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Optimization of Scoring Rules*

Yingkai Li[†]

Jason D. Hartline[‡] Liren Shan[§]

Yifan Wu[¶]

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Abstract

This paper introduces an objective for optimizing proper scoring rules. The objective is to maximize the increase in payoff of a forecaster who exerts a binary level of effort to refine a posterior belief from a prior belief. In this framework we characterize optimal scoring rules in simple settings, give efficient algorithms for computing optimal scoring rules in complex settings, and identify simple scoring rules that are approximately optimal. In comparison, standard scoring rules in theory and practice – for example the quadratic rule, scoring rules for the expectation, and scoring rules for multiple tasks that are averages of single-task scoring rules – can be very far from optimal.

1 Introduction

This paper provides a framework for a principal to optimize over proper scoring rules. Proper scoring rules are mechanisms that incentivize a forecaster to reveal her true beliefs about a probabilistic state. Proper scoring rules are well studied in theory and widely used in practice. The optimization framework of the paper is relevant for applications that include peer grading, peer prediction, and exam scoring.

Proper scoring rules incentivize a forecaster to reveal her true belief about an unknown and probabilistic state. The principal publishes a scoring rule that maps the reported belief and the realized state to a reward for the forecaster. The forecaster reports her belief about the state. The state is realized and the principal rewards the forecaster according to the scoring rule. A scoring rule is proper if the forecaster's optimal strategy, under any belief she may possess, is to report that belief. Proper scoring rules are also designed for directly eliciting a statistic of the distribution such as its expectation.

Not all proper scoring rules work well in any a given scenario. This paper considers a mathematical program for optimization of scoring rules where (a) the objective captures the incentive for the forecaster to exert effort and (b) the boundedness constraints prevent the principal from scaling the scores arbitrarily. For (a), we focus on a simple binary model of effort where the forecaster does or does not exert effort and with this effort the forecaster obtains a refined posterior distribution from the prior distribution on the unknown state (e.g., by obtaining a signal that is correlated with the state). We adopt the objective that takes the perspective of the forecaster at the point of the

^{*}The order of the authors are certified random. The records are available in https://www.aeaweb.org/journals/policies/random-author-order/search?RandomAuthorsSearch%5Bsearch%5D=4FJdnUr4sE80.

[†]Department of Computer Science, Northwestern University. Email: yingkai.li@u.northwestern.edu.

[‡]Department of Computer Science, Northwestern University. Email: hartline@northwestern.edu.

[§]Department of Computer Science, Northwestern University. Email: lirenshan2023@u.northwestern.edu.

[¶]Department of Computer Science, Northwestern University. Email: yifan.wu@u.northwestern.edu.

decision with knowledge of both the prior and the distributions of posteriors that is obtained by exerting effort. We want a scoring rule that maximizes the difference in expected scores for the posterior distribution and prior distribution. For (b), we impose the ex post constraint that the score is in a bounded range, i.e., without loss, between zero and one. Notice that this program would be meaningless without a constraint on the scores - otherwise the score could be scaled arbitrarily - and it would be meaningless without considering the difference in scores between posterior and prior - otherwise any bounded scoring rule scaled towards zero plus a constant close to the upper bound would be near optimal.

The first step in pursuing the optimization of scoring rules (according to the program of the previous paragraph) is characterizing the incentive constraint of the forecaster, i.e., that the forecaster's optimal strategy in either the effort or no-effort case is to report her true belief. McCarthy (1956) characterizes proper scoring rules as ones that induce a convex utility function for the forecaster, as a function of her belief, where the scoring rule can be expressed in terms of this utility function and its pseudo-gradients. This characterization is similar to the incentive compatible characterization of Rochet (1985) for multi-dimensional mechanism design which has enabled the design of revenue optimal mechanisms (e.g., Daskalakis, Deckelbaum, and Tzamos, 2017). Our optimization framework, with this characterization, enables the study of optimal scoring rules under many paradigms that have proven to be important for mechanism design.

There are several potentially important paradigms for optimization of proper scoring rules (cf. the paradigms for mechanism design). With a family of Bayesian environments for optimizing scoring rules:

- Characterization. Characterize the optimal scoring rule for any environment in the family (cf. Myerson, 1981).
- **Computation.** Give a polynomial time algorithm for identifying and executing the optimal scoring rule for any given environment in the family (cf. Cai, Daskalakis, and Weinberg, 2012a,b; Alaei, Fu, Haghpanah, Hartline, and Malekian, 2019).
- **Simple Approximation.** For any environment in the family, identify a simple scoring rule that approximates the optimal scoring rule (cf. Hartline and Roughgarden, 2009).
- **Prior-independent Optimization.** Give a single proper scoring rule that approximates the optimal scoring rule for any environment in the family (cf. Dhangwatnotai, Roughgarden, and Yan, 2015).
- **Sample Complexity.** As a function of the desired precision, bound the number of samples needed for the principal to identify a scoring rule with objective value that is within the desired precision of the optimal scoring rule (cf. Cole and Roughgarden, 2014).

This paper focuses primarily on providing characterizations of optimal scoring rules, computing optimal scoring rules, and identifying simple approximately optimal scoring rules. In addition, first results on prior-independence and sample complexity are given.

Results. We show that the problem of optimizing scoring rules for general beliefs over a finite set of states reduces to the problem of optimizing scoring rules for reporting the expectation of a multi-dimensional state. In scoring rules for the expectation of a multi-dimensional state, the forecaster is simultaneously reporting the marginal expectations of the state in all dimensions.

We solve for the optimal scoring rule for reporting the expectation in single-dimensional space. As we expect for single-dimensional mechanism design problems for an agent with linear utility Myerson (1981), the optimal scoring rule is a step function (which induces a V-shaped scoring rule with its lower tip at the expectation of the prior belief). To implement this V-shaped scoring rule, it is sufficient for the designer to know the prior mean instead of the details on the distribution over posteriors. We also demonstrate a first result for prior-independent analysis of scoring rules. Among scoring rules for reporting the expectation, the quadratic scoring rule is within a constant factor of optimal.

For multi-dimensional forecasting when the distribution over posterior means and the state space are given explicitly, we provide a polynomial time algorithm that computes the optimal scoring rule. For multi-dimensional forecasting with symmetric distributions, we give an analytical characterization of the optimal scoring rule as inducing a V-shaped utility function. For multidimensional forecasting without a symmetry assumption, we identify a V-shaped scoring rule that gives an 8-approximation. This scoring rule can be interpreted as scoring the dimension for which the agent's posterior in the optimal single-dimensional scoring rule gives the highest utility. Equivalently, it can be implemented by letting the agent select which dimension to score and only scoring that dimension (after exerting effort to learn the posterior mean of all dimensions). While optimal mechanisms generally depend on the distribution over posteriors, our approximation bounds are proved for simple mechanisms (V-shaped scoring rules) that depend only on the prior mean, and do not require detailed knowledge of the distribution over posteriors. For the peer grading example, e.g., it is sufficient to know that the mean grade is $0.8 \in [0, 1]$. In addition, due to the simple form of the V-shaped scoring rule, even when the designer is ignorant of the prior mean, the designer can estimate it using samples and the expected incentive loss for using the sample estimate is negligible. Finally, we show that the ad hoc of approach of scoring each dimension separately may have an multiplicative loss in incentives for effort that is linear in the size of the dimension.

An extensive discussion of future directions is given in Section 7.

Application to Peer Grading. Our framework for optimization of scoring rules firmly places the problem within the literature on mechanism design. A significant challenge for algorithmic methods in mechanism design is a lack of applications to which researchers can readily apply mechanism design results. Optimization of scoring rules, however, has application to peer grading and can be deployed in classrooms where algorithms researchers teach. The questions of this paper were in fact motivated by the failure of classical approaches to scoring rules in this context.

While peer grading may be employed to reduce effort of course staff, a primary concern is in improving learning outcomes. For peers to learn from peer reviewing they must be incentivized to put in effort, i.e., the peer reviews themselves must be graded. One way to algorithmically grade peer reviews is to compare the peer's marks to ground truth marks provided by the teaching staff. Specifically, a peer can be asked to review the submission and forecast the true marks.

If the grading rubric has multiple elements (denoted by n), the natural approach from the literature would be to score each dimension separately and then take the sum. In contrast the optimal multi-dimensional rule is not the sum over separate rules but the maximum over separate scoring rules. For a prior such that independently for each dimension, the signal reveals the state with probability $\frac{1}{n}$, these two are significantly different. Specifically, the incentives for effort for the separate scoring rule is $O(\frac{1}{n})$ while the incentives for effort for optimal scoring rule is O(1). Thus optimal scoring rule can be unboundedly better than separate scoring rule. For further details, see Theorem 4.6.

Related Work. Characterizations of scoring rules for eliciting the mean and for eliciting a finitestate distribution play a prominent role in our analysis. Previous works show, in various contexts, that scoring rules are proper if and only if their induced utility functions are convex. McCarthy (1956) characterized proper scoring rules for eliciting the full distribution on a finite set of states. Osband and Reichelstein (1985) characterized continuously differentiable scoring rules that elicit multiple statistics of a probability distribution. Lambert (2011) characterized the statistics that admit proper scoring rules and characterized the uniformly-Lipschitz-continuous scoring rules for the mean of a single-dimensional state. Abernethy and Frongillo (2012) characterized the proper scoring rules for the marginal means of multi-dimensional random states in the interior of the report space. We augment this characterization by showing that the induced utility function converges to a limit on the boundary of the report space. This augmentation enables us to write the mathematical program that optimizes over the whole report space.

Most of the prior work looking at incentives of eliciting information considers a fundamentally different model from ours. This prior work typically focuses on the incentives of the forecaster to exert effort to obtain a signal (a.k.a., a data point), but then assumes that this data point is reported directly (and cannot itself be misreported). In this space, Cai, Daskalakis, and Papadimitriou (2015) considers the learning problem where the principal aims to acquire data to train a classifier to minimize squared error less the cost of eliciting the data points from individual agents. The mechanism for soliciting the data from the agents trades off cost (in incentivizing effort) for accuracy of each individual point. Chen, Immorlica, Lucier, Syrgkanis, and Ziani (2018) and Chen and Zheng (2019) consider the estimation of the mean of a population data. Their objective is to minimize the variance of the resulting estimator subject to a budget constraint on the cost of procuring the data (from incentivizing effort).

A few papers have considered incentivizing effort under a proper scoring rule for a singledimensional state. Osband (1989) considers incentivizing the forecaster to reduce variance under constraints that result in the optimal scoring rule being quadratic. Zermeno (2011) considers a slightly different model and derives that the optimal scoring rule has V-shaped utility; our work begins with such a result for our model. Neyman, Noarov, and Weinberg (2021) consider a forecaster with access to costly samples of a Bernoulli distribution and characterizes optimal scoring rules in the limit as the sample cost approaches zero. Detailed discussion of these results is deferred to Appendix A. Our main contrasting result is the approximate optimality of the V-shaped scoring rule for binary effort and forecasts over multi-dimensional state spaces.

There are several papers on optimizing scoring rules following the model proposed in our paper. Hartline et al. (2021a) extend the framework to the setting where the agent's effort is multidimensional (e.g., corresponding to independent tasks) and the agent can independently exert effort in each dimension. The main result of this extension is that the intuition that linking incentives across different dimensions is beneficial generalizes. The authors propose a generalization of the V-shaped scoring rule that is approximately optimal, which requires the agent to predict k states correctly instead of one (where k is a constant depending on the primitives). Hartline et al. (2021b) extend the framework to the setting where the agent's effort is continuous (but single-dimensional) and the cost of the agent's effort is private to the agent. In this case the principal benefits from offering several scoring rules (and agents with different costs choose different ones), each offered scoring rule is V-shaped. The model also allows for the principal to have negative utility for payments to the agent. Chen and Yu (2021) consider our objective of maximizing the incentives of binary effort in a max-min design framework. For example, they show that the quadratic scoring rule is max-min optimal over a large family of distributional settings. Kong (2021) generalizes the framework from single-agent scoring rules to multi-agent peer prediction, i.e., without ground truth. In peer prediction, the designer needs to cross reference the reports of different agents to verify the informativeness of the report.

Scoring rules are also widely studied in the literature on peer prediction where ground truth

is unknown and agent reports must be compared to each other. Frongillo and Witkowski (2017) considers the optimization goal of incentive for effort in single-task peer prediction. The differences in this model result in incomparable results.

With broad strokes, our work connects the studies of optimal mechanisms and optimal scoring rules. A few points of connection are especially pertinent. Characterizions of incentives in scoring rules and multi-dimensional mechanisms are similar. The multi-dimensional characterization for mechanism design is given by Rochet (1985). One of our main results shows that a good scoring rule for a multi-dimensional state is the max-over-separate scoring rule, while averaging over separate scoring rules is far from optimal. This result parallels the main contribution of Jackson and Sonnenschein (2007), that linking independent decisions improves incentives in mechanism design. This result also connects simple scoring rules to simple mechanisms like the bundlingor-selling-separately mechanism of Babaioff, Immorlica, Lucier, and Weinberg (2014). Finally, the polynomial time algorithms we give for computing optimal scoring rules (in the cases where we do not provide simple analytic charcterizations) are based on Briest, Chawla, Kleinberg, and Weinberg (2015).

Organization. Section 2 formally defines the program for optimizing proper scoring rules. This program is simplified by appropriate characterizations of proper scoring rules which are adapted from the prior literature. Section 3 considers scoring rules for eliciting the posterior mean of a single-dimensional random state. It characterizes the optimal scoring rule for any distribution over posteriors, it shows that the quadratic scoring rule can be an arbitrarily bad approximation to the optimal scoring rule, but it shows that, nonetheless, the quadratic scoring rule is within a constant factor of the optimal prior-independent scoring rule. Section 4 considers scoring rules for eliciting the marginal poster means of a multi-dimensional random state. It gives a polynomial time algorithm for computing the optimal rule, it characterizes the optimal rule when the distribution of posterior means is symmetric, it gives a simple scoring rule that is approximately optimal without symmetry, and it shows that the average of separate scoring rules for each dimension can be a linear approximation in worst case. Section 5 considers scoring rules for eliciting the full distribution over a finite state space. It shows that there is a polynomial time algorithm for computing the optimal scoring rule and it shows that scoring rules that elicit the mean can be arbitrarily far from optimal. We apply our framework for optimization of scoring rules to peer grading in Section 6. We provide interpretations of the scoring rules we designed from Section 4 in this application. Finally, Section 7 overviews a number directions for future research that may be promising.

2 Preliminaries

In this section, we present a formal program for the optimization of proper scoring rules for multidimensional random states. Section 2.1 describes the basic setting for scoring rules and provides an informal description of the optimization problem for scoring rules that elicit the marginal means of the distribution. In Section 2.3, we discuss the characterization of proper scoring rules for eliciting the mean with a weak regularity condition. Section 2.2 gives the formal program for optimizing scoring rules for the mean.

A reason for our focus on scoring rules for eliciting the mean is that, even for continuous state spaces, the communication requirements of eliciting the mean are reasonable. Moreover, as we show in Appendix B.3, the problems of optimizing scoring rules for eliciting the full distribution reduce to problems of optimizing scoring rules for the mean by augmenting the state space.

2.1 The Scoring Rule Optimization Problem

This paper considers the problem of optimizing scoring rules. A scoring rule maps an agent's reported belief about a random state and the realized state to a payoff for the agent. Our model allows the agent to refine her prior belief by exerting a binary effort. Our objective is to maximize the agent's perceived benefit from exerting effort, i.e., the expected difference in score from reporting the prior and posterior distributions.

There is a prior distribution $D \in \Delta(\Theta)$ over the true state $\theta \in \Theta$ where $\Theta \subseteq \mathbb{R}^n$ is any n dimensional space. The distribution D is public information for both the agent and the principal, and in addition, the agent may privately observe a signal about the true state, which induces a posterior G. We denote the probability the agent will obtain the posterior G by f(G). We focus on scoring rules that elicit the mean of the posterior, i.e., the scoring rule asks the agent to report the marginal means of her posterior, and scores the agent based on her report and the realized state. Let μ_G be the mean of posterior G and μ_D be the mean of the prior distribution D. Let $R \subseteq \mathbb{R}^n$ be the report space including all possible posterior means μ_G and let $r \in R$ be the report of the agent. A simple property of means, the report space is the convex hull of the state space. Two constraints on the scoring rules are the boundedness constraint and the proper constraint¹.

Definition 1. A scoring rule $S(r, \theta)$ is proper² for eliciting mean if for any distribution G and report $r \in R$, we have

$$\mathbf{E}_{\theta \sim G}\left[S(\mu_G, \theta)\right] \geq \mathbf{E}_{\theta \sim G}\left[S(r, \theta)\right].$$

Definition 2. A scoring rule $S(r, \theta)$ is bounded by B in space $R \times \Theta$ if $S(r, \theta) \in [0, B]$ for any report $r \in R$ and state $\theta \in \Theta$.

The goal for the principal is to design a bounded proper scoring rule that maximizes the difference in expected score between agents who exert effort and those who do not. Next, we will informally define the optimization program.

Informal program. The problem of maximizing the difference in expected score given the maximum score of B, the state space Θ , the report space which is the convex hull of the state space, i.e., $R = \operatorname{conv}(\Theta)$, and the distribution over posteriors f can be written as the following optimization program:³

$$\begin{array}{ll}
\max_{S} & \mathbf{E}_{G \sim f, \theta \sim G} \left[S(\mu_{G}, \theta) - S(\mu_{D}, \theta) \right] \\
\text{s.t.} & S \text{ is a proper scoring rule for eliciting the mean,} \\
& S \text{ is bounded by } B \text{ in space } R \times \Theta.
\end{array}$$
(1)

The above program aims to optimize the incentive for the agent to exert effort. Consider the situation where the agent has a private stochastic cost for obtaining a signal of the true state. If

¹These two constraints are natural and standard in the scoring rule literature. For eliciting the mean, the restriction on proper scoring rules is not without loss in the optimization program (1). For eliciting the full distribution in Section B.3 and 5, it is without loss to consider the proper constraint.

²Our notion of proper scoring rule is weakly proper rather than strictly proper. Most of the literature on scoring rules does not have an objective and to obtain non-trivial results requires scoring rules to be strictly proper. When optimizing scoring rules there is no meaningful difference between strictly proper and proper as the strictness can be arbitrarily small and therefore provide insignificant additional benefit. Note that any weakly proper scoring rule can also be made strictly proper by taking an arbitrarily small convex combination with a strictly proper scoring rule.

 $^{^{3}}$ In Appendix F, we provide characterizations for a similar model where the ex post bounded score constraint is replaced with the bounded in expectation constraint.

the agent chooses to pay the cost, she sees the realized signal, forms a posterior about the true state, and optimizes according to the posterior. The agent will only choose to pay the cost if her expected gain from obtaining the signal, i.e., the objective value in Program (1), is higher than her cost. By designing the optimal scoring rule for Program (1), we also maximize the probability that the agent chooses to pay the cost. This paper will not formally model such costs.

2.2 Eliciting the Mean with Canonical Scoring Rules

There is a canonical approach for constructing proper scoring rules. In this section we specify Program (1) to canonical proper scoring rules. In the next section we show that this specification is without loss for the program. The following definition and proposition are straightforward from first-order conditions and can be found, e.g., in Abernethy and Frongillo (2012). We defer the proof of Proposition 2.1 to Appendix B.

Definition 3. A canonical scoring rule for the mean S is defined by convex utility function $u : R \to \mathbb{R}$ on report space R, subgradient $\xi : R \to \mathbb{R}^n$ of u, and function $\kappa : \Theta \to \mathbb{R}$ on state space Θ as

$$S(r,\theta) = u(r) + \xi(r) \cdot (\theta - r) + \kappa(\theta).$$
⁽²⁾

Proposition 2.1. Canonical scoring rules are proper.

The following two lemmas allow the objective and the boundedness constraint of Program (1) to be simplified. The first lemma justifies referring to u as the agent's utility function and its proof was observed in the proof of Proposition 2.1.

Lemma 2.2. For any canonical scoring rule for the mean S (defined by u, ξ , and κ), the expected utility from belief G and truthfully report of μ_G is

$$\mathbf{E}_{\theta \sim G}\left[S(\mu_G, \theta)\right] = u(\mu_G) + \mathbf{E}_{\theta \sim G}\left[\kappa(\theta)\right].$$
(3)

Lemma 2.3. Fixing utility function u and subgradients ξ and setting the state-function κ to minimize the score bound B, the canonical scoring rule S (defined by u, ξ and κ) satisfies

$$u(\theta) - u(r) - \xi(r) \cdot (\theta - r) \le B \tag{4}$$

for any report $r \in R$ and state $\theta \in \Theta$.

We now derive the simplified program for canonical scoring rules. The following notation is sufficient to describe this simplified program and is adopted throughtout the paper. For proper scoring rules for eliciting the mean, the posterior mean and report are denoted by r in report space R. The distribution over posterior beliefs induces a distribution over posterior means, slightly abusing notation, we denote both distributions by f. Specifically, $f(r) = \int_{G:\mu_G=r} f(G) \, \mathrm{d}G$, i.e., the density at posterior mean r is equal to the cumulative density of posteriors G with mean $\mu_G = r$. The prior mean of the distribution μ_D is equal to the mean of the posterior means, denoted μ_f , i.e., $\mu_D = \mathbf{E}_{\theta\sim D} [\theta] = \mathbf{E}_{r\sim f} [r] = \mu_f$.

By Lemma 2.2, the objective function in Program (1) for canonical scoring rules can be simplified as

$$\mathbf{E}_{G \sim f, \theta \sim G} \left[S(\mu_G, \theta) - S(\mu_D, \theta) \right] = \int_{\Delta(\Theta)} \left[u(\mu_G) - u(\mu_D) \right] f(G) \, \mathrm{d}G = \int_R \left[u(r) - u(\mu_f) \right] f(r) \, \mathrm{d}r.$$

Note that the simplified objective function does not depend on subgradient ξ or state function κ , the latter of which is cancelled in the score difference. Thus, the value of the objective function is uniquely determined by the utility function u and the distribution over posterior means f. We denote the performance of utility function u given the distribution over posteriors f by

$$Obj(u, f) = \int_{R} u(r) f(r) dr - u(\mu_f).$$
(5)

Combining Lemma 2.3 with the simplified objective function (5), and shifting the utility function by a constant such that $u(\mu_f) = 0$, we get the following optimization program for optimizing over canonical scoring rules. In the next section we show that the restriction to canonical scoring rules is without loss.

$$OPT(f, B, \Theta) = \max_{u} \qquad \int_{R} u(r)f(r) dr \qquad (6)$$

s.t. u is a continuous and convex function, and $u(\mu_{f}) = 0$,
 $\xi(r) \in \nabla u(r), \quad \forall r \in R,$
 $u(\theta) - u(r) - \xi(r) \cdot (\theta - r) \le B, \quad \forall r \in R, \theta \in \Theta,$
 $R = \operatorname{conv}(\Theta).$

Note that for any distribution f and state space Θ , the optimal objective $OPT(f, B, \Theta)$ is a linear function of the maximum score B. In most of the paper, we normalize B = 1 and mainly consider the state space $\Theta = [0, 1]^n$. To simplify the notation, we let $OPT(f) = OPT(f, 1, [0, 1]^n)$. We will write $OPT(f, B, \Theta)$ explicitly in Section 4 when we discuss general state spaces with bound $B \neq 1$.

2.3 Sufficiency of Canonical Scoring Rules

This section provides a partial converse to Proposition 2.1 and shows that the restriction to canonical scoring rules is without loss, i.e., Program (1) and Program (6) are equivalent. The converse will require a weak technical restriction on the set of scoring rules considered.⁴ With this restriction, Abernethy and Frongillo (2012) provide a converse to Proposition 2.1 for reports in the relative interior of the report space. We generalize their observation to the boundary of the report space when the scoring rule is bounded. The detailed discussion is deferred in Appendix B. Formally, we have the following result establishing that Program (1) and Program (6) are equivalent.

Definition 4 (Abernethy and Frongillo, 2012). A scoring rule S is μ -differentiable if all directional derivatives of $\mathbf{E}_{\theta \sim G}[S(\mu_G, \theta)]$ exists for all posteriors G with mean μ_G in the relative interior of R.

Theorem 2.1. For optimization of the incentive for exerting a binary effort via a bounded and μ -differentiable scoring rule for the mean, it is without loss to consider canonical scoring rules, i.e., Program (1) and Program (6) are equivalent.

3 Eliciting a Single-dimensional Mean

In this section, we focus on the special case of single dimensional state spaces. We characterize the optimal single dimensional scoring rules for eliciting the mean and show that the optimal scoring

 $^{^{4}}$ The literature on scoring rules for eliciting the mean, to the best of our knowledge, obtains converses to Proposition 2.1 only with restrictions. For example, Lambert (2011) assumes the scoring rules are continuously differentiable in the agent's report. The restriction we employ is weaker than differentiability.



Figure 1: The figure on the left hand side illustrates the bounded constraint for proper scoring rule for single dimensional states. The figure on the right hand side characterizes the optimal scoring rule (solid line) for single dimensional states. In this figure, for any convex function u (dotted line) that induces a bounded scoring rule, there exists another convex function \tilde{u} (solid line) which also induces a bounded scoring rule and weakly improves the objective.

rules are simple and only depend on the prior mean of the distribution. We compare the quadratic scoring rule to the optimal scoring rule and show that the quadratic scoring rule, though it can be far from optimal for specific distributions over posteriors, it is approximately optimal in the prior-independent setting.

In this section we normalize the state space Θ so that its convex hull, i.e., the report space R, is [0,1] and the boundedness constraint is given by B = 1.

3.1 Characterization of Optimal Scoring Rules

In this part, we characterize the optimal proper scoring rules for a single dimensional state. First note that for single dimensional scoring rules, the boundedness constraint of Program (6) can be further simplified.

Lemma 3.1. For state space Θ with convex hull [0,1] and any utility function u, there exists a μ -differentiable proper scoring rule induced by function u which is bounded by B = 1 if and only if there exists a set of subgradients $\xi(r) \in \nabla u(r)$ such that

$$u(1) - u(0) - \xi(0) \le 1$$
 and $u(0) - u(1) + \xi(1) \le 1$.

Proof. By Lemma B.3, it is sufficient to consider only convex function u such that there exists a set of subgradients $\xi(r)$ satisfying constraints that for any $r, \theta \in [0, 1]$

$$u(\theta) - u(r) - \xi(r) \cdot (\theta - r) \le 1.$$

By convexity of utility u and the monotonicity of subgradients ξ on report space R = [0, 1], it is straightforward to observe that the left-hand side of the boundedness constraint is maximized at $\theta \in \{0, 1\}$ with $r = 1 - \theta$ (see Figure 1a).

With Lemma 3.1, Program (6) can be written as

$$\max_{u} \int_{0}^{1} u(r)f(r) dr$$
s.t. $u(r)$ is convex and $u(\mu_{f}) = 0$,
 $\xi(r) \in \nabla u(r), \forall r \in [0, 1],$
 $u(1) - u(0) - \xi(0) \le 1,$
 $u(0) - u(1) + \xi(1) \le 1.$
(7)

The main result of this section is the following characterization of the optimal solutions to Program (7).

Definition 5. A function u is V-shaped at μ if there exists parameters a and b such that $u(r) = a(r - \mu)$ for $r \leq \mu$ and $u(r) = b(r - \mu)$ for $r \geq \mu$.

Utility functions that are V-shaped at prior mean μ_f are induced by scoring rules with the following simple form. If the agent reports the prior mean her score is zero. For reports above the prior mean, the score is equal to $b(\theta - \mu_f)$; and for reports below the prior mean, the score is equal to $a(\theta - \mu_f)$. I.e., as discussed in Section 2.2, the agent's report picks out the supporting hyperplane of the utility function on which to evaluate the state. Note that the implementation of the V-shaped scoring rule only needs the knowledge of the prior mean μ_f , and does not need the distribution over posteriors. We show the following theorem on the optimal solutions of Program (7). We defer its proof to Appendix C.

Theorem 3.1. For any distribution f over the posterior means with expectation μ_f and state space Θ with convex hull [0,1], the optimal solutions of Program (7) are V-shaped at μ_f with parameters $b = a + 1/\max\{\mu_f, 1 - \mu_f\}$ and objective value $\text{OPT}(f) = \mathbf{E}_{r\sim f} [\max(r - \mu_f, 0)]/\max(\mu_f, 1 - \mu_f).^5$

As mentioned above, we see from Theorem 3.1 that the set of utility functions that optimizes Program (7) only depends on the prior mean μ_f and not the general shape of the distribution over posterior means f.

An important special case for our subsequent analyses is when the mean of the posteriors is in the center of the report space, i.e., $\mu_f = 1/2$ for report space [0,1]. In this case, an optimal utility function u is V-shaped at 1/2 with u(0) = u(1) = 1/2. In fact, the symmetric case where fis the uniform distribution on the extremal poster means $\{0,1\}$ obtains the highest objective value for Program (7) with OPT(f) = 1/2. These two observations are fomalized in the following two corollaries.

Corollary 3.2. For any distribution f over the posterior means with expectation $\mu_f = 1/2$, one of the optimal solution of Program (7) is symmetric and V-shaped at 1/2 with u(0) = u(1) = 1/2.

Corollary 3.3. The objective value of any utility function u that is feasible for Program (7) on distribution f of posterior means is at most 1/2, i.e., $Obj(u, f) \leq 1/2$.

Proof of Corollary 3.3 is deferred to Appendix C.2.

3.2 The Quadratic Scoring Rule and Prior-independent Approximation

The previous section showed that the optimal single-dimensional scoring rule depends on the distribution over posteriors and, more specifically, on the mean of this distribution. On the other hand, standard scoring rules in theory and practice, like the quadradic scoring rule, are prior-independent, i.e., they do not depend on the principal's prior distribution (over posterior distributions of the agent), cf. Dhangwatnotai, Roughgarden, and Yan (2015). This section focuses on the quadratic scoring rule. It gives the characterization in terms of utility of the quadratic scoring rule for eliciting the mean of a single-dimensional state. It analizes the approximation factor of the quadratic scoring rule with respect to the optimal scoring rule, and shows that the performance of the former is quadratic in the performance of the latter. Specifically, the ratio of performances is unbounded as the performance of the optimal scoring rule approaches zero (and such a sequence of prior distributions exists). Thus, we conduct the prior-independent analysis on families of priors which give the

⁵By slightly perturbing the utility function u, the V-shaped scoring rule can be transformed into a strictly proper scoring rule with an arbitrarily close objective value.

same performance of the optimal scoring rule (cf. the "max/max ratio" of Ben-David and Borodin, 1994). Within each such family, the quadratic rule is approximately optimal among all prior-independent scoring rules.

The following observations will be useful in our analysis of the quadratic and other priorindependent scoring rules. First, for prior-independent analysis, the designer does not know the prior mean μ_f of the distribution. Therefore, we consider Program (7) equivalently with the agent's utility for reporting the prior mean $u(\mu_f)$ subtracted from the objective and without the constraint $u(\mu_f) = 0$. Second, in the worst case it is sufficient to only consider posterior distributions that are uniformly drawn as one of two deterministic points. This latter result is formalized in the following lemma.

Lemma 3.4. For any distribution f over posterior means, there exists another distribution \tilde{f} over posterior means with 2 point masses that satisfies $OPT(\tilde{f}) = OPT(f)$ and for any convex function u, $Obj(u, \tilde{f}) \leq Obj(u, f)$.

Proof. For any distribution f with prior mean μ_f , let \tilde{f} be the distribution that has

- a point mass at $\mathbf{E}_f[r'|r' < \mu_f]$ with probability $\mathbf{Pr}_f[r' < \mu_f]$;
- a point mass at $\mathbf{E}_f[r'|r' \ge \mu_f]$ with probability $\mathbf{Pr}_f[r' \ge \mu_f]$.

By Theorem 3.1, it is easy to verify that the optimal does not change, i.e., $OPT(f) = OPT(\tilde{f})$, and for any convex u, by Jensen's Inequality, we have $Obj(u, \tilde{f}) \leq Obj(u, f)$.

The quadratic scoring rule that is the focus of this section is defined as follows.

Definition 6. The [0,1]-bounded quadratic scoring rule for eliciting the mean with state and report spaces $\Theta = R = [0,1]$ is $S_q(r,\theta) = 1 - (\theta - r)^2$. For functions $u_q(r) = r^2$ and $\kappa_q(\theta) = 1 - \theta^2$ the quadratic scoring rule is $S_q(r,\theta) = u_q(r) + u'_q(r) \cdot (\theta - r) + \kappa_q(\theta)$.

Lemma 3.4 enables the identification of the worst-case performance the quadratic scoring rule. Recall that, by Corollary 3.3, the optimal objective value is at most 1/2, i.e., $OPT(f) \in (0, 1/2]$.

Theorem 3.2. Let \mathcal{F}_c be the set of distributions such that the objective value of the optimal scoring rule is $c \in (0, 1/2]$, i.e., OPT(f) = c for any $f \in \mathcal{F}_c$. We have that for utility function u_q of quadratic scoring rule,

$$\min_{f \in \mathcal{F}_c} \operatorname{Obj}(u_q, f) = c^2.$$

As will be evident from the proof of Theorem 3.2, for any $c \in (0, 1/2]$ there is a non-trivial family of distributions \mathcal{F}_c for which OPT(f) = c. Since the worst-case performance of the quadratic scoring rule on \mathcal{F}_c is $\min_{f \in \mathcal{F}_c} Obj(u_q, f) = c^2$, the prior-independent approximation factor of the quadratic scoring rule is unbounded. In fact, as we show next, this result is not a limitation of the quadratic scoring rule. For the family of distributions \mathcal{F}_c , any prior-independent scoring rule can at most guarantee a worst-case objective value of $O(c^2)$. Thus, the quadratic rule is within a constant factor of the prior-independent optimal rule. We defer the proof of Theorem 3.2 to Appendix C.3, and Theorem 3.3 to Appendix C.4.

Theorem 3.3. Let \mathcal{F}_c be the set of distributions over posterior means such that the objective value of the optimal scoring rule is $c \in (0, 1/2]$, i.e., OPT(f) = c for any $f \in \mathcal{F}_c$. For any convex and bounded utility function u, we have

$$\min_{f \in \mathcal{F}_c} \operatorname{Obj}(u, f) \le \min(\frac{1}{2}, \frac{8c^2}{(1-4c)^2}) \le 32c^2.$$

Combining Theorem 3.2 with Theorem 3.3, the quadratic scoring rule approximates any priorindependent scoring rule in terms of worst case payoff.

Theorem 3.4. For any constant $c \in (0, 1/2]$, let \mathcal{F}_c be the set of distributions such that the objective value of the optimal scoring rule is c, i.e., OPT(f) = c for any $f \in \mathcal{F}_c$. Let \mathcal{U} be the set of convex and bounded utility functions u. For quadratic utility function u_q , we have

$$\min_{f \in \mathcal{F}_c} \operatorname{Obj}(u_q, f) \ge \frac{1}{32} \max_{u \in \mathcal{U}} \min_{f \in \mathcal{F}_c} \operatorname{Obj}(u, f).$$

Note that in Theorem 3.4, the quadratic scoring rule does not exploit the extra information that OPT(f) = c and still achieves a constant approximation to the optimal max-min scoring rule in worst case.

Although the quadratic scoring rule is approximately max-min optimal, the approximation ratio between the quadratic scoring rule and the optimal scoring rule can still grow unboundedly as the optimal objective value OPT(f) vanishes to zero. In the following theorem, we will show that for any fixed distribution over posterior mean with variance σ^2 , the performance of the quadratic scoring rule is an approximation of the optimal solution within a factor of the standard deviation σ . That is, the quadratic scoring rule is approximately optimal when the distribution over posterior mean is sufficiently disperse. We defer the proof of Theorem 3.5 to Appendix C.5.

Theorem 3.5. For any $\sigma \in [0, 1]$, any distribution over posterior mean f with variance σ^2 , we have

$$\operatorname{Obj}(u_q, f) \ge \sigma \cdot \operatorname{OPT}(f).$$

4 Elicitation of a Multi-dimensional Mean

In this section, we focus on the case when the state space is multi-dimensional. We first give a polynomial time algorithm that identifies the optimal scoring rule for the problem when the posterior distribution and the set of realizable states are given explicitly. Then we characterize the optimal scoring rule for symmetric distributions over posterior means, and propose a simple scoring rule that is approximately optimal for asymmetric distributions. Finally, we show that the standard approach in both theory and practice of scoring the agents separately in each dimension is not a good approximation to the optimal multi-dimensional scoring rule.

4.1 Computing the Optimal Scoring Rule

We adopt an approach from Briest, Chawla, Kleinberg, and Weinberg (2015) and show that when the state space and the support of the posterior means are finite, there exists a polynomial time algorithm that solves the optimal scoring rule for eliciting the marginal means of a posterior.

Theorem 4.1. Given any n-dimensional state space Θ with $|\Theta| = d$ states and any distribution f with support size m over posterior means, there exists an algorithm that computes the optimal proper bounded scoring rule for eliciting the mean in time polynomial in n, m, and d.

To prove this theorem, we introduce a proposition stating the equivalence of Bayesian auction design and the design of proper scoring rules. With this equivalence result, we can solve Program (6) with finite reports using a linear program with (n+1)(m+d+1) variables and a quadratic number of constraints. We defer the proof of Proposition 4.1 to Appendix D.1, and the proof of Theorem 4.1 to Appendix D.2.

Proposition 4.1. A function u is the utility function of a μ -differentiable B-bounded proper scoring rule for eliciting the mean on report space $R = \operatorname{conv}(\Theta)$ and n-dimensional state space Θ if and only if there exists allocation and payment functions $x(\cdot)$ and $p(\cdot)$ satisfying

- 1. Bayesian incentive compatible: $x(r) \cdot r p(r) \ge x(r') \cdot r p(r')$, for any report $r, r' \in R$;
- 2. bounded utility difference: $x(\theta) \cdot \theta p(\theta) \leq B + x(r) \cdot \theta p(r)$, for any report $r \in R$ and state $\theta \in \Theta$;
- 3. induced utility is $u(r) = x(r) \cdot r p(r)$ for any $r \in R$.

Note that the bounded utility difference property means the utility loss for misreporting r with true state θ is at most B.

4.2 Optimal Scoring Rules for Symmetric Distributions

This section characterizes the optimal multi-dimensional scoring rule when the distribution over posteriors is symmetric about its center. Program (6) is optimized by a symmetric V-shaped utility function. This characterization affords a simple interpretation for rectangular report and state spaces, specifically, the optimal scoring rule can be calculated by taking the maximum score over optimal single-dimensional scoring rules for each dimension, i.e., it is a max-over-separate scoring rule. As these single-dimensional scoring rules depend only on the prior mean, so does the optimal multi-dimensional scoring rule. We first give the characterization and then give the interpretation.

Definition 7. A n-dimensional distribution f is center symmetric if there exists a center in the report space, i.e., $C \in R$ such that for any $r \in R$, f(r) = f(2C - r).

Note that for any center symmetric distribution f over posterior means, the mean of the prior coincides with the center of the space, i.e., $\mu_f = C$. The following definition generalizes symmetric V-shaped functions to multi-dimensional state and report spaces.

Definition 8. A function u is symmetric V-shaped in report and state space $R = \Theta$ with nonempty interior and center C if utility is zero at the center, i.e., u(C) = 0, utility is 1/2 on the boundary, i.e., u(r) = 1/2 for $r \in \partial R$, and all other points linearly interpolate between the center and the boundary, i.e., $u(\alpha \cdot r + (1 - \alpha) \cdot C) = \frac{\alpha}{2}$ for any $\alpha \in [0, 1]$ and $r \in \partial R$.

V-shaped utility functions on convex and center symmetric spaces are bounded and convex, i.e., they are feasible solutions to Program (6). The proof of Lemma 4.2 is deferred to Appendix D.3.

Lemma 4.2. For any convex and center symmetric report and state space $R = \Theta$ with non-empty interior, the center symmetric utility function is convex and bounded for B = 1.

The following theorem is proved by following a standard approach in multi-dimensional mechanism design, e.g., Armstrong (1996) and Haghpanah and Hartline (2015). The problem is relaxed onto single-dimensional paths, solved optimally on paths, and it is proven that the solution on paths combine to be a feasible solution on the whole space. Note that in relaxing the problem onto paths, constraints on pairs of reports that are not on the same path are ignored. The full proof of Theorem 4.2 is deferred to Appendix D.4. Similar to the single dimensional V-shaped scoring rule, the implementation of multi-dimensional V-shaped scoring rule only requires the knowledge of the prior mean μ_f .

Theorem 4.2. For any center symmetric distribution f over posterior means in convex report and state space $R = \Theta$, the optimal solution for Program (6) is symmetric V-shaped.

In the remainder of this section we give an interpretation of scoring rules that correspond to V-shaped utility functions on rectangular report and state spaces. On such spaces, these optimal scoring rules can be implemented as the maximum over separate scoring rules (for each dimension). Intuitively, the max-over-separate scoring rule rewards the agent only on the dimension the the agent will receive highest expected payment according to his posterior belief.

The definition of max-over-separate scoring rule is formally introduced in Definition 9, and it is easy to verify that a max-over-separate scoring rule is proper and bounded if is based on single dimensional scoring rules that are proper and bounded.

Definition 9. A scoring rule S is max-over-separate if there exists single dimensional scoring rules $(\hat{S}_1, \ldots, \hat{S}_n)$ such that

- 1. For any dimension i, $\hat{S}_i(r_i, \theta_i) = \hat{u}_i(r_i) + \hat{\xi}_i(r_i) \cdot (\theta_i r_i) + \hat{\kappa}_i(\theta_i)$ where $\hat{\xi}_i(r_i)$ is a subgradient of convex function $\hat{u}_i(r_i)$ and $\hat{\kappa}_i(\theta_i) = \beta_i$ is a constant.
- 2. the score is $S(r, \theta) = \hat{S}_i(r_i, \theta_i)$ where $i = \arg \max_j \hat{S}_j(r_j, r_j)$.

The incentives of max-over-separate is ensured by the equality of $\hat{S}_j(r_j, r_j)$ (from condition 2) and $\mathbf{E}_{\theta_j \sim G_j}[S_j(r_j, \theta_j)]$ for any marginal posterior distribution G_j on dimension j with mean r_j . Specifically, since the function $\hat{\kappa}_j$ is a constant function of the state, all posteriors G_j with the same mean induce the same expected score.

We conclude the section by showing that, for rectangular report and state spaces, symmetric V-shaped utility functions, which are shown to be optimal by Theorem 4.2, can be implemented by max-over-separate scoring rules.

Lemma 4.3. Symmetric V-shaped function u in n-dimensional rectangle report and state space $R = \Theta = \bigotimes_{i=1}^{n} [a_i, b_i]$ with function $\kappa(\theta) = 1/2$ can be implemented as max-over-separate scoring rule with single dimensional bounded proper scoring rules $\{\hat{S}_i\}_{i=1}^n$ where

$$\hat{S}_{i}(r_{i},\theta_{i}) = \begin{cases} -\frac{1}{b_{i}-a_{i}}(\theta_{i}-\mu_{D_{i}}) + \frac{1}{2} & \text{for } r_{i} \leq \mu_{D_{i}}, \\ \frac{1}{b_{i}-a_{i}}(\theta_{i}-\mu_{D_{i}}) + \frac{1}{2} & \text{for } r_{i} \geq \mu_{D_{i}}, \end{cases}$$

where $\mu_{D_i} = (a_i + b_i)/2$ is the *i*th coordinate of the prior mean μ_f .

Proof. First, it is easy to verify that the single dimensional scoring rules \hat{S}_i are proper and bounded in [0, 1]. For each dimension i, the utility function for each single dimensional scoring rule \hat{S}_i is V-shaped with

$$\hat{u}_i(r_i) = \begin{cases} -\frac{1}{b_i - a_i} (r_i - \mu_{D_i}) & r_i \le \mu_{D_i} \\ \frac{1}{b_i - a_i} (r_i - \mu_{D_i}) & r_i \ge \mu_{D_i} \end{cases}, \text{ and } \hat{\kappa}_i(\theta_i) = 1/2.$$

By Definition 9, the max-over-separate scoring rule S is $S(r, \theta) = \hat{S}_i(r_i, \theta_i)$ where $i \in \arg \max_j \hat{u}_j(r_j)$, and hence the utility function for max-over-separate scoring rule S can be computed as $u(r) = \max_{i \in [n]} \hat{u}_i(r_i)$, which coincides with the symmetric V-shaped function u.

Corollary 4.4. For any center symmetric distribution f over posterior means in rectangular report and state space $R = \Theta$, a max-over-separate scoring rule is optimal.

Finally, these max-over-separate scoring rules have an indirect choose-and-report implementation where the agent reports the dimension to be scored on and the mean for that dimension. This indirect implementation has a practical advantage that when the communication between



Figure 2: This figure depicts a two-dimensional state space. The state space $\Theta = [0, 1]^2$ and its point reflection around the prior mean μ_f are shaded in gray. The extended report and state space are depicted by the region within the thick black rectangle.

the principal and the agent is costly since in n-dimensional spaces, it requires only reporting two rather than n numbers. Note that choose-and-report and max-over-separate are essentially the same scoring rule, with different implementations.

Definition 10. A scoring rule S is choose-and-report if there exists single dimensional scoring rules $(\hat{S}_1, \ldots, \hat{S}_n)$ such that the agent reports dimension i and mean value r_i , and receives score $S((i, r_i), \theta) = \hat{S}_i(r_i, \theta_i).$

An agent's optimal strategy in the choose-and-report scoring rule for proper single-dimensional scoring rules $(\hat{S}_1, \ldots, \hat{S}_n)$ is to choose the dimension *i* with the highest expected score according to the posterior distribution, i.e., $i = \arg \max_j \mathbf{E}_{\theta_j \sim G_j} \left[\hat{S}_j(\mu_{G_j}, \theta_j) \right]$, and to report the mean of the posterior for that dimension, i.e., μ_{G_i} . As described above, the advantage of such an indirect scoring rule is that it only requires the agent to report two values to the principal. Lemma 4.5 illustrate a nice properties of choose-and-report scoring rules, with proof deferred to Appendix D.5.

Lemma 4.5. The choose-and-report scoring rule S defined by proper and bounded single-dimensional scoring rules $(\hat{S}_1, \ldots, \hat{S}_n)$ is itself proper and bounded.

4.3 Approximately Optimal Scoring Rules for General Distributions

When the distribution is not symmetric, max-over-separate scoring rules may not be optimal for Program (6). However, we show that the optimal max-over-separate scoring rule is approximately optimal for any asymmetric and possibly correlated distribution over a high dimensional rectangular space.

To show this, we symmetrize the distribution over posteriors, and construct a V-shaped scoring rule on the symmetrized distribution. This V-shaped scoring rule can be implemented as a maxover-separate scoring rule on the original problem, which only requires the knowledge of prior mean.

Theorem 4.3. For any distribution f over posterior means in n-dimensional rectangular report and state space $R = \Theta = \bigotimes_{i=1}^{n} [a_i, b_i]$, the utility function u of optimal max-over-separate scoring rule for Program (6) achieves at least 1/8 of the optimal objective value, i.e. $\operatorname{Obj}(u, f) \geq 1/8 \cdot \operatorname{OPT}(f, B, \Theta)$.

In the following discussion, we assume without loss of generality that $\mu_{D_i} \ge (a_i + b_i)/2$ for every dimension *i*. The proof of this theorem introduces the following constructs:

• The extended report and state space are $\widetilde{R} = \widetilde{\Theta} = \bigotimes_{i=1}^{n} [a_i, 2\mu_{D_i} - a_i]$. These are rectangular and contain the original report and state spaces $R = \Theta$. See Figure 2.

• The symmetric extended distribution of f on the extended report space is $\tilde{f}(r) = \frac{1}{2}(f(r) + f(2\mu_f - r))$. Note in this definition that the original distribution f satisfies f(r) = 0 for any $r \in \tilde{R} \setminus R$.

Theorem 4.3 now follows by combining the following five lemmas, with proofs provided in Appendices D.6 and D.7.

Lemma 4.6. Evaluated on any distribution over posterior means f, the optimal max-over-separate scoring rule for the distribution f and the state space Θ is at least as good as the optimal scoring rule for the extended distribution \tilde{f} and the extended state space $\tilde{\Theta}$.

Lemma 4.7. The symmetric optimizer \tilde{u} for the symmetric extended distribution f and extended state space $\tilde{\Theta}$ attains the same objective value on the original distribution f, i.e., $Obj(\tilde{u}, f) = OPT(\tilde{f}, B, \tilde{\Theta})$.

Lemma 4.8. On extended state space $\widetilde{\Theta}$, the optimal value of Program (6) for the symmetric extended distribution \tilde{f} is at least half that for the original distribution f, i.e., $OPT(\tilde{f}, B, \widetilde{\Theta}) \geq \frac{1}{2}OPT(f, B, \widetilde{\Theta})$.

Lemma 4.9. For any distribution over posterior means f, the optimal value of Program (6) on the extended state space $\widetilde{\Theta}$ is at least a quarter of that of the original state space Θ , i.e., $OPT(f, B, \widetilde{\Theta}) \geq \frac{1}{4}OPT(f, B, \Theta)$ or equivalently $OPT(f, 4B, \widetilde{\Theta}) \geq OPT(f, B, \Theta)$.

4.4 Robustness to Distributional Knowledge

By Theorem 4.3, the optimal max-over-separate scoring rule is approximately optimal, and to implement such a scoring rule, it is sufficient to know the prior mean of the distribution. In this section, we show that we can even relax the assumption of exact knowledge of the prior mean, and show that the designer can approximately attain the performance of the optimal max-over-separate scoring rule by having an estimate of the prior mean. To simplify the presentation, we will focus on the state space $\Theta = X_{i=1}^{n}[0,1]$ and score bound B = 1. The results can be directly extended to general rectangular state spaces and any score bound B > 0.

Theorem 4.4. For any $\epsilon > 0$, any distribution f with prior mean μ_D in state space $\Theta = \bigotimes_{i=1}^{n} [0,1]$, for any μ such that $\|\mu - \mu_D\|_{\infty} \leq \epsilon$, the incentive for effort of the V-shaped scoring rule for μ is at least that of the V-shaped scoring rule for μ_D less 3ϵ .

The proof of Theorem 4.4 is deferred to Appendix D.8. Note that in the following theorem we show that the prior mean can be estimated efficiently using samples.

Theorem 4.5. For any $\epsilon > 0, \delta > 0$, any distribution f with prior mean μ_D in state space $\Theta = \bigotimes_{i=1}^{n} [0,1]$, letting μ be the empirical mean with $\frac{1}{\epsilon^2} \cdot \log \frac{n}{\delta}$ samples, with probability at least $1 - \delta$, we have $\|\mu - \mu_D\|_{\infty} \leq \epsilon$.

Proof. By Chernoff-Hoeffding inequality, we have that for any sequence of k independent random variables $\{r_i\}_{i=1}^k$ bounded in [0, 1] with the same mean m, we have

$$\Pr\left[\left|\frac{1}{k}\sum_{i=1}^{k}r_{i}-m\right| \geq \epsilon\right] \leq 2\exp(-2n\epsilon^{2}).$$

Thus, with $\frac{1}{\epsilon^2} \cdot \log \frac{n}{\delta}$ samples, by union bound, we have that with probability at least $1 - \delta$, $\|\mu - \mu_D\|_{\infty} \leq \epsilon$.

Remark: In the proof of Theorem 4.5, we do not require the samples are drawn from i.i.d. distributions. Instead we only impose the constraint of independence with the same mean. This is particularly helpful if our estimate of the prior mean is from historical reports from different agents as the distribution of reports may vary from agent to agent as their abilities for acquiring information vary. However, all these distributions have the same mean by Bayesian plausibility.

Note that in the case the estimated mean is far from the prior mean, which occurs with probability at most δ , the loss in incentive for effort is at most 1. Combining Theorems 4.4 and 4.5, by setting $\delta = \epsilon$, we have the following corollary.

Corollary 4.10. For any $\epsilon > 0$, any distribution f with prior mean μ_D in state space $\Theta = \\ \times_{i=1}^{n} [0,1]$, letting μ be the empirical mean with $\frac{1}{\epsilon^2} \cdot \log \frac{n}{\epsilon}$ samples, the expected incentive for effort of the V-shaped scoring rule for μ is at least that of the V-shaped scoring rule for μ_D less 4ϵ .

4.5 Inapproximation by Separate Scoring Rules

One way to design the scoring rule for an *n*-dimensional space is to average independent scoring rules for the marginal distributions of each dimension. In this section we show that the worstcase multiplicative approximation of scoring each dimension separately and scoring optimally is $\Theta(n)$. Moreover, the upperbound O(n) holds for general correlated report distributions, while the lowerbound $\Omega(n)$ holds for independent distributions. The proof of Theorem 4.6 is deferred to Appendix D.9.

Definition 11. A scoring rule S is a separate scoring rule if there exists single dimensional scoring rules (S_1, \ldots, S_n) such that $S(r, \theta) = \sum_i S_i(r_i, \theta_i)$.

Theorem 4.6. In *n*-dimensional rectangular report and state spaces, the worst-case approximation factor of scoring each dimension separately is $\Theta(n)$.

5 Eliciting the Full Distribution

In Appendix B.3, we gave a reduction from the problem of optimal scoring rules for eliciting the full distribution over a finite state space to the problem of optimal scoring rules for eliciting the marginal means over a multi-dimensional state space. This reduction is based on representing the state space by an indicator vector. In this section, we first observe that the optimal scoring rule can be found in polynomial time when the distribution of posteriors is given explicitly. This result is a simple corollary of Theorem B.2 and Theorem 4.1. Second, we show that even for single dimensional state space with finite size, the gap in performance between the optimal scoring rule for eliciting the mean and the optimal scoring rule for eliciting the full distribution is unbounded.

Corollary 5.1. Given any finite state space Θ with $|\Theta| = d$ and any distribution f with support size m over posteriors, there exists an algorithm that computes the optimal proper bounded scoring rule for eliciting the full distribution in time polynomial in m and d.

Proof. This result follows from combining Theorem B.2 (the reduction from full distribution reporting to reporting the mean) and Theorem 4.1 (polynomial time computation of the optimal scoring rule for the mean). \Box

We now show that the multiplicative gap between the optimal proper scoring rule for eliciting the full distribution and the optimal proper scoring rule for eliciting the mean is unbounded, even when the size of the state space is a constant. The proof of Theorem 5.1 is deferred to Appendix E.1. **Theorem 5.1.** For any $\epsilon \in (0, \frac{1}{2}]$, there exists a state space Θ with size $|\Theta| = 4$, and a distribution f over the posteriors on space Θ such that the objective value of optimal scoring rule for eliciting full distribution is at least 1/4, while the objective value of optimal scoring rule for eliciting mean is at most ϵ .

6 Application to Peer Grading

Peer grading systems manage the the assignment, collection, and aggregation of peer reviews, e.g., CrowdGrader (de Alfaro and Shavlovsky, 2013) and Mechanical TA (Wright et al., 2015). The primary focus in peer grading algorithms has been in aggregating scores from peer reviewers to produce accurate grades for submissions (e.g., Karger et al., 2014; Zhang et al., 2016). There is also an important non-algorithmic literature on measuring impact of peer grading on learning outcomes (e.g. Sadler and Good, 2006; Gielen et al., 2010). Unfortunately, these papers do not carefully consider the incentives of the peers to produce high quality peer reviews. In this section, we apply our framework of optimization of scoring rules to the peer grading scenario for incentivizing the peers to exert effort.

In this application, there are m submissions of the assignment and each student is required to grade one of the submissions. We assume that the instructor has access to the true grade of each submission, and the goal of the instructor is to incentivize the students to exert effort through grading their evaluation of the assignments. That is, students are incentivized to have a better understanding of the course material through evaluating other students' submissions. Note that if the instructor only has access to the truth grades for a subset of submissions, the technique of spot checking can be applied here, while the incentive for effort decreases linearly in probability of spot checking.

The instructor announces the *n*-dimensional marking criteria. This criteria can be in the form of a grading rubric or it can be tags of common mistakes. E.g., for an inductive proof tags might include "missing base case" or "unclear induction hypothesis". (For example, such a system for tagging common mistakes has been previously employed for automated feedback systems, e.g., Stephens-Martinez et al., 2017.) Given the marking criteria, each peer will mark the submissions by submitting a prediction $\mu_i \in [0, 1]$ for whether the submission satisfies each criterion $i \in \{1, \ldots, n\}$.

These predictions are then be graded by a scoring rule with ground truth marks for the submission that are provided by course staff. Naturally such a scoring rule needs to satisfy the boundedness constraint. We assume that the effort of the peer is binary in this setting, either she can read the submission to have a refined prediction of the true mark for each criterion, or her optimal strategy is to submit her prior belief, i.e., the class average mark for the criterion across all submissions.

Theorem 4.3 suggests that the max-over-separate scoring rule approximately maximizes the incentives for the peers to exert effort. This scoring rule assumes knowledge of the prior probabilities for each criterion; however, these quantities can be estimated efficiently by aggregating the true grades and the reports from the peers for the m-1 remaining submissions with negligible loss on the incentives (Corollary 4.10).

In this peer grading context the max-over-separate scoring rule has a natural interpretation. Consider criterion *i* with $\mu_{D_i} > 1/2$ and $r_i < \mu_{D_i}$. The utility from the single dimensional scoring rule for *i* given posterior belief r_i is

$$\hat{u}_i(r_i) = \frac{1}{2\mu_{D_i}} \cdot (\mu_{D_i} - r_i) = \frac{1}{2} - \frac{r_i}{2\mu_{D_i}}$$

Specifically, the more surprising the report is (i.e., the smaller r_i is and the bigger μ_{D_i} is) the higher the peer's expected score. According to the max-over-separate scoring rule, the peer is only

scored against the realized criteria for which her utility according to her posterior is the highest. Interpreting the criteria as common mistakes, good scoring rules are ones that reward the peer for spotting the mistakes that occur more rarely.

7 Conclusions

In this paper, we develop a framework for optimizing scoring rules. Our objective is to maximize the incentive for the forecaster to exert a binary level of effort subject to a boundedness constraint on the ex post score of the scoring rule. We characterize the optimal scoring rule for eliciting the mean in single-dimensional state spaces and in multi-dimensional state spaces with center symmetric distributions. More generally, we give a polynomial time algorithm for computing the optimal scoring rule when the posterior distributions are given explicitly. We also show that the simple max-over-separate scoring rule is a constant approximation to the optimal scoring rule for eliciting the mean for any asymmetric distribution. Our novel scoring rules contrast with standard scoring rules in theory and practice which are far from optimal in our model.

There are a number of open directions for the study of the model proposed in the paper. First, can an (approximately) prior-independent optimal scoring rule be identified for multi-dimensional state spaces? Second, is the max-over-separate scoring rule a constant approximation to the optimal scoring rule for eliciting the full distribution over a finite set of states? The difficulty of the latter question comes from the fact that a distribution is a point in the simplex and our analysis of the approximation of max-over-separate uses the fact that the report space is a rectangular region.

We have shown that the optimal prior-independent scoring rule, e.g., for single-dimensional states, does not obtain a constant approximation to the optimal scoring rule. This result suggests that, for practical implementation of good scoring rules, a theory of sample complexity for scoring rules is needed. Such a theory would enable the principal to identify a pretty good scoring rule from samples of the posterior distributions.

There are a number potentially interesting extensions to our model that could be considered. Towards prior-independent scoring rules, it would be interesting to consider settings with multiple forecasters. For example, Osband (1989) considers an extension of the basic one-forecaster model where there are multiple forecasters and only the one whose report is closest to the true state receives a reward. Could it be that there is a Bulow and Klemperer (1996) style result where there is a prior-independent scoring rule for several forecasters that outperforms the optimal scoring rule for fewer forecasters, e.g., one forecaster?

An important generalization of our model is to one with non-binary levels of effort. In Osband (1989) and Neyman, Noarov, and Weinberg (2021) the forecaster exerts a single-dimensional effort for learning more accurate prior. The objective of these papers is the accuracy of the forecast as is measured by its prediction error. It would be interesting to evaluate our max-over-separate scoring rule with richer levels of possible effort and compare it to the optimal scoring rule. One of our main motivations for considering multi-dimensional states is the case where these dimensions correspond to different elicitation tasks, e.g., peer graders evaluating different submissions, peer prediction of different labeling tasks, or exam answers of different questions. For these tasks the effort is also multi-dimensional. Optimization of scoring rules with multi-dimensional effort is a critical problem for these applications.

Another variation to the model considers the case where the principal has to pay a cost to observe the distinct dimensions of a realized state. For example, in the peer grading application, the multi-dimensional state corresponds to a peer's reviews of different submissions. It is costly to obtain ground truth labels for the submission grades, e.g., by having an instructor grade the submission, and a natural approach is to use spot checking. However, the most natural approach, namely spot checking one of the submissions and assigning a grade based on an optimal scoring rule for that submission, corresponds to averaging over separate scoring rules which we have shown is very far from optimal. Are there scoring rules that are near optimal but are more frugal with costly evaluation of the state?

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A Detailed Discussion of Related Work.

In this section we give a detailed discusson of some of the most related works.

The early work of Osband (1989) is close to ours in that it assumes that the agent has a prior and, with a continuous level effort, can receive a signal from which the prior is updated to a posterior. The principal then aims at optimizing a quadratic loss function while incentivizing the agent to both put in effort and truthfully report the posterior. Osband (1989) imposes additional constraints on the scoring rule such that the restricted optimal scoring rule is quadradic. In our setting of binary effort, we impose no constraint on the scoring rule except the ex post boundedness, and we find that the optimal scoring rule for incentivizing effort is V-shape instead of quadratic.

Zermeno (2011) considers the optimization of scoring rules in the binary state setting, and he shows that among all scoring rules that induces a certain level of effort, the V-shaped scoring rule is the one that minimizes the expected transfer from the principal to the agent. This objective is qualitatively different from ours, where we consider the objective of maximizing the agent's expected surplus for exerting effort, subject to the ex post boundedness constraint. In addition, the model in Zermeno (2011) is restricted in the following two ways: 1) it only considers single dimensional (i.e., binary states) optimization problem; 2) agent's cost of effort is known to the principal. In our paper, we show that the V-shaped scoring rule is optimal in the single dimensional problem even when the agent has private cost of effort, and more importantly, in the multi-dimensional problem, the V-shaped scoring rule is approximately optimal for eliciting effort.

Contemporaneously with and independently from our work, Neyman, Noarov, and Weinberg (2021) consider the optimization of scoring rules for a binary state setting with uniform prior. The forecaster has access to costly samples and solves the optimal stopping problem given the cost and the scoring rule. They show that all scoring rules can be ranked by an incentivization index such that when the cost of the forecaster's samples converges to zero, the scoring rule with higher incentivization index induces lower prediction error given that the forecaster optimizes his expected reward net the cost. The authors characterize the scoring rule that maximizes the incentivization index. The main difference between their paper and ours is: in their model, different scoring rules only lead to prediction error with lower order terms that vanishes to zero, and under equilibrium the forecaster acquires almost perfect information about the state. In contrast, in our model, scoring rule plays a crucial rule for incentivizing effort, and both the additive gap and the multiplicative gap between the optimal scoring rule and heuristic scoring rules (e.g., quadratic scoring rules) for providing incentives can be large.

Frongillo and Witkowski (2017) considers the same optimization goal of maximizing incentive in the different single-task peer prediction setting. In the peer prediction model, the designer does not have access to a sample of the ground truth and must cross reference the reports from different agents to elicit the truthful report. Thus, the truthful peer prediction mechanism is unique up to positive affine transformations. Their optimization program reduces to the optimization of the parameters for affine transformations, which is significantly different from the optimization of scoring rules.

B Missing Proofs in Section 2

B.1 Proofs in Section 2.2

Proof of Proposition 2.1. Canonical scoring rules have the following simple interpretation. By making a report r, the agent selects the supporting hyperplane of u at r on which to evaluate the state. This supporting hyperplane has gradient $\xi(r)$ and contains point (r, u(r)). The agent's utility is

equal to the value of the realized state θ on this hyperplane (plus constant $\kappa(\theta)$ which is independent of the agent's report). With utility given by a random point on a hyperplane, the expected utility is equal to its mean on the hyperplane. When the agent's true posterior belief is that the state has mean r, the agent's expected utility is u(r) (plus a constant equal to the expected value of $\kappa(\cdot)$ under the agent's posterior belief; summarized below as Lemma 2.2). Misreporting r' with belief r gives a utility equal to the value of r on the supporting hyperplane with gradient $\xi(r')$ at r'. By convexity of u, a report of r gives the higher utility of u(r).

Proof of Lemma 2.3. Similar to the proof of Proposition 2.1, canonical scoring rules (Definition 3) can be interpreted via supporting hyperplanes of the utility function. The first term on the left-hand side of (4) upper bounds the utility that an agent can obtain at state θ , specifically, it is the utility from reporting state θ . The remainder of the left-hand side subtracts the utility that the agent obtains from report r in state θ , i.e., it evaluates, at state θ , the supporting hyperplane of u at report r. Thus, the boundedness constraint requires the difference between the utility function and the value of any supporting hyperplane of the utility function to be bounded at all states $\theta \in \Theta$. Figure 1(a) illustrates this bound.

The subgradient in $\{\xi(r) : r \in R\}$ that maximizes the right-hand side of the inequality identifies the range of ex post score of the agent for this scoring rule. To enforce that the score is within [0, B], we select $\kappa(\theta)$ equal to the negative of the lower endpoint of this range so that the score is 0 for the report with the worst score at state θ .

Of course, since the score bound is B, this inequality is tight for some $r \in R$ and $\theta \in \Theta$.

B.2 Proof of Theorem 2.1

In this section, we will formally prove Theorem 2.1. In the subsequent discussion, the boundary of the report space is denoted by ∂R and the interior of the report space by relint $(R) = R \setminus \partial R$.

Lemma B.1 (Abernethy and Frongillo, 2012). Any proper and μ -differentiable scoring rule for eliciting the mean S coincides with a canonical scoring rule (defined by u, ξ , and κ) at reports in the relative interior of the report space, i.e., it satisfies equation (2) for all $r \in \operatorname{relint}(R)$.

The main new results need to show that canonical scoring rules are without loss for Program (1) are extensions of Lemma B.1 to the boundary of the report space ∂R . The form of scoring rules considered enters the program in two places: the objective and the boundedness constraint. The two lemmas below show that canonical scoring rules are without loss in these two places in the program.

Lemma B.2. Any μ -differentiable, bounded, and proper scoring rule S for eliciting the mean is equal in expectation of truthful reports to a canonical scoring rule (defined by u, ξ , and κ), i.e., it satisfies equation (3).

Lemma B.3. For any μ -differentiable and proper scoring rule S for eliciting the mean that induces utility function u (via Lemma B.2) and satisfies score bounded in [0, B], there is a canonical scoring rule defined by u (and some ξ and κ) that satisfies the same score bound, i.e., it satisfies equation (4).

Note that Lemma B.2 implies that the utility function u corresponding to any μ -differentiable scoring rule S can be identified (via the equivalent canonical scoring rule); thus, the assumption of Lemma B.3 is well defined. Lemma B.2 and Lemma B.3 combine to imply that Program (1) and Program (6) are equivalent.

Next, we will formally prove Lemma B.2 and B.3. First we show that when the scoring rule is bounded, the corresponding functions $u(r), \xi(r), \kappa(\theta)$ in the characterization of Lemma B.1 are bounded in the interior as well.

Lemma B.4. For any bounded scoring rule S, there exist convex function $u : R \to \mathbb{R}$ and function $\kappa : \Theta \to \mathbb{R}$ such that for any report $r \in \operatorname{relint}(R)$ and any state $\theta \in \Theta$,

$$S(r,\theta) = u(r) + \xi(r) \cdot (\theta - r) + \kappa(\theta)$$

where $\xi(r) \in \partial u(r)$ is a subgradient of u, and functions $u(r), \xi(r), \kappa(\theta)$ are bounded for any report $r \in \operatorname{relint}(R)$ and any state $\theta \in \Theta$.

Proof. Since scoring rule S is bounded, let $\overline{B}_{\theta} = \sup_{r \in \operatorname{relint}(R)} S(r, \theta)$ and $\underline{B}_{\theta} = \inf_{r \in \operatorname{relint}(R)} S(r, \theta)$. Let $\hat{r} \in \operatorname{relint}(R)$ be a report in the interior such that both $u(\hat{r})$ and $\xi(\hat{r})$ are finite. Note that for any state $\theta \in \Theta$, state θ locate on the boundary of the report space, i.e., $\theta \in \partial R$, and the report space is a linear combination of the state space.

For any report $r \in \operatorname{relint}(R)$, by the convexity of function u, we have

$$u(r) \ge u(\hat{r}) - \xi(\hat{r}) \cdot (r - \hat{r})$$

and hence u(r) is bounded below.

Next we show that u(r) is bounded above for any report $r \in \operatorname{relint}(R)$. We first show that fixing any state θ , any report r which is a linear combination of θ and \hat{r} has bounded utility u(r). If $u(r) \leq u(\hat{r})$, then naturally u(r) is bounded above. Otherwise, note that

$$\bar{B}_{\theta} - \underline{B}_{\theta} \ge S(r,\theta) - S(\hat{r},\theta) = u(r) + \xi(r) \cdot (\theta - r) - u(\hat{r}) - \xi(\hat{r}) \cdot (\theta - \hat{r}) \\ \ge (u(r) - u(\hat{r})) \cdot \frac{\|\theta - \hat{r}\|}{\|\hat{r} - r\|} + u(\hat{r}) - u(\hat{r}) - \xi(\hat{r}) \cdot (\theta - \hat{r}) \ge u(r) - u(\hat{r}) - \xi(\hat{r}) \cdot (\theta - \hat{r})$$

where the first inequality holds because the scoring rule is bounded. The second inequality holds because the convex function u projected on line (θ, \hat{r}) is still a convex function. The last inequality holds because report r lies in between θ and \hat{r} . Therefore, we have that u(r) is bounded above for report r lies in between θ and \hat{r} . For any state $\theta \in \Theta$, let $\hat{u}(\theta) = \lim_{k\to\infty} u(r^k)$ where $\{r^k\}_{k=1}^{\infty}$ is a sequence of report on line (θ, \hat{r}) that converges to θ . Since $u(r^k)$ are bounded for any r^k , we have that $\hat{u}(\theta)$ is bounded as well. Since the report space is a subset of the convex hull of the state space, we have that for any report $r \in \operatorname{relint}(R)$, u(r) is upper bounded by the convex combination of $\hat{u}(\theta)$, which is also bounded by above.

For any state $\theta \in \Theta$, we have

$$S(\hat{r},\theta) = u(\hat{r}) + \xi(\hat{r}) \cdot (\theta - \hat{r}) + \kappa(\theta),$$

which implies $\kappa(\theta)$ is bounded since all other terms are bounded.

Finally, for any report $r \in \operatorname{relint}(R)$ and any state $\theta \in \Theta$,

$$S(r,\theta) = u(r) + \xi(r) \cdot (\theta - r) + \kappa(\theta),$$

which implies $\xi(r) \cdot (\theta - r)$ is bounded. Since the boundedness holds for all directions, the subgradient $\xi(r)$ must also be bounded.

Lemma B.5. Given any state space Θ and report space R with non-empty interior, for any distribution $G \in \Delta(\Theta)$ with mean μ_G , there exists a sequence of posteriors $\{G^k\}$ such that for any bounded function $\phi(\theta)$ in space Θ , we have $\{\mathbf{E}_{\theta \sim G^k} [\phi(\theta)]\}$ converges to $\mathbf{E}_{\theta \sim G} [\phi(\theta)]$. *Proof.* Since space R has a non-empty interior, let \tilde{G} be a distribution with mean $\mu_{\tilde{G}}$ in the interior of R. Let the sequence of posteriors $G^k = (1 - 1/k) \cdot G + 1/k \cdot \tilde{G}$. For any bounded function $\phi(\theta)$ in space Θ , we have

$$\lim_{k \to \infty} \mathbf{E}_{\theta \sim G^k} \left[\phi(\theta) \right] = \lim_{k \to \infty} \left[(1 - 1/k) \cdot \mathbf{E}_{\theta \sim G} \left[\phi(\theta) \right] + 1/k \cdot \mathbf{E}_{\theta \sim \widetilde{G}} \left[\phi(\theta) \right] \right] \to \mathbf{E}_{\theta \sim G} \left[\phi(\theta) \right].$$

Proof of Lemma B.2. By Lemma B.1, for μ -differentiable proper scoring rule S, there exists convex function $u: R \to \mathbb{R}$ and function $\kappa: \Theta \to \mathbb{R}$ such that for any report $r \in \operatorname{relint}(R)$ and any state $\theta \in \Theta$, we have

$$S(r,\theta) = u(r) + \xi(r) \cdot (\theta - r) + \kappa(\theta)$$

where $\xi(r) \in \nabla u(r)$ is a subgradient of u. By Lemma B.4, since the scoring rule is bounded, function u is convex and bounded and hence continuous in the interior. Thus, we can well define the value of u on the boundary as its limit from the interior, i.e., set $u(r) = \lim_{k \to \infty} u(r^k)$ for any ron the boundary of the report space R and $\{r^k\}_{k=1}^{\infty}$ as a sequence of interior reports converging to r. Thus we can replace the convex function u with continuous and convex function u for bounded scoring rules and the characterization still holds in the interior.

For any bounded proper scoring rule, we have that u(r) is bounded for any report $r \in \operatorname{relint}(R)$ and $\kappa(\theta)$ is bounded for any state $\theta \in \Theta$. Given any posterior G such that $\mu_G \in \partial R$, let $\{G^k\}$ be the sequence of posteriors constructed in Lemma B.5.

- 1. The identity function $\phi(\theta) = \theta$ is bounded. Therefore, the mean of the posteriors converges, i.e., $\lim_{k\to\infty} \mu_{G^k} = \mu_G$. And all means $\{\mu_{G^k}\}$ are in the interior of R.
- 2. Function $\kappa(\theta)$ is bounded. Therefore, the expected value for function κ converges. That is, $\lim_{k\to\infty} \mathbf{E}_{\theta\sim G^k} [\kappa(\theta)] = \mathbf{E}_{\theta\sim G} [\kappa(\theta)].$
- 3. The expost score $S(r, \theta)$ is bounded. Therefore, the expected score for reporting μ_G converges, i.e., $\lim_{k\to\infty} \mathbf{E}_{\theta\sim G^k} [S(\mu_G, \theta)] = \mathbf{E}_{\theta\sim G} [S(\mu_G, \theta)].$

Moreover, considering the sequence of expected score for reporting μ_{G^k} with distribution G, we have

$$\lim_{k \to \infty} \mathbf{E}_{\theta \sim G} \left[S(\mu_{G^k}, \theta) \right] = \lim_{k \to \infty} \left[u(\mu_{G^k}) + \mathbf{E}_{\theta \sim G} \left[\xi(\mu_{G^k}) \cdot (\theta - \mu_{G^k}) \right] + \mathbf{E}_{\theta \sim G} \left[\kappa(\theta) \right] \right]$$
$$= \lim_{k \to \infty} \left[u(\mu_{G^k}) + \mathbf{E}_{\theta \sim G^k} \left[\kappa(\theta) \right] \right] = \lim_{k \to \infty} \left[\mathbf{E}_{\theta \sim G^k} \left[S(\mu_{G^k}, \theta) \right]$$

where the second equality holds because $\lim_{k\to\infty} \mathbf{E}_{\theta\sim G^k}[\kappa(\theta)] = \mathbf{E}_{\theta\sim G}[\kappa(\theta)]$ and $\lim_{k\to\infty} \mu_{G^k} = \mu_G$. Combining the equalities, we have

$$\begin{aligned} \mathbf{E}_{\theta \sim G} \left[S(\mu_G, \theta) \right] &= \lim_{k \to \infty} \mathbf{E}_{\theta \sim G^k} \left[S(\mu_G, \theta) \right] \leq \lim_{k \to \infty} \mathbf{E}_{\theta \sim G^k} \left[S(\mu_{G^k}, \theta) \right] \\ &= \lim_{k \to \infty} \mathbf{E}_{\theta \sim G^k} \left[S(\mu_{G^k}, \theta) \right] = \lim_{k \to \infty} \mathbf{E}_{\theta \sim G} \left[S(\mu_{G^k}, \theta) \right] \leq \mathbf{E}_{\theta \sim G} \left[S(\mu_G, \theta) \right] \end{aligned}$$

where the inequalities holds by the properness of the scoring rule. Therefore, all inequalities must be equalities, and hence

$$\mathbf{E}_{\theta \sim G}\left[S(\mu_G, \theta)\right] = \lim_{k \to \infty} \mathbf{E}_{\theta \sim G^k}\left[S(\mu_{G^k}, \theta)\right] = \lim_{k \to \infty} \mathbf{E}_{\theta \sim G^k}\left[u(\mu_{G^k}) + \kappa(\theta)\right] = u(\mu_G) + \mathbf{E}_{\theta \sim G}\left[\kappa(\theta)\right].$$

where the last equality hold since function u is continuous.

Finally, given any bounded, continuous and convex function u with bounded subgradients and any bounded function κ , the corresponding canonical scoring rule is proper, bounded, and the expected score coincides.

Proof of Lemma B.3. If a proper scoring rule S is induced by function u and bounded by B in space Θ , by Lemma B.1, there exists function $\kappa : \Theta \to \mathbb{R}$ such that for any report $r \in \operatorname{relint}(R)$ and any state $\theta \in \Theta$,

$$S(r,\theta) = u(r) + \xi(r) \cdot (\theta - r) + \kappa(\theta)$$

where $\xi(r) \in \nabla u(r)$ is a subgradient of u. Moreover, the score $S(r, \theta) \in [0, B]$ for any report and state $r \in R, \theta \in \Theta$. Thus, it holds that for any report and state $r \in \operatorname{relint}(R), \theta \in \Theta$

$$S(\theta, \theta) - S(r, \theta) = u(\theta) - u(r) - \xi(r)(\theta - r) \le B.$$

For any report $R \in \partial R$, there exists a sequence of reports r_i such that $\{r_k\}$ converges to r and $\xi(r) = \lim_{k\to\infty} \xi(r_k)$ is a subgradient at report r. Thus, it holds that for any report $r \in \partial R$ and state $\theta \in \Theta$,

$$S(\theta, \theta) - S(r, \theta) = u(\theta) - u(r) - \lim_{k \to \infty} \xi(r_k)(\theta - r) \le B.$$

Therefore, the canonical scoring rule defined by u with the same function κ is proper and bounded in [0, B].

B.3 Proper Scoring Rules for Optimally Eliciting the Full Distribution

The previous discussions in this section focused on scoring rules for eliciting the mean of the posterior distribution. Note that elicitation of the mean is a restriction on scoring rules and in general, the principal could solicit the full distribution and reward the agent accordingly. In this section, we will show that, with respect to optimization and approximation, the problems of eliciting the full posterior distribution over a finite state space reduces to problems of eliciting the mean of a multi-dimensional state space.

Note that the sum of probabilities for all states is 1. So the report space is a $|\Theta|$ -dimensional simplex, i.e., $R = \{r \in [0,1]^{|\Theta|} : \sum_i r_i = 1\}$. For simplicity, for any posterior distribution G, we also use G to denote the $|\Theta|$ -dimensional vector of probabilities for the posterior distribution. For any finite state space $\Theta = \{\theta^{(j)}\}_{j=1}^{|\Theta|}$, we rewrite state $\theta^{(j)}$ as $|\Theta|$ -dimensional vectors, i.e., $\theta_i^{(j)} = 1$ if i = j and $\theta_i^{(j)} = 0$ otherwise. It is easy to verify that $R = \operatorname{conv}(\Theta)$. Next we introduce the characterization of proper scoring rules for eliciting the full distribution.

Definition 12. A scoring rule S is proper for eliciting the full distribution in space Θ if for any distribution $G \in \Delta(\Theta)$ and any report $G' \in R$, we have

$$\mathbf{E}_{\theta \sim G}\left[S(G,\theta)\right] \geq \mathbf{E}_{\theta \sim G}\left[S(G',\theta)\right].$$

Theorem B.1 (McCarthy, 1956). For any finite state space Θ and corresponding report space R, a scoring rule S is proper for eliciting the full distribution in space Θ if and only if there exists a convex function $u : R \to \mathbb{R}$ such that for any report $G \in R$ and any state $\theta \in \Theta$, we have

$$S(G,\theta) = u(G) + \xi(G) \cdot (\theta - G),$$

where $\xi(G) \in \nabla u(G)$ is a subgradient of u.

Similar to Lemma B.2, if the scoring rule is bounded, then the utility function u in Theorem B.1 is bounded and continuous. The proof of continuity is the same as Lemma B.2 and hence omitted here.

Note that there is no function $\kappa(\theta)$ in the characterization of Theorem B.1. The reason is that here for any finite state space Θ , any scoring rule $S(G, \theta) = u(G) + \xi(G) \cdot (\theta - G) + \kappa(\theta)$, there exists another convex function \hat{u} such that $S(G, \theta) = \hat{u}(G) + \xi(G) \cdot (\theta - G)$, where $\xi(G) \in \nabla \hat{u}(G)$ is a subgradient of \hat{u} . The objective value for reporting the full distribution with distribution f and scoring rule S is

$$\operatorname{Obj}(u, f) = \mathbf{E}_{G \sim f, \theta \sim G} \left[S(G, \theta) - S(D, \theta) \right] = \int_{R} \left[u(G) - u(D) \right] f(G) \, \mathrm{d}G.$$

Thus the form of the objective function for reporting the full distribution coincides with the objective function for reporting the mean. Moreover, it is easy to verify that the bounded constraint coincides as well. This result follows because distributions with finite state space Θ can be viewed as $|\Theta|$ -dimensional perfectly negatively correlated distributions with Bernoulli marginals. One important property of Bernoulli distributions is that reporting the full distribution is equivalent to reporting the mean of the distribution. Since reporting the full distribution and reporting the mean have the same characterization in this case, by viewing the distribution as $|\Theta|$ -dimensional correlated distribution, we have the following theorem.

Theorem B.2. For any finite state space Θ , report space $R = \operatorname{conv}(\Theta)$, and any distribution $f \in R$ over posteriors, scoring rule S is optimal for eliciting the full distribution if and only if it is optimal for eliciting the mean.

C Missing Proofs in Section 3

C.1 Proof of Theorem 3.1

Proof of Theorem 3.1. Consider any feasible solution u(r) of Program (7). We construct a V-shaped utility function $\tilde{u}(r)$ as

$$\tilde{u}(r) = \begin{cases} -\frac{u(0)}{\mu_D}(r - \mu_D) & \text{for } r \le \mu_D, \\ \frac{u(1)}{1 - \mu_D}(r - \mu_D) & \text{for } r \ge \mu_D. \end{cases}$$

The construction of \tilde{u} is illustrated in Figure 1b. It is easy to see that \tilde{u} is convex, $\tilde{u}(\mu_D) = 0$ and $\tilde{u}(r) \ge u(r)$ for any $r \in [0, 1]$. Therefore, the objective value for function \tilde{u} is higher than objective value for function u. Moreover, we have $\tilde{u}(0) = u(0)$, $\tilde{u}(1) = u(1)$, $\tilde{u}'(0) \ge \xi(0)$ and $\tilde{u}'(1) \le \xi(1)$, which implies \tilde{u} is also a feasible solution to Program (7). Thus, an optimal solution is V-shaped.

Next we focus on finding the optimal V-shaped function \tilde{u} for Program (7). Let $a = -u(0)/\mu_D = \tilde{u}'(0)$ and $b = u(1)/(1 - \mu_D) = \tilde{u}'(1)$. Since function \tilde{u} satisfies the constraints in Program (7), we get

$$b(1 - \mu_D) = \tilde{u}(1) \le 1 + \tilde{u}(0) + \tilde{u}'(0) = 1 - a \cdot \mu_D + a,$$

$$b(1 - \mu_D) = \tilde{u}(1) \ge \tilde{u}'(1) + \tilde{u}(0) - 1 = b - a \cdot \mu_D - 1,$$

which implies $b \leq a + 1/(1 - \mu_D)$ and $b \leq a + 1/\mu_D$. If $b < a + 1/\max\{\mu_D, 1 - \mu_D\}$, then we can either increase b or decrease a to get a better feasible V-shaped utility function. Suppose we fix parameter a, the objective value is pointwise maximized for any report r when $b = a + 1/\max\{\mu_D, 1 - \mu_D\}$.

Next we fix the optimal choice for parameter b. Note that the objective value given any parameter a is

$$\int_{0}^{1} u(r)f(r) \, \mathrm{d}r = \int_{0}^{\mu_{D}} a(r-\mu_{D})f(r) \, \mathrm{d}r + \int_{\mu_{D}}^{1} \left(a + \frac{1}{\max(\mu_{D}, 1-\mu_{D})}\right)(r-\mu_{D})f(r) \, \mathrm{d}r$$
$$= \frac{1}{\max(\mu_{D}, 1-\mu_{D})} \int_{\mu_{D}}^{1} (r-\mu_{D})f(r) \, \mathrm{d}r, \tag{8}$$

which invariant of parameter a. Therefore, any V-shaped utility function with parameters satisfying $b = a + 1/\max\{\mu_D, 1 - \mu_D\}$ is optimal and obtains objective value given by equation (8).

C.2 Proof of Corollary 3.3

Proof of Corollary 3.3. In the characterization of the optimal performance of Theorem 3.1, i.e.,

$$OPT(f) = \mathbf{E}_{r \sim f} \left[\max(r - \mu_D, 0) \right] / \max(\mu_D, 1 - \mu_D),$$

it is easy to see that the numerator is maximized and the denominator is minimized in when the distibution of posterior means f is uniform on the extreme points $\{0, 1\}$. For this distribution, the numerator is 1/4 and the denominator is 1/2. Thus, OPT(f) = 1/2.

C.3 Proof of Theorem 3.2

Proof of Theorem 3.2. Suppose the distribution over report f(r) has two point masses, which is a with probability p, and b > a with probability 1 - p. Then, we have the mean of prior is $\mu_D = pa + (1 - p)b$ and $a < \mu_D < b$. Without loss of generality, we can assume that $\mu_D \leq \frac{1}{2}$. By Theorem 3.1, it holds that

$$c = OPT(f) = \frac{1}{\max\{\mu_D, 1 - \mu_D\}} \cdot (1 - p)(b - \mu_D) = \frac{p(1 - p)(b - a)}{\max\{\mu_D, 1 - \mu_D\}}.$$
(9)

For quadratic scoring rule with utility function $u_q(r) = r^2$ (Definition 6), we have

$$Obj(u_q, f) = \mathbf{E}_{r \sim f} \left[u_q(r) \right] - u_q(\mu_D) = p(a^2 - \mu_D^2) + (1 - p)(b^2 - \mu_D^2) = p(1 - p)(b - a)^2.$$
(10)

Combining equations (9) and (10), we have

Obj
$$(u_q, f) = (\max\{\mu_D, 1 - \mu_D\})^2 \cdot \frac{c^2}{p(1-p)}$$

The worst case ratio is achieved when $Obj(u_q, f)$ is minimized, i.e., $\mu_D = \frac{1}{2}$ and $p = \frac{1}{2}$, which gives $\min_{f \in \mathcal{F}_c} Obj(u_q, f) = c^2$.

C.4 Proof of Theorem 3.3

To simplify the proof of Theorem 3.3, we define the benchmark \widetilde{OPT} as an approximate upperbound on OPT:

$$\widetilde{\operatorname{OPT}}(f) = 2\max(\mu_D, 1 - \mu_D)\operatorname{OPT}(f) = 2\mathbf{E}_{r \sim f}\left[\max(r - \mu_D, 0)\right]$$

Notice that $\max(\mu_D, 1 - \mu_D) \in [1/2, 1]$; thus, $OPT(\mu_D) \leq \widetilde{OPT}(\mu_D) \leq 2 OPT(\mu_D)$. Thus, approximation of benchmark \widetilde{OPT} is equivalent to approximation of OPT up to a factor of two. Theorem 3.3 is obtained from Lemma C.1 and the bound of $c \leq \tilde{c} \leq 2c$.

Lemma C.1. Let $\mathcal{F}_{\tilde{c}}$ be the set of distributions over posterior means such that benchmark \widetilde{OPT} is $\tilde{c} \in (0, 1/2]$. For any convex and bounded utility function u, we have

$$\min_{f \in \mathcal{F}_c} \operatorname{Obj}(u, f) \le \min(\frac{1}{2}, \frac{2\tilde{c}^2}{(1-2\tilde{c})^2}) \le 8\tilde{c}^2.$$

Proof. A convex and bounded utility function u has monotone derivative u' and, by Lemma 3.1, the amount this derivative increases on its [0, 1] domain is u'(1) - u'(0) bounded by 2. Consider any positive integer d and partition the [0, 1] domain of u into d intervals of width 1/d. By the pigeon

hole principle, one part must contain at most the average increase of u', i.e., there exists interval [a, b = a + 1/d] with $u'(b) - u'(a) \le 2/d$.

Consider distribution f_d defined as the uniform distribution over deterministic points a and b with mean $\mu_d = a + 1/2d$. By the definition of benchmark \widetilde{OPT} :

$$\widetilde{\operatorname{OPT}}(f_d) = 2\mathbf{E}_{r \sim f_d} \left[\max(r - \mu_d, 0) \right] = \frac{1}{2d}.$$

Calculating the objective value of utility function u, we have

$$Obj(u, f_d) = \frac{u(a) + u(b)}{2} - u(\mu_d) \le \frac{u'(b) - u'(a)}{2} \cdot \frac{b - a}{2} = \frac{1}{2d^2}$$

where the inequality follows from identifying an optimal utility u satisfying $u'(b) - u'(a) \leq 2/d$. It is u'(r) = -1/d for $r \in [a, \mu_d)$ and u'(r) = 1/d for $r \in (\mu_d, b]$. Combining the two bounds with $\widetilde{OPT}(f_d) = \tilde{c}$ we see that $Obj(u, f_d) \leq 2\tilde{c}^2$ for $\tilde{c} \in \{1/2d : d \in \{1, \ldots\}\}$.

To extend this bound to all $\tilde{c} \in [0, 1/2]$, observe that the bound on $Obj(u, f_d)$ easily extends to $Obj(u, f_{d'})$ for non-integral $d' \geq d$, while the value of $\widetilde{OPT}(f_{d'})$ holds as calculated for nonintegral d'. Thus, we can obtain bounds for non-integral d' by combining bounds on $\widetilde{OPT}(f_{d+1})$ and $Obj(u, f_d)$. Solving for the bound on $Obj(u, f_d)$ in terms of $\tilde{c} = \widetilde{OPT}(f_{d+1})$: for any $\tilde{c} \in (0, 1/2]$ there exists $f \in \mathcal{F}_{\tilde{c}}$ with $Obj(u, f) \leq \min(\frac{1}{2}, \frac{2\tilde{c}^2}{(1-2\tilde{c})^2}) \leq 8\tilde{c}^2$. The first inequality holds by substituting $d = 1/2\tilde{c} - 1$ into the formula of $Obj(u, f_d)$, the second inequality uses $Obj(u, f) \leq 1/2$ and notes that the bound of the first inequality is trivial until $\tilde{c} \leq 1/4$, and thereafter the denominator is lower bounded by 1/4.

C.5 Proof of Theorem 3.5

Proof. By Theorem 3.1, there is an optimal utility function that is V-shaped at μ_f with parameters $|a|, |b| \leq 1$. Thus, we have

$$OPT(f) = \int_0^{\mu_D} a(r - \mu_D) f(r) \, \mathrm{d}r + \int_{\mu_D}^1 b(r - \mu_D) f(r) \, \mathrm{d}r \le \mathbf{E}_{r \sim f} \left[|r - \mu_D| \right].$$

By Definition 6, the objective value of the quadratic scoring rule is

$$Obj(u_q, f) = \mathbf{E}_{r \sim f} \left[u_q(r) - u_q(\mu_D) \right] = \mathbf{E}_{r \sim f} \left[(r - \mu_D)^2 \right].$$

By Jensen's inequality, we have

$$\mathbf{E}_{r\sim f}\left[|r-\mu_{f}|\right] = \mathbf{E}_{r\sim f}\left[\sqrt{(r-\mu_{f})^{2}}\right] \leq \sqrt{\mathbf{E}_{r\sim f}\left[(r-\mu_{f})^{2}\right]} = \frac{\mathbf{E}_{r\sim f}\left[(r-\mu_{f})^{2}\right]}{\sigma},$$

where the last equality is due to $\mathbf{E}_{r \sim f} \left[(r - \mu_f)^2 \right] = \sigma^2$.

D Missing Proofs in Section 4

D.1 Proof of Proposition 4.1

Proof of Proposition 4.1. For the "if" direction: if the allocation x and payment p satisfies the above conditions, by Rochet (1985) and the Bayesian incentive compatibility, the utility function u

is continuous and convex, and $\xi(r) = x(r)$ is a feasible subgradient of the utility function. By the bounded utility difference, we have that

$$u(\theta) - u(r) - \xi(r) \cdot (\theta - r) = x(\theta) \cdot \theta - p(\theta) - x(r) \cdot r + p(r) - x(r) \cdot (\theta - r)$$
$$= x(\theta) \cdot \theta - p(\theta) - x(r) \cdot \theta + p(r) \le B,$$

which implies utility function u corresponds to a μ -differentiable B-bounded proper scoring rule.

For the "only if" direction: given a utility function u of a μ -differentiable bounded proper scoring rule for eliciting the mean, by Lemma B.3, there exists a set of subgradients $\xi(r) \in \partial u(r)$ such that

$$u(\theta) - u(r) - \xi(r) \cdot (\theta - r) \le B$$

for any report $r \in R$ and state $\theta \in \Theta$. Setting the allocation as $x(r) = \xi(r)$, and the payment as $p(r) = r \cdot \xi(r) - u(r)$, it is easy to verify that this allocation and payment satisfy all three conditions above.

D.2 Proof of Theorem 4.1

Proof of Theorem 4.1. Denote the finite set of state space as $\Theta = \{\theta_j\}_{j=1}^d$, Let the support of distribution f over posterior means be $\{r_i\}_{i=1}^m$. Denote the probability that posterior mean r_i happens as f_i . For simplicity, denote $r_0 = \mu_D$ as the mean of the prior and $r_{m+j} = \theta_j$ as the report for pointmass distribution on states for any $j \in [d]$. Program (6) is equivalent to the following program.

$$\max_{\{x_i, p_i\}_{i \in \{0, \dots, m+d\}}} \sum_{i \in [m]} (x_i \cdot r_i - p_i) f_i$$
s.t. $x_0 \cdot r_0 - p_0 = 0,$
 $x_i \cdot r_i - p_i \ge x_{i'} \cdot r_i - p_{i'}, \quad \forall i, i' \in \{0, \dots, m+d\},$
 $(x_i \cdot r_i - p_i) - (x_{i'} \cdot r_i - p_{i'}) \le B \quad \forall i \in \{m+1, \dots, m+d\}, i' \in \{0, \dots, m+d\}.$
(11)

Note that Program (11) is a linear program with number of variables and constraints polynomial in n, m, and d; and hence there exists a polynomial time algorithm that optimally solves it. Next we will formally prove the equivalence of Program (6) and Program (11).

For one direction: For any utility function u that is a feasible solution to Program (6), by Proposition 4.1, there exists corresponding allocation and payment functions x and p. Let the variables in Program (11) be $x_i = x(r_i)$, $p_i = p(r_i)$, for any $i \in \{0, \ldots, m+d\}$. It is easy to verify that this is a feasible solution to Program (11) with the same objective value.

For the other direction: For any feasible solution $\{x_i, p_i\}_{i \in \{0,...,m+d\}}$ to Program (11), define the utility function

$$u(r) = \max_{i \in \{0,\dots,m+d\}} x_i \cdot r - p_i$$

for any report $r \in R$. We show that this utility function u satisfies Program (6) and has the same objective value. Obviously, the utility function u is continuous and convex. For any $i \in \{0, \ldots, m + d\}$, the utility function $u(r_i) = x_i \cdot r_i - p_i$ by the definition of Bayesian incentive compatibility, and hence the objective value of Program (6) given by this utility u equals the objective value of Program (11). Moreover, for any report $r \in R$, letting $i' = \arg \max_{i \in \{0, \ldots, m+d\}} x_i r - p_i$, the allocation $x_{i'}$ is a subgradient of the utility function u(r) at report r. Thus, we have for any state $\theta^{(j)} \in \Theta$

$$u(\theta^{(j)}) - u(r) - \xi(r) \cdot (\theta^{(j)} - r) = (x_{m+j} \cdot \theta^{(j)} - p_{m+j}) - (x_{i'} \cdot r - p_{i'}) - x_{i'} \cdot (\theta^{(j)} - r)$$
$$= (x_{m+j} \cdot \theta^{(j)} - p_{m+j}) - (x_{i'} \cdot \theta^{(j)} - p_{i'}) \le B,$$

where the last inequality holds by the bounded utility difference property. Therefore, utility function u is a feasible solution to Program (6), which establishes the equivalence of two programs.

D.3 Proof of Lemma 4.2

Proof. The following geometry of the utility function is easy verify. First, convexity of report space R implies convexity of u. Second, consider the n + 1 dimensional space $R \times [-1/2, 1/2]$, where the n+1st dimension represents the utility u. The utility function defines a truncated convex cone with vertex equal to $(\mu_D, 0)$ and base at height 1/2 with cross section R. Consider the point reflection, henceforth, the reflected cone, of this convex cone around its vertex $(\mu_D, 0)$. By basic properties of cones and their point reflections, this reflected cone has the same supporting hyperplanes as the original cone. By the symmetry assumption of R around μ_D , the reflected cone is equal to the mirror reflection of the original cone with respect to the u = 0 plane. Consequently, the base of the reflected cone at u = -1/2 has cross section equal to R.

We now argue that the utility function satisfies the boundeness constraint, restated for convenience (with report $r \in R$ and state $\theta \in \Theta$):

$$u(\theta) - u(r) - \nabla u(r) \cdot (\theta - r) \le 1.$$

By definition of the V-shaped utility, we know that the first term is at most 1/2. The second and third terms, together, can be viewed as subtracting the evaluation, at state θ , of the supporting hyperplane of u at r. The highest point in the reflected cone for any $\theta \in R$ is $-u(\theta)$ and this point lower bounds the value of θ in any of the reflected cones supporting hyperplanes (which are the same as the original cones supporting hyperplanes). By definition, the reflected cone satisfies $-u(\theta) \geq -1/2$ for $\theta \in R$. We conclude, as desired, that the difference between the first term and the second and third terms is at most 1.

D.4 Proof of Theorem 4.2

Proof. Consider relaxing the optimization problem on the general space solve it independently on lines through the center. Specifically, consider the conditional distribution of f on the line segment through the center μ_D and the boundary points r and $2\mu_D - r$ on ∂R . Center symmetry implys symmetry on this line segment. By Corollary 3.2, the solution to this single-dimensional problem is symmetric V-shaped, i.e., with $u(r) = u(2\mu_D - r) = 1/2$ and $u(\mu_D) = 1/2$.

The solutions on all lines through the center μ_D coincide at μ_D with $u(\mu_D) = 0$. They can be combined, and the resulting utility function u is a symmetric V-shaped function (Definition 8). Lemma 4.2 implies that u is convex and bounded and, thus feasible for the original program. Since it optimizes a relaxation of the original program, it is also optimal for the original program.

D.5 Proof of Lemma 4.5

Proof. Given posterior distribution G, let i be the dimension that maximizes the agent's expected utility under separate scoring rules $\hat{S}_1, \ldots, \hat{S}_n$, i.e., $i = \arg \max_j \mathbf{E}_{\theta_j \sim G_j} \left[\hat{S}_j(\mu_{G_j}, \theta_j) \right]$, and let

 $r_i = \mu_{G_i}$ be the mean of the posterior on dimension *i*. For report $r = (i, r_i)$ and any other report $r' = (i', r'_i)$, we have

$$\mathbf{E}_{\theta \sim G}\left[S(r,\theta)\right] = \mathbf{E}_{\theta_i \sim G_i}\left[\hat{S}_i(r_i,\theta_i)\right] \geq \mathbf{E}_{\theta_{i'} \sim G_{i'}}\left[\hat{S}_{i'}(\mu_{G_{i'}},\theta_{i'})\right] \geq \mathbf{E}_{\theta_{i'} \sim G_{i'}}\left[\hat{S}_{i'}(r'_{i'},\theta_{i'})\right] = \mathbf{E}_{\theta \sim G}\left[S(r',\theta)\right]$$

The first and last equality hold by the definition of choose-and-report proper scoring rules, and the first inequality holds by the definition of dimension i. The second inequality holds since each single dimensional scoring rule is proper. Thus the choose-and-report scoring rule S is proper. Moreover, if each single dimensional proper scoring rule \hat{S}_i is bounded, it is easy to verify that the choose-and-report scoring rule S is also bounded.

D.6 Proofs of Lemma 4.6-Lemma 4.8

Proof of Lemma 4.6. This result follows because the extended distribution is symmetric on the extended state space, thus, its optimal scoring rule is max-over-separate (Corollary 4.4). This scoring rule can be applied to the original space where it is still max-over-separate. The optimal max-over-separate scoring rule for the original space is no worse.

Proof of Lemma 4.7. Let \tilde{u} be the optimal utility function corresponding to $OPT(\tilde{f}, B, \tilde{\Theta})$. Since the distribution \tilde{f} is center symmetric, by Theorem 4.2, the utility function \tilde{u} is symmetric Vshaped. Thus, we have

$$OPT(\tilde{f}, B, \tilde{\Theta}) = \int_{\tilde{R}} \tilde{u}(r) \tilde{f}(r) dr$$
$$= \frac{1}{2} \int_{R} \tilde{u}(r) f(r) dr + \frac{1}{2} \int_{R} \tilde{u}(2\mu_{D} - r) f(r) dr$$
$$= \int_{R} \tilde{u}(r) f(r) dr = Obj(\tilde{u}, f).$$

Proof of Lemma 4.8. Let \hat{u} be the optimal solution of Program (6) with distribution f and state space Θ , i.e., $Obj(\hat{u}, f) = OPT(f, B, \Theta)$. On the other hand, utility function \hat{u} may not be optimal for distribution \tilde{f} , thus, $OPT(\tilde{f}, B, \Theta) \ge Obj(\hat{u}, \tilde{f})$. We have,

$$\begin{aligned} \operatorname{OPT}(\tilde{f}, B, \widetilde{\Theta}) &\geq \operatorname{Obj}(\hat{u}, \tilde{f}) = \int_{\widetilde{R}} \hat{u}(r) \, \tilde{f}(r) \, \mathrm{d}r = \frac{1}{2} \int_{R} \tilde{u}(r) \, f(r) \, \mathrm{d}r + \frac{1}{2} \int_{R} \tilde{u}(2\mu_{D} - r) \, f(r) \, \mathrm{d}r \\ &\geq \frac{1}{2} \int_{R} \tilde{u}(r) \, f(r) \, \mathrm{d}r = \frac{1}{2} \operatorname{OPT}(f, B, \widetilde{\Theta}) \end{aligned}$$

where the final inequality follows from convexity of \hat{u} , $\int_{R} (2\mu_D - r) f(r) dr = \mu_D$, Jensen's Inequality, and $\hat{u}(\mu_D) = 0$.

D.7 Proof of Lemma 4.9

The approach to proving Lemma 4.9, i.e., $OPT(f, B, \widetilde{\Theta}) \geq \frac{1}{4}OPT(f, B, \Theta)$, is as follows. Let u be the optimal utility corresponding to $OPT(f, B, \Theta)$. We construct \tilde{u} that (a) exceeds u at all point $r \in R$ and (b) is feasible for $OPT(f, 4B, \widetilde{\Theta})$. The utility function $\tilde{u}/4$, thus, has objective value at least $\frac{1}{4}OPT(f, B, \Theta)$ and is feasible for $OPT(f, B, \widetilde{\Theta})$. The optimal utility is only better.

The proof of the lemma introduces the following constructs.



Figure 3: The figure on the left hand side illustrates a hyperplane for report r' on the boundary of the report space, which is shifted from a tangent plane of u at the boundary r'. The figure on the right hand side illustrates the extended utility function \tilde{u} that takes the supremum over all hyperplanes shifted from the feasible tangent planes to intersect with the $(\mu_D, 0)$ point.

• The extended utility function \tilde{u} for program $OPT(f, 4B, \tilde{\Theta})$ given utility function u for the program $OPT(f, B, \Theta)$ is defined as follows.

Feasibility of u for Program (6) defines subgradients $\{\xi(r) : r \in R\}$ that satisfy the boundedness condition. Let \mathcal{G}_u be the set of all subgradients of u that satisfy the boundedness constraint. Clearly the latter set contains the former set. Define the extended utility function \tilde{u} as the convex function defined by the supremum of the supporting hyperplanes given by the subgradients \mathcal{G}_u shifted to intersect with the $(\mu_D, 0)$ point. See Figure 3.

Convexity of u implies that its supporting hyperplane at r with subgradient $\xi(r)$ is below $u(\mu_D) = 0$ at μ_D . Thus, relative to the supporting hyperplanes of u these supporting hyperplanes of \tilde{u} are shifted upwards.

The extended utility function \tilde{u} is *convex-conical* as it is defined by supporting hyperplanes that all contain point $(\mu_D, 0)$.

• The extended state spaces are $\Theta \subset \widetilde{\Theta}' \subset \widetilde{\Theta}'' \subset \widetilde{\Theta}$. State space $\widetilde{\Theta}'$ is the union of the original state space and its point reflection about μ_D as $\widetilde{\Theta}' = \Theta \cup \{2\mu_D - \theta : \theta \in \Theta\}$, state space $\widetilde{\Theta}''$ is the convex hull of $\widetilde{\Theta}'$, and state space $\widetilde{\Theta}$ (as previously defined) is the extended rectangular state space containing $\widetilde{\Theta}''$.

Lemma 4.9, i.e., $OPT(f, 4B, \widetilde{\Theta}) \ge OPT(f, B, \Theta)$, follows by combining the following lemmas.

Lemma D.1. For any feasible solution u for Program (6), the extended utility function \tilde{u} is at least u, i.e., $\tilde{u}(r) \ge u(r)$ for any report $r \in R$.

Lemma D.2. For any feasible solution u for Program (6) with score bound B and state space Θ , the extended utility function \tilde{u} is a feasible solution of Program (6) with score bound 2B and state space Θ .

Lemma D.3. Any convex-conical utility function \tilde{u} that is a feasible solution of Program (6) with score bound 2B and state space Θ is a feasible solution to Program (6) with bound 2B and state space $\tilde{\Theta}'$.

Lemma D.4. Any convex-conical utility function \tilde{u} that is a feasible solution of Program (6) with score bound 2B and state space $\tilde{\Theta}'$ is a feasible solution to Program (6) with bound 2B and state space $\tilde{\Theta}'' = \operatorname{conv}(\tilde{\Theta}')$.

Lemma D.5. Any convex-conical utility function \tilde{u} that is a feasible solution of Program (6) with score bound 2B and state space $\tilde{\Theta}''$ is a feasible solution to Program (6) with bound 4B and state space $\tilde{\Theta}$.

Proof of Lemma D.1. Since the supporting hyperplanes of \tilde{u} are shifted upwards relative to u, we have $\tilde{u}(r) \geq u(r)$ at all $r \in R$. Thus, \tilde{u} obtains at least the objective value of u, i.e., $Obj(f, \tilde{u}) \geq Obj(f, \tilde{u})$.

Proof of Lemma D.2. First, the subgradients of \tilde{u} are a subset of the subgradients of u that satisfy the boundedness constraint. Lemma D.6 (stated and proved at the end of this subsection) shows that the set of subgradients \mathcal{G}_u of u that satisfy the boundedness constraint is closed. As \tilde{u} is defined the supremum over these hyperplanes, closure of the set implies that the supremum at any report $r \in R$ is attained on one of these hyperplanes.

Now observe that in the construction of \tilde{u} , the supporting hyperplanes of u are shifted up by at most B. The boundedness constraint corresponding to state μ_D and the report r with subgradient $\xi(r) \in \nabla u(r)$ implies that the supporting hyperplane corresponding to $\xi(r)$ at r has value at least -B at μ_D . Thus, in the construction of the extended utility function \tilde{u} , the hyperplane corresponding to $\xi(r)$ is shifted up by at most B and, at any state $\theta \in \Theta$, $\tilde{u}(\theta) \leq u(\theta) + B$.

Finlly, the boundedness constraint is the difference between the utility at a given state and the value of any supporting hyperplane of the utility evaluated at that state. From u to \tilde{u} the former has increased by at most B and the latter is no smaller; thus, \tilde{u} satisfies the boundedness constraint on state space Θ with bound 2B.

Proof of Lemma D.3. The lemma follows by the geometries of the boundedness constraint and convex cones. The boundedness constraint requires a bounded difference between the utility at any state (in the state space) and the value at that state on any supporting hyperplane of the utility function (corresponding to any report in the report space). For convex-conical utility functions, the supporting hyperplanes are also supporting hyperplanes of the cone defined by the point reflection of the utility function around its vertex (μ_D , 0), henceforth, the reflected cone. Thus, the boundedness constraint for convex-conical utility function requires that the difference between the original cone and the reflected cone be bounded at all states in the state space.

The original space Θ and the reflected state space $\{2\mu_D - \theta : \theta \in \Theta\}$ are symmetric with respect to the original cone and the reflected cone. Thus, if states in the original state space are bounded, by comparing a state on the cone to the same state on the reflected cone; then states in the reflected state space are bounded by comparing its reflected state (in the original state space) on the reflected cone to its reflected state on the original cone.

Thus, if a boundedness constraint holds on Θ it also holds on the reflected state space $\{2\mu_D - \theta : \theta \in \Theta\}$ and their union.

Proof of Lemma D.4. Consider the cone and reflected cone defined in the proof of Lemma D.3 and the geometry of the boundedness constraint. Notice that, by convexity of the cone defining the utility function \tilde{u} and concavity of the reflected cone, the convex combination of the bounds, i.e., the difference of values of states on these two cones, of any set of states is at least the bound of the convex combination of the states. Hence, if the boundedness constraint holds on state space $\widetilde{\Theta}'$, then it holds on its convex hull $\widetilde{\Theta}'' = \operatorname{conv}(\widetilde{\Theta}')$. Proof of Lemma D.5. Consider any ray from μ_D . Since the utility \tilde{u} is a convex cone, the utility on this ray is a linear function of the distance from μ_D . The same holds for this ray evaluated on the point reflection of the utility at μ_D . The difference between these utilities is also linear. Thus, by the geometry of the boundedness constraint for convex-conical utility functions, on any ray from μ_D , the bound is linear. Considering the state space $\tilde{\Theta}''$ and $\tilde{\Theta}$, if the former is scaled by a factor of two around μ_D , then it contains the latter (by simple geometry, see Figure 2). Thus, if the convex-conical utility function \tilde{u} satisfies bound 2B on state space $\tilde{\Theta}''$ it satisfies bound 4B on state space $\tilde{\Theta}$.

Lemma D.6. For any feasible solution u for Program (6), the set \mathcal{G}_u of all subgradients of u satisfying the bounded constraints is a closed set.

Proof. By Lemma B.2, any feasible solution u for Program (6) is convex, bounded and continuous with bounded subgradients. For any convex, bounded and continuous function u, let $\{\xi^k(r^k)\}_{k=1}^{\infty} \subseteq \mathcal{G}_u$ be a convergent sequence of subgradients in set \mathcal{G}_u , where r^k is the report corresponds to the k^{th} subgradient. Let $\xi^* = \lim_{k \to \infty} \xi^k(r^k)$ be the limit of the subgradients. Since the report space is a closed and bounded space, there exists a subsequence of reports $\{r^{k_j}\}_{j=1}^{\infty} \subseteq \{r_k\}_{k=1}^{\infty}$ such that $\{r^{k_j}\}_{j=1}^{\infty}$ converges. Letting report $r = \lim_{j\to\infty} r^{k_j}$, we have report r is in the report space, i.e., $r \in R$. Moreover, we have $\lim_{j\to\infty} \xi^{k_j}(r^{k_j}) = \lim_{k\to\infty} \xi^k(r^k) = \xi^*$. Next we show that ξ^* is a subgradient for some report $r \in R$ such that the bounded constraints of the induced scoring rule are satisfied for any state $\theta \in \Theta$, i.e., $\xi^* \in \mathcal{G}_{u,r}$.

First for any state θ , we have

$$u(r) + \xi^* \cdot (\theta - r) = \lim_{j \to \infty} [u(r^{k_j}) + \xi^* \cdot (\theta - r^{k_j})]$$
$$= \lim_{j \to \infty} [u(r^{k_j}) + \xi^{k_j}(r^{k_j}) \cdot (\theta - r^{k_j})] \le u(\theta),$$

where the first equality holds because function u and function $\xi^* \cdot r$ are continuous and bounded in reports. The inequality holds because $\xi^{k_j}(r^{k_j})$ is a subgradient for report r^{k_j} . Thus ξ^* is subgradient for report r. Next we show that the scoring rule induced by subgradient ξ^* is bounded for report r. For any state θ , we have

$$u(\theta) - u(r) - \xi^* \cdot (\theta - r) = u(\theta) - \lim_{j \to \infty} [u(r^{k_j}) + \xi^{k_j}(r^{k_j}) \cdot (\theta - r^{k_j})] \le u(\theta) - (u(\theta) - B) = B,$$

where the inequality holds because the subgradient $\xi^{k_j}(r^{k_j})$ satisfies the bounded constraint for report r^{k_j} at state θ , i.e., $\xi^{k_j}(r^{k_j}) \in \mathcal{G}_{u,r^{k_j}}$ and $u(r^{k_j}) + \xi^{k_j}(r^{k_j}) \cdot (\theta - r^{k_j}) \ge u(\theta) - B$. Therefore, $\xi^* \in \mathcal{G}_{u,r} \subset \mathcal{G}_u$, which implies the set \mathcal{G}_u is a closed set.

D.8 Proof of Theorem 4.4

Proof. Note that by definition, it is easy to verify that the utility function u_{μ_D} satisfies

$$u_{\mu_D}(r) = \max_i \frac{1}{2 \max\{\mu_{D_i}, 1 - \mu_{D_i}\}} |r_i - \mu_{D_i}|$$

and hence

$$Obj(u_{\mu_D}, f) = \mathbf{E}_{r \sim f} \left[\max_{i} \frac{1}{2 \max\{\mu_{D_i}, 1 - \mu_{D_i}\}} |r_i - \mu_{D_i}| \right]$$

Moreover, we have

$$\begin{aligned} \operatorname{Obj}(u_{\mu}, f) &- \operatorname{Obj}(u_{\mu_{D}}, f) \\ &= \mathbf{E}_{r \sim f} \left[\max_{i} \frac{1}{2 \max\{\mu_{i}, 1 - \mu_{i}\}} \left| r_{i} - \mu_{i} \right| - \max_{i} \frac{1}{2 \max\{\mu_{D_{i}}, 1 - \mu_{D_{i}}\}} \left| r_{i} - \mu_{D_{i}} \right| \right] - u_{\mu}(\mu_{D}) \\ &\geq -3\epsilon, \end{aligned}$$

which implies that the incentive for effort of the V-shaped scoring rule for μ is at least that of the V-shaped scoring rule for μ_D less 3ϵ , and the theorem holds. Note that the last inequality holds because

$$u_{\mu}(\mu_{D}) = \max_{i} \frac{1}{2 \max\{\mu_{i}, 1 - \mu_{i}\}} |\mu_{D_{i}} - \mu_{i}| \le \max_{i} |\mu_{D_{i}} - \mu_{i}| \le \epsilon$$

and for any dimension $i \in [n]$,

$$\begin{aligned} &\frac{1}{2\max\{\mu_{D_i}, 1-\mu_{D_i}\}} \left| r_i - \mu_{D_i} \right| \leq \frac{1}{2\max\{\mu_{D_i}, 1-\mu_{D_i}\}} (|r_i - \mu_i| + \epsilon) \\ &\leq \frac{1}{2\max\{\mu_{D_i}, 1-\mu_{D_i}\}} \left| r_i - \mu_i \right| + \epsilon \leq \frac{1}{2\max\{\mu_i, 1-\mu_i\}} \left| r_i - \mu_i \right| + 2\epsilon. \end{aligned}$$

D.9 Proof of Theorem 4.6

Proof. We first argue the upper bound that scoring separately in rectangular report and state spaces guarantees an O(n) approximation. By Theorem 4.3, there exists proper and bounded single-dimensional proper scoring rules (S_1, \ldots, S_n) such that the induced max-over-separate S is an 8-approximation to the optimal scoring rule. Let \hat{S} be the separate scoring rule induced by single-dimensional proper scoring rules $(\frac{1}{n}S_1, \ldots, \frac{1}{n}S_n)$. It is easy to verify that scoring rule \hat{S} is bounded, with objective value at least $\frac{1}{n}$ fraction of that for scoring rule S. Thus, separate scoring rule \hat{S} is an O(n) approximation to the optimal scoring rule.

We now give an example of a symmetric distribution over posteriors over the space $R = \Theta = [0, 1]^n$ such that the approximation is $\Omega(n)$. Consider the i.i.d. distribution over posterior means f with marginal distribution f_i dimension i defined by

$$r_i = \begin{cases} 1 & \text{w.p. } ^{1/2n}, \\ \frac{1}{2} & \text{w.p. } 1 - \frac{1}{n}, \\ 0 & \text{w.p. } \frac{1}{2n}. \end{cases}$$

The prior mean for each dimension is 1/2 and by Corollary 3.2, the optimal scoring rule for each dimension i has V-shaped utility function \hat{u}_i with $\hat{u}_i(0) = \hat{u}_i(1) = 1/2$ and $\hat{u}_i(1/2) = 0$. Thus, the expected objective value for the optimal scoring rule of dimension i is $1/2 \operatorname{\mathbf{Pr}}_{r_i \sim f_i} [r_i \in \{0, 1\}] = 1/2n$. Any average of optimal separate scoring rules, thus, has objective value 1/2n.

Now consider the max-over-separate scoring rule which has a (multi-dimensional) symmetric V-shaped utility function u and is optimal (see Definition 8 and Theorem 4.2). The objective value is $\mathbf{E}_{r\sim f}[u(r)]$. Importantly u(r) = 0 if $r = (1/2, \ldots, 1/2)$ and, otherwise, u(r) = 1/2. Thus,

OPT(f) =
$$\frac{1}{2} \mathbf{Pr}_{r \sim f} [r \neq (1/2, \dots, 1/2)]$$

= $\frac{1}{2} (1 - (1 - \frac{1}{n})^n) \ge \frac{1}{2} (1 - \frac{1}{e}).$

Thus, the approximation ratio of optimal separate scoring to optimal scoring is at least en/e-1 (and this bound is tight in the limit of n).

E Missing Proofs in Section 5

E.1 Proof of Theorem 5.1

Proof. Consider the following single dimensional problem with state space $\Theta = \{0, 1/2 - \epsilon, 1/2 + \epsilon, 1\}$. The distribution over posteriors is

- 1. pointmass distributions at state 0 and 1 with probability $\epsilon/2$ each.
- 2. pointmass distributions at state $\frac{1}{2} \epsilon$ and $\frac{1}{2} + \epsilon$ with probability $\frac{(1-\epsilon)}{2}$ each.

Thus, the prior mean is $\mu_D = 1/2$ and by Corollary 3.2 the optimal scoring rule for reporting the mean is V-shaped with u(0) = u(1) = 1/2 and u(1/2) = 0. Utility is linear above and below the mean with magnitude of its slope equal to 1; thus, $u(1/2 \pm \epsilon) = \epsilon$. The expected utility under the above distribution is

$$\mathbf{E}_{r \sim f} \left[u(r) \right] = \frac{1}{2} \epsilon + \epsilon \left(1 - \epsilon \right) \le \epsilon,$$

assuming $\epsilon \leq 1/2$.

Consider the following mechanism for reporting the full distribution. The designer combines the low states as $L = \{0, 1/2 - \epsilon\}$ and the high states as $H = \{1/2 + \epsilon, 1\}$ and uses a scoring rule for the indicator variable that the state θ is high, i.e., the variable is 1 if $\theta \in H$ and 0 if $\theta \in L$. Note that for Bernoulli distributions, reporting the distribution is equivalent to reporting the mean of the distribution. The mean of the posteriors of this indicator variable is $\mu_D = 1/2$. For the indicator on high states, the symmetric V-shaped utility function of Corollary 3.2 is optimal. Its performance is

$$\mathbf{E}_{r\sim f}\left[u(\mathbf{1}[r\in H])\right] = 1/2.$$

Combining these two analyses, the approximation factor of the optimal scoring rule for the mean is at least $2/\epsilon$. As ϵ approaches zero, the approximation ratio is unbounded.

F Eliciting the Mean with an Expected Score Bound

In this section, we provide the optimal scoring rule for eliciting the single-dimensional mean under a boundedness constraint on the expected score. We consider the following optimization program:

$$\max_{S} \qquad \mathbf{E}_{G \sim f, \theta \sim G} \left[S(\mu_{G}, \theta) - S(\mu_{D}, \theta) \right]$$
(12)
s.t. S is a proper scoring rule for eliciting the mean,
 S is non-negative in space $R \times \Theta$,
$$\mathbf{E}_{G \sim f, \theta \sim G} \left[S(\mu_{G}, \theta) \right]$$
 is upper bounded by B .

We consider this optimization program with restriction to canonical scoring rules. By Definition 3, we write the single dimensional version of this optimization program as follows:

$$\max_{u} \qquad \int_{R} u(r)f(r) \, dr \tag{13}$$
s.t. u is a continuous and convex function,
 $u(\mu_{D}) = 0,$
 $u(r) + u'(r) \cdot (\theta - r) + \kappa(\theta) \ge 0, \quad \forall r \in [0, 1], \theta \in [0, 1],$
 $\mathbf{E}_{G \sim f, \theta \sim G} [u(r) + \kappa(\theta)] \le 1.$

Theorem F.1. The optimal solution for Program (12) is V-shaped.

To prove Theorem F.1, we show there is a feasible V-shaped utility function that gives the same objective.

Proof. Consider any feasible solution u of Program (12). We construct a V-shaped utility function \tilde{u} as follows:

$$\tilde{u}(r) = \begin{cases} -\int_{0}^{\mu_{D}} u(x)f(x)dx / \int_{0}^{\mu_{D}} xf(x)dx \cdot (r-\mu_{D}) & \text{for } r \leq \mu_{D}, \\ \int_{\mu_{D}}^{1} u(x)f(x)dx / \int_{\mu_{D}}^{1} xf(x)dx \cdot (r-\mu_{D}) & \text{for } r \geq \mu_{D}. \end{cases}$$

This V-shaped utility function \tilde{u} has the same objective value as the utility function u. We then show that this V-shaped utility function \tilde{u} is a feasible solution of Program (12).

It is easy to see that \tilde{u} is a continuous and convex function and $\tilde{u}(\mu_D) = 0$. We now show that there exists a function κ such that the scoring rule defined by \tilde{u} and κ is bounded in expectation and non-negative in space $R \times \Theta$. Since u is a feasible solution, there exists a function κ such that function u and κ satisfies constraints in Program (13). Thus, we have for any $\theta \in [0, 1]$

$$\kappa(\theta) \ge \max_{r \in [0,1]} \{-u(r) - u'(r) \cdot (\theta - r)\}.$$

Since function u is convex, we have for any $\theta \in [0, 1]$

$$\max_{r \in [0,1]} \{ -u(r) - u'(r) \cdot (\theta - r) \} = \max\{ -u(1) - u'(1) \cdot (\theta - 1), -u(0) - u'(0) \cdot \theta \}.$$

We then show that $u(0) \geq \tilde{u}(0)$ and $u'(0) \leq \tilde{u}'(0)$. Note that the V-shaped utility function satisfies $\int_0^{\mu_D} u(r)f(r)dr = \int_0^{\mu_D} \tilde{u}(r)f(r)dr$. If $u(0) < \tilde{u}(0)$, then by the convexity of function u, we have for any $r \in [0, \mu_D]$

$$u(r) \le (1 - r/\mu_D)u(0) < (1 - r/\mu_D)\tilde{u}(0) = \tilde{u}(r),$$

which contradicts with $\int_0^{\mu_D} u(r)f(r)dr = \int_0^{\mu_D} \tilde{u}(r)f(r)dr$. If $u'(0) > \tilde{u}'(0)$, then by the convexity of function u, we have $u(r) > \tilde{u}(r)$ for any $r \in [0, \mu_D]$, which also contradicts with $\int_0^{\mu_D} u(r)f(r)dr = \int_0^{\mu_D} \tilde{u}(r)f(r)dr$.

Similarly, we have $u(1) \ge \tilde{u}(1)$ and $u'(1) \ge \tilde{u}'(1)$. Thus, we have for $\theta \in [0, \mu_D]$

$$-\tilde{u}(1) - \tilde{u}'(1) \cdot (\theta - 1) = -\tilde{u}'(1) \cdot (\theta - \mu_D) \le -u'(1) \cdot (\theta - \mu_D) \le -u(1) - u'(1) \cdot (\theta - 1),$$

where the last inequality is due to the convexity of function u. Similarly, we have for $\theta \in [\mu_D, 1]$

$$-\tilde{u}(0) - \tilde{u}'(0) \cdot \theta = -\tilde{u}'(0) \cdot (\theta - \mu_D) \le -u'(0) \cdot (\theta - \mu_D) \le -u(0) - u'(0) \cdot \theta$$

Combining these two inequalities, we derive that

$$\max\{-\tilde{u}(1) - \tilde{u}'(1) \cdot (\theta - 1), -\tilde{u}(0) - \tilde{u}'(0) \cdot \theta\} \le \max\{-u(1) - u'(1) \cdot (\theta - 1), -u(0) - u'(0) \cdot \theta\},\$$

which means the same function κ also satisfies constraints of Program (13) for the V-shaped utility function \tilde{u} . Therefore, this V-shaped function \tilde{u} is also a feasible solution, which completes the proof.