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ABSTRACT

Listing dense subgraphs in large graphs plays a key task in varieties of network analysis applications like community detection. Clique, as the densest model, has been widely investigated. However, in practice, communities rarely form as cliques for various reasons, e.g., data noise. Therefore, k-plex, - graph with each vertex adjacent to all but at most k vertices, is introduced as a relaxed version of clique. Often, to better simulate cohesive communities, an emphasis is placed on connected k-plexes with small k. In this paper, we continue the research line of listing all maximal k-plexes and maximal *k*-plexes of prescribed size. Our first contribution is algorithm *ListPlex* that lists all maximal k-plexes in $O^*(\gamma^D)$ time for each constant k, where γ is a value related to k but strictly smaller than 2, and D is the degeneracy of the graph that is far less than the vertex number *n* in real-word graphs. Compared to the trivial bound of 2^n , the improvement is significant, and our bound is better than all previously known results. In practice, we further use several techniques to accelerate listing k-plexes of a given size, such as structural-based prune rules, cache-efficient data structures, and parallel techniques. All these together result in a very practical algorithm. Empirical results show that our approach outperforms the state-of-the-art solutions by up to orders of magnitude.

CCS CONCEPTS

• Information systems → Web mining; • Theory of computation \rightarrow Graph algorithms analysis.

KEYWORDS

Listing maximal k-plexes, Graph algorithms, Worst-case time guarantee, Community detection, Parallelization

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1.1 Motivation

Finding cohesive groups (or communities) has received a lot of attention from various areas such as social network analysis and web mining, and is also a fundamental problem in graph algorithms. The community can be modeled in many ways. For example, the notion of *clique* is the strictest and arguably the most studied community model. A clique is a subgraph in which vertices are pairwise connected, i.e., a complete subgraph. A large body of literature dedicated to related problems has emerged, e.g., enumerating cliques in graphs [29], sparse graphs [9, 18], uncertain graphs [22], limited main memory [12], and optimizing the running time as a function of the output [17]. The clique model has also been applied in many domains such as data mining [12], bio-informatics [8] and ad-hoc wireless network [10].

In real-world graphs, due to various reasons such as the existence of data noise, communities rarely appear in the form of cliques [1, 14, 15]. Therefore, other forms of relaxed cliques are proposed as relaxations of the notion of clique. For example, the k-core [11] relaxes the vertex degree, k-club [23] relaxes pairwise distance of vertices and k-clique densest subgraph [25] relaxes density of induced subgraph. In this paper, we continue on this line of research by focusing on k-plex, - the notion that has been receiving increasing attention and popularity in recent years [14, 15, 28, 31].

A k-plex is a relaxed clique model first proposed in [24]. A k-plex is a graph in which each vertex's degree is at least n - k, where n is the number of vertices in the graph. In other words, a k-plex allows every vertex missing at most k links to other vertices (including itself) compared to the clique. Note that a 1-plex is just a clique. A k-plex in a graph is called *maximal* if and only if it is not a subgraph of any larger k-plex.

Listing maximal k-plexes. In this paper, we will study the problem of listing all maximal k-plexes from a given graph. It would seem that the listing of maximal k-plexes will be also useful in applications where maximal clique listing is applied. Additionally, the *k*-plex listing has other potential applications like link prediction.

From the theoretical point of view, listing all maximal k-plexes is hard. In fact, for any given k, this problem is NP-hard [1] and it is known that the number of maximal *k*-plexes is exponential in the

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worst-case [21]. Therefore, a large number of existing studies focus on the design of practically efficient methods. The majority of these algorithms have been derived and motivated by the Bron-Kerbosch algorithm [7], though it was originally designed to only list maximal cliques. Wu and Pei (2007) adapted the Bron-Kerbosch algorithm to list maximal k-plexes with a few new rules to prune unnecessary searches [27]. Wang et al. (2017) integrated more heuristic pruning rules and applied multi-thread parallelization technique [26]. Zhou et al. (2020) devised a novel branch heuristic with a worst-case time complexity proof. With their branch heuristic, the running time of Bron-Kerbosch algorithm is improved from $O^*(2^n)$ to $O^*(\gamma_k^n)$ where γ_k is related to k but strictly smaller than 2¹. Aside from the Bron-Kerbosch variants, there is another type of algorithms which have bounded *delay* between the output of two consecutive solutions. Berlowitz et al. (2020) initialized such kind of study by providing a polynomial-time delay algorithm for the problem [4, 13].

Listing large maximal k-plexes. We also study the problem of listing large maximal k-plexes, i.e., listing maximal k-plexes which have at least l vertices, l being a large number, say at least 2k-1. The problem was originally proposed to amend two issues that arise in modeling the communities by maximal k-plexes [14–16, 31]. First, it is observed that there are enormous maximal k-plexes in real-world graphs, and empirically most of them are small or even unconnected. However, in community detection application, communities should be large and densely connected subgraphs. Second, existing maximal k-plex listing algorithms can only handle graphs with hundreds to thousands of vertices in days. But large sparse graphs are ubiquitous these days, e.g., the webbase-2001 webgraph has more than a hundred million vertices and more than a billion edges [6]).

Fortunately, by requiring that the output *k*-plexes must be at least larger than a threshold l ($l \ge 2k - 1$), the two issues can be alleviated. Due to the structural property of k-plexes (Property 3 in [28]), a k-plex with at least 2k - 1 vertices is densely connected, i.e., the shortest length of paths between every two vertices is not larger than 2. Therefore, the first issue does not exist. For the second issue, with a lower bound requirement on the size of output k-plexes, the performance of listing algorithm can be also accelerated with many powerful strategies [14-16]. For instance, in [15, 16], Conte et al. (2017) took advantage of the size constraint to remove a large portion of unfruitful vertices from the input graph, which made Berlowitz et al.'s listing algorithm possible to run on graphs of millions of vertices. Conte et al. (2018) further used decomposition and parallel techniques, leading to a listing algorithm capable of running on some web-scale graphs, e.g., the it-2004 graph. Zhou et al. (2020) used the same decomposition framework as in [14] so that their Bron-Kerbosch pivot heuristic can accommodate large real-world graphs.

1.2 Contributions

Motivated by the aforementioned studies, we develop the most efficient algorithm for listing both maximal *k*-plexes and large maximal *k*-plexes from sparse real-world graphs.

1. We propose ListPlex, an algorithm that lists all maximal kplexes with provably worst-case running time. The general idea of ListPlex is a marriage of new decomposition scheme and an efficient Bron-Kerboch search. Our analysis discloses that for each constant k, ListPlex has a worst-case time bound $O^*(\gamma_k^D)$ where D is the degeneracy number of the input graph and γ_k is related to k but strictly smaller than 2. As far as we know, it is the first algorithm that reduces the exponent of running time from n to D. Due to the power-law distribution of most real-world graphs, $D \ll n$ in most cases, e.g., the webbase-2001 has more than a thousand million vertices but its degeneracy number is only 1506. To some extent, this bound provides theoretical evidence for the good performance of our algorithm.

2. We optimize the practical performance of ListPlex for listing large maximal k-plexes of size at least 2k - 1. It is known that listing large maximal k-plexes is of more real-world importance than purely listing all maximal k-plexes. Thus, we study efficient implementation techniques from multiple perspectives. From algorithmic perspective, we suggest strong prune rules to reduce the search space of our algorithm. From the computational system perspective, we propose new data structures to reduce cache misses and increase parallelism. All optimization techniques bring evident speedup for processing large sparse real-world graphs.

Our experiments show that ListPlex outperforms the state-ofthe-art approaches in terms of both problems. For example, our parallel algorithm can list all large maximal 2-plexes (with l = 800) from the huge webbase-2001 graph with over one billion edges in 1 minutes. This is almost an order of magnitude speedup compared to the best-known parallel approach.

All codes are available at https://github.com/joey001/ListPlex.git.

2 BACKGROUNDS

2.1 Basic notations

Let G = (V, E) be a simple and undirected input graph, where V and E are the sets of vertices and edges, respectively. We will let n = |V| and m = |E| in this paper. For $v \in V$ and a positive integer k, we use $N_G^k(v)$ to denote the set of vertices with distance exactly k to v in G. The vertices in $N_G^k(v)$ are also called k-hop neighbors of v. The set $N_G^1(v)$ may be simply written as $N_G(v)$ and 1-hop neighbors may be simply called neighbors. The degree of a vertex v is $|N_G(v)|$. The maximum degree among all vertices in G is denoted by Δ . When the underlying graph G is clear from the content, we may ignore the subscript G and write $N_G^k(v)$ as $N^k(v)$. Given a vertex set $P \subseteq V$, let G[P] be the subgraph induced by P. The diameter of G is the maximum distance among all pairs of vertices in G.

A permutation of vertices $v_1 ldots v_n$ is called a *degeneracy ordering* (or *core ordering*) of the graph *G* if for each *i*, vertex v_i has the minimum degree in the induced subgraph $G[\{v_i, ..., v_n\}]$. The degeneracy ordering of a graph can be computed in linear time by the algorithm that repeatedly removes a node with the minimum degree until the graph becomes empty [2]. For a degeneracy ordering $\eta = v_1 \dots v_n$, the degree of v_i in $G[\{v_i, ..., v_n\}]$ is called the *core number of* v_i . It is known that for any degeneracy ordering of the same graph, the largest core number among all vertices is the same and is called as *degeneracy* (or core number). We denote it by *D*.

¹The notation O^* omits the polynomial factors.

As defined above, a k-plex is a graph such that each vertex is not adjacent to at most k vertices (including itself) in the graph. Thus, a 1-plex is a clique, i.e., a complete graph. A subgraph G' of G is called a *maximal* k-plex if G' is not a subgraph of any larger k-plex. So a maximal k-plex is always an induced subgraph. In this paper, we are interested in listing all maximal k-plexes of a graph.

PROBLEM 1 (LISTING MAXIMAL k-PLEXES). Given a graph G = (V, E), a positive integer k, list all maximal k-plexes of G.

2.2 Some properties

We present basic properties of *k*-plexes. These are important for our algorithm design. Proofs of the lemmas below as well as missing proofs in the rest of the paper are left in the Appendix.

LEMMA 1. Any induced subgraph of a k-plex is still a k-plex.

This property is known in the literature [13, 24, 30]. It will be frequently used in our algorithm. For example, we can validate the maximality of a k-plex, i.e., a k-plex G[P] is maximal if there is no vertex that can be added into G[P] such that G[P] is still a k-plex.

LEMMA 2. Any k-plex with at least 2k - 1 vertices is a connected graph with the diameter at most 2. A k-plex with at most 2k - 2 vertices may be disconnected.

Lemma 2 is also known in the literature [15, 29]. It shows 2k - 1 is a key boundary between connectedness and unconnectedness.

In practice, the *k*-plex is closely related to the community detection problem which asks for dense and large communities from a large network [14, 32]. Using maximal *k*-plex as a graph model of the community, we translate the community detection as listing maximal *k*-plexes that are at least connected, and with prescribed number of vertices. By Lemma 2, any *k*-plex of size at least 2k - 1 must be connected and even diameter-2 bounded. Therefore, it is rational to form the practical community detection as finding all maximal *k*-plexes of size at least l, where l is a given lower bound value and l must be at least 2k - 1.

PROBLEM 2 (LISTING LARGE MAXIMAL k-PLEXES). Given a graph G = (V, E), two positive integers k and l where $l \ge 2k - 1$, list all maximal k-plexes with at least l vertices.

2.3 Existing Bron-Kerbosch based algorithms

Before we present our algorithm, we introduce the Bron-Kerbosch algorithm and its variants as they are closely related to ours.

2.3.1 The fundamental Bron-Kerbosch Algorithm. Many existing algorithms for listing maximal *k*-plexes, as in [3, 26, 27], stem from the Bron-Kerbosch algorithm that was originally designed from listing maximal cliques [7, 12]. We review the main idea of the Bron-Kerbosch algorithm for listing *k*-plexes.

The algorithm is recursive. We leave the pseudo-code in Alg. 2 in the Appendix. It calls a recursive procedure *BKRec* with three disjoint sets as parameters, i.e., *P*, *C* and *X*. *P* represents the set

of vertices that should be contained in the k-plex in the current stage. *C* includes the remaining *candidate vertices* for enumerating. *X* contains *excluded vertices*. They are excluded from the k-plex to avoid non-maximal solutions.

BKRec lists all maximal *k*-plexes G[P'] satisfying the following three properties: (i) $P \subseteq P'$, (ii) $P' \subseteq P \cup C$, and (iii) $\forall v \in X$, the subgraph $G[\{v\} \cup P']$ is not a *k*-plex.

Given a graph G = (V, E) and an integer k > 0, the algorithm calls BKRec initialized with $P = X = \emptyset$ and C = V. Then the algorithm iteratively branches on a vertex in *C* by including it to either *P* or *X*. We will use BKPlex(*G*, *k*) to denote this algorithm.

Complexity. As mentioned in [31], the Bron-Kerbosch requires $O^*(2^n)$ time in the worst-case, where *n* is the number of vertices in the input graph. Although several pruning rules were suggested for the Bron-Kerbosch in [26, 27], but the worst-case running time bound was not improved.

2.3.2 Zhou et al.'s Pivot Heuristic. Zhou et al. (2020) improved the Bron-Kerbosch algorithm with a *pivot* heuristic [31]. They observed that for any graph G, either G is a k-plex or there is a vertex v not adjacent to at least k + 1 vertices in G, including itself. As such, they designed a pivot heuristic which always branches on the vertex of minimum degree in the graph.

Complexity. The pivot heuristic can reduce the total number of branches and then improve the worst-case running time from $O^*(2^n)$ to $O^*(\gamma_k^n)$, where γ_k is a number related to k but strictly smaller than 2.

2.3.3 Conte et al.'s Decomposition Algorithm. In [14], Conte et al. proposed a decomposition-based algorithm, namely D2K, for listing k-plexes with the diameter at most 2. D2K first sorts the vertices of G by degeneracy ordering v_1, \ldots, v_n . Then, for each v_i , D2K builds a subgraph $G_i = (V_i, E_i)$ induced by $\{v_i\} \cup N_{>_{\eta}}(v_i) \cup N_{>_{\eta}}^2(v_i)$. The Bron-Kerbosch algorithm is then called to search all maximal k-plexes in G_i . However, a maximal k-plex $G_i[P]$ of G_i is not a maximal k-plex of the original graph if a vertex preceding v_i can form a larger k-plex with P. Hence, for every maximal k-plex $G_i[P]$ emitted by the Bron-Kerbosch search algorithm, D2K further validates that no other vertex in v_1, \ldots, v_{i-1} can form a k-plex with P before outputting it.

Complexity. D2K restricts the search space to each subgraph $G_i = (V_i, E_i)$ and so the search size is bounded by $O^*(\sum_i 2^{|V_i|})$. Recall that D is the degeneracy of the input graph G and Δ is the maximum degree of G. It holds that $|V_i| \leq D\Delta$ for each i. Thus, $\sum_i 2^{|V_i|} \leq n2^{D\Delta}$. Due to the sparsity of many real-world graphs, Δ and D are normally small values. The algorithm thus performs better than the Bron-Kerbosch algorithm in these large graphs.

3 LISTING ALL MAXIMAL *k*-PLEXES

We present our algorithm, ListPlex, for listing all maximal k-plexes.

3.1 The main Structure

Our algorithm contains two parts that are to list maximal *k*-plexes of size at most 2k-2 vertices and at least 2k-1 vertices, respectively. As mentioned in Lemma 2, maximal *k*-plexes of size at most 2k-2

may not be connected and this kind of *k*-plex is not interesting in practice. In fact, usually the parameter *k* is also small and most previous algorithms only tested the cases of $k \le 5$. In our algorithm, we will modify the Bron-Kerbosch algorithm by adding the size constraint to find all maximal *k*-plexes of size at most 2k - 2.

Next, we will focus on listing maximal *k*-plexes of size at least 2k - 1. By Lemma 2, we know that maximal *k*-plexes of size at least 2k - 1 are connected graphs with the diameter at most 2. So following the idea of Conte et al.'s decomposition algorithm, we list maximal *k*-plexes containing a vertex v_i by only considering the local subgraph induced by $\{v_i\} \cup N_{>\eta}(v_i) \cup N_{>\eta}^2(v_i)$. However, we further use some techniques to reduce the search space again and get a significantly improved running time bound.

3.2 Listing maximal k-plexes larger than 2k - 2

In this subsection, we focus on listing all maximal k-plexes of size at least 2k - 1. The pseudo-code corresponds to the second part in Alg. 1. We will explain the idea and each step of the algorithm.

First, ListPlex sorts the V by a degeneracy ordering $\eta = v_1 \dots v_n$. From v_1 to v_n , ListPlex iteratively lists maximal v_i -leaded k-plexes with at least 2k - 1 vertices.

DEFINITION 1. Given an ordering $\eta = v_1 \dots v_n$ of the vertices of G, $a v_i$ -leaded k-plex is a k-plex G[S] such that v_i is in S and it holds that $v_i \prec_{\eta} u$ for each vertex $u \in S \setminus \{v_i\}$. A v_i -leaded k-plex is maximal if it is not a subgraph of any k-plex in $G[\{v_i, ..., v_n\}]$.

Note that a maximal v_i -leaded k-plex may not be maximal in the original graph G. So in our algorithm, when a maximal v_i -leaded k-plex is found, we also check its maximality in G.

The core part of the algorithm is to find all maximal v_i -leaded k-plexes. Instead of using a brute force method, we dramatically reduce the search space by utilizing the structural properties.

LEMMA 3. Given an ordering η of the vertices of G, let G[P] be a v_i -leaded k-plex induced by P and $|P| \ge 2k - 1$. Then G[P] must be a subgraph of $G[\{v_i\} \cup N_{>_{\eta}}(v_i) \cup N_{>_{\eta}}^2(v_i)]$. Furthermore, P contains at most k - 1 vertices from $N_{>_{\eta}}^2(v_i)$.

Let us call $G_i = G[\{v_i\} \cup N_{>_{\eta}}(v_i) \cup N^2_{>_{\eta}}(v_i)]$ as the seed graph of v_i . Given a v_i and a subset $S \subseteq N^2_{>_{\eta}}(v_i)$ such that $|S| \le k - 1$, let us call $P_s = \{v_i\} \cup S$ as a seed set. For set $P_s = \{v_i\} \cup S$, by Lemma 3, we call BKPivot to search maximal v_i -leaded k-plexes that must contain P_s . The elaboration of BKPivot is left to the next subsection. In the current stage, we specify that for each seed set $P_s = \{v_i\} \cup S$, BKPivot emits all maximal k-plexes that must include P_s , possibly include some vertices in $N_{>_{\eta}}(v_i)$ and must not include vertices in $N^2_{>_{\eta}}(v_i) \setminus S$.

For each maximal v_i -leaded k-plex G[P] found by BKPivot, List-Plex further tests the maximality of G[P] in the input graph G. That is to say, if a vertex in $N_{\leq \eta}(v_i)$ and $N^2_{\leq \eta}(v_i)$ can form a larger k-plex with P, then G[P] is not maximal in G. Otherwise, G[P] is maximal and P is emitted.

3.3 The BKPivot algorithm

We introduce BKPivot. It is also a branching algorithm following the style of the basic Bron-Kerbosch algorithm and it accepts three

Alg	gorithm 1: Our maximal <i>k</i> -plex listing algorithm
1 Li	stPlex(G,k)
2 be	egin
3	Part I: Use the basic Bron-Kerbosch algorithm to list all
	the maximal k-plexes of size at most $2k - 2$.
4	Part II:
5	Sort V by degeneracy ordering as $\{v_1,, v_n\}$
6	for $i \leftarrow 1,, n$ do
7	Build seed graph $G_i = G[\{v_i\} \cup N_{\geq_{\eta}}(v_i) \cup N^2_{\geq_{\eta}}(v_i)]$
8	for any $S \subseteq N^2_{>_n}(v_i)$ that $ S \le k - 1$ do
9	Build seed set
	$P_s = \{v_i\} \cup S, C_s \leftarrow N_{\succ_\eta}(v_i), X_s \leftarrow N^2_{\succ_\eta}(v_i) \setminus S$
10	Call BKPivot(G_i, k, P_s, C_s, X_s)
11	for each P emitted by BKPivot do
12	if $ P > 2k - 2$ and $\nexists u \in N_{\prec_{\eta}}(v_i) \cup N^2_{\prec_{\eta}}(v_i)$
	that $G[\{u\} \cup P]$ is a k-plex in G then
13	emit P
L	-

disjoint sets P, C and X playing the same roles as those in the Bron-Kerbosch algorithm. However, it additionally integrates some ideas into its branch scheme to reduce more vertices.

The pseudo-code is given in Alg. 3 in the Appendix. The recursive procedure, BKPivot(G, k, P, C, X), lists all maximal k-plexes that must subsume P, possibly include vertices in C and must not contain any vertex in X. The idea relies on the fact that, if $G[P \cup C]$ is a k-plex, then no further branches will be produced. Otherwise, there is a vertex in $P \cup C$ that has at least k + 1 non-neighbors in $G[P \cup C]$, including itself. In detail, BKPivot first checks the maximality of P. Afterwards, a vertex u_p of minimum degree in $G[P \cup C]$ is selected as pivot and BKPivot branches as follows:

- If u_p is not adjacent to at most k vertices in $P \cup C$, then $G[P \cup C]$ is a k-plex. In this case, we check if $G[P \cup C]$ is maximal in G. If so, emit $P \cup C$ and stop the current branch.
- Otherwise, u_p is not adjacent to q vertices in $P \cup C$, where $q \ge k + 1$. The consecutive branches are generated with respect to either $u_p \notin P$ and $u_p \in P$.
 - If $u_p \notin P$, we generate two branches by either moving u_p from *C* to *X* or moving u_p from *C* to *P*. The latter case will fall into the next case.
 - If $u_p \in P$, let $|P \setminus N(u_p)| = q_1$ and $|C \setminus N(u_p)| = q_2$. Then $q_1 + q_2 = q$. It is not hard to prove $q_1 < k$ and let $k' = k - q_1$. Thus, at most k' vertices in $C \setminus N(u_p)$ can be included in the *k*-plex. Denote $C \setminus N(u_p)$ as $\{u_1 \cdots u_{q_2}\}$ by an arbitrary order. we generate k' + 1 branches:
 - (a) In the first branch, u_1 is moved from *C* to *X*;
 - (b) In the second branch, u₁ is moved from C to P and u₂ is moved from C to X;
 - (c) In the *i*th branch where *i* is from 3 to k', {u₁, ..., u_{i-1}} are moved from C to P, and u_i is moved from C to X.
 - (d) In the last branch, {u₁, ..., u_{k'}} are moved from C to P and {u_{k'+1}, ..., u_{q2}} are moved from C to X.

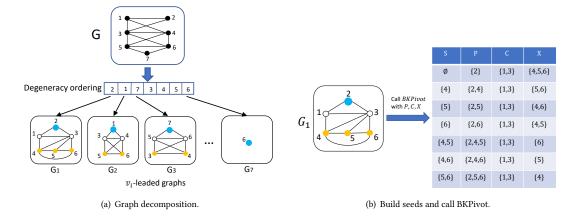


Figure 1: An example of the ListPlex algorithm. (a) sort V in degeneracy ordering η and induce seed graphs G_i for each $v_i \in \eta$. (b) enumerate $S \subseteq N_{\geq_n}^2(v_i)$ with bound $|S| \le k - 1$ (k = 3) and call BKPivot with P_s, C_s, X_s .

Correctness relies on Steps (a)-(d). Each maximal *k*-plex will fall into one case of (a)-(d). In the last case (d), the maximal *k*-plexes that include $\{u_1, ..., u_{k'}\}$ are visited. Because u_p and $\{u_1, ..., u_{k'}\}$ are in *P*, so $\{u_{k'+1}, ..., u_{q_2}\}$ can be excluded from further consideration since at most k' vertices in $C \setminus N(u_p)$ can be included in the *k*-plex. Fig. 2 shows an example of the branch scheme.

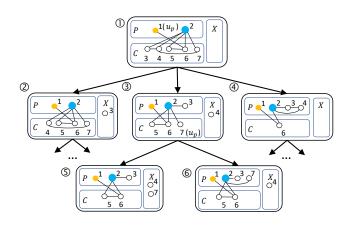


Figure 2: An example of BKPivot's branch scheme with k = 4. In node 1, pivot $u_p = 1 \in P$, $P \setminus N(u_p) = \{1, 2\}$ and $C \setminus N(u_p) = \{3, 4, 5, 7\}$. At most $k' = k - |P \setminus N(u_p)| = 2$ non-neighbors of u_p can be moved from C to P. Node 1 generates three branches, i.e., node 2, node 3 and node 4. In node 3, there are several vertices of minimum degree. Assume pivot $u_p = 7 \in C$, node 3 further generates two branches by moving u_p to X or P, i.e., node 5 and node 6.

3.4 Complexity analysis

The main complexity result is below. See Appendix for the proof.

THEOREM 1. Given a graph G = (V, E) with maximum degree Δ and degeneracy D, ListPlex(G, k) lists all maximal k-plexes without

repetition in time $O(n^{2k} + n(D\Delta)^{k+1}\gamma_k^D)$, where $\gamma_k < 2$ is the largest root of $1 = x^{-1} + \cdots + x^{-k-1}$.

Remark. Note that for each k, the exponential part of the running time of our algorithm is γ_k^D and γ_k is bounded by $O(2 - \frac{1}{2^{k+1}})$. The exponential part for the Bron-Kerbosch algorithm is 2^n . The exponential part for Conte et al.'s decomposition algorithm is $2^{D\Delta}$, The exponential part for Zhou et al.'s algorithm is $\gamma_k^{D\Delta}$. Hence, our algorithm provides a significant improvement of the previously known state-of-the-art algorithms. By keeping the status of at most k vertices at each branch, the *BKPivot* also greatly optimizes the space complexity of Zhou et al.'s pivot heuristic.

4 LISTING LARGE MAXIMAL *k*-PLEXES

In order to list large maximal *k*-plexes, i.e., maximal *k*-plexes of size at least l ($l \ge 2k - 1$), ListPlex can be reused by simply prohibiting the output of *k*-plexes smaller than *l*. However, this mildly changed algorithm is previewed to be inefficient in practice. In fact, it is possible to prune some branches early and improve the practical performance due to the import of this size constraint. For example, because $l \ge 2k - 1$, the search for maximal *k*-plexes of size at most 2k - 2 (Part I of Alg 1) can be simply dropped. For more stronger pruning techniques, let us first introduce an important observation.

LEMMA 4. Assume G[P] is a k-plex of G = (V, E), $|P| \ge l$. For any two vertices $u, v \in P$, if $(u, v) \in E$, then $|N(u) \cap N(v) \cap P| \ge l - 2k$, otherwise $|N(u) \cap N(v) \cap P| \ge l - 2k + 2$.

Note that this property was also observed in [14, 31].

4.1 Pruning seed graph G_i

Suppose the degeneracy ordering of G = (V, E) is $\eta = v_1, ..., v_n$. Recall that when we search the maximal v_i -leaded k-plexes, we build a seed graph G_i which is an induced graph of $\{v_i\} \cup N_{>_{\eta}}(v_i) \cup N_{>_{\eta}}^2(v_i)$. Denote the vertex and edge sets of G_i are V_i and E_i , respectively. We show rules to reduce the scale of G_i .

PRUNE RULE 1. Assume $u \in V_i$, if u satisfies

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- *u* ∈ N_{>η}(v_i) and |N(u) ∩ N(v_i) ∩ V_i| < l − 2k,
 or *u* ∈ N²_{>η}(v_i) and |N(u) ∩ N(v_i) ∩ V_i| < l − 2k + 2,

then u can be excluded from G_i without affecting the correctness of ListPlex.

4.2 Excluding unfruitful seed sets

Intuitively, if we can identify some unfruitful seed sets P_s , i.e., sets that are impossible to be a part of large k-plexes, we can save the forthcoming exponential search in G_i . With this in mind, we make use of the following pruning rule.

PRUNE RULE 2. Given a seed graph $G_i = (V_i, E_i)$, a seed set $P_s =$ $\{v_i\} \cup S \text{ where } S \subseteq N^2_{\geq_{\eta}}(v_i) \text{ and } |S| \leq k-1. \text{ Denote } C_s = N_{\geq_{\eta}}(v_i).$ For any two vertices $u, v \in S$, if

- $(u, v) \in E \text{ and } |N_{G_i}(u) \cap N_{G_i}(v) \cap C_s| < l 2k max(k-3, 0),$
- or $(u, v) \notin E$ and $|N_{G_i}(u) \cap N_{G_i}(v) \cap C_s| < l 2k + 2 2k +$ max(k - 3, 0).

then P_s is not in any maximal v_i -leaded k-plexes of size at least l.

It turns out that this prune rule dramatically improves the performance of our algorithm. In Fig. 3, we show the comparison between the algorithm using Prune Rule 2 and the one without it.

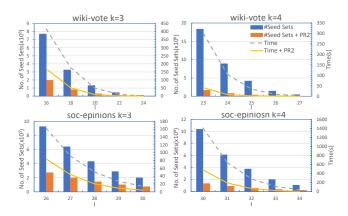


Figure 3: The number of seed sets and running time with and without Prune Rule 2.

5 **IMPLEMENTATION TECHNIQUES**

We present important techniques to implement ListPlex on modern computers: computers with multi-level caches and multiple cores.

5.1 Reducing cache misses

In an initial implementation, the algorithm searches maximal kplexes by visiting G_i and G alternatively. When a maximal v_i -leaded k-plex G[P] is found from G_i , ListPlex revisits the input graph *G* to validate if a vertex in $v_1, ..., v_{i-1}$ forms a larger *k*-plex with that solution. This results in a high amount of cache misses when checking the maximality of G[P]. Clearly, it is partially caused by the fact that the data of G is swapped out from the cache.

In order to reduce cache misses, we further make use of the diameter-2 property of large k-plexes. For each vertex v_i in ordering η , we build a bipartite graph $B_i = (L_i, R_i, F_i)$ where $L_i = N_{\prec_{\eta}}(v_i) \cup$ $N_{\prec_{\eta}}^{2}(v_{i}), R_{i} = \{v_{i}\} \cup N_{\succ_{\eta}}(v_{i}) \cup N_{\succ_{\eta}}^{2}(v_{i}) \text{ and edge set } F_{i} \subseteq L_{i} \times R_{i}$ is induced from G. When BKPivot finds a maximal v_i -leaded k-plex G[P] on G_i , we further validate if for each vertex $u \in L_i$,

then G[P] is maximal in G. With B_i , to check the maximality of a k-plex, we only need to visit G_i and B_i . Though the vertex numbers of G_i and B_i are both $O(D\Delta)$, in real-world graphs, the vertex numbers of G_i and B_i are far less than |V|, implying good locality. We compare the time and cache misses between the algorithm using B_i and the one without B_i in Figure 4.

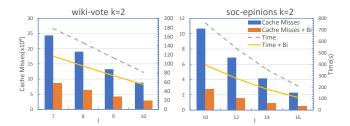


Figure 4: The total number of data cache misses and the running time with and without using bipartite graph B_i .

5.2 Parallelization

ListPlex also owns appealing parallel features. We introduce a shared-memory parallel version of ListPlex in this subsection.

It is observed that searches of maximal v_i -leaded k-plexes are independent for each v_i . Thus, for each vertex v_i , we create a task, say T_i , to process the search of all maximal v_i -leaded k-plexes. T_i owns its private seed graph G_i and bipartite graph B_i . Tasks $T_1, ..., T_n$ can be executed in parallel.

However, it could happen that most tasks stop but a few heavy tasks are still running. Specifically, when the number of running tasks is less than the number of available cores, computational resources are wasted. In such case, we split the branches of a running task T_i into new subtasks for the idle cores. Assume that T_n has been dispatched for execution but T_i (i < n) is still staying in the BKPivot(G_i, k, P, C, X) procedure. Then, when T_i detects some idle cores, it spawns recursive calls to BKPivot (G_i, k, P, C, X) as subtasks of T_i and dispatches them to idle cores. A subtask of T_i owns its sets P, C and X but shares G_i and B_i with T_i . Indeed, the schedule follows the work-stealing scheduling algorithm which accommodates well with the Bron-Kerbosch algorithm [5].

Fine-Grained Task. In parallel computing, the granularity of subtasks substantially affects the performance. Empirically, simple small tasks should not be spawned due to the overhead. In our implementation, we measure the complexity of a subtask, i.e., the time of executing BKPivot(G_i, k, P, C, X), by the size of C. Particularly, if |C| > 10 and there are some idle cores, we spawn new subtasks and assign them to available cores.

6 EXPERIMENTS

Experiments setup. The codes are written in C++11 and compiled by g++-9.3.0 with optimization option '-O3'. All experiments are conducted on a computer with a Ubuntu20.04 operating system, two-way Intel Xeon Gold 6130 CPUs (2.1GHz, 22MB L3-cache, 2 CPU chips and 32 physical cores in total), a 132G RAM and a 1T SSD. We also disable hyper-threading and turbo techniques. ListPlex is parallelized with the OpenMP library.

Dataset. In Table 1, we report basic information of benchmark graphs, including the number of vertices n, number of undirected edges m, maximum degree Δ and degeneracy D. These graphs are taken from Stanford Large Network Dataset Collection (SNAP) [20] and Laboratory for Web Algorithmics (LAW)². As we can see, the size of these graphs broadly ranges. Like [14], we divide them into three categories, i.e., small, medium and large graphs. Large graphs have more than ten million nodes, medium graphs are those with more than ten thousand nodes while the remaining graphs are classified as small graphs.

Table 1: Considered networks and their properties

Network	n	m	Δ	D
jazz	198	2742	100	29
ca-grqc	5241	14484	81	43
gnutella08	6301	41554	97	10
wiki-vote	7116	100763	1065	53
lastfm	7624	55612	216	20
as-caida	26475	53381	2628	22
soc-epinions	75888	405739	3044	67
soc-slashdot	82144	500480	2548	54
email-euall	265214	365569	7636	37
amazon0505	410236	2439436	2760	10
in-2004	1353703	13126172	21869	488
soc-pokec	1632803	22301964	14854	47
as-skitter	1696415	11095298	35455	111
soc-livejournal	4847571	68993773	14815	360
arabic-2005	22744080	639999458	575628	3247
uk-2005	39459925	936364282	1372171	584
it-2004	41291594	1150725436	1243927	3209
webbase-2001	118142155	1019903190	816127	1506

6.1 Listing all maximal k-plexes

In this section, we evaluate the performance of our ListPlex for listing all maximal *k*-plexes. We compare our ListPlex with the fastest known algorithm BKPivot [31] and the traditional Bron-Kerbosch algorithm BKPlex. Note that the competitive D2K [14] solver only outputs large maximal *k*-plexes, i.e., *k*-plexes of size at least *l* where l > 2k - 2. The recent solvers *GP* [26] and *Enum* [4] are not as time-efficient as BKPivot, see [31]. In case a solver cannot finish in 12 hours (43200 seconds) for an instance, we imperatively stop it. In the table, we mark the unfinished instances with OOT.

In Table 2, we show the time performance of these listing algorithms. We also report the parallel running time of ListPlex with 16 threads and the parallel speedup. Due to the huge amount of maximal *k*-plexes, neither of these algorithms is able to list all of them on medium or large graphs in 12 hours, even setting k = 2.

In terms of time, ListPlex outperforms both competitors for all these instances. For cases like wiki-vote with k = 2, ListPlex runs like 7× faster than the other algorithms. ListPlex also achieves a nearly perfect speedup for almost all cases except very simple ones. Unexpectedly, BKPlex runs faster than BKPivot for the last two larger graphs when k = 2.

Network	k	#k-plexes		Smaadum				
Network	^		BKPlex	BKPivot	ListPlex	ListPlex(16)	Speedup	
jazz	2	35214	648.864	0.29	0.086	0.408	0.211	
jazz	3	3602575	772.826	17.55	6.477	0.832	7.785	
jazz	4	193056583	3226.746	829.40	417.646	26.187	15.949	
ca-grqc	2	13718439	OOT	1858.02	649.985	40.880	15.899	
gnutella08	2	19866959	1500.208	3627.57	1117.858	70.207	15.922	
wiki-vote	2	66193264	10356.553	10671.92	1526.884	95.656	15.962	
lastfm	2	29086855	2643.394	6676.89	1989.701	124.525	15.978	

Table 2: Listing all maximal k-plexes in small graphs

6.2 Listing large maximal k-plexes

We evaluate the problem of listing large maximal k-plexes, i.e., maximal k-plexes that have at least l vertices. There are a rich number of solvers, e.g., GP [26], LP [15], D2K [14] and CommuPlex [31] for the problem. According to their empirical results, D2K and CommuPlex outperform earlier GP and LP in terms of practical running time. Thus, we compare our ListPlex with D2K and CommuPlex in this subsection. Also, D2K only outputs diameter-2 bounded maximal k-plexes. By setting l at least 2k - 1, we make sure that three compared algorithms output the same set of k-plexes. Also, we set a cut-off time of 12 hours for each instance.

The Sequential Performance. Let us first compare the sequential versions of D2K, CommuPlex and ListPlex. In Table 3, we show the sequential running time of different algorithms. For small networks, we set k = 2, 3 and 4, and l = 12, 20 and 30. For medium networks, we also set k = 2, 3 and 4 but we change l for different graphs, mainly because all three algorithms cannot list all the 2 to 4-plexes even k = 30 in 12 hours. As for the large networks, we leave the test in the parallel environment. These large graphs contain a dramatic number of maximal k-plexes that cannot be efficiently listed by these sequential algorithms.

ListPlex is the best performing algorithm for these instances. Exceptions can only be observed in graphs which contain very few maximal *k*-plexes, e.g., wiki-vote with k = 2 and l = 30. For the rest of these instances, ListPlex achieves a 4-100× speedup over CommuPlex and a 3-420× speedup over D2K. For example, ListPlex is able to list all 4-plexes with l = 20 for wiki-vote in half an hour but CommuPlex and D2K cannot finish in 12 hours. For some instances like soc-slashdot with k = 4 and l = 30, ListPlex is the only algorithm that lists all maximal *k*-plexes of size at least *l*. It is worth observing that, the running time of D2K and CommuPlex contrasts in different scenarios, e.g., D2K runs 10× faster than CommuPlex in in-2004 with k = 2 but CommuPlex performs much better in soc-epinions with k = 2 or 3. In total, the results show the great superiority of ListPlex over the existing algorithms.

The Parallel Performance. It is known that D2K also provides a parallel version that achieves almost linear speedup for many

²http://law.di.unimi.it/

Table 3: The running time of listing large maximal k-plexes from small and medium graphs by CommuPlex, D2K and ListPlex.

Graph (V , E)		l	#k-plexes	The running time (s)		Graph		,	#k-plexes	The running time (s)			
				CommuPlex	D2K	ListPlex	(V , E)	k		#K=piexes	CommuPlex	D2K	ListPlex
jazz (198, 2742)	4	12	2745953	25.218	33.054	4.498			12	2919931	75.871	115.757	17.653
lastfm (7624, 55612)	4	12	1827337	20.724	23.991	4.586		2	20	52	4.52	11.289	0.591
as-caida	3	12	281251	5.684	13.421	0.867			30	0	1.033	0.027	0.091
(26475, 53381)	4	12	15939891	300.388	785.506	47.98	wiki-vote		12	458153397	OOT	OOT	2185.598
amazon0505	2	12	376	1.825	0.641	0.137	(7116, 100763)	3	20	156727	595.636	1852.186	9.384
(410236, 2439436)	3	12	6347	11.359	0.77	0.286	(7110, 100703)		30	0	1.072	0.029	0.1
(410230, 2437430)	4	12	105649	47.049	5.338	1.171	1	4	20	46729532	OOT	OOT	1174.2
	2	12	412779	8.793	11.199	1.946		4	30	0	9.17	3.627	0.112
email-euall	3	12	32639016	619.384	1043.266	91.62			12	7679906	1537.506	172.987	47.475
(265214, 365569)	3	20	2637	10.754	53.691	0.429		2	20	94184	1064.371	20.03	15.161
	4	20	1707177	825.126	3800.889	24.089			30	3	662.64	8.637	9.557
	2	12	27208777	376.071	213.141	59.42	soc-pokec (1632803, 22301964)	3	12	520888893	OOT	OOT	1607.285
		20	11411028	227.016	137.159	32.988			20	5911456	1470.536	856.393	46.262
soc-slashdot		30	453	10.77	16.481	0.688			30	5	717.425	9.993	10.127
(82144, 500480)		12	2807943240	OOT	26029.006	7813.045		4	20	318035938	34048.155	OOT	1825.216
(02144, 300400)	3	20	1303148522	28361.707	15308.777	4538.022			30	4515	1140.117	111.987	11.211
		30	1679468	699.876	2066.598	51.364			12	49823056	843.9	735.589	193.307
	4	30	502699966	OOT	OOT	6680.261		2	20	3322167	137.427	180.061	19.382
	2	50	47969775	OOT	OOT	520.884	soc-epinions		30	0	8.995	12.109	0.492
as-skitter	2	100	0	1.793	2.951	0.716	(75888, 405739)	3	20	548634119	27037.614	35525.693	3072.267
(1696415, 11095298)	3	50	21070497438	OOT	OOT	OOT		3	30	16066	546.69	2591.439	6.123
	3	100	0	2.37	3.285	0.718		4	30	13172906	OOT	OOT	661.103
	2	50	25855779	7663.843	576.06	150.212		2	340	650322	2284.435	OOT	109.382
in-2004	2	100	9978037	5899.638	256.225	72.063	com-livejournal	2	345	0	57.548	13589.487	6.914
(1353703, 13126172)	3	50	29045783792	OOT	OOT	OOT	(4847571, 68993773)	3	340	555718694	OOT	OOT	22863.467
	3	100	4257410159	OOT	OOT	28384.76		3	345	3963139	24861.871	OOT	826.183

Table 4: The parallel running time of large networks by List-Plex and D2K with 16 threads.

Graph		1	#k-plexes	The running time (s)			
(V , E)	k	ı	#K-piexes	D2K(16)	ListPlex(16)		
	2	800	224870903	2195.272	714.159		
arabic-2005	2	1000	236897	151.328	40.202		
(22744080, 639999458)	3	800	>25062182205	OOT	OOT		
	3	1000	34155502	587.967	128.737		
	2	250	106243475	OOT	355.855		
uk-2005	2	500	256406	318.118	35.001		
(39459925, 936364282)	3	250	>18336111409	OOT	OOT		
	3	500	28199814	9506.661	121.726		
	2	2000	675111	340.904	41.983		
it-2004	2	3000	675111	307.735	38.468		
(41291594, 1150725436)	3	2000	197679229	4254.456	724.979		
	3	3000	197679229	4235.389	715.002		
	2	800	1599005	374.134	54.19		
webbase-2001	2	1000	1164383	346.393	53.651		
(118142155, 1019903190)	3	800	1785341050	36116.817	5521.386		
	3	1000	1484341137	35005.343	6960.816		

instances. In Table 4, we run the parallel ListPlex and D2K with 16 threads for large networks. Still, ListPlex runs about $3-8\times$ faster than D2K in these tested instances. In Fig. 5, we show the speedup achieved by ListPlex for large graphs with different ks and ls. Clearly, ListPlex also can reach a nearly perfect speedup in these instances. As both ListPlex and D2K scale well in large graphs, the superiority of ListPlex may be achieved by doing fewer work.

7 CONCLUSION

We studied the problems of listing maximal *k*-plexes and maximal *k*-plexes of prescribed size. We proposed ListPlex, a fast and scalable algorithm that efficiently solves the two problems in real-world graphs. Especially, ListPlex combines a new decomposition scheme with the branching algorithm, achieving a better theoretical complexity. When maximal *k*-plexes of size at least l ($l \ge 2k - 1$) are asked, ListPLex can be also used for listing these large maximal

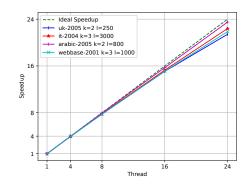


Figure 5: The speedup of ListPlex for the large graphs with different parameters.

k-plexes. For practical considerations, we designed some additional prune rules for listing large maximal k-plexes. These prune rules work very well in the context of large real-world graphs. Furthermore, we designed a new local bipartite graph to improve the cache performance of the algorithm, and parallel scheduling strategies to increase parallelism. Extensive empirical evaluations show the superiority of ListPlex over the state-of-the-art approaches for both problems.

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REFERENCES

- Balabhaskar Balasundaram, Sergiy Butenko, and Illya V Hicks. 2011. Clique relaxations in social network analysis: The maximum k-plex problem. Operations Research 59, 1 (2011), 133–142.
- [2] Vladimir Batagelj and Matjaz Zaversnik. 2003. An O(m) algorithm for cores decomposition of networks. arXiv preprint cs/0310049 (2003).
- [3] Matthias Bentert, Anne-Sophie Himmel, Hendrik Molter, Marco Marik, Rolf Niedermeier, and René Saitenmacher. 2018. Listing all maximal k-plexes in temporal graphs. In 2018 IEEE/ACM International Conference on Advances in Social Networks Analysis and Mining (ASONAM). IEEE, 41–46.
- [4] Devora Berlowitz, Sara Cohen, and Benny Kimelfeld. 2015. Efficient enumeration of maximal k-plexes. In Proceedings of the 2015 ACM SIGMOD International Conference on Management of Data. ACM, 431–444.
- [5] Jovan Blanuša, Radu Stoica, Paolo Ienne, and Kubilay Atasu. 2020. Manycore clique enumeration with fast set intersections. *Proceedings of the VLDB Endow*ment 13, 12 (2020), 2676–2690.
- [6] Paolo Boldi and Sebastiano Vigna. 2004. The WebGraph Framework I: Compression Techniques. In Proc. of the Thirteenth International World Wide Web Conference (WWW 2004). ACM Press, Manhattan, USA, 595–601.
- [7] Coen Bron and Joep Kerbosch. 1973. Algorithm 457: Finding All Cliques of an Undirected Graph. Commun. ACM 16, 9 (Sept. 1973), 575–577. https://doi.org/ 10.1145/362342.362367
- [8] Sergiy Butenko and Wilbert E Wilhelm. 2006. Clique-detection models in computational biochemistry and genomics. *European Journal of Operational Research* 173, 1 (2006), 1–17.
- [9] Lijun Chang, Jeffrey Xu Yu, and Lu Qin. 2013. Fast maximal cliques enumeration in sparse graphs. Algorithmica 66, 1 (2013), 173–186.
- [10] Y Chen, A Liestman, and Jiangchuan Liu. 2004. Clustering algorithms for ad hoc wireless networks. Ad hoc and sensor networks 28 (2004), 76.
- [11] James Cheng, Yiping Ke, Shumo Chu, and M Tamer Özsu. 2011. Efficient core decomposition in massive networks. In 2011 IEEE 27th International Conference on Data Engineering. IEEE, 51–62.
- [12] James Cheng, Linhong Zhu, Yiping Ke, and Shumo Chu. 2012. Fast algorithms for maximal clique enumeration with limited memory. In Proceedings of the 18th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining. ACM, 1240–1248.
- [13] Sara Cohen, Benny Kimelfeld, and Yehoshua Sagiv. 2008. Generating all maximal induced subgraphs for hereditary and connected-hereditary graph properties. J. Comput. System Sci. 74, 7 (2008), 1147–1159.
- [14] Alessio Conte, Tiziano De Matteis, Daniele De Sensi, Roberto Grossi, Andrea Marino, and Luca Versari. 2018. D2K: Scalable Community Detection in Massive Networks via Small-Diameter k-Plexes. In Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining. ACM, 1272– 1281.
- [15] Alessio Conte, Donatella Firmani, Caterina Mordente, Maurizio Patrignani, and Riccardo Torlone. 2017. Fast enumeration of large k-plexes. In Proceedings of the 23rd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining. ACM, 115–124.
- [16] Alessio Conte, Donatella Firmani, Maurizio Patrignani, and Riccardo Torlone. 2021. A meta-algorithm for finding large k-plexes. *Knowledge and Information Systems* (2021), 1–25.
- [17] Alessio Conte, Roberto Grossi, Andrea Marino, and Luca Versari. 2016. Sublinear-Space Bounded-Delay Enumeration for Massive Network Analytics: Maximal Cliques. In 43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016. 148:1-148:15.
- [18] David Eppstein and Darren Strash. 2011. Listing all maximal cliques in large sparse real-world graphs. In *International Symposium on Experimental Algorithms*. Springer, 364–375.
- [19] Fedor V. Fomin and Dieter Kratsch. 2010. Exact Exponential Algorithms (1st ed.). Springer-Verlag, Berlin, Heidelberg.
- [20] Jure Leskovec and Andrej Krevl. 2014. SNAP Datasets: Stanford Large Network Dataset Collection. http://snap.stanford.edu/data.
- [21] John W Moon and Leo Moser. 1965. On cliques in graphs. Israel journal of Mathematics 3, 1 (1965), 23-28.
- [22] Arko Provo Mukherjee, Pan Xu, and Srikanta Tirthapura. 2016. Enumeration of maximal cliques from an uncertain graph. *IEEE Transactions on Knowledge and Data Engineering* 29, 3 (2016), 543–555.
- [23] Foad Mahdavi Pajouh, Balabhaskar Balasundaram, and Illya V Hicks. 2016. On the 2-club polytope of graphs. Operations Research 64, 6 (2016), 1466–1481.
- [24] Stephen B Seidman and Brian L Foster. 1978. A graph-theoretic generalization of the clique concept. Journal of Mathematical sociology 6, 1 (1978), 139–154.
- [25] Charalampos Tsourakakis. 2015. The k-clique densest subgraph problem. In Proceedings of the 24th International Conference on World Wide Web. 1122–1132.
- [26] Zhuo Wang, Qun Chen, Boyi Hou, Bo Suo, Zhanhuai Li, Wei Pan, and Zachary G Ives. 2017. Parallelizing maximal clique and k-plex enumeration over graph data. J. Parallel and Distrib. Comput. 106 (2017), 79–91.

- [27] Bin Wu and Xin Pei. 2007. A parallel algorithm for enumerating all the maximal k-plexes. In Pacific-Asia Conference on Knowledge Discovery and Data Mining. Springer, 476–483.
- [28] Mingyu Xiao, Weibo Lin, Yuanshun Dai, and Yifeng Zeng. 2017. A fast algorithm to compute maximum k-plexes in social network analysis. In *Thirty-First AAAI Conference on Artificial Intelligence*. 919–925.
- [29] Mingyu Xiao and Hiroshi Nagamochi. 2017. Exact algorithms for maximum independent set. Information and Computation 255 (2017), 126–146.
- [30] Yi Zhou, Shan Hu, Mingyu Xiao, and Zhang-Hua Fu. 2021. Improving Maximum k-Plex Solver via Second-Order Reduction and Graph Color Bounding. In Proceedings of the AAAI Conference on Artificial Intelligence, Vol. 35. 12453–12460.
- [31] Yi Zhou, Jingwei Xu, Zhenyu Guo, Mingyu Xiao, and Yan Jin. 2020. Enumerating maximal k-plexes with worst-case time guarantee. In Proceedings of the AAAI Conference on Artificial Intelligence, Vol. 34. 2442–2449.
- [32] Jinrong Zhu, Bilian Chen, and Yifeng Zeng. 2020. Community detection based on modularity and k-plexes. *Information Sciences* 513 (2020), 127–142.

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A THE BRON-KERBOSCH ALGORITHM

Algorithm 2: The Basic Bron-Kerbosch Algorithm

```
1 BKPlex(G, k)
2 begin
     BKRec(G, k, \emptyset, V, \emptyset)
3
4 BKRec(G, k, P, C, X)
5 begin
6
         C \leftarrow \{v \in C : G[P \cup \{v\}] \text{ is a } k\text{-plex}\}
         X \leftarrow \{v \in X : G[P \cup \{v\}] \text{ is a } k\text{-plex}\}
7
         if C = \emptyset then
8
               if X = \emptyset then
 9
                    emit P
10
              return
11
         else
12
              for u \in C do
13
                    C \leftarrow C \setminus \{u\}
14
                    BKRec(G, k, P \cup \{u\}, C, X)
15
16
                    X \leftarrow X \cup \{u\}
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B THE BKPIVOT ALGORITHM

Algorithm 3: The Bron-Kerbosch algorithm with pivot heuristic for listing all maximal *k*-plexes.

1 BKPivot(G, k, P, C, X)2 begin $C \leftarrow \{v : v \in C \text{ and } G[\{v\} \cup P] \text{ is a } k\text{-plex}\}$ 3 $X \leftarrow \{v : v \in X \text{ and } G[\{v\} \cup P] \text{ is a } k\text{-plex}\}$ 4 if $C = \emptyset$ then 5 if $X = \emptyset$ then 6 emit P7 return 8 Find a vertex of minimum degree u_p in $G[P \cup C]$ 9 10 if $|N(u_p)| \ge |P| + |C| - k$ then if $\nexists v \in X$ that $G[P \cup C \cup \{v\}]$ is a k-plex then 11 emit $P \cup C$ 12 else if $u_p \in P$ then 13 Let $u_1, ..., u_{q_2}$ be an arbitrary ordering of $C \setminus N(u_p)$ 14 $k' \leftarrow k - |P \setminus N(u_p)|$ 15 $BKPivot(G, k, P, C \setminus \{u_1\}, X \cup \{u_1\})$ 16 for $i \in \{2, ..., k'\}$ do 17 $BKPivot(G, k, P \cup \{u_1, ..., u_{i-1}\}, C \setminus$ 18 $\{u_1, ..., u_i\}, X \cup \{u_i\})$ $BKPivot(G, k, P \cup \{u_1, ..., u'_k\}, C \setminus \{u_1, ..., u_{q_2}\}, X)$ 19 else 20 $BKPivot(G, k, P, C \setminus \{u_p\}, X \cup \{u_p\})$ 21 $BKPivot(G, k, P \cup \{u_p\}, C \setminus \{u_p\}, X)$ 22

C MISSING PROOFS

Proof of Lemma 2

PROOF. Let *u* and *v* be any pair of nonadjacent vertices in a *k*-plex. There are at most k - 1 vertices not adjacent to *u* and at most k - 1 vertices not adjacent to *v*. If the graph has more than 2k - 2 vertices, then there exists a vertex *w* that is adjacent to both of *u* and *v*. So the graph is connected and the diameter is at most 2.

Here is an example of a disconnected *k*-plex of size 2k - 2. The graph consists of two cliques of size k - 1. We can see that the graph is a *k*-plex since each vertex is not adjacent to *k* vertices (including itself). The number of vertices in the graph is 2k - 2.

Proof of Lemma 3.

PROOF. By Lemma 2, we know that the diameter of G[P] is bounded by 2. Since G[P] contains v_i , we know that G[P] can only be a subgraph of $G[\{v_i\} \cup N_{\geq_{\eta}}(v_i) \cup N^2_{\geq_{\eta}}(v_i)]$. The second claim holds due to the definition of *k*-plexes.

Proof of Lemma 4.

PROOF. Let us denote $O = P \setminus \{u, v\}$. Then $|O| \ge q - 2$.

- If *u* and *v* are adjacent, there are at most 2(*k*−1) vertices that are not common neighbors of *u* and *v* in *O*. Thus, |*N*(*u*) ∩ *N*(*v*) ∩ *P*| ≥ |*O*| − 2(*k* − 1) ≥ (*l* − 2) − (2*k* − 2) = *l* − 2*k*.
- If u and v are not adjacent, then there are at most 2(k-2) nonneighbors in O. Thus $|N(u) \cap N(v) \cap P| \ge |O| - 2(k-2) \ge (l-2) - (2k-4) = l - 2k + 2.$

Proof of Theorem 1.

PROOF. For the first part, the running time is bounded by the number of subsets of size at most 2k - 2 times the time to check its maximality. There are at most $O(\binom{n}{2k-2}) = O(n^{2k-2})$ subsets of size at most 2k - 2. By [31], the time to check the maximality of a k-plex is bounded by $O(n^2)$. So the running time is $O(n^{2k})$.

Before analyzing the part for listing maximal *k*-plexes of size at least 2k - 1, we first consider the running time bound of the procedure BKPivot. When $C = \emptyset$, we do not need to branch anymore. So we analyze our branching operations by measuring the number of vertices removed from *C*. The branching operation for the case $u_p \in P$ will generate k' + 1 subbranches. In the first subbranch, one vertex u_1 is removed from *C*. In the second subbranch, two vertices $\{u_1, u_2\}$ are removed from *C*. In the *i*th branch for $3 \le i \le k'$, exactly *i* vertices $\{u_1, \ldots, u_i\}$ are removed from *C*. In the last branch, q_2 vertices $\{u_1, \ldots, u_{q_2}\}$ are removed from *C*, where $q_2 \ge k' + 2$. If we use T(c) to denote the running time of BKPivot working on *C* with c = |C|, then we get the following recurrence

$$T(c) \leq T(c-1) + \dots + T(c-k') + T(c-q_2).$$

When $u_p \notin P$ ($u_p \in C$), we generate two branches each of which will remove one vertex u_p from *C*. In the latter case, we will follow with the above recurrence. Combining them together, we have

$$T(c) \leq T(c-1) + \cdots + T(c-k'-1) + T(c-q_2-1).$$

Note that $k' \le k - 1$ and $q_2 \ge k' + 1$. For the worst case that k' = k - 1 and $q_2 = k' + 1$, we get the recurrence

$$T(c) \le T(c-1) + \dots + T(c-k) + T(c-k-1)$$

Let γ_k be the largest root of function $1 = x^{-1} + \cdots + x^{-k-1}$. Then the running time bound of the algorithm is bounded by $O(\gamma_k^{|C|})$. In our algorithm, initially C is $N(v_i)$ and then $|C| \leq D$, where D is the degeneracy of the graph. We also note that γ_k is strictly smaller than 2. For example, when k = 1, 2, 3, 4 and $5, \gamma_k = 1.618, 1.839, 1.928, 1.966$ and 1.984, respectively. Details on solving recurrence relations and time analysis can be found in [19].

Next, we analyze the algorithm for listing maximal *k*-plexes of size at least 2k - 1. Note that computing the degeneracy order of a graph *G* is in O(m) [2]. For each vertex v_i in the degeneracy order, we find all maximal v_i -leaded *k*-plexes in the subgraph $G_i = G[\{v_i\} \cup N_{>_{\eta}}(v_i) \cup N_{>_{\eta}}^2(v_i)]$. Hereby, we enumerate all subsets $S \subseteq N_{>_{\eta}}^2(v_i)$ with size $|S| \leq k - 1$ and for each *S* we include it to *P* to generate an instance. So we will generate at most $|N_{>_{\eta}}^2(v_i)|^k$ instances. For each instance, we will call BKPivot with running time $O(\gamma_k^{|N_{>_{\eta}}(v_i)|})$. Additionally, in order to validate the maximality of a maximal v_i -leaded *k*-plex in *G*, the algorithm tries if any vertex in $N_{<_{\eta}}(v_i) \cup N_{<_{\eta}}^2(v_i)$ can form a *k*-plex with *P*. So, this will at most

add a factor of $|N_{\prec_{\eta}}(v_i)| + |N_{\prec_{\eta}}^2(v_i)| \le D + D\Delta$. In total, the running time is in $O((D + D\Delta) \sum_i |N_{\succ_{\eta}}^2(v_i)|^k \gamma_k^{|N(v_i)|}) = O(n(D\Delta)^{k+1} \gamma_k^D)$.

Proof of Prune Rule 1.

PROOF. Fix v with the leading vertex v_i in Lemma 4.

- If $u \in N_{>\eta}(v_i)$, then $(u, v_i) \in E$. Thus for any vertex $u \in P$, $|N(u) \cap N(v_i) \cap V_i| \ge |N(u) \cap N(v_i) \cap P| \ge l - 2k$,
- If $u \in N^2_{\geq_{\eta}}(v_i)$, then $(u, v_i) \notin E$. Thus for any vertex $u \in P$, $|N(u) \cap N(v_i) \cap V_i| \ge |N(u) \cap N(v_i) \cap P| \ge l - 2k + 2.$

Proof of Prune Rule 2.

PROOF. It is clear $|N_{G_i}(u) \cap N_{G_i}(v) \cap \{v_i\}| + |N_{G_i}(u) \cap N_{G_i}(v) \cap S| + |N_{G_i}(u) \cap N_{G_i}(v) \cap C_s| \ge |N(u) \cap N(v) \cap P|$. Because $u, v \in S$, then $|N_{G_i}(u) \cap N_{G_i}(v) \cap \{v_i\}| = 0$ and $|N_{G_i}(u) \cap N_{G_i}(v) \cap S| \le k - 1 - 2 = k - 3$. Thus, $|N_{G_i}(u) \cap N_{G_i}(v) \cap C_s| \ge |N(u) \cap N(v) \cap P|$ - max(k - 3, 0). According to Lemma 4, $|N(u) \cap N(v) \cap P|$ has a lower bound depending on whether $(u, v) \in E$ or not. Combining that, we present Prune Rule 2 as above.