left[x_{0}, x_{1}, \ldots, x_{n}\right]$, has a common solution on $\mathbb{P}^{n}$. The homogenization does not change the solutions of the original system on $\mathbb{C}^{n}$, i.e., for $p \in \mathbb{C}^{n}$, if $f\left(p_{1}, \ldots, p_{n}\right)=0$, then $f^{h}\left(1, p_{1}, \ldots, p_{n}\right)=0$. However, it might introduce new solutions at infinity, i.e., $\mathbb{P}^{n} \backslash \mathbb{C}^{n}:=\left\{\left[p_{0}: \cdots\right.\right.$ : $\left.\left.p_{n}\right] \in \mathbb{P}^{n}: p_{0}=0\right\}$. In an analogous way, the sparse resultant $\operatorname{Res}_{A}$ of a sparse system will vanish if and only if a homogenization of the system vanishes over a normal projective toric variety $X$. This toric variety is a compact variety containing $\left(\mathbb{C}^{*}\right)^{n}$ which is constructed from the Minkowski sum $P:=\sum_{i} P_{i}$; see, e.g., [35, Chp. 2.3] for more details on this construction. As in the classical setting, the homogenization of the system has the same solutions as the original system over $\left(\mathbb{C}^{*}\right)^{n}$, but might have extra zeros on $X \backslash\left(\mathbb{C}^{*}\right)^{n}$, which we will call zeros at infinity. More precisely, for each $P_{i}$, we can find a torus-invariant nef Cartier divisor $D_{i}$ such that the global sections of its associated line bundle, denoted by $O_{X}\left(D_{i}\right)$, can be identified with the sparse polynomials whose Newton polytope is equal to (or contained in) $P_{i}$ [35, Chp. 6], that is,

$$
\begin{equation*}
H^{0}\left(X_{P}, O_{X}\left(D_{i}\right)\right)=\left\{g \in \mathbb{C}\left[x_{1}^{ \pm}, \ldots, x_{n}^{ \pm}\right]: \mathrm{NP}(g) \subseteq P_{i}\right\} \tag{1}
\end{equation*}
$$

This identification corresponds to the homogenization. Somehow, if we have a global section $s \in H^{0}\left(X_{P}, O_{X}(D)\right)$ identified with $g:=\sum_{\alpha \in Q} c_{\alpha} x^{\alpha}$, then for every $p \in\left(\mathbb{C}^{*}\right)^{n}, s(p)=0$ if and only if $g(p)=0$. Using this intuition, we consider the incidence variety

$$
\Omega^{h}:=\left\{\left(p,\left(F_{0}, \ldots, F_{n}\right)\right) \in X_{P} \times \mathbb{P}^{A}:(\forall i) \operatorname{hom}\left(F_{i}\right)(p)=0\right\}
$$

where hom $\left(F_{i}\right)$ denotes the global section of $O_{X}(D)$ identified with $F_{i}$. If $\pi$ now denotes the projection onto $\mathbb{P}^{A}, \pi\left(\Omega^{h}\right)$ is an algebraic variety (because $X$ is projective) which we can prove agrees with $\overline{\pi(\Omega)}$. We refer the interested reader to [34, Chp. 7.3] for a more detailed explanation in the case of unmixed systems ${ }^{5}$.

[^2]Remark 2.1. To define a homogenization in terms of coordinates, as we did over $\mathbb{P}^{n}$, we need to introduce a coordinate ring for $X$. However, such a ring depends on the way (embedding) that we use to think about $X$. For example, we can think about $X$ as an almost geometric quotient and use its associated ring, the Cox ring [31], to homogenize the polynomials; see [35, Chp. 5]. Alternatively, we can use the so-called Cayley trick [63, Chp. 8], where we choose nef line bundles on $X$ associated to each $P_{i}$ and consider the embedding of $X$ on the (weighted) multi-projective space defined by them. We will use the latter approach in Sec. 3.

### 2.2 Computing the sparse resultant

2.2.1 The determinant of a matrix. A classical way of computing the resultant is a factor of the determinant of a matrix. Sylvester [94] showed that we can compute the resultant of two univariate polynomials $f_{0}, f_{1} \in \mathbb{C}\left[x_{0}, x_{1}\right]$ by considering the determinant of the Sylvester matrix, which linearizes the (Sylvester) map $\left(g_{0}, g_{1}\right) \mapsto$ $g_{0} f_{0}+g_{1} f_{1}$, where $g_{0}$ is a polynomial of degree at $\operatorname{most} \operatorname{deg}\left(f_{1}\right)-1$ and $g_{1}$ a polynomial of degree at $\operatorname{most} \operatorname{deg}\left(f_{0}\right)-1$. This construction was later generalized by Macaulay [81], who showed how to compute the classical (projective) resultant of $n+1$ polynomials $f_{0}, \ldots, f_{n} \in \mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ of degree $d_{0}, \ldots, d_{n}$. He also showed that there is a way of constructing monomial sets $B_{0}, \ldots, B_{n}$ such that each $x^{\beta} \in B_{i}$ has degree at most $\sum_{j \neq i} d_{j}-n$ and $\sum_{i} \# B_{i}$ agrees with the number of monomials of degree at most $\sum_{j} d_{j}-n$ in $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$. With these considerations, we can construct the Macaulay matrix $M$, whose columns we index with monomials in $\mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$ of degree at most $\sum_{j} d_{j}-n$ and the rows with pairs $\left\{\left(i, x^{\beta}\right): i \in\{0, \ldots, n\}, x^{\beta} \in B_{i}\right\}$. The element in $M$ associated with the column indexed by $x^{\alpha}$ and row indexed by (i, $x^{\beta}$ ) corresponds to the coefficient of the monomial $x^{\alpha}$ in $x^{\beta} f_{i}$. Macaulay proved that the resultant of $\left(f_{0}, \ldots, f_{n}\right)$ is a factor of the determinant of $M$, i.e., $\operatorname{det}(M)=\operatorname{Res}\left(f_{0}, \ldots, f_{n}\right) \cdot E$. Moreover, he showed that the polynomial $E$, usually called the extra factor, is a specific minor of $M$. Observe that the matrix $M$ is the transpose of a matrix representing the Sylvester map $\left(g_{0}, \ldots, g_{n}\right) \mapsto \sum_{i} f_{i} g_{i}$, where the support of each $g_{i}$ is contained in $B_{i}$. For this reason, we say that his construction is a Sylvester formula for the resultant. Besides Sylvester formulas, we observe that other maps can be used to compute resultants. We refer the reader to [48] and references therein.

Canny and Emiris [25] generalized Macaulay's construction to the sparse case. They presented a Sylvester formula for the resultant constructed out of mixed subdivisions of the Newton polytopes of the input polynomials. The interested reader can find more details and references in [34, Chp. 7.6]. However, Canny and Emiris did not characterize the extra factor appearing in their constructions. Different authors studied this question, e.g., [37, 38, 66], and very recently D'Andrea, Jeronimo, and Sombra [38] presented a complete characterization of the extra factor which allowed them to recover Macaulay's original results.
2.2.2 The determinant of a complex. A more general and elaborated way of computing the resultant is as the determinant of a complex of vector spaces. This approach is attributed to Cayley and was extended to study general kinds of resultants, such as the sparse one; the interested reader can find more details in [63, Chp. 3.4]. It turns out that the aforementioned Sylvester maps are
just the last maps appearing in certain Koszul complexes that we can use to compute resultants; see, e.g., [63, Chp. 13]. Moreover, we can use other kinds of complexes to construct smaller matrices from which to compute the resultant. In the best case scenario, the complex will involve only two non-zero modules and so its determinant will agree with the determinant of the maps between these modules. In this case, we will be able to construct determinantal formulas, that is, maps whose determinant is exactly the resultant, i.e., the extra factor $E$ is a non-zero constant independent of the specific system.
Following [63], to construct the resultant as the determinant of a complex, we use sheaf cohomology. Let $X$ be the normal projective toric variety constructed from the polytope $P$ detailed before and consider the nef line bundles $O_{X}\left(D_{0}\right), \ldots, O_{X}\left(D_{n}\right)$. Consider global sections $\tilde{f}_{0}, \ldots, \tilde{f}_{n}$ such that $\tilde{f}_{i} \in H^{0}\left(X, O_{X}\left(D_{i}\right)\right)$. If $\left(\tilde{f}_{0}, \ldots, \tilde{f}_{n}\right)$ has no common zeros on $X$, the Koszul complex of sheaves $\mathcal{K}_{\bullet}$ associated with the Sylvester map, defined locally at $U \subset X$ as $\left.\operatorname{Sylv}\left(g_{0}, \ldots, g_{n}\right) \mapsto \sum_{i} g_{i} \tilde{f}_{i}\right|_{U}$, is exact;

$$
\begin{align*}
\mathcal{K}_{\bullet}: 0 & \rightarrow O_{X}\left(-\sum_{i} D_{i}\right) \rightarrow \bigoplus_{i} O_{X}\left(-\sum_{j \neq i} D_{j}\right) \rightarrow \ldots \\
& \cdots \rightarrow \bigoplus_{j \neq i} O_{X}\left(-D_{i}-D_{j}\right) \rightarrow \bigoplus_{j} O_{X}\left(-D_{j}\right) \xrightarrow{\text { Sylv }} O_{X} \rightarrow 0, \tag{2}
\end{align*}
$$

where $O_{X}(D-B)=O_{X}(D) \otimes O_{X}(B)^{-1}$, for any two Cartier divisors $D$ and $B$. We will transform this complex of sheaves into a complex of vector spaces by considering its sheaf cohomology. Sheaf cohomology of line bundles on the toric variety $X$ is a well-understood subject; for more details, see [35, Chp. 9] and [1].

Observe that, for any Cartier divisor $D$, the twisted complex $\mathcal{K}_{\bullet} \otimes O_{X}(D)$ is exact, because $O_{X}(D)$ is locally invertible. Hence, we consider the exact complex $\mathcal{K}_{\bullet} \otimes O_{X}\left(\sum_{i} D_{i}\right)$. Every module in the twisted complex is a direct sum of sheaves of the form $O_{X}\left(\sum_{i \in I} D_{i}\right)$, for some $I \subset\{0, \ldots, n\}$, so they are all nef. Hence, all the higher sheaf cohomologies of the modules in $\mathcal{K}_{\bullet} \otimes O_{X}\left(\sum_{i} D_{i}\right)$ vanish [35, Thm. 9.2.3] and so, the associated complex of global sections is exact [63, Lem. 2.2.4],

$$
\begin{align*}
& H^{0}\left(X, \mathcal{K}_{\bullet} \otimes O_{X}\left(\sum_{i} D_{i}\right)\right): 0 \rightarrow H^{0}\left(X, O_{X}\right) \rightarrow \ldots \\
& \cdots \rightarrow \bigoplus_{j} H^{0}\left(X, O_{X}\left(\sum_{i \neq j} D_{i}\right)\right) \xrightarrow{\text { sylv }} H^{0}\left(X, O_{X}\left(\sum_{i} D_{i}\right)\right) \rightarrow 0 . \tag{3}
\end{align*}
$$

The global sections of the line bundle $O_{X}\left(\sum_{i \in I} D_{i}\right)$, for $I \subset\{0, \ldots, n\}$, correspond to the sparse polynomials with Newton polytope equal to (or contained in) $\sum_{i \in I} P_{i}$. Hence, the last map of the exact complex of global sections is the Sylvester map $\left(g_{0}, \ldots, g_{n}\right) \mapsto \sum_{i} g_{i} \tilde{f}_{i}$, where $g_{i}$ is a sparse polynomial such that $\mathrm{NP}\left(g_{i}\right) \subset \sum_{j \neq i} P_{j}$. Canny and Emiris' matrix is a square submatrix of the matrix associated with this Sylvester map ${ }^{6}$. Using this construction, we can also recover a matrix related to Macaulay's construction. In this case, we consider $X=\mathbb{P}^{n}$ and so, its Picard group is $\mathbb{Z}$. We identify $D_{i}$ with the degree of $\tilde{f}_{i}$ and consider the complex $\mathcal{K}_{\bullet} \otimes O_{X}\left(\sum_{i} d_{i}-n\right)$. By Serre's vanishing theorem, the invertible sheaves appearing in this complex have no higher sheaf cohomologies and so, we can consider the exact complex of vector spaces $H^{0}\left(X, \mathcal{K}_{\bullet} \otimes O_{X}\left(\sum_{i} d_{i}-n\right)\right)$. The last map of this complex is the Sylvester map $\left(g_{0}, \ldots, g_{n}\right) \mapsto$

[^3]$\sum_{i} g_{i} f_{i}$, where $g_{i}$ has degree $\sum_{j \neq i} \operatorname{deg}\left(f_{j}\right)$. Macaulay's matrix is a maximal square submatrix of this map.

We can also use this construction to understand why the Sylvester matrix gives us a determinantal formula. In this case, we have that $n=1$ and $X=\mathbb{P}^{1}$. The aforementioned complex reduces to

$$
0 \rightarrow \mathbb{C}\left[x_{0}, x_{1}\right]_{d_{1}-1} \oplus \mathbb{C}\left[x_{0}, x_{1}\right]_{d_{0}-1} \xrightarrow{\text { Sylv }} \mathbb{C}\left[x_{0}, x_{1}\right]_{d_{0}+d_{1}-1} \rightarrow 0,
$$

where $d_{i}=\operatorname{deg}\left(f_{i}\right)$. The exactness of the Sylvester map determines the exactness of the full complex and agrees with Sylvester's construction. Hence, there is no extra-factor in his construction. ${ }^{7}$

In the general case, Weyman proposed an approach to derive from $\mathcal{K}_{\bullet}$ an exact complex whose modules are direct sums of sheaves cohomologies of the form $H^{i}\left(X, O_{X}(B)\right)$ and such that the maps only depend on the coefficients of $\left(\tilde{f}_{0}, \ldots, \tilde{f}_{n}\right)$ [102]. By twisting Weyman's complex by certain line bundles and studying the vanishing of the sheaf cohomologies appearing in it, several authors constructed smaller formulas for computing the sparse resultant, e.g. [39, 41, 103]. Moreover, by studying the case when the Weyman complex only involves two non-zero modules, this construction was used to derive determinantal formulas for the resultant of multihomogeneous systems, e.g., [6, 7, 23, 46, 47, 103]. In what follows, we present an example taken from [46, Ex. 5.3] of a determinantal formula obtained from Weyman's complex which is not a Sylvester formula.

Example 2.2 (Koszul-type formula). We consider bilinear forms $f_{0}, f_{1}, f_{2} \in \mathbb{C}\left[x_{0}, x_{1}\right]_{1} \otimes \mathbb{C}\left[y_{0}, y_{1}\right]_{1}$. Their resultant vanishes if and only if they have a common solution on $\mathbb{P}^{1} \times \mathbb{P}^{1}$;

$$
\left\{\begin{array}{l}
f_{0}=\left(a_{0,0} x_{0}+a_{1,0} x_{1}\right) y_{0}+\left(a_{0,1} x_{0}+a_{1,1} x_{1}\right) y_{1}  \tag{4}\\
f_{1}=\left(b_{0,0} x_{0}+b_{1,0} x_{1}\right) y_{0}+\left(b_{0,1} x_{0}+b_{1,1} x_{1}\right) y_{1} \\
f_{2}=\left(c_{0,0} x_{0}+c_{1,0} x_{1}\right) y_{0}+\left(c_{0,1} x_{0}+c_{1,1} x_{1}\right) y_{1}
\end{array}\right.
$$

To construct a determinantal formula, we consider the bilinear map

$$
\begin{gathered}
\star: \mathbb{C}\left[y_{0}, y_{1}\right]_{1} \times\left(\mathbb{C}\left[x_{0}, x_{1}\right]_{1} \otimes \mathbb{C}\left[y_{0}, y_{1}\right]_{1}\right) \rightarrow \mathbb{C}\left[x_{0}, x_{1}\right]_{1} \\
y_{i} \star\left(x_{k} y_{j}\right):=\left\{\begin{array}{cc}
x_{j} & \text { if } i=j, \\
0 & \text { otherwise } .
\end{array}\right.
\end{gathered}
$$

The determinant of $\phi:\left(\mathbb{C}\left[y_{0}, y_{1}\right]_{1}\right)^{3} \rightarrow\left(\mathbb{C}\left[x_{0}, x_{1}\right]_{1}\right)^{3}$ is the resultant of the system;

$$
y_{i} \boldsymbol{e}_{j} \mapsto \phi\left(y_{i} \boldsymbol{e}_{j}\right):=\left(y_{i} \star f_{J_{2}}\right) \boldsymbol{e}_{J_{1}}-\left(y_{i} \star f_{J_{1}}\right) \boldsymbol{e}_{J_{2}}, \quad(J=\{0,1,2\} \backslash\{j\})
$$

This formula is not a Sylvester formula, but a Koszul one.

|  | $x_{0} e_{0}$ | $x_{1} e_{0}$ | $x_{1} e_{2}$ | $x_{1} e_{1}$ | $x_{0} e_{2}$ | $x_{0} e_{1}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $y_{0} e_{0}$ | 0 | 0 | $b_{1,0}$ | $-c_{1,0}$ | $b_{0,0}$ | $-c_{0,0}$ |
| $y_{1} e_{0}$ | 0 | 0 | $b_{1,1}$ | $-c_{1,1}$ | $b_{0,1}$ | $-c_{0,1}$ |
| $y_{1} e_{1}$ | $-c_{0,1}$ | $-c_{1,1}$ | $a_{1,1}$ | 0 | $a_{0,1}$ | 0 |
| $y_{1} e_{2}$ | $-b_{0,1}$ | $-b_{1,1}$ | 0 | $a_{1,1}$ | 0 | $a_{0,1}$ |
| $y_{0} e_{1}$ | $-c_{0,0}$ | $-c_{1,0}$ | $a_{1,0}$ | 0 | $a_{0,0}$ | 0 |
| $y_{0} e_{2}$ | $-b_{0,0}$ | $-b_{1,0}$ | 0 | $a_{1,0}$ | 0 | $a_{0,0}$ |

[^4]
### 2.3 Solving sparse systems

In an analogous way to what we can do with the classical (projective) resultant, we can use the sparse resultant to compute all the solutions of a sparse square system. In this section, we will discuss eigenvalue methods to solve polynomial systems. Other solving strategies as the hidden-variable approach or U-resultants [34, Chp. 3.5] can be also extended; see, e.g., [40].

The following method was proposed by Emiris and Rege to recover multiplication maps using the sparse resultant [49]. Their approach generalizes ideas by Auzinger and Stetter for solving classical homogeneous systems [2]. Consider a sparse polynomial system $f_{1}, \ldots, f_{n}$ such that their number of solutions on $\left(\mathbb{C}^{*}\right)^{n}$ is the mixed volume of $\operatorname{NP}\left(f_{1}\right), \ldots, \operatorname{NP}\left(f_{n}\right)$ and this volume is not zero. Let $f_{0}$ be a linear form in the variables $x_{1}, \ldots, x_{n}$ with Newton polytope $\Delta_{n}$. We consider the matrix constructed by Canny and Emiris' algorithm for the sparse resultant of $\left(f_{0}, \ldots, f_{n}\right)$ [25]. This construction will generate monomial sets $B_{0}, \ldots, B_{n}, A$ such that $\sum \# B_{i}=\# A$, for any $i$ and $x^{\beta} \in B_{i}$, for which $\operatorname{Supp}\left(x^{\beta} f_{i}\right) \in A$, $\# B_{0}=\operatorname{MV}\left(P_{1}, \ldots, P_{n}\right)$ and $B_{0} \subset A$. From these sets, we consider the Macaulay matrix $M$ whose rows are indexed by monomials in $A$ and its columns by pairs $\left(i, x^{\beta}\right)$ where $x^{\beta} \in B_{i}$, and where in the position corresponding to the row $x^{\alpha}$ and column ( $i, x^{\beta}$ ), the element is the coefficient of the monomial $x^{\alpha}$ in $x^{\beta} f_{i}$. We reorder $M$ such that the bottom rows correspond to the pairs ( $0, x^{\beta}$ ) and the left-more columns corresponds to the monomials in $B_{0}$, and we split the matrix accordingly:

$$
\begin{array}{rl||l|}
\left\{x^{\beta} f_{i}: i>0, x^{\beta} \in B_{i}\right\}  \tag{6}\\
\left\{x^{\beta} f_{0}: x^{\alpha} \in B_{0}\right\} \\
\left\{\begin{array}{ll|l|}
M_{1,1} & M_{1,2} \\
M_{2,1} & M_{2,2}
\end{array}\right.
\end{array}
$$

Emiris and Rege showed that if the system $\left(f_{1}, \ldots, f_{n}\right)$ is genericenough then the matrix $M_{1,1}$ is invertible and so the Schur complement of $M_{2,2}$, i.e. the matrix $M_{2,2}-M_{1,2} M_{1,1}^{-1} M_{2,1}$, is the multiplication matrix of $f_{0}$ in $\mathbb{C}\left[x_{1}^{ \pm}, \ldots, x_{n}^{ \pm}\right] /\left\langle f_{1}, \ldots, f_{n}\right\rangle$. The latter holds because this matrix allows us to rewrite elements of the form $f_{0}\left(\sum_{\beta \in B_{0}} c_{\beta} x^{\beta}\right)$ as $\sum_{\beta \in B_{0}} \bar{c}_{\beta} x^{\beta}+g$, where $g \in\left\langle f_{1}, \ldots, f_{n}\right\rangle$. If the resultant of $\left(f_{0}, \ldots, f_{n}\right)$ does not vanish, then the map is invertible, and so $B_{0}$ is a monomial basis of $\mathbb{C}\left[x_{1}^{ \pm}, \ldots, x_{n}^{ \pm}\right] /\left\langle f_{1}, \ldots, f_{n}\right\rangle$ as its cardinal $\operatorname{MV}\left(P_{1}, \ldots, P_{n}\right)$ agrees with the dimension of the quotient ring [90]. The generic assumptions of the aforementioned construction can be relaxed [11, 91, 96] and the construction can be extended to solve overdetermined sparse systems [10, 83]. Recently, special attention has been given to the numerical aspects of this approach [10, 11, 96]. In general, we can use determinantal formulas to compute matrices which are similar to multiplication matrices, i.e., they share the same eigenvalues [6]. In contrast to the Sylvester formulas, the eigenvectors of the determinantal formulas are not well-understood, with the exception of the Koszul formulas [6].

Example 2.3 (Solving using determinantal formulas). Following our Example 2.2, we fix $f_{0}, f_{1}, f_{2} \in \mathbb{C}\left[x_{0}, x_{1}\right]_{1} \otimes \mathbb{C}\left[y_{0}, y_{1}\right]_{1}$ as in (4)
and partition the determinantal formula as in (5),

$$
\left[\begin{array}{l|l}
M_{1,1} & M_{1,2} \\
\hline M_{2,1} & M_{2,2}
\end{array}\right]
$$

We will show that, if the coordinates $x_{0}$ and $y_{0}$ of every solution of ( $f_{1}, f_{2}$ ) are different to zero, then we can solve $\left(f_{1}, f_{2}\right)$ by computing the eigenvalues of the Schur complement of $M_{2,2}$.

By assumption, the system ( $x_{0} y_{0}, f_{1}, f_{2}$ ) has no solution and so the matrix $M_{1,1}$ is invertible as the resultant of ( $x_{0} y_{0}, f_{1}, f_{2}$ ) is $\operatorname{det}\left(M_{1,1}\right) \neq 0$. We introduce a new variable $\lambda$ and consider the resultant of the system ( $f_{0}-\lambda x_{0}, y_{0}, f_{1}, f_{2}$ ). By the Poisson formula [40], the resultant is a polynomial in $\lambda$ whose roots are of the form $\frac{f_{0}}{x_{0} y_{0}}$ evaluated at the solutions of $\left(f_{1}, f_{2}\right)$. Hence, the determinantal formula for ( $f_{0}-\lambda x_{0} y_{0}, f_{1}, f_{2}$ ) given in (5), can be written as

$$
\left[\begin{array}{c|c}
M_{1,1} & M_{1,2} \\
\hline M_{2,1} & M_{2,2}-\lambda I
\end{array}\right]
$$

where $I$ is the $2 \times 2$ identity matrix. Hence, the resultant of ( $f_{0}-$ $\lambda x_{0} y_{0}, f_{1}, f_{2}$ ) agrees with the determinant of the Schur complement of $M_{2,2}-\lambda$ I. Equivalently, the eigenvalues of the Schur complement of $M_{2,2}$ are the evaluations of $\frac{f_{0}}{x_{0} y_{0}}$ at the solutions of $\left(f_{1}, f_{2}\right)$.

## 3 GRÖBNER BASES

The objective of this section is to compute Gröbner bases for sparse polynomial systems. Gröbner bases algorithms usually can incorporate a priori information of the system to speed up computations, e.g., syzygy module [45,55] or Hilbert series [100]. The main issue when we compute with sparse polynomials is that they behave differently from dense systems, and we do not have much prior information on them. We will explain how toric geometry can help us understand the systems better and so, speed up computations.

### 3.1 Embedding of a toric variety

In this section, we will fix polytopes $P_{1}, \ldots, P_{m}$ and consider the projective normal toric variety $X$ associated with $P:=P_{1}+\cdots+P_{m}$ detailed in the previous section. We assume $P$ is full-dimensional. Given an integer polytope $Q$, we say it is a $\mathbb{N}$-Minkowski summand of $P$ if there is $k \in \mathbb{N}$ and an integer polytope $S$ such that $Q+S=k P$. To each $\mathbb{N}$-Minkowski summand $Q$ of $P$ there is a nef Cartier divisor $D_{Q}$ such that we can identify the global sections of the nef line bundle $O_{X}\left(D_{Q}\right)$ with the polynomials whose Newton polytope is equal to (or contained in) $Q$, as in (1) [35, Chp. 6]. Moreover, we have that for two $\mathbb{N}$-Minkowski summands $Q, S$, the global sections of $O_{X}\left(D_{Q}+D_{S}\right)$ correspond to the polynomials with Newton polytope $Q+\bar{S}$. Given $f_{i}$ with Newton polytope contained in $P_{i}$, we will homogenize $f_{i}$ and write it as $\tilde{f}_{i} \in H^{0}\left(X, O_{X}\left(D_{P_{i}}\right)\right)$. The common zeros of $\tilde{f}_{1}, \ldots, \tilde{f}_{m}$ will determine a subscheme of $X$, that we denote by $Y$. The subscheme $Y \cap\left(\mathbb{C}^{*}\right)^{n}$ agrees with the subscheme defined by the original $f_{1}, \ldots, f_{m}$ on $\left(\mathbb{C}^{*}\right)^{n}$. Particularly, when $Y \subset\left(\mathbb{C}^{*}\right)^{n}$, $Y$ is zero-dimensional and we can study the solutions of $f_{1}, \ldots, f_{m}$ on $\left(\mathbb{C}^{*}\right)^{n}$ by studying $Y$.

Example 3.1 (Intersection theory for toric varieties). If $m=n$ and $Y$ is zero-dimensional, the mixed volume $\operatorname{MV}\left(P_{1}, \ldots, P_{n}\right)$ is actually the intersection number of $D_{P_{1}}, \ldots, D_{P_{n}}$; see [61, Chp. 5]. Hence, the number of solutions of $f_{1}, \ldots, f_{n}$ is bounded by the degree of

Y, and so by this mixed volume. Moreover, generically, the system $\left(\tilde{f}_{1}, \ldots, \tilde{f}_{n}\right)$ has no solutions on $X \backslash\left(\mathbb{C}^{*}\right)^{n}$ and so the BKK bound is tight. In general, the BKK bound is tight when we count the solutions at infinity.

In what follows, we assume that there are $\mathbb{N}$-Minkowski summands $Q_{1}, \ldots, Q_{r}$ of $P$ and vectors $d_{1}, \ldots, d_{m} \in \mathbb{Z}^{r}$ such that $P_{i}=$ $\sum_{k} d_{i, k} Q_{k}$. We consider the $\mathbb{N}^{r}$-graded algebra given by

$$
R^{h}:=\bigoplus_{b \in \mathbb{N}^{r}} H^{0}\left(X, O_{X}\left(\sum_{k} b_{k} D_{Q_{k}}\right)\right)=\bigoplus_{\substack{b \in \mathbb{N}^{r} \\ \alpha \in\left(\sum_{k} b_{k} Q_{k}\right) \cap \mathbb{Z}^{n}}} \mathbb{C} x^{(\alpha, b)}
$$

The Multi-Proj of $R^{h}$ is the embedding of $X$ in a (weighted) multiprojective space.

Hence, given polynomials $f_{1}, \ldots, f_{m}$ with Newton polytopes $P_{1}, \ldots, P_{m}$, we can homogenize them as $\left(\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right)$ such that, for each $i, \tilde{f}_{i} \in$ $S_{d_{i}}^{h}$. If the polynomials $f_{1}, \ldots, f_{m}$ are generic enough, we can predict the properties of $\left(\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right)$ on $R^{h}$. For this reason, we will compute Gröbner bases over $R^{h}$ and use them to compute Gröbner bases over other rings. To formalize this, we introduce the notion of Gröbner bases over pointed affine semigroup algebras.

### 3.2 Gröbner bases for semigroup algebras

In this section, we will only consider affine semigroups with identity, that is, a semigroup $(S,+)$ with identity 0 , isomorphic to a finitely-generated submonoid of $\mathbb{Z}^{k}$. We say that $(S,+)$ is pointed if $\alpha \in S \backslash\{0\},-\alpha \notin S$, where $-\alpha$ is defined in the smallest group containing $S$. Given a semigroup, we define the semigroup algebra $\mathbb{C}[S]$ as the monomial $\mathbb{C}$-algebra generated by the monomials $\left\{x^{\alpha}: \alpha \in S\right\}$.

If ( $S,+$ ) is a pointed semigroup, it is possible to define a monomial ordering $<$ for $\mathbb{C}[S]$; see [53, Def. 3.1]. For example, if $S \subset \mathbb{N}^{s}$, for some $s \in \mathbb{N}$, then any monomial ordering for the standard algebra $\mathbb{C}\left[x_{1}, \ldots, x_{s}\right]$ induces a monomial ordering on $\mathbb{C}[S]$. Hence, we can define Gröbner bases for ideals in $\mathbb{C}[S]$ using the standard definition, i.e., $G \subset \mathbb{C}[S]$ is a Gröbner basis for an ideal $I \subset \mathbb{C}[S]$ with respect to a monomial ordering $<$ if $G \subset I$ and, for every $f \in I$, there is $g \in G$ and $x^{\alpha} \in \mathbb{C}[S]$ such that $x^{\alpha} L M_{<}(g)=L M_{<}(f)$, where $L M_{<}(\cdot)$ is the leading monomial in $\mathbb{C}[S]$ with respect to $<$.

Given $\alpha \in \mathbb{R}^{n} \times \mathbb{R}^{r}$, let $\pi_{1}$, $\pi_{2}$ be its projections onto $\mathbb{R}^{n}$ and $\mathbb{R}^{r}$, respectively. Clearly, as $S$ is a semigroup, $\pi_{1}(S)$ and $\pi_{2}(S)$ are also semigroups. In what follows, we assume that, for the affine pointed semigroup $S$, it holds that $\pi_{1}(S)$ and $\pi_{2}(S)$ are also an affine pointed semigroups and the preimage of each $\gamma \in \pi_{2}(S)$ is finite. Somehow, we will think about $\pi_{2}(S)$ as a grading of $S$. In this case, we say that a monomial ordering $<$ for $S$ is graded if there is another monomial ordering $<_{1}$ for $\pi_{1}(S)$ and $<_{2}$ for $\pi_{2}(S)$ such that

$$
x^{\alpha}<x^{\beta} \Longleftrightarrow\left\{\begin{array}{l}
x^{\pi_{2}(\alpha)}<2 x^{\pi_{2}(\beta)} \text { or } \\
x^{\pi_{2}(\alpha)}=x^{\pi_{2}(\beta)} \text { and } x^{\pi_{1}(\alpha)}<_{1} x^{\pi_{1}(\beta)}
\end{array}\right.
$$

The algebra $R^{h}$ defined in Subsection 3.1 corresponds to the semigroup algebra associated with the semigroup

$$
S^{h}:=\left\{(\alpha, b) \in \mathbb{Z}^{n} \times \mathbb{Z}^{r}: \alpha \in\left(\sum_{i} b_{i} Q_{i}\right) \cap \mathbb{Z}^{n}\right\}
$$

If 0 is a vertex of each $Q_{i}$ and of $\sum_{i} Q_{i}$, we can see that $S^{h}$ satisfies the assumptions on the projections $\pi_{1}$ and $\pi_{2}$ detailed above. As the solution set of $\left(f_{1}, \ldots, f_{m}\right)$ over $\left(\mathbb{C}^{*}\right)^{n}$ does not change if we multiply the polynomials by monomials, equivalently, if we translate their Newton polytopes, we will assume with no loss of generality than 0 is a vertex of the aforementioned polytopes. Therefore, we consider the semigroup algebra

$$
R:=\mathbb{C}\left[\pi_{1}\left(S^{h}\right)\right] \subset \mathbb{C}\left[x_{1}^{ \pm}, \ldots, x_{n}^{ \pm}\right]
$$

We define $\pi$ as the homomorphism between $R^{h}$ and $R$ taking $x^{\alpha} \mapsto$ $\pi\left(x^{\alpha}\right):=x^{\pi_{1}(\alpha)}$. This morphism acts as a dehomogenization and it relates to graded orders as follow. Given a Gröbner basis $G$ of an ideal of $I \subset R^{h}$ with respect to a graded order $<$, the set $\pi(G)$ is a Gröbner basis for $\pi(I) \subset R$ with respect to $<_{1}$ [53, Prop. 3.5].

### 3.3 Computing Gröbner bases

In this section we discuss an algorithm to compute Gröbner bases for an ideal $\left\langle f_{1}, \ldots, f_{r}\right\rangle \subset R$ with respect to a monomial ordering $<_{1}$. For this, we will compute Gröbner bases for $\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{r}\right\rangle \subset R^{h}$ with respect to a graded monomial ordering < associated with $<_{1}$. In what follows, we ignore the subindex from the ordering $<_{1}$.

For each multidegree $b \in \mathbb{N}^{r}$, we will construct a Macaulay matrix $M_{b}$ where the columns are indexed by the monomials of degree $b$ in $R^{h}$, the rows by the pairs $\left(i, x^{\beta}\right)$, where $x^{\beta}$ is a monomial of degree $b-d_{i}$. The rows are sorted in decreasing order with respect to $<$ and the columns are sorted in decreasing order in the following way: if $i>j$, or $i=j$ and $x^{\alpha}>x^{\beta}$, then $\left(i, x^{\alpha}\right)<\left(j, x^{\beta}\right)$. The element of the matrix $M_{b}$ of column index $x^{\alpha}$, and row $\left(i, x^{\beta}\right)$ is the coefficient of the monomial $x^{\alpha}$ in $x^{\beta} \tilde{f}_{i}$. As Lazard realized [80], if we perform Gaussian elimination with no pivoting, we obtain new rows representing elements in the Gröbner basis of $\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right\rangle$ of degree $b$. As the Gröbner basis is finite, if we perform this computation for sufficiently many different degrees, we recover the complete Gröbner basis.

A classical bottleneck of Gröbner bases computations are reductions to zero. In the sketched algorithm, these reductions to zero correspond to the rows reducing to zero after performing Gaussian elimination. An established way of speeding up computations is to avoid these reductions using the F5 criterion [45, 55]. In this context, the F5 criterion is called Matrix-F5 [45, Sec. 3] and translates into skipping some rows from the construction of the matrix. The F5 criterion tells us that the row indexed by $\left(i, x^{\beta}\right)$ will reduce to zero after performing Gaussian elimination if and only if $x^{\beta}$ is the leading monomial of a polynomial of degree $b-d_{i}$ in the colon ideal $\left(\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{i-1}\right\rangle: f_{i}\right)$, i.e., $x^{\beta} \in L M\left(\left(\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{i-1}\right\rangle: f_{i}\right)\right)_{b-d_{i}}$. We can organize our computation in such a way that we can recover from it $L M\left(\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{i-1}\right\rangle\right)_{b-d_{i}}[45$, Sec. 3]. Hence, when these two sets agree, we can skip every reduction to zero, that is, the F5 criterion is optimal. A classical setting where this happens is when $f_{i}$ is not a zero divisor in $R^{h} /\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{i-1}\right\rangle$. Hence, when $\left(\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right)$ is a regular sequence, i.e., when the previous condition holds for every $i$, then we can avoid every reduction to zero. In the unmixed case, we have that $r=1$ and the ring $R^{h}$ is $\mathbb{N}$-graded. It can be shown that this is a Cohen-Macaulay ring and that, if $m \leq n$,
a generic unmixed sparse system $\left(\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right)$ forms a regular sequence on $R^{h}$ [69]. This last fact was exploited by Faugère, Spaenlehauer, and Svartz [53] to compute Gröbner bases for unmixed systems. However, in the mixed case, this is not true anymore, that is, our system $\left(\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right)$ usually does not form a regular sequence on $R^{h}$. To solve this problem, together with Faugère and Tsigaridas [8, 9], we observed that the F5 criterion is optimal at certain degree $b$ whenever $\left(\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{i-1}\right\rangle: f_{i}\right)_{b-d_{i}}=\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{i-1}\right\rangle_{b-d_{i}}$, for each $i$. This condition can be enforced by the vanishing of the first homology of the Koszul complex of $\left(\tilde{f}_{1}, \ldots, \tilde{f}_{i}\right)$ at degree $b$, for each $i$. We can prove that, if the subscheme defined by $\left(\tilde{f}_{1}, \ldots, \tilde{f}_{i}\right)$ on $X$ is a complete intersection, then the first homology of the Koszul complex of $\left(\tilde{f}_{1}, \ldots, \tilde{f}_{i}\right)$ vanishes at any degree $b \geq \sum_{j \leq i} d_{i}$. Moreover, this bound can be improved for special systems. For example, for multihomogeneous systems over $\mathbb{P}^{n_{1}} \times \cdots \times \mathbb{P}^{n_{r}}$, we have that $b \geq \sum_{j \leq i} d_{i}-\left(n_{1}, \ldots, n_{r}\right)$ suffices [8] ${ }^{8}$. Hence, we can predict all the reductions to zero appearing at "big-enough" degree. Moreover, if our objective is to only compute Gröbner bases over $R$, we will just avoid the degrees at which we cannot predict every reduction.

Remark 3.2. This strategy generalizes partially previous work on computing Gröbner bases for special sparse systems such as bilinear [58] and weighted homogeneous systems [59].

To derive complexity bounds, we need to construct a bound of the maximal degree of an element in a Gröbner bases of $\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right\rangle$. However, the monomial orderings defined over semigroup algebras, in general, does not have the dehomogenization property of GRevLex [5, Ex. 7.2.1]. Moreover, we cannot talk about generic coordinates for the solutions, as random change of coordinates destroys the sparsity. Hence, we cannot use strategies as [3] to construct bounds for the maximal degrees of the elements in the Gröbner bases. As we will show in the next section, we can work around this problem in the zero-dimensional case, i.e., when the subscheme associated to $\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right\rangle$ has dimension zero.

### 3.4 Zero-dimensional systems

In this section, we will construct bounds for computing Gröbner bases of zero-dimensional ideals. Classically, to compute Gröbner bases for zero-dimensional systems, we first calculate a Gröbner basis with respect to GRevLex and then we use FGLM to recover the Gröbner basis with respect to the ordering we want. We do so because, with respect to GRevLex, the Gröbner basis involves elements of smaller degrees, for which we may find upper bounds. Generally, the maximal degree of an element in a Gröbner basis of a zero-dimensional ideal is given by the Castelnuovo-Mumford regularity of its homogenization [27, Cor. 3]. In the semigroup setting,

[^5]where there usually does not exist a GRevLex ordering, the maximal degree of an element in the Gröbner basis could be higher than the regularity; see [5, Sec. 8.3.1]. To work around this problem, in [8, 9], we presented a way to recover the multiplication maps by truncating our computation of Gröbner bases at a certain degree. By doing so, we bounded the complexity of our computation. In what follows, we explain the idea behind this strategy in the general case of mixed sparse polynomial systems and refer the reader to [5, Chp. 8] for improvements in special cases.

Following the notation from the previous section, we consider an affine sparse system $\left(f_{1}, \ldots, f_{m}\right)$ such that the subscheme $Y$ associated with $\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right\rangle \subset R^{h}$ has only solutions on $\left(\mathbb{C}^{*}\right)^{n}$ (in particular, $Y$ is finite as $X$ is compact). That is, it has no solutions at infinity. We assume without loss of generality that $1, x_{1}, \ldots, x_{n} \in R .{ }^{9}$ In this case, we can prove [9, Lem. 4.11],

$$
R / \pi\left(\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right\rangle\right) \simeq \mathbb{C}\left[x_{1}, \ldots, x_{n}\right] /\left(\left\langle f_{1}, \ldots, f_{n}\right\rangle:\left\langle\Pi_{i} x_{i}\right\rangle^{\infty}\right)
$$

Our strategy will be to perform FGLM using multiplication maps over $R / \pi\left(\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right\rangle\right)$ to recover the Gröbner basis of $\left\langle f_{1}, \ldots, f_{m}\right\rangle$ : $\left\langle\Pi_{i} x_{i}\right\rangle^{\infty}$ in $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$. In what follows, we focus on computing these multiplication maps by adapting Emiris and Rege [49].

Consider $b=\sum_{i} d_{i}$, where $d_{i}$ is the multidegree of $\tilde{f}_{i}$. Let $d_{0}$ be a multidegree such that $\Delta_{n} \cap \mathbb{N}^{n} \subset \pi\left(R_{d_{0}}^{h}\right)$, that is, after dehomogenization, we can obtain a linear form in $R$. At the degrees $b$ and $b+$ $d_{0}$, the ideal $R^{h} /\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right\rangle$ is a vector space of dimension $\delta$ equal to the number of points in $Y$, counting multiplicities [83, Thm. 3]. Let $B_{0}$ be the set of monomials of degree $d$ which are not leading monomials in $L M\left(\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right\rangle\right)$. The cardinality of $B_{0}$ is $\delta$. Moreover, also by [83, Thm. 3], as there are no solutions outside $\left(\mathbb{C}^{*}\right)^{n}$, for any monomial $x^{\alpha} \in R_{d_{0}}^{h}, R_{b+d_{0}}^{h}=\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{m}, x^{\alpha}\right\rangle_{b+d_{0}}$, and using the F5 criterion, it is easy to see that the monomial set $\left\{x^{\alpha} x^{\beta}\right.$ : $\left.x^{\beta} \in B_{0}\right\}$ forms a basis of the vector space $\left(R^{h} /\left\langle\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right\rangle\right)_{b+d_{0}}$. Let $x^{\alpha_{0}}$ be the monomial of degree $b+d_{0}$ such that $\pi\left(x^{\alpha_{0}}\right)=1$. Then, we can use our Macaulay matrix at degree $b+d_{0}$ related to $\left(\tilde{f}_{1}, \ldots, \tilde{f}_{m}\right)$ to rewrite any product $x^{\beta} f_{0}$, where $f_{0} \in R_{d_{0}}^{h}$ and $x^{\beta} \in$ $B_{0}$, as a linear combination of monomials in $\left\{x^{\alpha} x^{\beta}: x^{\beta} \in B_{0}\right\}$. By dehomogenizing this relation, we recover the multiplication map of $\pi\left(f_{0}\right)$ over $R /\left\langle\pi\left(\tilde{f}_{1}\right), \ldots, \pi\left(\tilde{f}_{m}\right)\right\rangle$, with respect to the basis $\pi\left(B_{0}\right)$. As the monomials $1, x_{1}, \ldots, x_{n}$ belong to $\pi\left(B_{0}\right)$, we can recover every multiplication map in $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right] /\left(\left\langle f_{1}, \ldots, f_{n}\right\rangle:\left\langle\Pi_{i} x_{i}\right\rangle^{\infty}\right)$.

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[^0]:    $\overline{{ }^{1} \text { The evaluation }}$ of $g \in \mathbb{C}\left[x_{1}, \ldots, x_{n}\right] /\langle f\rangle$ at a point $p \in \mathbb{C}^{n}$ corresponds to $\bar{g}(p)$, where $\bar{g} \in \mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ is such that $\bar{g}$ belongs to the class of $g$ in $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right] /\langle f\rangle$.

[^1]:    ${ }^{2}$ Heuristically, implementations of the F4 algorithm to compute Gröbner bases [60] can benefit partially from the sparsity of the inputs; see, e.g., [13, 84].
    ${ }^{3}$ We identify the systems using the multi-projective space because multiplying the polynomials by non-zero constants does not change their solution set.

[^2]:    ${ }^{4}$ We follow Esterov's [51] and D'Andrea \& Sombra's [40] definition of the sparse resultant. That is, the polynomial $\operatorname{Res}_{A}$ might not be irreducible, but it is the $d$-power of an irreducible polynomial, the eliminant, where $d$ is the degree of $\pi$ restricted to $\Omega$; see [40, Sec. 2]. Classically [63, 93], the sparse resultant was defined as the eliminant.
    ${ }^{5}$ Beware that the toric variety constructed in [34, Chp. 7.3] and $X$ might not agree ( $X$ is its normalization; see [35, Chp. 3.A]), but the argument extends to $X$.

[^3]:    ${ }^{6}$ However, it is not a submatrix of maximal size; see [83] for more details.

[^4]:    ${ }^{7}$ Beware that the arguments at our exposition are not formal. To prove that the determinant of the complex is the resultant we need to consider the Koszul (or, more generally, the Weyman) complex not over $X$, but over $X \times \mathbb{P}^{A}$, i.e., the construction has to be done using systems with generic (symbolic) coefficients; see [63].

[^5]:    ${ }^{8} \mathrm{~A}$ spectre is haunting this argument - a spectral sequence associated to the Koszul complex of sheaves mentioned in the previous section. Normal projective toric varieties are Cohen-Macaulay, so if $\left(\tilde{f}_{1}, \ldots, \tilde{f}_{i}\right)$ defines a complete intersection $Y$ on $X$, the Koszul complex gives a resolution for the structure sheaf $O_{Y}$. In this case, we can transform this complex into a resolution of $R^{h} /\left\langle f_{1}, \ldots, f_{i-1}\right\rangle$ at degree $b \in \mathbb{N}^{r}$, if for each $I \subseteq\{1, \ldots, i\}, H^{t}\left(X, O_{X}\left(\sum_{j}\left(b_{j}-\sum_{k \in I} d_{j, k}\right) D_{Q_{j}}\right)\right)=0$, for $t>0$. It is easy to construct a priori bounds for $b$ as $\sum_{j} d_{i}$ using vanishing theorems for the sheaf cohomology of line bundles over toric varieties; see, e.g., [11, Sec. 4]. In our papers [8, 9], we skipped this discussion by defining the concept of Koszul regular systems as the systems such that, for every subsystem, the first homology of its associated Koszul complex vanishes at certain degrees, which we can predict from the previous analysis.

[^6]:    ${ }^{9}$ If the monomials do not belong to $R$, we change the toric variety $X$ by another one associated with $P+\Delta_{n}$. As $Y \subset\left(\mathbb{C}^{*}\right)^{n}$, this change does not affect $Y$ and adds a degree $d_{0}$ to $R^{h}$ as wanted; see [ 9, Sec. 4].

