A Machine-Checked Direct Proof of the Steiner-Lehmus Theorem

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ABSTRACT

A direct proof of the Steiner-Lehmus theorem has eluded geometers for over 170 years. The challenge has been that a proof is only considered direct if it does not rely on *reductio ad absurdum*. Thus, any proof that claims to be direct must show, going back to the axioms, that all of the auxiliary theorems used are also proved directly. In this paper, we give a proof of the Steiner-Lehmus theorem that is guaranteed to be direct. The evidence for this claim is derived from our methodology: we have formalized a constructive axiom set for Euclidean plane geometry in a proof assistant that implements a constructive logic and have built the proof of the Steiner-Lehmus theorem on this constructive foundation.

CCS CONCEPTS

• **Theory of computation** → **Constructive mathematics**; **Logic and verification**.

KEYWORDS

constructive logic, proof assistants, constructive geometry, foundations of mathematics

1 INTRODUCTION

The Steiner-Lehmus theorem, which states that *if two internal angle bisectors of a triangle are equal then the triangle is isosceles,* was posed by C. L. Lehmus in 1840. Since the publication of Jakob Steiner's 1844 proof of the theorem, it has become somewhat infamous for the many failed attempts of a *direct* proof; that is, one that does not use *reductio ad absurdum.* Numerous allegedly direct proofs have appeared over the years, only to later be discredited due to their reliance on *reductio ad absurdum* by means of auxiliary theorems with indirect proofs. Given the many failures to provide a direct proof, attempts have been made to prove that a direct proof can't possibly exist [6, 14]. Recently, we have been reassured that a direct proof does exist [13], but we have yet to see one. Thus, the history of the Steiner-Lehmus theorem serves as 177 years of evidence that a human can't account for *all* instances of the use of particular rule of logic, even in the proof of a theorem that many would consider to be rather elementary.

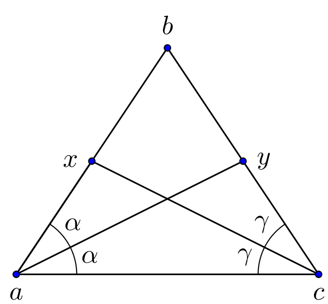


Figure 1: The Steiner-Lehmus theorem: if $ay \cong cx$, $\angle cay \cong \angle yab$, and $\angle bcx \cong xca$ then $ab \cong cb$.

In this paper, we provide a direct proof of the Steiner-Lehmus theorem. Our guarantee of directness is obtained on modern terms: by formalizing a constructive axiom set for Euclidean plane geometry in the Nuprl proof assistant [5] and building a proof of the Steiner-Lehmus theorem on this constructive foundation.

Our direct proof of the Steiner-Lehmus theorem is given in Section 5 and can be found in the Nuprl library[1]. Of the many indirect proofs published since the theorem was first posed by Lehmus in 1840, ours is superficially the most similar to the proof given by R.W. Hogg in 1982 [10]. Like many of the former proofs of the Steiner-Lehmus theorem, the proof given by Hogg uses *reductio ad absurdum* both explicitly and implicitly; the implicit use is hidden in auxiliary constructions. In comparison, our proof is completely free from the use of *reductio ad absurdum.* The atomic relations, axioms, and definitions used in our direct proof of the Steiner-Lehmus theorem are described in Sections 3–4. The soundness of our axioms with respect to the constructive reals provides additional assurance that the axioms themselves do not harbor any hidden instances of *reductio ad absurdum.* The basis for our model in the constructive reals is given in Section 7.

Before introducing the necessary axioms and definitions in Sections 3–4, we first provide background on the methods of constructive logic that clarify how a direct proof of the Steiner-Lehmus theorem was obtained.

2 CONSTRUCTIVE PROOF, STABILITY, AND DECIDABILITY

When the Steiner-Lehmus theorem was first posed in 1840, the field of logic was not fully formed. Perhaps if it had been, and constructive logic had flourished, geometers would have realized that the Steiner-Lehmus theorem is an example of a *proof of negation.* In particular, the notion of a triangle being isosceles is constructively understood to be a negative statement about a strict notion of inequality of segment lengths. While it is generally assumed that the use of case distinctions is rejected in constructive reasoning, the proof of a negation is an instance where reasoning by cases is constructively valid.

[1]The entire formalization of geometry can be found at http://www.nuprl.org/LibrarySnapshots/Published/Version2/Mathematics/euclidean!plane!geometry/index.html and the Steiner-Lehmus theorem can be found at http://www.nuprl.org/LibrarySnapshots/Published/Version2/Mathematics/euclidean!plane!geometry/Steiner-LehmusTheorem.html.

Proof by contradiction (reductio ad absurdum) is a classically admissible reasoning principle that allows one to prove a proposition P by assuming $\neg P$ and deriving absurdity. A *proof of negation* is superficially similar, as one provides proof of the proposition $\neg P$ by assuming P and deriving absurdity. The validity of the two is clearly differentiated in constructive reasoning by the general rejection of the law of double negation elimination for arbitrary propositions P: in constructive logic, a proof of negation remains perfectly valid, while a proof by reductio ad absurdum, which is classically equivalent to the law of double negation elimination, does not.

While the law of double negation elimination is not constructively valid for arbitrary propositions P, it is provably true for some propositions. Propositions for which double negation elimination is provably true are referred to as *stable*:

Definition 2.1. (Stability)

$$\text{A proposition } P \text{ is stable if } \neg\neg P \rightarrow P \text{ holds.}$$

In constructive logic, all negative propositions are stable. Specifically,

Theorem 2.2. *For all propositions P, $\neg\neg(\neg P) \rightarrow \neg P$.*

Furthermore, the proof of a stable proposition permits case distinctions. In particular, the double negation of the law of excluded middle for arbitrary propositions is constructively valid:

Theorem 2.3. *For all propositions P, $\neg\neg(P \vee \neg P)$.*

Thus, in the proof of a stable proposition P, one need only introduce the double negation of the desired case distinction as a hypothesis and apply the appropriate elimination rules to obtain the desired cases.

Finally, a strictly stronger notion than stability is decidability:

Definition 2.4. (Decidability)

$$\text{A proposition } P \text{ is decidable if } P \vee \neg P \text{ holds.}$$

While the decidability of arbitrary propositions does not hold constructively, it is a provable property for many propositions P, specifically those for which there is an algorithm for deciding which of P or $\neg P$ holds.

2.1 Stable Relations in Constructive Geometry

We do not take the stability or decidability of any of our atomic or defined relations as axioms. This choice clearly distinguishes our theory from those that take the stability of equality, betweenness, or congruence as axioms [1, 2, 11]. Instead, we define equivalence, collinearity, betweenness, and congruence (Definitions 3.5, 3.6, 3.7, and 3.10) negatively in terms of a strictly positive atomic relation. It follows from Theorem 2.2 that these are stable relations. Our axioms therefore clearly distinguish the geometric propositions that constructively permit case distinctions from those that do not.

Finally, the choice to not take the stability of equality as an axiom is driven by the desired model, which is the Nuprl implementation of the constructive reals. If we were to take the stability of equality as an axiom, then equality and equivalence would coincide, which does not hold in the Nuprl implementation of the constructive reals [3, p. 3].

3 CONSTRUCTIVE GEOMETRIC PRIMITIVES AND RELATIONS

Our axioms rely on two atomic relations on points: a quaternary relation representing an ordering on segment lengths and a ternary relation for plane orientation. These relations are introduced using the formalism of type theory to parallel their implementation in the Nuprl proof assistant. For example, the statement $a : \text{Point}$ is to be read as "a of type Point."

3.1 Segments in Type Theory

The segment type is defined as the Cartesian product of two points. The elements of a Cartesian product are pairs, denoted $\langle a, b\rangle$. If a has type Point and b has type Point, then $\langle a, b\rangle$ has type $\text{Point} \times \text{Point}$:

$$\frac{a : \text{Point} \quad b : \text{Point}}{\langle a, b\rangle \; : \; \text{Point} \times \text{Point}.}$$

We will abbreviate segment pairs $\langle a, b\rangle$ by simply writing ab:

$$\frac{a : \text{Point} \quad b : \text{Point}}{ab : \text{Segment}}$$

When it is necessary to decompose the points constituting a segment ab, we may write $\texttt{fst}(ab)$ and $\texttt{snd}(ab)$ for a and b respectively.

3.2 Atomic Relations and Apartness

Constructive geometry traditionally utilizes a binary *apartness* relation in place of equality [9, 11, 15, 16]. A notable exception is the axiom set presented by Lombard and Vesley [12], which uses an atomic six place relation and defines a binary apartness relation in terms of the atomic six place relation. In this work, we use an atomic quaternary *strictly greater than* relation to define a binary apartness relation. In particular, given the four points a, b, c, and d, if the length of the segment ab is *strictly greater than* the length of the segment cd, then the atomic ordering relation on points will be denoted by $ab > cd$. A binary *apartness* relation on points can then be defined using the atomic *strictly greater than* relation as follows.

Definition 3.1 (Apartness of points). The points a and b satisfy an *apartness* relation if the length of the segment ab is strictly greater than the length of the null segment aa:

$$a\#b := ab > aa.$$

The *strictly greater than* relation is used to define two additional quaternary relations on points: apartness of segment lengths and a non-strict ordering of segment lengths.

Definition 3.2 (Apartness of segment lengths). The length of the segments ab and cd satisfy a *length apartness* relation if either the length of the segment ab is strictly greater than the length of the segment cd or the length of the segment cd is strictly greater than the length of the segment ab:

$$ab\#cd := ab > cd \;\vee\; cd > ab.$$

Definition 3.3 (Non-strict order of segment lengths). The length of ab is *greater than or equal to* the length of cd if the length of cd is not strictly greater than the length of ab:

$$ab \geq cd := \neg cd > ab.$$

The atomic relation for *plane orientation* used in this work is adopted from the constructive axiom set for Euclidean plane geometry introduced in [11]: given the three points a, b, and c, if the point a lies to the left of the segment bc, then the atomic *leftness* relation on points will be denoted by $\mathrm{Left}(a, bc)$. We use the atomic *leftness* relation to define an apartness relation between a point a segment as follows.

Definition 3.4 (Apartness of a point and a segment). The point a lies apart from the segment bc if it is either to the left of the segment bc or to the left of the segment cb :

$$a\#bc := \mathrm{Left}(a, bc) \ \vee \ \mathrm{Left}(a, cb).$$

3.3 The Constructive Interpretation of Classical Geometric Relations

The classical relations of *equivalence*, *collinearity*, *betweenness*, and *congruence* are defined using the atomic relations of *leftness* and *strictly greater than.* In this section, we give the definitions of these relations, and provide the proof of a useful theorem as a simple example of proving stable propositions using constructive logic.

Definition 3.5 (Equivalence on points). The points a and b are *equivalent* if they do not satisfy the binary apartness relation on points (Definition 3.1):

$$a \equiv b := \neg a\#b.$$

As is mentioned in Section 2.1, equivalence and equality do not coincide: while equality on points implies equivalence, equivalence does not imply equality.

Definition 3.6 (Collinearity). The points a, b, and c are *collinear* if they do not satisfy the apartness relation between a point and a segment:

$$\mathrm{Col}(abc) := \neg(a\#bc).$$

Definition 3.7 (Betweenness). The point b lies *between* the points a and c if a, b, and c are collinear and the length of the segment ac is not strictly greater than the lengths of ab and bc:

$$B(abc) := \mathrm{Col}(abc) \ \wedge \ ac \geq ab \ \wedge \ ac \geq bc.$$

Note that the above definition coincides with what is referred to as *non-strict* betweenness. That is, the points a, b, and c may be equivalent.

Theorem 3.8 (Collinear Cases). *Any three collinear points satisfy a weak betweenness relation.*

$$\forall a, b, c : \mathrm{Point}\, .\ \mathrm{Col}(abc) \Rightarrow$$
$$\neg\neg(B(abc) \ \vee \ B(cab) \vee B(bca) \ \vee a \equiv b \ \vee a \equiv c \ \vee \ b \equiv c).$$

Proof. The stability of the conclusion allows for reasoning by cases on

$$\neg\neg(a\#b \ \vee \ \neg a\#b),$$

and similarly for $a\#c$ and $b\#c$. Consider the case where $a\#b$, $a\#c$, and $b\#c$. Assume

$$\neg(B(abc) \ \vee \ B(cab) \ \vee \ B(bca) \ \vee \ a \equiv b \ \vee a \equiv c \ \vee \ b \equiv c),$$

and prove false. Observe that $\neg B(abc) \ \wedge \ \neg B(cab) \ \wedge \ \neg B(bca)$ follows from the assumption. From $\neg B(abc)$ it follows that

$$\neg\neg(a\#bc \ \vee \ ab > ac \ \vee \ bc > ac).$$

Stability of the conclusion allows for elimination of the double negation for each betweenness relation, and expanding the disjunctions results in absurdity. □

Definition 3.9 (Strict Betweenness). The point b lies *strictly between* the points a and c if the point b lies between the points a and c, and the points a, b, and c satisfy apartness relations:

$$SB(abc) := B(abc) \ \wedge \ a\#b \ \wedge \ b\#c.$$

Definition 3.10 (Congruence). The segments ab and cd are *congruent* if they do not satisfy the apartness relation on segment lengths (Definition 3.2):

$$ab \cong cd := \neg ab\#cd.$$

Definition 3.11 (Out). The point p lies *out* along the segment ab if it is separated from both a and b and satisfies some weak betweenness relation with a and b. Observe that this definition can be viewed as using an constructive interpretation of the classical disjunction used in Definition 6.1 of [17].

$$out(p, ab) := p\#a \ \wedge \ p\#b \ \wedge \ \neg(\neg B(pab) \ \wedge \ \neg B(pba))$$

The universally quantified axioms introduced in Section 4 imply that *collinearity*, *betweenness*, and *congruence* are equivalence relations.

3.4 Angle Relations

Our proof of the Steiner-Lehmus theorem required constructive definitions for angle congruence, the sum of two angles, and angle ordering. The following definition of angle congruence is taken from Tarski [17], but has been modified to use the appropriate constructive relations.

Definition 3.12 (Congruent Angles). The angles abc and xyz are *congruent* if the segments of each angle are distinct and there exist points making the corresponding segments of the two angles congruent:

$$abc \cong_a xyz :=$$
$$a\#b \ \wedge \ b\#c \ \wedge \ x\#y \ \wedge \ y\#z \wedge$$
$$(\exists a', c', x', z' : \text{Point} \,.\, B(baa') \ \wedge \ B(bcc') \ \wedge \ B(yxx') \wedge$$
$$B(yzz') \ \wedge \ ba' \cong yx' \ \wedge \ bc' \cong yz' \ \wedge \ a'c' \cong x'z').$$

The set of Axioms U, introduced in Section 4, imply that angle congruence is an equivalence relation.

Definition 3.13 (Sum of two angles).

$$abc + xyz = def :=$$
$$\exists p, p', d', f' : \text{Point} \,.\, abc \cong_a dep \ \wedge \ fep \cong_a xyz \wedge$$
$$B(ep'p) \ \wedge \ out(edd') \ \wedge \ out(eff') \ \wedge \ SB(d'p'f')$$

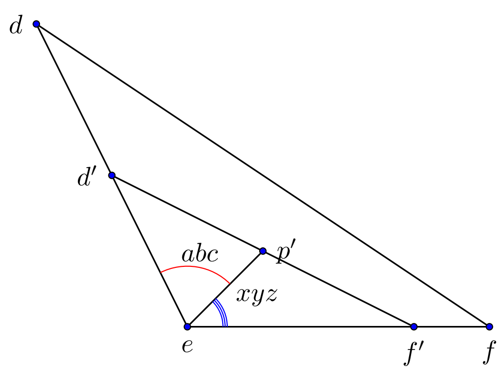


Figure 2: A diagram of Definition 3.13: $abc + xyz = def$ **with** $p' = p$**.**

Definition 3.14 (Angle Inequality).

$$abc <_a xyz =: \neg out(yxz) \wedge$$
$$\exists \, p, p', x', z' : \text{Point} \,.\, abc \cong_a xyp \wedge$$
$$B(yp'p) \ \wedge \ out(yxx') \ \wedge \ out(yzz') \wedge$$
$$\neg B(xyp) \ \wedge \ B(x'p'z)' \ \wedge \ p'\#z'$$

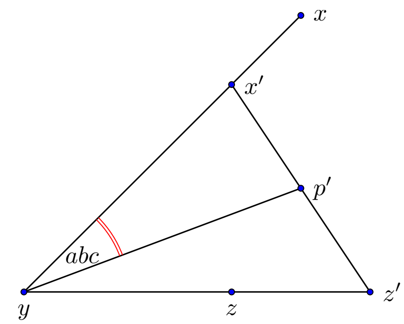
Our axioms imply that angle inequality is a transitive relation for angles satisfying the ternary apartness relation on points (Definition 3.4).

3.5 Parallel Segments

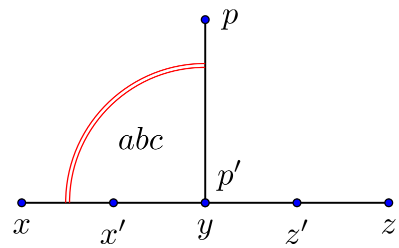
The following definition of parallel segments was essential to our proof of the Steiner-Lehmus theorem.

Definition 3.15 (Parallel Segments). The segments ab and cd are *parallel* if $a\#b$ and $c\#d$ and there do not exist points x and y collinear with ab such that x and y lie on opposite sides of cd:

$$ab \parallel cd := a\#b \ \wedge \ c\#d \ \wedge \ \neg(\exists x, y : \text{Point} \,.\, \text{Col}(xab) \wedge$$
$$\text{Col}(yab) \ \wedge \text{Left}(x, cd) \ \wedge \ \text{Left}(y, dc)).$$



(a) Definition 3.14 for typical angles, with $p = p'$**.**



(b) Definition 3.14 for a straight angle.

Figure 3: Definition 3.14, $abc <_a xyz$**.**

According to our axioms introduced in the following section, parallelism is a symmetric and reflexive relation but not a transitive relation. Transitivity of parallelism is known to be equivalent to the parallel postulate [4], which is not an axiom of the theory presented in this paper.

4 CONSTRUCTION POSTULATES AND AXIOMS

The axioms are introduced here in two separate groups: Axioms U and Axioms C. Axioms U are universally quantified and contain no disjunctions or existential quantifiers. The application of any one of these axioms does not result in a geometric construction. Axioms C are constructor axioms relying on disjunctions and existential quantifiers. As a result, the axioms in group C have a convenient functional reading which may be used in proofs.

From the Axioms listed below, Axioms U7–U13 and Axioms C1–C3 are also used in a previous set of constructive axioms for Euclidean geometry [11]. Unlike the previous system, the relations of betweenness, congruence, and apartness are not primitives of our theory, they are instead defined as described in Section 3.3. Furthermore, as previously noted, the stability of congruence and betweenness are not taken as axioms in the current work. Finally, the axioms C4 and C5 have been simplified to remove the assertion of the existence of redundant points.

4.1 Universally Quantified Axioms

Axiom U1. $\forall a, b, c : \text{Point} \,.\, bc \geq aa$

Axiom U2. $\forall a, b, c, d : \text{Point} \,.\, ab > cd \Rightarrow ab \geq cd$

Axiom U3. $\forall a, b, c : \text{Point} .\ ba > ac \Rightarrow b\#c$

Axiom U4.
$\forall a, b, c, d, e, f : \text{Point} .\ ab > cd \Rightarrow cd \geq ef \Rightarrow ab > ef$

Axiom U5.
$\forall a, b, c, d, e, f : \text{Point} .\ ab \geq cd \Rightarrow cd > ef \Rightarrow ab > ef$

Axiom U6. $\forall a, b, c : \text{Point} .\ B(abc) \Rightarrow b\#c \Rightarrow ac > ab$

Axiom U7. $\forall a, b, c : \text{Point} .\ \text{Left}(a, bc) \Rightarrow \text{Left}(b, ca)$

Axiom U8. $\forall a, b, c : \text{Point} .\ \text{Left}(a, bc) \Rightarrow b\#c$

Axiom U9. $\forall a, b, c, d : \text{Point} .\ B(abd) \Rightarrow B(bcd) \Rightarrow B(abc)$

We take an constructive versions of Tarski's Five-Segment axiom and Upper Dimension axiom [17].

Axiom U10 (Five-Segment).

$$\forall a, b, c, d, w, x, y, z : \text{Point} .\ (a\#b \ \wedge\ B(abc) \ \wedge\ B(wxy) \ \wedge \\ ab \cong wx \ \wedge\ bc \cong xy \ \wedge\ ad \cong wz \ \wedge\ bd \cong xz) \Rightarrow \\ cd \cong yz$$

Axiom U11 (Upper Dimension).

$$\forall a, b, c, x, y : \text{Point} .\ ax \cong ay \Rightarrow bx \cong by \Rightarrow \\ cx \cong cy \Rightarrow x\#y \Rightarrow \text{Col}(abc)$$

Axiom U12 (Convexity of Leftness).

$$\forall a, b, x, y, z : \text{Point} .\ \text{Left}(x, ab) \ \wedge \text{Left}(y, ab) \ \wedge B(xzy) \Rightarrow \\ \text{Left}(z, ab)$$

Axiom U13.

$$\forall a, b, c, y : \text{Point} .\ a\#bc \Rightarrow y\#b \Rightarrow \text{Col}(yab) \Rightarrow y\#bc$$

4.2 Construction Postulates

Axiom C1 (Cotransitivity of separated points:).

$$\forall a, b, c : \text{Point} .\ a\#b \Rightarrow a\#c \ \vee\ b\#c$$

Axiom C2 (Plane Separation). *If the points u and v lie on opposite sides of the segment ab, then the point x, collinear with ab, exists between u and v.*

$$\forall a, b, u, v : \text{Point} .\ (\ \text{Left}(u, ab) \wedge\ \text{Left}(v, ba) \ \Rightarrow \\ \exists x : \text{Point} .\ Col(abx) \ \wedge\ B(uxv))$$

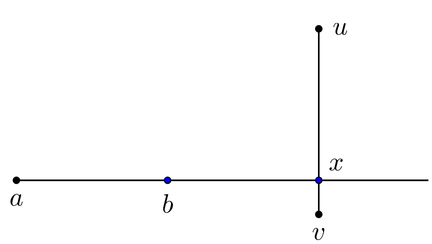


Figure 4: Axiom C2: plane separation.

Axiom C3 (Non-triviality).

$$\exists a, b : \text{Point} .\ a\#b$$

Axiom C4 (Straightedge-Compass). *The straight-edge compass axiom constructs a single point of intersection between a circle and a segment (see Figure 5):*

$$\forall a, b, c, d : \text{Point} .\ (a\#b \wedge B(cbd)) \Rightarrow \\ \exists u : \text{Point} .\ cu \cong cd \ \wedge\ B(abu) \wedge (b\#d \Rightarrow b\#u)$$

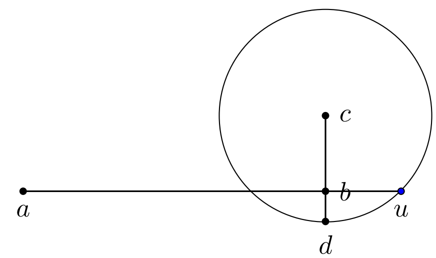


Figure 5: Axiom C4, the straight-edge compass construction.

Axiom C5 (Compass-Compass). *The compass-compass axioms constructs a single point of intersection between two circles (see Figure 6):*

$$\forall a, b, c, d : \text{Point} .\ a\#c \wedge \\ (\exists\, p, q : \text{Point} .\ ab \cong ap \wedge cd > cp \wedge cd \cong cq \wedge ab > aq) \Rightarrow \\ \exists u : \text{Point} .\ ab \cong au \wedge cd \cong cu \wedge\ \text{Left}(u, ac)$$

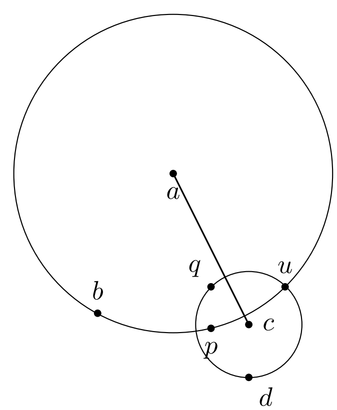


Figure 6: Axiom C5, the compass-compass construction.

5 THE STEINER-LEHMUS THEOREM

The conclusion of the Steiner-Lehmus theorem is stable and so it suffices to prove the double negation of auxiliary theorems with constructive content. Thus, rather than proving a lemma stating that

from two points along the sides of any triangle, a parallelogram can be constructed such that one side of the parallelogram lies along one side of the triangle, we prove the following lemma:

Lemma 5.1.

$$\forall a, b, c, x, y : \text{Point} . (a\#bc \ \wedge \ SB(axb) \ \wedge \ SB(cyb) \Rightarrow \neg\neg(\exists t : \text{Point} . yt \parallel ax \ \wedge \ xt \parallel ay \ \wedge \ ax \cong yt \ \wedge \ xt \cong ay \ \wedge \ t\#bc).$$

Proof. Construct the midpoint m along the segment xy using Euclid I.10 (Theorem 6.5), and extend the segment am to construct the point t such that $am \cong mt$ by Lemma 6.1. Now, the angle congruence $xma \cong_a ymt$ follows from Euclid I.15 (Theorem 6.6), and the congruence relations $ax \cong yt$ and $xt \cong ay$ follow from Euclid I.4 (Theorem 6.2) and Axiom U10, respectively. The angle congruence $axy \cong_a tyx$ then follows by definition, and from Euclid I.27 (Theorem 6.9) it follows that $ax \parallel yt$ and $xt \parallel ay$. Finally, stability of the conclusion allows for reasoning by cases on $t\#bc$ or $\mathrm{Col}(tbc)$.

If $\mathrm{Col}(tbc)$ then by Lemma 6.10 the point t must be the point p such that $SB(bpc)$ and $SB(amp)$; p is guaranteed to exist by construction using Lemma 6.14. Without loss of generality, from $a\#bc$ assume $\mathrm{Left}(a, cb)$. From Lemma 6.11, it follows that $\mathrm{Left}(c, xa)$. Now, construct the point q by Lemma 6.1 such that $SB(cbq)$ and $SB(ypq)$. It follows from Lemma 6.13 that $\mathrm{Left}(q, ax)$, contradicting $ax \parallel yt$. □

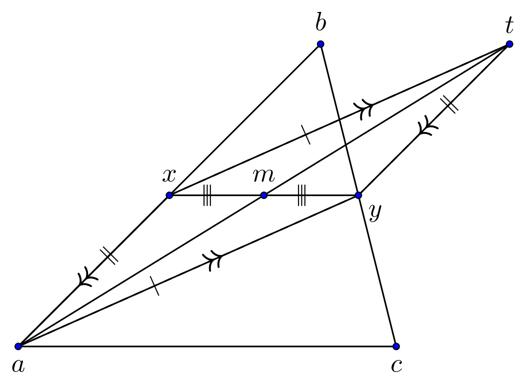


Figure 7: Lemma 5.1

Theorem 5.2 (Steiner-Lehmus).

$$\forall a, b, c, x, y : \text{Point} . (a\#bc \ \wedge \ SB(axb) \ \wedge \ SB(cyb) \ \wedge \ ay \cong cx \ \wedge \ xay \cong_a cay \ \wedge \ ycx \cong_a acx \Rightarrow ab \cong cb).$$

Proof. Construct the parallelogram $ayxt$ by Lemma 5.1. From Euclid I.5 (Theorem 6.3) it follows that $xct \cong_a xtc$. The angle sum relations $xty + ytc \cong_a xtc$ and $xcy + yct \cong_a xct$ follow by definition from construction of the point q using Axiom C2 such that $SB(qyc)$, $B(tyy)$, $SB(xqt)$, and $B(cqq)$. Now, stability of the conclusion allows for reasoning by cases on $cy > ax$ or $\neg(cy > ax)$.

If $cy > ax$, then $cy > yt$ by definition of the parallelogram $ayxt$. From Euclid I.25 (Theorem 6.8) it follows that $acx <_a cay$, and therefore $xcy <_a xty$. It then follows from Euclid I.18 (Theorem 6.7) that $tcy <_a ytc$, which, along with Lemma 6.15 and the angle sum relations $xty + ytc \cong_a xtc$ and $xcy + yct \cong_a xct$, yields the contradiction $xty <_a xcy$.

If $\neg(cy > ax)$, it follows that $\neg\neg(ax > cy \vee ax \cong cy)$: stability of the conclusion allows for elimination of the double negation, so that we can reason by cases on $ax > cy$ or $ax \cong cy$. A contradiction is reached for $ax > cy$ by the same reasoning used for $cy > ax$.

Finally, if $ax \cong cy$, it follows from Definition 3.12 that $xac \cong_a yca$. Theorem 6.4 then yields $ab \cong cb$, as desired. □

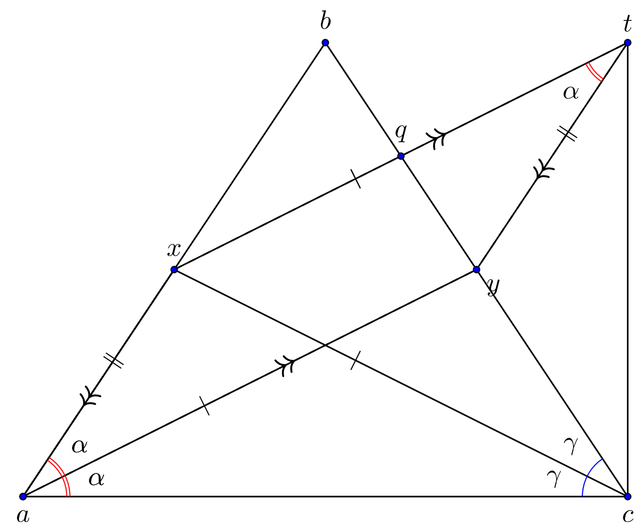


Figure 8: The Steiner-Lehmus Theorem 5.2

6 ESSENTIAL AUXILIARY THEOREMS

This section contains only the statements of the auxiliary theorems used in the proof of the Steiner-Lehmus theorem (Theorem 5.2) and Lemma 5.1. The names given to the theorems in this section match their names in the Nuprl library[2]. Some definitions used in the Nuprl statement of a theorem may occur unfolded in the following theorem statements for clarity.

Theorem 6.1 (geo-extend-exists).

$$\forall q, a, b, c : \text{Point} . q\#a \Rightarrow \exists x : \text{Point} . B(qax) \ \wedge \ ax \cong bc.$$

Theorem 6.2 (Euclid-Prop4). *If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.*

$$\forall a, b, c, x, y, z : \text{Point} . a\#b \ \wedge a\#c \ \wedge b\#c \ \wedge \ x\#y \ \wedge \ x\#z \ \wedge \ y\#z \ \wedge \ ab \cong xy \ \wedge \ bc \cong yz \ \wedge \ abc \cong_a xyz \Rightarrow ac \cong xz \ \wedge \ bac \cong_a yxz \ \wedge \ bca \cong_a yzx.$$

[2] http://www.nuprl.org/LibrarySnapshots/Published/Version2/Mathematics/euclidean!plane!geometry/index.html

THEOREM 6.3 (EUCLID-PROP5). *In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.*

$$\forall a, b, c, x, y : \text{Point} \,.\, ab \cong ac \wedge a\#bc \wedge SB(abx) \;\wedge\; SB(acy) \Rightarrow abc \cong_a acb \wedge xbc \cong_a ycb.$$

THEOREM 6.4 (EUCLID-PROP6). *If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.*

$$\forall a, b, c : \text{Point} \,.\, c\#ab \Rightarrow cab \cong_a cba \Rightarrow ca \cong cb.$$

THEOREM 6.5 (EUCLID-PROP10). *To bisect a given straight line.*

$$\forall a, b : \text{Point} \,.\, a\#b \;\Rightarrow\; \exists d : \text{Point} \,.\, SB(adb) \;\wedge\; ad \cong db.$$

THEOREM 6.6 (VERT-ANGLES-CONGRUENT). *If two straight lines cut one another, then they make the vertical angles equal to one another.*

$$\forall a, b, c, x, y : \text{Point} \,.\, SB(abx) \;\wedge\; SB(cby) \;\Rightarrow\; abc \cong_a xby.$$

THEOREM 6.7 (EUCLID-PROP18). *In any triangle the angle opposite the greater side is greater.*

$$\forall a, b, c : \text{Point} \,.\, a\#bc \;\wedge\; ac > ab \;\Rightarrow\; bca <_a abc.$$

THEOREM 6.8 (EUCLID-PROP25). *If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have the one of the angles contained by the equal straight lines greater than the other.*

$$\forall a, b, c, d, e, f : \text{Point} \,.\, a\#bc \;\wedge\; d\#ef \;\wedge\; ab \cong de \;\wedge\; ac \cong df \;\wedge\; bc > ef \Rightarrow edf <_a bac.$$

In the following theorem, the Left relation is used in the antecedent to capture the notion of "alternate angles."

THEOREM 6.9 (EUCLID-PROP27). *If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another:*

$$\forall a, b, c, d, x, y : \text{Point} \,.\, (\text{Col}(xab) \wedge \text{Col}(ycd) \;\wedge\; a\#b \;\wedge\; c\#d \;\wedge\; \text{Left}(a, yx) \;\wedge\; \text{Left}(c, xy) \;\wedge\; axy \cong_a cyx) \Rightarrow ab \parallel cd.$$

LEMMA 6.10 (GEO-INTERSECTION-UNICITY).

$$\forall a, b, c, d, p, q : \text{Point} \,.\, \neg\, \text{Col}(abc) \;\wedge\; c\#d \;\wedge\; \text{Col}(abp) \;\wedge\; \text{Col}(abq) \;\wedge\; \text{Col}(cdp) \;\wedge\; \text{Col}(cdq) \Rightarrow p \equiv q.$$

LEMMA 6.11 (LEFT-CONVEX). *Given a segment ab and a point x lying to the left of it, the point y lying* out *from x that along the segment ax or bx is in the same half-plane as x.*

$$\forall a, b, x, y : \text{Point} \,.\; \text{Left}(x, ab) \;\wedge\; (out(axy) \;\vee\; out(bxy)) \Rightarrow \text{Left}(y, ab)$$

LEMMA 6.12 (GEO-LEFT-OUT). *Given a segment ab and a point c lying* out *from b along ab, if the point x lies to the left of ab, then x also lies to the left of ac.*

$$\forall a, b, c, x : \text{Point} \,.\; \text{Left}(x, ab) \;\wedge\; out(abc) \Rightarrow \text{Left}(x, ac)$$

LEMMA 6.13 (STRICT-BETWEEN-LEFT-RIGHT).

$$\forall a, b, c, x, y : \text{Point} \,.\; \text{Left}(x, ab) \;\wedge\; \text{Col}(abc) \;\wedge\; SB(xcy) \Rightarrow \text{Left}(y, ba)$$

THEOREM 6.14 (OUTER-PASCH-STRICT).

$$\forall a, b, c, x, q : \text{Point} \,.\, x\#bq \;\wedge\; SB(bqc) \;\wedge\; SB(qxa) \Rightarrow \exists p : \text{Point} \,.\, SB(bxp) \;\wedge\; SB(cpa).$$

LEMMA 6.15 (HP-ANGLE-SUM-LT4). *If the sum of the strict angles abc and xyz is equal to the sum of the strict angles $a'b'c'$ and $x'y'z'$, and $x'y'z'$ is less than xyz, then it must be the case that the angle abc is less than the angle $a'b'c'$.*

$$\begin{aligned} &\forall a, b, c, x, y, z, i, j, k : \text{Point} \,. \\ &\forall a', b', c', x', y', z', i', j', k' : \text{Point} \,. \\ &abc + xyz \cong ijk \;\wedge\; a'b'c' + x'y'z' \cong i'j'k' \;\wedge \\ &ijk \cong_a i'j'k' \;\wedge\; a'\#b'c' \;\wedge\; x'\#y'z' \;\wedge\; x\#yz \;\wedge\; i\#jk \;\wedge \\ &x'y'z' < xyz \Rightarrow abc < a'b'c'. \end{aligned}$$

7 A MODEL ON THE CONSTRUCTIVE REALS

The soundness of our axioms with respect to the Nuprl implementation of the constructive reals [3] is implied by the following interpretations of our primitives[3].

Definition 7.1. If $x \in \mathbb{R}$ is the length of the segment ab and $y \in \mathbb{R}$ is the length of the segment cd, x is *strictly greater* than y if and only if there exists a natural number n such that the nth rational terms of x and y differ by more than four:

$$x >_{\mathbb{R}} y := \exists n \in \mathbb{N}.\, x(n) >_{\mathbb{Q}} y(n) + 4.$$

Note that the ordering relation $>_{\mathbb{Q}}$ on the rational numbers is decidable (Definition 2.4) while the ordering relation $>_{\mathbb{R}}$ on the constructive reals is not.

[3]The proofs of soundness for the Axiom sets U and C can be found at http://www.nuprl.org/LibrarySnapshots/Published/Version2/Mathematics/reals!model!euclidean!geometry as the theorems *r2-basic-geo-axioms* and *r2-eu_wf*, respectively.

Definition 7.2. Given the real coordinates $(x_0, y_0, 1)$, $(x_1, y_1, 1)$, $(x_2, y_2, 1)$ of the points a, b and c, respectively, the point a lies *left of* the segment bc if and only if the determinant of the matrix formed by the points a, b and c is strictly positive:

$$\mathrm{Left}(a, bc) := \begin{vmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} >_{\mathbb{R}} 0.$$

The soundness of our axioms with respect to the constructive reals provides additional assurance that our axioms do not use *reductio ad absurdum.* Although the axioms presented in this paper differ (as described in Section 4) from those presented in [11], only minor modifications were necessary for the soundness proofs.

Finally, while the constructive real model for our axioms guarantees that a direct proof of the Steiner-Lehmus theorem exists in the constructive reals, it says nothing about the existence of direct proof in the classical reals.

8 CONCLUSION

We have introduced here for the first time a proof of the Steiner-Lehmus theorem that is entirely absent of the use of *reductio ad absurdum* and can therefore be considered *fully direct.* This theorem was proved in the constructive logic of the Nuprl proof assistant using a novel axiomatization of Euclidean plane geometry without the parallel postulate. The crux of the proof is the realization that congruence in constructive geometry is a *stable relation*, and that the proof of a stable relation permits double negation elimination and therefore also case distinctions.

Finally, we conclude by addressing the suggestion that the many years of failed attempts to find a direct proof of the Steiner-Lehmus theorem was cause to celebrate the indispensability of *reductio ad absurdum.* In particular, a discussion of the Steiner-Lehmus theorem given in a geometry textbook by Coxeter and Greitzer [7] includes the popular quote of G. H. Hardy [8]: *Reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons.* We instead propose the following:

Double negation is one of a mathematician's finest weapons, and a proof assistant one of her most steadfast companions.

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