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ABSTRACT

In this paper, we develop tools to analyze stability properties of discrete-time switched linear systems driven by switching signals belonging to a given ω- regular language. More precisely, we assume switching signals to be generated by a Büchi automaton where the alphabet corresponds to the modes of the switched system. We define notions of attractivity and uniform stability for this type of systems and also of uniform exponential stability when the considered Büchi automaton is deterministic. We then provide sufficient conditions to check these properties using Lyapunov and automata theoretic techniques. For a subclass of such systems with invertible matrices, we show that these conditions are also necessary. We finally show an example of application in the context of synchronization of oscillators over a communication network.

CCS CONCEPTS

• **Computing methodologies** → *Computational control theory*; • **Theory of computation** → *Automata over infinite objects*.

KEYWORDS

Switched systems, Büchi automata, Stability, Lyapunov methods

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1 INTRODUCTION

Switched systems are dynamical systems with several modes of operations where the active mode is determined by a switching signal. In this paper, we consider discrete-time switched systems where each mode corresponds to a linear system. This class of systems is broadly considered in the literature as modeling framework for cyber-physical systems as it makes it possible to describe faithfully the interaction between the physical dynamics and the cyber components such as shared computing resources and communication networks (see e.g. [1, 6, 13]).

Stability analysis of switched linear systems has been the object of numerous studies. Early works focus on proving stability of switched systems driven by arbitrary switching signals or by switching signals with dwell-time conditions [11, 12, 17]. More recent works have considered systems with constrained switching signals where the switching signals are generated by labeled graphs [2, 5, 10, 15, 16].

Shuffled switching signals is also a class of constrained switching signals that has been considered in the literature [7, 8, 18]. A switching signal is said to be shuffled if all the modes of the switched systems are activated infinitely often. The set of shuffled switching signals cannot be generated by the labeled graphs considered in the works above (see [7]) but constitutes an example of an ω-regular language, which can always be characterized using Büchi automata [3]. Since ω-regular languages are frequently used to specify properties, such as fairness, in scheduling algorithms or communication protocols, e.g through linear temporal logic specifications [3], it is of interest for some cyber-physical systems applications, where multiple components must be granted access to a shared resource infinitely often, to analyze the stability of switched systems driven by switching signals belonging to a given ω-regular language. Moreover, a practical application of systems driven by ω-regular languages is consensus/synchronization over time-varying (undirected) graphs, which can be seen as a switched system. It is well known (see e.g. [4, 14]) that consensus/synchronization can be reached if and only if, at every time instant, the union of future interaction graphs is connected. This connectivity condition cannot be described using labeled graphs as in [16] but can be specified using deterministic Büchi automata.

Hence, in this paper, we consider discrete-time switched linear systems whose switching signals are generated by a given Büchi automaton. The main contributions of the paper are as follows. First, we define for this class of systems the notions of attractivity, uniform stability, and in the case of deterministic Büchi automaton, of uniform exponential stability. We then establish sufficient stability conditions using Lyapunov functions. For a particular class of such systems with invertible matrices, we show that these conditions are also necessary with a converse Lyapunov result. Our approach is illustrated using a simple example of oscillator synchronization over a communication network.

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The work presented in this paper can be seen as a generalization of [7] from the particular case of shuffled switching signals to the general case of arbitrary ω-regular languages. Although the proof of the sufficient conditions is easily adapted from [7], the proofs of the converse result require some novel techniques such as the construction of a labeled graph based on accepting states of a non-deterministic Büchi automaton, the lifting of results of [16] to analyze a resulting constrained system, and the construction of a Lyapunov function. Our paper also builds on the results of [16] that are paramount to prove our converse Lyapunov result. In this work, the authors consider switched systems where the switching signals are generated by labeled graphs, which correspond to non-deterministic Büchi automata where all states are accepting. However, not all ω-regular languages can be generated using labeled graphs. Hence, our results subsume those of [7] and [16] and also apply to systems that cannot be handled by any of these methods. Finally, stability analysis of discrete-time switched linear systems constrained by ω-regular languages have already been considered in the literature in [18] where it is shown that the stability is equivalent to the stability of a lifted system driven by shuffled switching signals. In comparison, our approach works directly on the original state-space, and thus results in more tractable conditions. Our Lyapunov functions also resembles that considered in [9]. However, in that work, the connection to ω-regular languages has not been investigated.

The rest of the paper is organized as follows. Section 2 introduces the class of systems under consideration and the associated stability notions. In Section 3, we provide sufficient stability conditions given by the existence of a Lyapunov function. In Section 4, we establish a converse Lyapunov result for a particular class of systems. An illustrative example is shown in Section 5.

Notations: We denote by $\emptyset$ the empty set, by $\mathbb{R}$, $\mathbb{R}_0^+$ and $\mathbb{N}$ the sets of real numbers, nonnegative real numbers and nonnegative integers, respectively. $\|.\|$ denotes an arbitrary norm on $\mathbb{R}^n$ and the associated induced matrix norm defined for $M \in \mathbb{R}^{n\times n}$ by $\|M\| = \sup_{x\neq 0} \frac{\|Mx\|}{\|x\|}$. $I_n \in \mathbb{R}^{n\times n}$ denotes the n dimensional identity matrix. We use $|.|$ to denote the cardinality of a finite set. Given an alphabet Σ, a word $\sigma = \sigma_0\sigma_1\dots$ is a finite or infinite sequence of elements of the alphabet, $\sigma_i \in \Sigma$ for all $i = 0, 1, \dots$; Σ^+ denotes the set of all finite words over the alphabet Σ exempted from the empty word ϵ; Σ^ω denotes the set of all infinite words over the alphabet Σ.

2 STABILITY OF SWITCHED SYSTEMS WITH ω-REGULAR SWITCHING SEQUENCES

In this section, we first introduce the class of systems under study and define several associated notions related to stability.

A non-deterministic Büchi automaton (NBA) is a tuple $\mathcal{B} = (Q, \Sigma, \delta, Q_0, F)$ where Q is a finite set of states, Σ is the alphabet, $\delta : Q\times\Sigma \to 2^Q$ is a transition function, Q_0 is the set of initial states and F is the set of accepting states. A run associated with a finite or infinite word $\sigma \in \Sigma^+ \cup \Sigma^\omega$ is a sequence of states $q_0q_1q_2\dots$ such that $q_0 \in Q_0$ and $q_{i+1} \in \delta(q_i, \sigma_i)$ for all $i = 0, 1, \dots$. A run $q_0q_1q_2\dots$ associated with an infinite word $\sigma \in \Sigma^\omega$ is said to be accepting if $q_i \in F$ for infinitely many indices $i \in \mathbb{N}$. The language of $\mathcal{B}$, denoted by $Lang(\mathcal{B})$, is the set of all infinite words over the alphabet Σ which have an accepting run. We note that an NBA is called deterministic Büchi automaton (DBA) if $|\delta(q, i)| \le 1$ for all $q \in Q$, $i \in \Sigma$ and $|Q_0| = 1$.

Let us consider a discrete-time switched linear system in which the switching sequences are infinite words accepted by a given NBA. Specifically, given a Büchi automaton $\mathcal{B} = (Q, \Sigma, \delta, Q_0, F)$ where the alphabet is the set $\Sigma = \{1, \dots, m\}$, given a finite set of matrices $\mathcal{A} = \{A_1, \dots, A_m\}$ with $A_i \in \mathbb{R}^{n\times n}$, $i \in \Sigma$, the discrete-time switched linear system with ω-regular switching sequences $(\mathcal{A}, \mathcal{B})$ is described by the equation

$$x(t+1) = A_{\theta(t)}x(t),$$

where $t \in \mathbb{N}$, $x(t) \in \mathbb{R}^n$ is the state and $\theta : \mathbb{N} \to \Sigma$ is the switching signal with $\theta \in Lang(\mathcal{B})$ where, by abuse of notation, we say that $\theta \in Lang(\mathcal{B})$ if the infinite word $\theta(0)\theta(1)\cdots \in Lang(\mathcal{B})$. We note that, given an initial condition $x_0 \in \mathbb{R}^n$, and a switching signal $\theta \in Lang(\mathcal{B})$, the trajectory with $x(0) = x_0$ is unique, denoted by $\mathbf{x}(., x_0, \theta)$ and given by

$$\forall t \ge 1 : \mathbf{x}(t, x_0, \theta) = \prod_{i=0}^{t-1} A_{\theta(i)}x_0,$$

where $\prod_{i=0}^{t-1} A_{\theta(i)} = A_{\theta(t-1)} \times \cdots \times A_{\theta(0)}$.

Now we introduce the following running Assumption:

Assumption 1. *All the states of the Büchi automaton $\mathcal{B}$ are reachable from at least one initial state and for any finite run $q_0q_1q_2\dots q_k$ there exists an infinite sequence of states $q_{k+1}q_{k+2}\dots$ such that $q_0q_1q_2\dots$ is an accepting run.*

Note that there is no loss of generality to suppose that Assumption 1 holds true since it can be shown easily that for any NBA (resp. DBA), there exists an NBA (resp. DBA) with the same language and satisfying Assumption 1. Hence, in the rest of the paper, Assumption 1 is always supposed to be satisfied.

We start by defining some stability notions.

Definition 1. *The system $(\mathcal{A}, \mathcal{B})$ is **globally attractive (GA)** if for all switching signals $\theta \in Lang(\mathcal{B})$ and for all initial conditions $x_0 \in \mathbb{R}^n$, we have*

$$\lim_{t\to\infty} \|\mathbf{x}(t, x_0, \theta)\| = 0.$$

Definition 2. *The system $(\mathcal{A}, \mathcal{B})$ is **globally uniformly stable (GUS)** if there exists a scalar $\alpha \ge 1$ such that for all switching signals $\theta \in Lang(\mathcal{B})$ and for all initial conditions $x_0 \in \mathbb{R}^n$, we have*

$$\|\mathbf{x}(t, x_0, \theta)\| \le \alpha\|x_0\|, \forall t \in \mathbb{N}.$$

Now, we define some quantities specific to a DBA $\mathcal{B}$. For $\theta \in Lang(\mathcal{B})$, we denote by $q_0q_1\dots$ its unique accepting run in $\mathcal{B}$ and we define the sequence of *return instants* $(\tau_k^\theta)_{k\in\mathbb{N}}$, by $\tau_0^\theta = 0$, and for all $k \in \mathbb{N}$

$$\tau_{k+1}^\theta = \min\{t > \tau_k^\theta | q_t \in F\}.$$

The *return index* $\kappa^\theta(t) : \mathbb{N} \to \mathbb{N}$ is defined by

$$\kappa^\theta(t) = \max\{k \in \mathbb{N} | \tau_k^\theta \le t\}.$$

Intuitively, τ_k^θ is the first instant where the run associated with θ has visited an accepting state k times, and $\kappa^\theta(t)$ is the number of times the run associated with θ has visited an accepting state in $\mathcal{B}$ up to time t. Since $\theta \in Lang(\mathcal{B})$, the set F of accepting states will be visited infinitely often, so τ_k^θ is well defined for every $k \in \mathbb{N}$ and $\lim_{t\to\infty} \kappa^\theta(t) = \infty$. We note that an infinite word can have multiple runs in an NBA, therefore $\kappa^\theta(t)$ and τ_k^θ cannot be defined as above for an NBA.

Let us define the following stability notion for a DBA.

Definition 3. *The system $(\mathcal{A}, \mathcal{B})$, where $\mathcal{B}$ is a DBA, is **globally uniformly exponentially stable (GUES)** if there exist a scalar $C \geq 1$ and a scalar $0 < \lambda < 1$ such that for all switching signals $\theta \in Lang(\mathcal{B})$ and for all initial conditions $x_0 \in \mathbb{R}^n$, we have*

$$\|\mathbf{x}(t, x_0, \theta)\| \leq C\lambda^{\kappa^\theta(t)}\|x_0\|, \forall t \in \mathbb{N}.$$

It is clear that if the system $(\mathcal{A}, \mathcal{B})$ is **GUES** then it is **GA** and **GUS**, we note that this type of stability cannot be defined for an NBA since $\kappa^\theta(t)$ is not defined for these automata.

We have seen in this section several notions of stability of switched linear systems with ω-regular switching sequences, some of them concern the general case of an NBA, and the rest concerns the DBA only. In the next sections, we will develop sufficient conditions for stability using a Lyapunov approach and we will give a converse result for a specific class of systems.

3 SUFFICIENT CONDITIONS FOR STABILITY

In this section, for a system $(\mathcal{A}, \mathcal{B})$, we establish sufficient conditions for the notions of stability defined in the previous section based on the following type of Lyapunov functions:

Definition 4. *For the system $(\mathcal{A}, \mathcal{B})$, the function $V : Q \times \mathbb{R}^n \to \mathbb{R}_0^+$, is called **Lyapunov function** if there exist scalars $\alpha_1, \alpha_2 > 0$ and $0 < \rho < 1$ such that for all $x \in \mathbb{R}^n$, the following hold:*

$$\alpha_1\|x\| \leq V(q, x) \leq \alpha_2\|x\|, \qquad q \in Q \qquad (1)$$

$$V(q', A_i x) \leq V(q, x), \qquad q \in Q, i \in \Sigma, q' \in \delta(q, i) \setminus F \qquad (2)$$

$$V(q', A_i x) \leq \rho V(q, x), \qquad q \in Q, i \in \Sigma, q' \in \delta(q, i) \cap F \qquad (3)$$

Theorem 1. *If there exists a Lyapunov function for the system $(\mathcal{A}, \mathcal{B})$ then $(\mathcal{A}, \mathcal{B})$ is **GA** and **GUS**. If in addition $\mathcal{B}$ is a DBA then $(\mathcal{A}, \mathcal{B})$ is **GUES**.*

Proof. Let us consider an initial condition $x_0 \in \mathbb{R}^n$ and a switching signal $\theta \in Lang(\mathcal{B})$, let $q_0q_1q_2\ldots$ be an accepting run associated with θ. Let $t_0 = 0$ and $0 < t_1 < t_2 < \ldots$ be the time instants where $q_{t_i} \in F$, for all $i \geq 1$. We denote $x(.) = \mathbf{x}(., x_0, \theta)$ and we define the function $W : \mathbb{N} \to \mathbb{R}_0^+$ by $W(t) = V(q_t, x(t))$ for all $t \in \mathbb{N}$. It follows from (2) and (3) that $W(t+1) \leq W(t)$ for all $t \in \mathbb{N}$. From the monotonicity of W, we get that

$$\forall t \in \mathbb{N} : W(t) \leq W(0).$$

Therefore, from (1) we conclude that

$$\forall t \in \mathbb{N} : \|x(t)\| \leq \frac{\alpha_2}{\alpha_1}\|x_0\|,$$

and we get that the system $(\mathcal{A}, \mathcal{B})$ is **GUS**.

On the other hand, from (3), we get that

$$\forall k \geq 1 : W(t_k) \leq \rho W(t_k - 1).$$

From the monotonicity of W, we deduce that

$$\forall k \geq 1 : W(t_k) \leq \rho W(t_{k-1}).$$

By induction on k, we get that

$$\forall k \in \mathbb{N} : W(t_k) \leq \rho^k W(0).$$

Since W is non-increasing and since $W(t) \geq 0$ for all $t \in \mathbb{N}$ and $0 < \rho < 1$, we get that

$$\lim_{t\to\infty} W(t) = 0.$$

Therefore, from (1) we get that the system $(\mathcal{A}, \mathcal{B})$ is **GA**.

Now if $\mathcal{B}$ is a DBA, the sequence $(t_k)_{k\in\mathbb{N}}$ defined above coincides with the sequence of return instants $(\tau_k^\theta)_{k\in\mathbb{N}}$. Therefore, we get that

$$\forall k \in \mathbb{N} : W(\tau_k^\theta) \leq \rho^k W(0).$$

Now let $t \in \mathbb{N}$, and let $k \in \mathbb{N}$ such that $t \in [\tau_k^\theta, \tau_{k+1}^\theta)$, then the return index is $\kappa^\theta(t) = k$. We get from the monotonicity of W that $W(t) \leq \rho^{\kappa^\theta(t)}W(0)$. Finally from (1), we get, for all $t \in \mathbb{N}, x_0 \in \mathbb{R}^n$ that

$$\|x(t)\| \leq \frac{\alpha_2}{\alpha_1}\rho^{\kappa^\theta(t)}\|x_0\|. \qquad (4)$$

Hence $(\mathcal{A}, \mathcal{B})$ is **GUES**. □

Let us remark that (4) provides an upper bound on the convergence rate of the state with respect to the number of visits to the accepting set F given by the return index $\kappa^\theta(t)$. If we restrain to Lyapunov functions of the form $V(q, x) = \sqrt{x^\top P_q x}$ where, for every $q \in Q$, P_q is a positive definite matrix, then the conditions in Definition 4 are equivalent to a set of linear matrix inequalities (LMI). In that case, stability of the switched system $(\mathcal{A}, \mathcal{B})$ can be verified by solving a convex optimization problem.

4 NECESSARY CONDITIONS FOR STABILITY

In this section we show that, if all matrices in $\mathcal{A}$ are invertible, then the existence of a Lyapunov function for $(\mathcal{A}, \mathcal{B})$ is not only sufficient but also necessary for the attractivity and uniform stability of the system. We introduce some quantities related to the NBA $\mathcal{B}$. We define $\mathcal{L}_{qq_f}$ as the set of all words in Σ^+ corresponding to a run starting from $q \in Q$ and reaching $q_f \in F$ without visiting any accepting state between q and q_f. Formally

$$\mathcal{L}_{qq_f} = \left\{ \sigma_1 \ldots \sigma_k \in \Sigma^+ \;\middle|\; \begin{array}{c} q_{i+1} \in \delta(q_i, \sigma_i) \setminus F, 1 \leq i < k \\ \text{where } q_1 = q \text{ and } q_f \in \delta(q_k, \sigma_k) \end{array} \right\}.$$

Now we define $\mathcal{L}_{qq_f}^\times$ as the set of all products of matrices in $\mathcal{A}$ associated with words in $\mathcal{L}_{qq_f}$:

$$\mathcal{L}_{qq_f}^\times = \left\{ A_{\sigma_k} \times \cdots \times A_{\sigma_1} \;\middle|\; \sigma_1 \ldots \sigma_k \in \mathcal{L}_{qq_f} \right\}.$$

Given a word $l \in \mathcal{L}_{qq_f}$ the corresponding matrix in $\mathcal{L}_{qq_f}^\times$ is denoted M_l.

In order to establish necessary conditions, we apply a result obtained in [16] based on labeled graphs. A labeled graph is a tuple $\mathcal{G} = (V, L, E)$ where V is a set of nodes, L is a set of labels, $E \subset V \times L \times V$ is a set of edges or transitions. For an edge $e = (v, l, v') \in E$, the node $v \in V$ is the origin, $l \in L$ is the label, $v' \in V$ is the end. A path in the labeled graph $\mathcal{G}$ is a set of consecutive

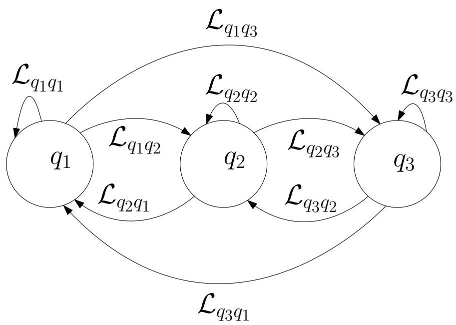

Figure 1: The labeled graph $\mathcal{G}$ corresponding to an NBA $\mathcal{B}$ with 3 accepting states q_1, q_2 and q_3.

edges $(v_0, l_1, v_1), (v_1, l_2, v_2), \dots$ where $v_i \in V$ for all $i = 0, 1, \dots,$ its label is the word $w = l_1 l_2 \dots$.

In order to analyze the dynamics of the system $(\mathcal{A}, \mathcal{B})$, we use the concept of constrained system [16]. A constrained system is formed from a labeled graph $\mathcal{G}$, and a set of matrices $\mathcal{M}$ where each matrix of $\mathcal{M}$ corresponds to an element of the labels of $\mathcal{G}$. In our case, with a switched system $(\mathcal{A}, \mathcal{B})$ one can associate a constrained system $(\mathcal{M}_{\mathcal{B}}^F, \mathcal{G})$, where $\mathcal{M}_{\mathcal{B}}^F = \bigcup_{q,p \in F} \mathcal{L}_{qp}^{\times}$ and the nodes of $\mathcal{G}$ are $V = F$, the labels $L = \bigcup_{q,p \in F} \mathcal{L}_{qp}$, and the set of edges is $E = \left\{(q, l, p) \,\middle|\, q, p \in F, l \in \mathcal{L}_{qp}\right\}$.

Remark 1. *In [16], the set of labels L is assumed to be finite. In our case, this set can be infinite.*

Figure 1 shows the labeled graph corresponding to an NBA $\mathcal{B}$ with three accepting states q_1, q_2 and q_3. In the figure, by an abuse of notation, for $q_i, q_j \in F$, the edge $(q_i, \mathcal{L}_{q_i q_j}, q_j)$, denotes the set of edges $\left\{(q_i, l, q_j) \,\middle|\, l \in \mathcal{L}_{q_i q_j}\right\}$. Note that, unlike [16], in our case the set of edges of the labeled graph is usually infinite. We consider the following stability notion.

Definition 5. *We say that the constrained system $(\mathcal{M}_{\mathcal{B}}^F, \mathcal{G})$ is attractive if for all paths $(q_{f_0}, l_1, q_{f_1}), (q_{f_1}, l_2, q_{f_2}), \dots$ we have*

$$\lim_{k \to \infty} \|M_{l_k} \cdots M_{l_1}\| = 0.$$

In the following we will make use of the following assumption.

Assumption 2. *All the matrices in $\mathcal{A}$ are invertible.*

The following lemma provides a uniform bound on the set of matrix products that correspond to transitions from any state $q \in Q$ to an accepting state $q_f \in F$.

Lemma 1. *Under Assumption 2, if the system $(\mathcal{A}, \mathcal{B})$ is **GUS**, then the set $\mathcal{L}_{qq_f}^{\times}$ is either empty or bounded for all $q \in Q, q_f \in F$.*

Proof. Let $q \in Q, q_f \in F$ such that $\mathcal{L}_{qq_f} \neq \emptyset$. Let $l \in \mathcal{L}_{qq_f}$, by Assumption 1 we know that there exist a switching signal $\theta(0)\theta(1) \cdots \in Lang(\mathcal{B})$ and a corresponding run $q_0 q_1 q_2 \dots$ such that $q_{t_0'} = q$ and $q_{t_0} = q_f$, for some $t_0' < t_0$ and $l = \theta(t_0') \dots \theta(t_0 - 1)$. Without loss of generality we may assume that $q_i \neq q_j$ for every $0 \leq i < j \leq t_0'$ since otherwise, if $q_i = q_j$, we can replace the switching signal θ with $\theta(0)\theta(1) \dots \theta(i-1)\theta(j) \dots$ and consider the corresponding run $q_0 q_1 q_2 \dots q_i q_{j+1} \dots$. Hence we may assume $t_0' \leq |Q| - 1$.

From the definition of global uniform stability we get that there exists $\alpha \geq 1$ such that

$$\|A_{\theta(t_0-1)} \cdots A_{\theta(t_0')} A_{\theta(t_0'-1)} \cdots A_{\theta(1)} A_{\theta(0)}\| \leq \alpha$$

Using the fact that, for A, B in $\mathbb{R}^{n \times n}$ with B invertible, one has $\|A\| \leq \|AB\| \|B^{-1}\|$ and since $M_l = A_{\theta(t_0-1)} \cdots A_{\theta(t_0')}$, we obtain from the previous inequality

$$\begin{aligned}
\|M_l\| &\leq \alpha \|(A_{\theta(t_0'-1)} \cdots A_{\theta(1)} A_{\theta(0)})^{-1}\| \\
&= \alpha \|(A_{\theta(0)})^{-1} (A_{\theta(1)})^{-1} \cdots (A_{\theta(t_0'-1)})^{-1}\| \\
&\leq \alpha \left(\max_{A \in \mathcal{A}} \|A^{-1}\| \right)^{t_0'} \leq \alpha \max \left\{ 1, \left(\max_{A \in \mathcal{A}} \|A^{-1}\| \right)^{|Q|-1} \right\}.
\end{aligned}$$

We have thus obtained a uniform bound on the set of matrices $\mathcal{L}_{qq_f}^{\times}$ whenever $\mathcal{L}_{qq_f} \neq \emptyset$. □

We will analyse the attractivity of the constrained system $(\mathcal{M}_{\mathcal{B}}^F, \mathcal{G})$ by making use of the concept of multinorm, defined below.

Definition 6. *(Definition 1 in [16]) A multinorm of the constrained system $(\mathcal{M}_{\mathcal{B}}^F, \mathcal{G})$, denoted by $\mathcal{H}$, is a set of $|F|$ norms in $\mathbb{R}^n$, that is $\mathcal{H} = \{\|.\|_q, q \in F\}$. The value of the multinorm $\gamma^*(\mathcal{H})$ is defined as*

$$\gamma^*(\mathcal{H}) = \inf \left\{ \gamma > 0 \;\middle|\; \begin{array}{c} \|Mx\|_p \leq \gamma \|x\|_q, \ \forall x \in \mathbb{R}^n, \\ \forall q, p \in F \text{ s.t. } \mathcal{L}_{qp} \neq \emptyset, \ \forall M \in \mathcal{L}_{qp}^{\times} \end{array} \right\}.$$

This definition coincides with Definition 1 in [16] except for the fact that here the matrix M takes values on a possibly infinite set.

Theorem 2. *Suppose that Assumption 2 holds true and that $(\mathcal{A}, \mathcal{B})$ is **GUS**. Then the constrained system $(\mathcal{M}_{\mathcal{B}}^F, \mathcal{G})$ is attractive iff it admits a multinorm $\mathcal{H}$ with value $\gamma^*(\mathcal{H}) < 1$.*

Proof. The result follows from Proposition 2.2 and Theorem 1.1 in [16], where these results are proven assuming that the number of labels going from a state to another is finite and that the graph $\mathcal{G}$ is strongly connected. The arguments of the proofs still apply in our case, although the set $\mathcal{L}_{qp}^{\times}$ is bounded but not necessarily finite and we do not require $\mathcal{G}$ to be strongly connected. □

Now we relate the attractivity of $(\mathcal{A}, \mathcal{B})$ with that of $(\mathcal{M}_{\mathcal{B}}^F, \mathcal{G})$ using the following lemma.

Lemma 2. *Under Assumption 2, if the switched system $(\mathcal{A}, \mathcal{B})$ is **GA** then the constrained system $(\mathcal{M}_{\mathcal{B}}^F, \mathcal{G})$ is attractive.*

Proof. Let $(q_{f_0}, l_1, q_{f_1}), (q_{f_1}, l_2, q_{f_2}), \dots$ be a path in $(\mathcal{M}_{\mathcal{B}}^F, \mathcal{G})$. By Assumption 1 there exist $q_0 \in Q_0$, a switching sequence $\theta \in Lang(\mathcal{B})$ and a sequence of instants $(t_k)_{k \in \mathbb{Z}^+}$ with $0 < t_1 < t_2 \dots$ such that

$$\forall k \in \mathbb{Z}^+, x_0 \in \mathbb{R}^n : \mathbf{x}(t_k, x_0, \theta) = M_{l_{k-1}} \cdots M_{l_1} M_{l_0} x_0,$$

where $M_{l_0} \in \mathcal{L}_{q_0 q_{f_0}}^{\times}$. Since $(\mathcal{A}, \mathcal{B})$ is **GA** we have

$$\lim_{k \to \infty} \|M_{l_k} \cdots M_{l_0}\| = 0.$$

By submultiplicativity we have

$$\|M_{l_k} \cdots M_{l_1}\| \leq \|M_{l_k} \cdots M_{l_0}\| \|M_{l_0}^{-1}\|$$

which implies

$$\lim_{k\to\infty} \|M_{l_k}\cdots M_{l_1}\| = 0,$$

concluding the proof of the lemma. □

We next provide a converse result to Theorem 1.

Theorem 3. *Under Assumption 2, if the switched system $(\mathcal{A},\mathcal{B})$ is* ***GUS*** *and* ***GA****, then it admits a Lyapunov function.*

Proof. Under the assumptions of the theorem we get from Theorem 2 and Lemma 2 that, for the constrained system $(\mathcal{M}^F_{\mathcal{B}},\mathcal{G})$, there exists a multinorm $\mathcal{H} = \{\|.\|_{q_f}, q_f \in F\}$ with a value strictly less than 1, that is $\gamma^*(\mathcal{H}) < 1$.

Consider the function $V : Q \times \mathbb{R}^n \to \mathbb{R}^+_0$ defined as follows

$$\forall q \in Q, x \in \mathbb{R}^n : V(q,x) = \max_{\substack{q_f\in F\\ \mathcal{L}_{qq_f}\neq\emptyset}} \sup_{M\in\mathcal{L}^\times_{qq_f}} \|Mx\|_{q_f}.$$

Note that V is well defined thanks to Assumption 1. Let us prove that V is a Lyapunov function for $(\mathcal{A},\mathcal{B})$, namely that V satisfies equations (1), (2) and (3) in Definition 4 for all $x \in \mathbb{R}^n$, for some positive constants α_1, α_2 and ρ such that $\gamma^*(\mathcal{H}) < \rho < 1$.

Concerning (1), since the system is **GUS** and thanks to Assumption 2 we know from Lemma 1 that the set $\mathcal{L}^\times_{qq_f}$ is bounded for all $q \in Q$ and $q_f \in F$. Hence there exists $B > 0$ such that $V(q,x) \le B \max_{q_f\in F} \|x\|_{q_f}$. Furthermore, from the equivalence of norms in $\mathbb{R}^n$ and since F is finite, we get that there exists a constant $\alpha_2 > 0$ such that $V(q,x) \le \alpha_2\|x\|$ for all $q \in Q$ and $x \in \mathbb{R}^n$.

On the other hand, for $q \in Q$, there exists $q_f \in F$ such that $\mathcal{L}_{qq_f} \neq \emptyset$. Taking $M_q \in \mathcal{L}^\times_{qq_f}$ we get, from the equivalence of norms in $\mathbb{R}^n$ and since F is finite, that there exists a scalar $\alpha > 0$ such that for all $x \in \mathbb{R}^n$ it holds

$$V(q,x) \ge \alpha\|M_q x\|.$$

From Assumption 2, we then obtain

$$V(q,x) \ge \alpha \frac{\|x\|}{\|M_q{}^{-1}\|} \ge \alpha \min_{q\in Q} \frac{1}{\|M_q^{-1}\|}\|x\|.$$

By taking $\alpha_1 = \alpha \min_{q\in Q} \frac{1}{\|M_q{}^{-1}\|}$ we get that (1) holds true.

We next show (2). Let $q \in Q$ and $q' \in \delta(q,i) \setminus F$ for some $i \in \Sigma$. For every $q_f \in F$ such that $\mathcal{L}_{q'q_f} \neq \emptyset$ and $M \in \mathcal{L}^\times_{q'q_f}$ we have that the product MA_i is an element of the set $\mathcal{L}^\times_{qq_f}$. In particular

$$\{q_f \in F \mid \mathcal{L}_{q'q_f} \neq \emptyset\} \subseteq \{q_f \in F \mid \mathcal{L}_{qq_f} \neq \emptyset\}$$

and

$$\sup_{M\in\mathcal{L}^\times_{q'q_f}} \|MA_ix\|_{q_f} \le \sup_{M'\in\mathcal{L}^\times_{qq_f}} \|M'x\|_{q_f}, \quad \forall x \in \mathbb{R}^n.$$

Then

$$\begin{aligned}\max_{\substack{q_f\in F\\ \mathcal{L}_{q'q_f}\neq\emptyset}} \sup_{M\in\mathcal{L}^\times_{q'q_f}} \|MA_ix\|_{q_f} &\le \max_{\substack{q_f\in F\\ \mathcal{L}_{q'q_f}\neq\emptyset}} \sup_{M'\in\mathcal{L}^\times_{qq_f}} \|M'x\|_{q_f}\\ &\le \max_{\substack{q_f\in F\\ \mathcal{L}_{qq_f}\neq\emptyset}} \sup_{M'\in\mathcal{L}^\times_{qq_f}} \|M'x\|_{q_f}\end{aligned}$$

Hence (2) is satisfied.

Finally, let us prove (3). Let $q \in Q$ and $q' \in \delta(q,i) \cap F$ for some $i \in \Sigma$. Let $q_f \in F, M \in \mathcal{L}^\times_{q'q_f}$. Letting ρ such that $\gamma^*(\mathcal{H}) < \rho < 1$ we get that $\|Mx\|_{q_f} \le \rho\|x\|_{q'}$ for all $x \in \mathbb{R}^n$. Since $A_i \in \mathcal{L}^\times_{qq'}$, then

$$\begin{aligned}\|MA_ix\|_{q_f} \le \rho\|A_ix\|_{q'} &\le \rho \sup_{M'\in\mathcal{L}^\times_{qq'}} \|M'x\|_{q'}\\ &\le \rho \max_{\substack{q'_f\in F\\ \mathcal{L}_{qq'_f}\neq\emptyset}} \sup_{M'\in\mathcal{L}^\times_{qq'_f}} \|M'x\|_{q'_f}\end{aligned}$$

Taking the supremum over $M \in \mathcal{L}^\times_{q'q_f}$ and then the maximum over all $q_f \in F$ such that $\mathcal{L}_{q'q_f} \neq \emptyset$ on the left-hand side, we get that (3) is satisfied, concluding the proof of the theorem. □

Corollary 1. *Under Assumption 2, let $\mathcal{B}$ be deterministic. Then, the $(\mathcal{A},\mathcal{B})$ is* ***GUES*** *if and only if it is* ***GA*** *and* ***GUS****.*

Proof. The fact that **GUES** implies **GA** and **GUS** follows directly from the definitions. Then, if $(\mathcal{A},\mathcal{B})$ is **GA** and **GUS**. Then from Theorem 3, there exists a Lyapunov function. Theorem 1 gives that $(\mathcal{A},\mathcal{B})$ is **GUES**. □

5 NUMERICAL EXAMPLE

We consider a multi-agent system consisting of 3 discrete-time oscillators whose dynamics is given by:

$$z_i(t+1) = Rz_i(t) + u_i(t), \ i = 1,2,3 \tag{5}$$

where $z_i(t) \in \mathbb{R}^2, u_i(t) \in \mathbb{R}^2$ and $R = \begin{pmatrix} \cos(\phi) & -\sin(\phi)\\ \sin(\phi) & \cos(\phi)\end{pmatrix}$ with $\phi = \frac{\pi}{6}$. The input $u_i(t)$ is used for synchronization purpose and is based on the available information at time t. There exist 3 communication channels between agent 1 and agent 2 (channel 1), 2 and 3 (channel 2) and 1 and 3 (channel 3). At each instant, only one of these channels is active and the active channel is selected by a switching signal $\theta : \mathbb{N} \to \Sigma = \{1,2,3\}$. Then, the input value is given as follows:

$$u_1(t) = \begin{cases}\gamma(z_2(t) - z_1(t)), & \text{if } \theta(t) = 1\\ 0, & \text{if } \theta(t) = 2\\ \gamma(z_3(t) - z_1(t)), & \text{if } \theta(t) = 3\end{cases}$$

$$u_2(t) = \begin{cases}\gamma(z_1(t) - z_2(t)), & \text{if } \theta(t) = 1\\ \gamma(z_3(t) - z_2(t)), & \text{if } \theta(t) = 2\\ 0, & \text{if } \theta(t) = 3\end{cases}$$

$$u_3(t) = \begin{cases}0, & \text{if } \theta(t) = 1\\ \gamma(z_2(t) - z_3(t)), & \text{if } \theta(t) = 2\\ \gamma(z_1(t) - z_3(t)), & \text{if } \theta(t) = 3\end{cases}$$

where $\gamma = 0.05$ is a control gain. Denoting the vector of synchronization errors as $x(t) = (x_1(t)^\top, x_2(t)^\top)^\top$ with $x_i(t) = z_{i+1}(t) - z_i(t)$, the error dynamics is described by a 4-dimensional switched linear system of the form:

$$x(t+1) = A_{\theta(t)}x(t)$$

where the 3 matrices describing the 3 modes of communication are given by:

$$A_1 = \begin{pmatrix} R-2\gamma I_2 & 0\\ \gamma I_2 & R\end{pmatrix}, A_2 = \begin{pmatrix} R & \gamma I_2\\ 0 & R-2\gamma I_2\end{pmatrix}, A_3 = \begin{pmatrix} R-\gamma I_2 & -\gamma I_2\\ -\gamma I_2 & R-\gamma I_2\end{pmatrix}.$$

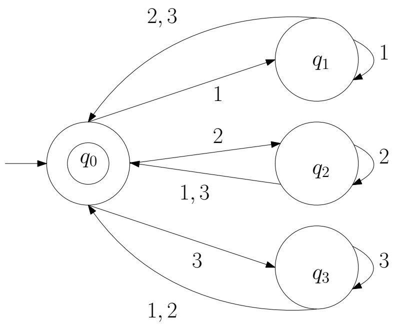

Figure 2: Deterministic Büchi automaton $\mathcal{B}$.

As for the communication protocol, we impose a fairness constraint that the switching signal cannot keep activating the same communication channel:

$$\forall t \in \mathbb{N}, \exists t' \geq t, \ \theta(t') \neq \theta(t).$$

Note that this property can be formulated as the following linear temporal logic formula:

$$\bigwedge_{i=1}^{3} \neg \left(\Diamond \Box (\theta = i) \right).$$

This is equivalently described by a deterministic Büchi automaton, $\mathcal{B}$, where the set of states is $Q = \{q_0, q_1, q_2, q_3\}$, the alphabet $\Sigma = \{1, 2, 3\}$, $Q_0 = \{q_0\}$, $F = Q_0$. Figure 2 shows the corresponding Büchi automaton which describes the switching logic in this system. We want to show that the agents synchronize if the state q_0 in $\mathcal{B}$ is visited infinitely often. This can be done by studying the stability of $(\mathcal{A} = \{A_1, A_2, A_3\}, \mathcal{B})$. We then look for a Lyapunov function of the form $V(q, x) = \sqrt{x^T P_q x}$ where P_q is a positive definite symmetric matrix. The conditions in Theorem 1 translate into the following linear matrix inequalities:

$$\begin{aligned} &I_4 \leq P_q, && q \in Q \\ &A_i^\top P_{q'} A_i \leq P_q, && q \in Q, i \in \Sigma, q' \in \delta(q, i) \setminus F \\ &A_i^\top P_{q'} A_i \leq \rho^2 P_q && q \in Q, i \in \Sigma, q' \in \delta(q, i) \cap F \end{aligned}$$

By solving these 16 LMIs, we find for $\rho = 0.96$:

$$P_{q_0} = \begin{pmatrix} 1.98I_2 & 0.98I_2 \\ 0.98I_2 & 1.98I_2 \end{pmatrix}, \qquad P_{q_1} = \begin{pmatrix} 2.26I_2 & 0.99I_2 \\ 0.99I_2 & 1.98I_2 \end{pmatrix},$$

$$P_{q_2} = \begin{pmatrix} 1.98I_2 & 0.99I_2 \\ 0.99I_2 & 2.26I_2 \end{pmatrix}, \qquad P_{q_3} = \begin{pmatrix} 2.22I_2 & 1.22I_2 \\ 1.22I_2 & 2.22I_2 \end{pmatrix}.$$

From Theorem 1, we get that the switched system $(\mathcal{A}, \mathcal{B})$ is **GUES** which means the oscillators synchronize well after sufficient time.

We now consider the following scenario: for the first 50 time units, the communication channel 1 is constantly active, then at $t = 50$ a switch occurs and for the next 50 time units the channel 2 is active, then at $t = 100$ another switch occurs and the communication channel 3 stays active for the next 50 time units. After $t = 150$, the switching signal randomly activates channel 1 and channel 2 with equal probability so that the accepting state q_0 is visited infinitely often. The simulation results are shown in Figure 3. It is interesting to remark that when the switching signal remains constant the synchronization error does not go to zero, however after $t = 150$, when q_0 is visited more frequently, the synchronization

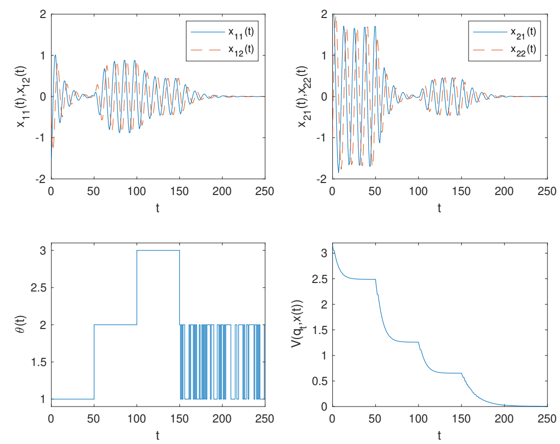

Figure 3: Time evolution of the synchronization error $x(t)$ (top figures), switching signal $\theta(t)$ (bottom left), and the Lyapunov function $V(q_t, x(t))$ (bottom right).

error starts to converge towards zero. As expected, the Lyapunov function $V(q_t, x(t))$ is non-increasing and, as soon as the state q_0 is visited frequently enough, it starts approaching zero.

6 CONCLUSION

In this paper, we established some results concerning the stability of discrete time switched systems where the switching signal is generated by a Büchi automaton. We developed sufficient conditions for attractivity and uniform stability for this type of systems and also of uniform exponential stability when the considered Büchi automaton is deterministic, all based on Lyapunov arguments, Note that these conditions can readily be extended to the case of nonlinear systems. Moreover, we proved that these conditions are also necessary for a subclass of such systems with invertible matrices. Finally, we have shown through a numerical example, how these Lyapunov functions can actually be computed using a convex optimization problem based on linear matrix inequalities.

The current work opens several research directions for the future. First, the development of numerical techniques to compute more complicated Lyapunov functions is necessary for cases where the simple linear matrix inequalities approach used in this paper proves unsuccessful. Finally, it should be possible to define a joint spectral radius for this class of switched systems. In this case, it would be interesting to investigate its properties and how it relates to the convergence of the system trajectories.

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