Multivariate Counterfactual Systems And Causal Graphical Models

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Abstract

Among Judea Pearl's many contributions to causality and statistics, the graphical *d*-separation criterion, and the *do*-calculus stand out. In this paper we show that *d*-separation provides direct insight into an earlier causal model originally described in terms of potential outcomes and event trees. In turn, the resulting synthesis leads to a simplification of the *do*-calculus that clarifies and separates the underlying concepts, and a simple counterfactual formulation of a complete identification algorithm in causal models with hidden variables.

Keywords: causal inference; identification; graphical models; potential outcomes

1. Introduction

For the last three decades Judea Pearl has been a leading advocate for the adoption of causal models throughout the sciences. Pearl (1995) introduced causal models based on non-parametric structural equation models (NPSEMs). NPSEMs encode direct causal relations between variables. More precisely each variable V is modeled as a function of its direct causes and an error term ε_V ; this is the "structural equation" for V; see Table 2. These causal relationships can be represented naturally by the directed arrows on a directed acyclic graph (DAG) in which there is an edge $X \to V$ if X is present in the structural equation for V. The resulting graph is often called a causal DAG or diagram. However, further probabilistic assumptions are required to link the NPSEM to the distribution of the data.

Pearl has often considered a submodel of an NPSEM, hereafter referred to as the NPSEM-IE, which assumes the independence of error terms. NPSEM-IEs typically include both observed and

^{1.} See also (Pearl, 2009, p.69). More recently Pearl has used the term (Structural) Causal Model (SCM) to refer to NPSEMs; see (Pearl, 2009, p.203, Def. 7.1.1). However, (S)CM is sometimes also used to denote NPSEMs in which, in addition, the error terms are assumed to be independent either explicitly, see (Pearl, 2009, p.44, Def. 2.2.2) (Forré and Mooij, 2019), or implicitly see (Lee et al., 2020). For this reason, we prefer to use Pearl's earlier terminology.

hidden variables.² Thus although these models assume that errors are independent they still allow a modeler to postulate non-causal dependence between observed variables X and Y by including a hidden variable $X \leftarrow H \rightarrow Y$ (instead of allowing errors ε_X and ε_Y to be dependent).

Under the NPSEM-IE the distribution over the factual (i.e. hidden plus observed) variables factorizes according to the causal DAG. This allows one to reason about conditional independence in the distribution for the factual variables via d-separation relations on the causal graph. Based on this insight, Pearl developed an influential reasoning system called the *do*-calculus which allows complex derivations to be made linking causal and observed quantities by appealing to d-separation in graphs derived from the causal DAG.

Causal graphs plus d-separation turn a difficult mathematical problem into a simple one of graph topology. The use of causal DAGs, as championed by Pearl, has revolutionized causal reasoning in many fields, including fields such as epidemiology and sociology, precisely because causal reasoning based on DAGs and d-separation is so "user-friendly." That is, individuals lacking the necessary mathematical background to understand probabilistic inference based on solely on an NPSEM-IE have been given a tool with which they can solve subtle problems in causal inference. In fact, even the mathematically sophisticated find causal reasoning with graphs to be much easier than algebraically manipulating the underlying structural equations. As Pearl emphasizes this is largely because causal DAGs faithfully represent the way humans, including scientists and mathematicians, encode causal relations.

The use of DAGs to encode causal relationships dates back to the work of the geneticist Sewall Wright Wright (1921) in the 1920s, who used a special case of the NPSEM associated with linear structural equations, and Gaussian errors for pedigree analysis among other applications in biology. These ideas were further developed and applied by Wright, Haavelmo, the Cowles Commission, Strotz & Wold, Fisher (Wright, 1921; Haavelmo, 1943; Simon, 1953; Strotz and Wold, 1960; Fisher, 1969, 1970).

In statistics, (non-graphical) causal inference models have a long history also dating back to the 1920s (Neyman, 1923; Rubin, 1974; Robins, 1986). These models are based on counterfactual variables (potential outcomes) that encode the value the variable would have if, possibly contrary to fact, a particular treatment had been given. Causal contrasts in these models compare the distributions of potential outcomes under two or more treatments.

In general, these counterfactual models considered treatments or exposures at a single point in time. Extending the framework introduced by Neyman to allow for treatment at multiple time-points, Robins introduced *causally interpretable structured tree graph* (CISTG) models. These counterfactuals models, which were represented using event tree graphs, extended the point treatment model of Neyman (1923) to longitudinal studies with time-varying treatments, direct and indirect effects and feedback of one cause on another.

Pearl has noted that a NPSEM (even without assumptions on the distribution of the errors) implies the existence of potential outcomes and thus an NPSEM model also allows reasoning about counterfactuals; see (Halpern and Pearl, 2001, 2005). Indeed, Robins and Richardson have shown that in fact a particular finest CISTG model ("as detailed as the data") is mathematically isomorphic to an NPSEM model in the sense that any such CISTG model can be written as an (acyclic) NPSEM model and vice-versa. A finest CISTG "as detailed as the data" is a counterfactual causal model in which all the underlying variables can be intervened on – an assumption that Pearl has sometimes also adopted.³ Other versions of CISTG models, unlike the NPSEM, assume that only a subset

^{2.} Causal DAG models with unobserved variables are also referred to as "semi-Markovian" by (Pearl, 2009, p.69 and p.76).

^{3.} See (Galles and Pearl, 1998), Definitions 2 and 3 and footnote 2, also (Pearl, 2009) Definitions 7.1.2 and 7.1.3. However, in more recent work, Pearl (2018, 2019) has made a further distinction between hypothetical interventions and a concept of causation based on variables that "listen to others." Pearl continues to assume that for every variable there are counterfactuals associated with applying the *do* operator to that variable. However, the model resulting from applying the *do* operator and removing structural equations need no longer correspond to an actual intervention.

of the variables can be thought of as treatments with associated counterfactuals; thus interventions and causal effects are only defined for this subset. Henceforth, unless stated otherwise, the term "CISTG model" will be used to denote a "finest CISTG model as detailed as the data."

Since counterfactual variables are not directly observed, assumptions are needed to link counterfactuals and their distributions to those of the factual data. A necessary assumption is consistency, which states that for a unit their observed outcome (Y) and their potential outcome (Y(a)) had a particular treatment a been assigned, will coincide, if in reality the treatment they received (A) is a. However, since both counterfactuals are not directly observed for any individual – the fundamental problem of causal inference – distributions of causal effects are not identified without additional assumptions, beyond consistency.

These assumptions typically take the form of Markov (conditional independence) assumptions that link the distribution of the factual data to that of the counterfactuals, as further discussed below. The simplest example is a randomized clinical trial which assigns treatment via the flip of a coin, and thus treatment is independent of the potential outcomes, so for all a, $A \perp\!\!\!\perp Y(a)$. Together, consistency and Markov assumptions allow population level causal contrasts to be identified from observed data. In contrast, individual level effects are not typically identifiable.

Under the NPSEM-IE model, the additional Markov assumptions follow from the assumption that the errors in the structural equation for each variable (hidden or observed) are independent of the errors in the structural equations for the other variables.

Robins (1986) similarly added independence assumptions to the CISTG model. Robins referred to the version of this model in which all variables can be intervened on as the "finest fully randomized CISTG model as detailed as the data," which we henceforth refer to as the "FFRCISTG model," unless stated otherwise. Interestingly, the NPSEM-IE implies many more counterfactual independence assumptions than does the corresponding FFRCISTG model. In fact, if we consider complete graphs on p binary variables then the difference between the number of assumptions implied by the NPSEM-IE and the FFRCISTG model grows at a doubly exponential rate.⁴

The NPSEM-IE allows the identification of certain causal effects – the pure and total direct and indirect effects and more generally path specific effects⁵ – by making use of additional independence assumptions that cannot be confirmed, even in principle, by any experiment conducted using the variables represented on the graph. In contrast under the less restrictive FFRCISTG model all counterfactual independence assumptions are in principle experimentally testable,⁶ the pure and total direct effects are not identifiable (from the variables on the graph). However, ordinary intervention distributions of the type that arise in Pearl's *do*-calculus are identifiable under the FFRCISTG model.

Many statisticians and econometricians exclusively use counterfactuals (without graphs) when carrying out causal data analyses. Pearl has developed purely graphical criteria to reason about confounding and many other causal questions. Since graphical criteria, such as Pearl's *do*-calculus, make no reference to counterfactuals, they can appear confusing to those unused to causal graphs. Indeed only factual variables typically appear on Pearl's causal diagrams so any connection between Pearl's graphical criteria and the statistician's counterfactual criteria appear at first glance to be obscure. This is true even though Pearl and others have shown mathematically that the two approaches to evaluation of confounding are effectively logically equivalent.

This leaves open the question as to whether there are predictions made by these removals and, if so, how can they be validated.

^{4.} With three binary variables, the difference in the dimension of the two models is 94, with four it is 32, 423 (Richardson and Robins, 2013).

^{5.} See (Robins et al., 2021) for more detail on these effects.

^{6.} This assumes that it is possible to observe the natural value of a variable and then intervene on it an instant later; see discussion in Section 2.

In this paper, we will describe an approach that unifies the graphical and counterfactual approaches to causality, via a graph known as a Single-World Intervention Graph (SWIG). The SWIG is defined by the counterfactual independencies implied by the FFRCISTG model. The nodes on a SWIG correspond to the counterfactual random variables present in these independences. Furthermore, Pearl's d-separation criterion can be applied to the SWIG to read off counterfactual independences implied by the FFRCISTG model. In fact, we will show that SWIGs lead directly to a simpler reformulation of the do-calculus in terms of potential outcomes that allows a considerable simplification of Rule 3. This reformulated calculus, that we term the potential outcome calculus or po-calculus is also strictly stronger than Pearl's in that it may be used to infer equalities that are not expressible in terms of the $do(\cdot)$ operator. We use the po-calculus to derive a new simple formulation of an extended version of the ID algorithm for identification of causal queries in the presence of hidden variables. The extended algorithm identifies joint distributions over sets of counterfactual outcomes, where some outcomes are the "natural" values that treatment variables would take were they not intervened on.

2. Graphs, Non-Parametric Structural Equation Models (NPSEMs) And The g-/do Operator

Fix a set of indices $V \equiv \{1,\ldots,K\}$ under a total ordering \prec , define the sets $\operatorname{pre}_i \equiv \{1,\ldots,i-1\}$. For each index $i \in V$, associate a random variable X_i with state space \mathfrak{X}_i ; the ordering here could be given by temporal ordering but need not be.⁸ Given $A \subseteq V$, we will denote subsets of random variables indexed by A as $X_A \in \mathfrak{X}_A \equiv \mathop{\textstyle \times}_{i \in A} \mathfrak{X}_i$. For notational conciseness we will sometimes use index sets to denote random variables themselves, using V and A to denote X_V and X_A , respectively, and similarly using lower case a to denote $x_A \in \mathfrak{X}_A$. Similarly, by extension, we will also use V_A to denote X_A and X_A to denote X_A and X_A to denote X_A and X_A to denote X_A .

We assume the existence of all one-step-ahead $potential\ outcome$ (also called counterfactual) random variables of the form $V_i(x_{\mathrm{pa}_i})$, where pa_i is a fixed subset of pre_i , and x_{pa_i} is any element in $\mathfrak{X}_{\mathrm{pa}_i}$. The variable $V_i(x_{\mathrm{pa}_i})$ denotes the value of V_i had the set V_{pa_i} of direct causes of V_i been set, possibly contrary to fact, to values pa_i . The existence of a total ordering \prec on indices and the fact that $\mathrm{pa}_i\subseteq\mathrm{pre}_i$ precludes the existence of cyclic causation. That is, we consider causal models that are recursive. $V_i(x_{\mathrm{pa}_i})$ may be conceptualized as the output of a structural equation $f_i:(x_{\mathrm{pa}_i},\epsilon_i)\mapsto x_i$, a function representing a causal mechanism that maps values of x_{pa_i} , as well as the value of a variable ϵ_i , to values of V_i . Specifically, we may define the error term ϵ_i to be the vector comprised of the set of random variables $\{V_i(x_{\mathrm{pa}_i})\mid x_{\mathrm{pa}_i}\in\mathfrak{X}_{\mathrm{pa}_i}\}$ and f_i to be such that $f_i(x_{\mathrm{pa}_i},\epsilon_i)\equiv(\epsilon_i)_{x_{\mathrm{pa}_i}}=V_i(x_{\mathrm{pa}_i})$.

^{7.} The approach taken here is inspired by, but distinct from, earlier approaches to combining graphs and counterfactuals such as Pearl's Twin-Network approach (Balke and Pearl, 1994), (Shpitser and Pearl, 2008), (Pearl, 2009, Section 7.1.4).

However, the d-separation criterion on twin-networks is not complete, as there are deterministic relationships that are present – but not represented graphically – among the variables in a twin-network. Consequently, it is possible for there to be a d-connecting path and yet the corresponding conditional independence holds for all distributions in the model; see the Appendix for a simple example. In contrast, d-separation is complete for a SWIG. However, it should be noted that twin-networks are addressing a harder problem than SWIGs since their goal is to determine all independencies implied by an NPSEM-IE model including "cross-world" independencies.

Finally, we note that twin-network graphs have not typically used (minimal) labelings, which turn out to be important in some applications; see Section 2.3.

If some variables do not affect variables later in time then many non-temporal orders may be used; see (Robins, 1986, Chapter 11) and later.

^{9.} pa here is short for "parent," which will be motivated later when we subsequently build a connection to directed graphs.

We define non-parametric structural equation causal models (NPSEMs) as sets of densities over the set of random variables

$$\mathbb{V} \equiv \{ V_i(x_{pa_i}) \mid i \in V, \ x_{pa_i} \in \mathfrak{X}_{pa_i} \}.$$

Note that \mathbb{V} includes variables V_i which have no parents, and which are thus factual. For simplicity of presentation, we assume \mathfrak{X}_i is always finite, and thus ignore the measure-theoretic complications that arise with defining densities over sets of random variables in the case where some state spaces \mathfrak{X}_{pa_i} are infinite.

Given a set of one-step-ahead potential outcomes \mathbb{V} , for any $A \subseteq V$ and $i \in V$, the potential outcome $V_i(a)$, the response of V_i had variables in V_A been set to $a \in \mathfrak{X}_A$, is the one step ahead counterfactual $V_i(\mathrm{pa}_i) \in \mathbb{V}$ if $V_A = V_{\mathrm{pa}_i}$, and is otherwise defined via recursive substitution:

$$V_i(a) \equiv V_i(a_{\mathrm{pa}_i}, V_{\mathrm{pa}_i \setminus A}(a)). \tag{1}$$

In words, this states that $V_i(a)$ is the potential outcome where variables in both pa_i and A are set to their corresponding values in a, and all elements of pa_i not in A are set to whatever values their recursively defined counterfactual versions would have had had V_A been set to a. This is well-defined because of of the requirement that $pa_i \subseteq pre_i$.

Equivalently, $V_i(a)$ is the random variable induced by a modified set of structural equations: specifically the set of functions f_j for V_j such that $A \cap \mathrm{pa}_j \neq \emptyset$ are replaced by modified functions $f_j^a: (x_{\mathrm{pa}_j} \setminus A, \varepsilon_j) \mapsto x_j$ that are obtained from $f_j: (x_{\mathrm{pa}_j}, \varepsilon_j) \mapsto x_j$ by always evaluating $\mathrm{pa}_j \cap A$ at the corresponding values in a.

We will extend our notational shorthand by using index sets to denote sets of potential outcomes themselves. Thus, for $B \subset V$, we let B(a) denote the set of potential outcomes $V_B(a)$. We denote by \mathbb{V}^* the set of all variables derived by (1) from \mathbb{V} for all possible choices of the set A (together with the set \mathbb{V} itself).

While the potential outcome and the structural equation formalisms both yield the same causal model, there are some differences regarding the way in which the frameworks are typically presented. Specifically, regarding which "objects" are taken as primitive and which are derived.

The counterfactual formalism here starts with one-step-ahead counterfactuals that intervene on every parent (direct cause) of every variable, and constructs all other counterfactuals by means of recursive substitution. Recursive substitution implies, in particular, that $A(a) \equiv A$. This accords with the substantive claim that it is possible to *first* learn the "natural" value a variable A would take on, and *then* an instant *later* intervene setting it to a specific value a, resulting in all subsequent variables V_i behaving as counterfactual variables $V_i(a)$.

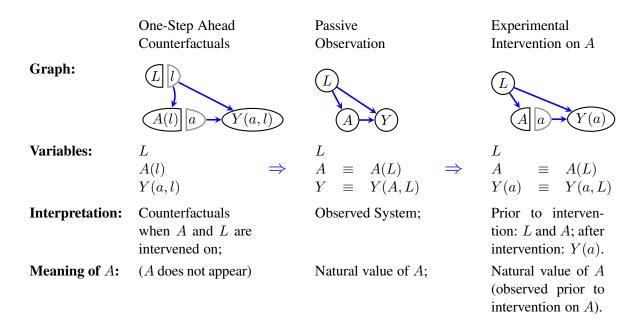
On the other hand, the structural equation formalism typically starts with a set of unaltered structural equations which yields the observed data distribution (via substitution). Counterfactual distributions representing an intervention that sets elements in A to a are generated by replacing structural equations corresponding to elements in A by degenerate functions that yield constants in a (Strotz and Wold, 1960; Pearl, 2009). The resulting modified equation system thus represents the set of variables (including A) after the action of setting A to the value a. Consequently, there are two subtle, but important notational (not conceptual) distinctions:

• Under the standard presentation of structural equation models, used by Pearl, the meaning of a variable such as A, Y or L, is dependent on the set of equations in which it appears. For example in Table 2, Y in the left (unmodified) display corresponds to the natural value; in the middle display Y corresponds to the value after intervening on a or Y(a) in counterfactual notation; in the right display Y denotes the value after intervening on A and L, or Y(a, l). In contrast, in the potential outcome framework, variables that are affected by an intervention take on a new name.

^{10.} The set V* corresponds to Robins' *Finest Causally Interpreted Structured Tree Graph as Detailed as the Data*. See Appendix C in (Richardson and Robins, 2013).

FFRCISTG/SWIG Potential Outcome Model Robins (1986); Richardson and Robins (2013)

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Single World No Confounding Assumption: for each pair a, l: $L \perp \!\!\!\perp A(l) \perp \!\!\!\perp Y(a, l)$

Table 1: Graphical causal models based on the SWIG/FFRCISTG counterfactual framework. Note that the meaning of a variable, such as A or Y does not depend on the graph in which it appears; compare to Table 2. The SWIG no confounding assumption is less restrictive than the assumption of independent errors.

Non-Parametric Structural Equation Model with Independent Errors (NPSEM-IE) Strotz and Wold (1960); Pearl (2009)

Passive Experimental Experimental Observation Intervention on A Intervention on A and L Graph: $\begin{array}{rcl}
L & = & f_L(\varepsilon_L) \\
A & = & f_A(L, \varepsilon_A)
\end{array} \implies$ Variables: $Y = f_Y(A, L, \varepsilon_Y)$ $= f_Y(A, L, \varepsilon_Y)$ **Interpretation:** Observed system; Variables in system L and A after each is intervened on; Y in which A is set to after both intervena; tions. **Meaning of** A: Natural value of A; Value of A after in-Value of A after intervention on A; tervention on A (and

Independent Errors No Confounding Assumption: $\varepsilon_L \perp \!\!\! \perp \varepsilon_A \perp \!\!\! \perp \varepsilon_Y$

Relationships to Counterfactuals:

7

Error terms: $\varepsilon_L = L;$ $\varepsilon_A = \{A(l) \text{ for all } l\};$ $\varepsilon_Y = \{Y(a,l) \text{ for all } l,a\}$

L).

Structural equations: $L = f_L(\varepsilon_L)$ $A(l) = f_A(l, \varepsilon_A)$ $Y(a, l) = f_Y(a, l, \varepsilon_Y)$

Table 2: Structural equation models and their relationship to counterfactuals. Note that the meaning of a variable such as A or Y is context specific, it depends on the graph in which it appears. This NPSEM is a strict sub-model of the SWIG given in Table 1. This is (solely) because the SWIG no confounding assumption is less restrictive than the assumption of independent errors.

• Second, in the standard presentation, the variable "A" in the modified set of equations represents the variable after it has been intervened on. Thus for Pearl, do(A = a) implies that A = a, a property he terms "effectiveness."

In this paper we will follow the notation conventions that are used in the potential outcome framework, but we stress that formally, non-parametric structural equations (NPSEMs) and one-step ahead counterfactuals are equivalent conceptually.¹² See the *Variables* rows in Tables 1 and 2 to see the correspondence between sets of one-step ahead counterfactuals and systems of structural equations; see (Pearl, 1995; Imbens, 2014) for further discussion of the representation of structural equations via potential outcomes.

Given a set $A\subseteq V$, the distribution on $V\setminus A$ resulting from setting A to a by interventions has been denoted in (Robins, 1986) by $p(V\setminus A\,|\,g=a)$, and subsequently as $p(V\setminus A\,|\,\operatorname{do}(a))$ (Pearl, 2009). The potential outcome view also allows us to consider distributions p(V(a)) for any $A\subseteq V$. In such a distribution, variables in A occur both as random and intervened on versions. We later consider identification theory for distributions of this sort, where the set of treatment variables and outcome variables may intersect.

Recursive substitution in NPSEMs provides a link between observed variables and potential outcomes. In particular, it implies the *consistency property*: for any disjoint $A, B \subseteq V$, $i \in V \setminus (A \cup B)$, $a \in \mathfrak{X}_A$, $b \in \mathfrak{X}_B$,

$$V_B(a) = b \text{ implies } V_i(a, b) = V_i(a).$$
(2)

See (Robins, 1986, 1987) and, for a proof using notation similar to this paper, (Richardson and Robins, 2013; Malinsky et al., 2019). Consistency is often phrased in a simpler form where $A = \emptyset$, yielding the identity $V_i(b) = V_i$ if $V_B = b$.

Equation (1) also implies the *causal irrelevance property*, namely that every $V_i(a)$ can be written as a function of a unique minimally causally relevant subset of a, as follows. (See Robins (1986); Richardson and Robins (2013) and, for a formulation similar to that used here, Malinsky et al. (2019).) Given \mathbb{V}^* derived from \mathbb{V} via (1), let $V_i(a) \in \mathbb{V}^*$, and let A^* be the maximal subset of A such that for every $j \in A^*$, there exists a sequence w_1, \ldots, w_m that does not intersect A, where $j \in \mathrm{pa}_{w_i+1}$, for $i=1,\ldots m-1$, and $w_m \in \mathrm{pa}_i$. Then, $V_i(a) = V_i(a^*)$.

As an example, given the indices $\{1,2,3\}$, under the ordering $1 \prec 2 \prec 3$, if $pa_2 = \{1\}$, and $pa_3 = \{2\}$, we have one step ahead potential outcomes $V_1, V_2(v_1), V_3(v_2)$, for any values v_1, v_2 . We can define other counterfactuals via (1), for example $V_3(v_1) \equiv V_3(V_2(v_1))$. Consistency implies statements of the form $V_1 = v_1 \Rightarrow V_2 = V_2(v_1)$, while causal irrelevance implies $V_3(v_2, v_1) = V_3(v_2)$.

Both consistency and causal irrelevance hold in any NPSEM, in the sense that these properties are implied by the existence of a total order on variables we wish to consider, the existence of one step ahead counterfactuals, and (1). While useful, these properties on their own fail to capture many of the hypotheses that arise in causal inference problems (either by design or assumption). These additional constraints correspond to conditional independence restrictions concerning the error terms in non-parametric structural equations. Although causal models are well-defined without reference to graphs, much conceptual clarity may be gained by viewing them graphically. Thus, before describing causal models in detail, we introduce graphs and graphical models.

^{11.} If we were to use A(a) to designate the value taken by a variable A after an intervention on A, then we could express this as A(a) = a. However, as noted, we use A(a) to designate the value taken by a variable A immediately prior to the intervention.

^{12.} However, as described below, structural equation models are often used under an additional (strong) assumption of independent errors. Since this is a stronger assumption than typically used in the potential outcome framework, we use the acronyms NPSEM and NPSEM-IE to distinguish whether this additional assumption is being made.

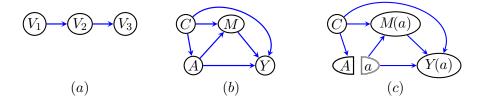


Figure 1: (a) A DAG representing a simple NPSEM. (b) A simple causal DAG \mathcal{G} , with a treatment A, an outcome Y, a vector C of baseline variables, and a mediator M. (c) A SWIG $\mathcal{G}(a)$ derived from (a) corresponding to the world where A is intervened on to value a.

2.1 Graphical Models

Statistical and causal models can be associated with graphs, where vertices represent variables, and edges represent (potential) statistical or causal relationships. Formally, random variables are indexed by vertices. However, when we depict graphs we will display them with the random variables as vertices.

We will consider graphs with either directed edges only (\rightarrow) , or mixed graphs with both directed and bidirected (\leftrightarrow) edges. Bidirected edges naturally arise as a way to represent (classes of) DAGs with latent variables; see Section 4.1 below. In all cases we will require the absence of directed cycles, meaning that whenever the graph contains a path of the form $V_i \rightarrow \cdots \rightarrow V_j$, the edge $V_j \rightarrow V_i$ cannot exist. Directed graphs with this property are called directed acyclic graphs (DAGs), and mixed graphs with this property are called Acyclic Directed Mixed Graphs (ADMGs). We will refer to graphs by $\mathcal{G}(V)$, where V is the set of random variables indexed by $\{1,\ldots,K\}$. We will write \mathcal{G} in place of $\mathcal{G}(V)$ when the vertex set is clear. We will use the following standard definitions for sets of vertices in a graph:

$$\begin{aligned} \operatorname{pa}_{i}^{\mathcal{G}} &\equiv \{j \mid V_{j} \to V_{i} \text{ in } \mathcal{G}\} \\ \operatorname{an}_{i}^{\mathcal{G}} &\equiv \{j \mid V_{j} \to \cdots \to V_{i} \text{ in } \mathcal{G}, \text{ or } V_{j} = V_{i}\} \\ \operatorname{de}_{i}^{\mathcal{G}} &\equiv \{j \mid V_{j} \leftarrow \cdots \leftarrow V_{i} \text{ in } \mathcal{G}, \text{ or } V_{j} = V_{i}\} \\ \operatorname{diss}_{i}^{\mathcal{G}} &\equiv \{j \mid V_{j} \leftrightarrow \cdots \leftrightarrow V_{i} \text{ in } \mathcal{G}, \text{ or } V_{j} = V_{i}\} \\ \operatorname{mb}_{i}^{\mathcal{G}} &\equiv \{j \mid V_{j} \leftrightarrow \cdots \leftrightarrow V_{i} \text{ in } \mathcal{G}\} \cup \\ \{j \mid V_{j} \to \circ \leftrightarrow \cdots \leftrightarrow V_{i} \text{ in } \mathcal{G}\} \end{aligned}$$

$$(\text{the Markov blanket of } V_{i}).$$

$$(3)$$

We will generally drop the superscript $\mathcal G$ if the relevant graph is obvious. By definition, $\operatorname{an}_i^{\mathcal G} \cap \operatorname{de}_i^{\mathcal G} \cap \operatorname{dis}_i^{\mathcal G} = \{V_i\}$. We define these relations on sets disjunctively. For example $\operatorname{an}_A^{\mathcal G} \equiv \bigcup_{V_i \in A} \operatorname{an}_i^{\mathcal G}$.

Given a DAG $\mathcal{G}(V)$, a statistical DAG model (also called a Bayesian network) associated with $\mathcal{G}(V)$ is a set of distributions that factorize (equivalently are Markov) with respect to $\mathcal{G}(V)$:

$$p(V) = \prod_{i=1}^{K} p(V_i | V_{pa_i^{\mathcal{G}}}). \tag{4}$$

^{13.} Other authors often define the Markov blanket for a variable to be the minimal set M that makes V_i m-separated from $V \setminus (\{V_i\} \cup M)$. Our definition corresponds to the minimal set M such that V_i is m-separated from its non-descendants.

Given a distribution p(V) that factorizes relative to a DAG $\mathcal{G}(V)$, conditional independence relations that are implied in p(V) by (4) can be derived using the well-known d-separation criterion Pearl (1988). More precisely, if p(V) is Markov relative to $\mathcal{G}(V)$, then the following global Markov property holds: for any disjoint X, Y, Z (where Z may be empty)

$$(X \perp \!\!\!\perp_d Y \mid Z)_{\mathcal{G}(V)} \Rightarrow (X \perp \!\!\!\perp Y \mid Z)_{p(V)}.$$

Here $(X \perp \!\!\! \perp_d Y \mid Z)_{\mathcal{G}(V)}$ denotes that X is d-separated from Y given Z in $\mathcal{G}(V)$; $(X \perp \!\!\! \perp Y \mid Z)_{p(V)}$ indicates that X is independent of Y given Z in p(V).

The global Markov property given by d-separation allows reasoning about conditional independence restrictions implied by the statistical DAG model using qualitative, visual reasoning on paths in the graph.

2.2 Causal Models Associated With DAGs

NPSEMs may be associated with directed graphs as well, by associating vertices with indices, and edges with relations given by $\operatorname{pa}_i, i \in \{1,\ldots,k\}$. Specifically, given a (recursive) non-parametric structural equation model defined on \mathbb{V} given the sets $\{\operatorname{pa}_i | i \in \{1,\ldots,k\}\}$, we construct a *causal diagram*, a directed acyclic graph (DAG) $\mathcal{G}(V)$ with a vertex for every $V_i, i \in \{1,\ldots,k\}$, and a directed edge from V_j to V_i if $j \in \operatorname{pa}_i$. In other words, $\mathcal{G}(V)$ is defined by the NPSEM by letting $\operatorname{pa}_i^{\mathcal{G}} \equiv \operatorname{pa}_i$ for every i. As an example, the NPSEM defined on the indices $\{1,2,3\}$ described in the previous section corresponds to the DAG in Figure 1 (a). See the *Graph* rows of Tables 1 and 2 for graphs corresponding to one-step ahead counterfactuals and structural equations.

Substantive knowledge may motivate additional independence assumptions relating to the set of one-step ahead counterfactuals \mathbb{V} . As we will show below, such assumptions may also allow causal effects to be identified even when hidden variables are present. Below we introduce two sets of such assumptions

NON-PARAMETRIC STRUCTURAL EQUATIONS WITH INDEPENDENT ERRORS

A non-parametric structural equation model with independent errors, or NPSEM-IE, is the set of distributions such that the K different sets of one-step ahead variables satisfy:

$$\{V_1\} \perp \!\!\!\perp \{V_2(x_{\mathrm{pa}_2}) \mid x_{\mathrm{pa}_2} \in \mathfrak{X}_{\mathrm{pa}_2}\} \perp \!\!\!\perp \cdots \perp \!\!\!\perp \{V_K(x_{\mathrm{pa}_K}) \mid x_{\mathrm{pa}_K} \in \mathfrak{X}_{\mathrm{pa}_K}\}$$

$$\tag{5}$$

so that they are mutually independent of one another. Phrased in terms of structural equations $f_i:(x_{\mathrm{pa}_j},\varepsilon_i)\mapsto x_i$ for each V_i , the NPSEM-IE states that the joint distribution of the disturbance terms factorizes into a product of marginals: $p(\epsilon_1,\ldots,\epsilon_K)=\prod_{i=1}^K p(\epsilon_i)$.

NPSEMs with independent errors arise naturally as putative data-generating processes for a closed system. For example, if we are simulating every variable in a model then, it is natural to do this in a step-wise process by specifying a set of structural equations. The equations provide recipes for generating a value for each variable in turn, given the previous values that have already been simulated plus an independently simulated error term. ¹⁴ See Table 2 for an example.

However, from an empiricist point of view the assumption of independent errors may be regarded as stronger than necessary. In particular, this assumption permits the identification of causal contrasts that are not subject to experimental verification even in principle; see the discussion in the companion paper (Robins et al., 2021, Section 1). At the same time, many causal contrasts of interest, including all intervention distributions, may be identified under a much smaller set of assumptions.

^{14.} Note that an NPSEM-IE may also contain unobserved variables, so that they include models described by Pearl as semi-Markovian (Pearl, 2009).

^{15.} Specifically, even if it were possible to carry out a randomized experiment manipulating any subset of the variables in the system, we could not directly observe certain counterfactual contrasts that are identified via an NPSEM-IE.

A LESS RESTRICTIVE MODEL: NON-PARAMETRIC STRUCTURAL EQUATIONS WITH SINGLE WORLD (FFRCISTG) INDEPENDENCES

The above observations motivate an alternative approach based on the *finest fully randomized* causally interpretable structured tree graph (as detailed as the data), or, FFRCISTG model of Robins Robins (1986).

The FFRCISTG model is ontologically liberal, but epistemologically conservative. Specifically, all the counterfactual queries that may be formulated within the scope of a non-parametric structural equation model, are still well-defined under this alternative, but, in contrast to the NPSEM-IE, only those contrasts that could in principle be experimentally verified by experiments on the variables in the system are identified.

A non-parametric structural equation model with FFRCISTG independences is the set of counterfactual distributions satisfying

For each
$$x_V \in \mathfrak{X}_V$$
, we have $V_1 \perp \!\!\! \perp V_2(x_{\mathrm{pa}_2}) \perp \!\!\! \perp \cdots \perp \!\!\! \perp V_K(x_{\mathrm{pa}_K});$ (6)

see (Robins and Richardson, 2010). Thus for each $x_V \in \mathfrak{X}_V$ there is a set of K random variables (the K one-step ahead counterfactuals associated with x_V) and the variables within each such set are assumed to be mutually independent. As V_1 is first in the ordering, it has no parents.

The FFRCISTG assumptions could be empirically verified in a set of randomized experiments, one for each x_V , under which we intervene on every variable in turn, setting V_i to the value x_i , but just before doing so, we are able to observe the random variable $V_i(x_{\mathrm{pa}_i})$, resulting from our earlier interventions. (Here it is assumed that because we intervene to set V_i to x_i an instant after it is measured, the value $V_i(x_{\mathrm{pa}_i})$ does not causally influence any subsequent variable.) Note the counterfactual random variables in (6) all refer to a specific set of values x_V , which thus correspond to a single counterfactual "world".

Note that (5) imposes all restrictions in (6), and in general exponentially many more. ¹⁶ Thus the FFRCISTG is less restrictive than the NPSEM-IE model, in other words, the NPSEM-IE is a strict submodel of the FFRCISTG.

As an example, the NPSEM associated with Figure 1 (b) is defined using one step ahead counterfactuals C, A(c), M(c,a), and Y(c,a,m), for every value set c,a,m. Then the FFRCISTG model restrictions for this NPSEM imply that

For each set of values
$$c, a, m, C \perp \perp A(c) \perp \perp M(a, c) \perp \perp Y(c, a, m),$$
 (7)

while the NPSEM-IE restrictions for the NPSEM state that

For each set of values
$$c, c', c'', a, a', C \perp \perp A(c) \perp \perp M(a, c') \perp \perp Y(c'', a', m)$$
. (8)

The restrictions in Equation (7) are a strict subset of the restrictions in Equation (8) which are themselves a subset of the restrictions defining the NPSEM-IE.

Interventional distributions of the form p(V(a)), for $A \subseteq V$ in both of the above models may be represented in graphical form by a simple splitting operation on DAGs. The graphs resulting from this operation will be called Single World Intervention Graphs (SWIGs) (Richardson and Robins, 2013) for reasons that will be described below.

2.3 Single World Intervention Graphs (SWIGs)

SWIGs were introduced in (Richardson and Robins, 2013) as graphical representations of potential outcome distributions that help unify the graphical and potential outcome formalisms. Given a

^{16.} In fact, in the case where all variables are binary, the fraction of experimentally untestable constraints implied by the NPSEM-IE rises at a doubly exponential rate in the number of variables. See (Richardson and Robins, 2013, Section 5) and Footnote 4.

set A of variables and an assignment a to those variables, a SWIG $\mathcal{G}(V(a))$ may be constructed from $\mathcal{G}(V)$ by splitting all vertices in A into a random half and a fixed half, with the random half inheriting all edges with an incoming arrowhead and the fixed half inheriting all outgoing directed edges. Then, all random vertices V_i are re-labelled as $V_i(a)$ or equivalently (due to causal irrelevance) as $V_i(a_{\mathrm{an}_i^*})$, where an_i^* consists of the fixed vertices that are ancestors of V_i in the split graph; the latter labelling is referred to as the *minimal labelling* of the SWIG. By using minimal labelling the SWIG encodes the property of causal irrelevance, so that, for example, if Y(x) appears in $\mathcal{G}(x,z)$ then Y(x,z)=Y(x).

Fixed nodes are enclosed by a double line. For an example of a SWIG representing the joint density p(Y(a), M(a), C(a), A(a)) = p(Y(a), M(a), C, A), under the FFRCISTG model (and thus under an NPSEM-IE) associated with the DAG of Figure 1 (b) see Figure 1 (c). If the vertex set V is assumed or obvious, we will denote $\mathcal{G}(V(a))$ by $\mathcal{G}(a)$, just as $\mathcal{G}(V)$ is denoted by \mathcal{G} .

Thus a SWIG $\mathcal{G}(V(a))$ is a DAG with vertex set $V(a) \cup a$; the vertices in V(a) correspond to random variables, while vertices in a are fixed, taking a specific value.

In a SWIG, every treatment variable has two versions: a fixed version representing the intervention on that treatment, and a random version (which corresponds to measuring the treatment variable just before the intervention took place). This feature of SWIGs allows them to directly express, using d-separation, independence restrictions linking observed versions of treatments, and counterfactual variables representing responses after treatments have been set.

Restrictions of this type, which generalize the well known conditional ignorability restriction ¹⁷ will be used later to reformulate the second rule of the do-calculus, using the language of SWIGs and potential outcomes.

Pearl's "mutilated graphs," which are an alternative graphical representation of interventional distributions, only contain the fixed versions of treatments. This makes it difficult to express restrictions such as conditional ignorability. Instead, the do-calculus uses a variant of the mutilated graph where certain outgoing edges are also removed. An additional difficulty with this variant, though it is formally correct, is that vertices on it are not labelled as counterfactual random variables.

Tables 2 and 1 illustrate, via simple examples, how SWIGs and mutilated graphs differ.

The edges among random variables on the SWIG encode the factorization of the joint distribution p(V(a)). More precisely, the FFRCISTG model (and thus the NPSEM-IE) imply that for any $A \subseteq V$, and $a \in \mathfrak{X}_A$, the distribution p(V(a)) factorizes with respect to $\mathcal{G}(V(a))$. In other words,

$$p(V(a)) = \prod_{i=1}^{K} p(V_i(a) \mid V_{\mathrm{pa}_i \setminus A}(a)).$$
(9)

Fixed nodes do not occur in the conditioning sets for the terms in (9) and thus the presence or absence of edges $(a_i \to V_i(a_j))$ from fixed nodes to random nodes in $\mathcal{G}(V(a))$ are not reflected in this expression (9). However, the fact that a random node is not a descendant of a fixed node does encode information about causal irrelevance. Specifically, if there is no directed path from the fixed node a_j to $V_i(a)$ then $V_i(a) = V_i(a_{A\setminus\{j\}})$, hence under minimal labeling a_j will not appear in the label for the vertex $V_i(a_{\operatorname{an}_i^*})$ in $\mathcal{G}(V(a))$. Thus, as noted earlier, by causal irrelevance, $V_i(a) = V_i(a_{\operatorname{an}_i^*})$, where an_i^* consists of the fixed vertices that are ancestors of $V_i(a)$ in $\mathcal{G}(V(a))$.

^{17.} Specifically, $Y(a) \perp \!\!\! \perp A \mid C$, for some set of baseline covariates C.

^{18.} In the Appendix we briefly consider using the SWIG to make inferences about weaker causal models, including the agnostic causal model, and models in which the absence of a directed edge corresponds to the absence of a population-level direct effect. In the latter models, the equality $V_i(a) = V_i(a \cap \operatorname{an}_i^*)$ would no longer hold, and minimal labelings are constructed using a (possibly strict) edge super-graph of the graph used for the factorization (9); see (Richardson and Robins, 2013, Section 7).

Thus (9) may be expressed as:

$$p(V(a)) = \prod_{i=1}^{K} p\left(V_i(a_{\mathrm{an}_i^*}) \mid \left\{ V_j(a_{\mathrm{an}_j^*}), \text{ for } j \in \mathrm{pa}_i \backslash A \right\} \right).$$

More generally, paths commencing with a fixed node but on which every other node is random also encode information about functional dependence. A path π in a SWIG $\mathcal{G}(a)$ is said to be *Markov relevant* if at most one endpoint is a fixed vertex, and every non-endpoint is random. A Markov relevant path π in $\mathcal{G}(a)$ is *d-connecting given* $V_Z(a)$ if every collider on π is an ancestor of a vertex in $V_Z(a)$ and every non-collider on π is not in $V_Z(a)$.

It follows directly from (9) that if $V_X(a)$ is d-separated from $V_Y(a)$ given $V_Z(a)$ in $\mathcal{G}(a)$ then $V_X(a) \perp V_Y(a) \mid V_Z(a)$ in p(V(a)), so that d-separation relations among random variables encode conditional independence. In addition, the absence of any d-connecting path in $\mathcal{G}(V(a))$ between a fixed node a_j and a set of random vertices $V_Y(a)$, given a (possibly empty) set of random variables $V_Z(a)$, encodes that $p(V_Y(a) \mid V_Z(a))$ does not depend on the value of a_j . Thus we allow d-separation statements of the form $(a_j \perp \!\!\!\perp_d V_Y(a) \mid V_Z(a))_{\mathcal{G}(V(a))}$. More generally, given three disjoint subsets $Y, X, Z \subseteq V$, where Z may be empty, and a set $A' \subseteq A$, then

$$(V_Y(a) \perp \!\!\! \perp_d V_X(a), a_{A'} \mid V_Z(a))_{\mathcal{G}(V(a))}$$
 (10)

if in the SWIG $\mathcal{G}(V(a))$ there is no path d-connecting a random vertex $V_i(a)$ with $i \in X$ or a fixed vertex a_j with $j \in A'$ to a random vertex in $V_j(a)$ with $j \in Y$ given $V_Z(a)$. Note that, by definition, fixed vertices may only arise on one side of a d-separation statement (10). Conversely, a possibly d-connecting path may only contain at most one fixed node in which case it is an endpoint vertex (thus, as in Richardson and Robins (2013), fixed nodes never arise as non-endpoint vertices on d-connecting paths).

Results for DAG models with fixed nodes Richardson et al. (2017) imply the following:

Proposition 1 (SWIG global Markov property)

Under the FFRCISTG for \mathcal{G} , for every set A, disjoint sets of random vertices $V_X(a)$, $V_Y(a)$, $V_Z(a)$ and a set of fixed nodes $a_{A'}$, where $A' \subseteq A$,

$$if (V_{Y}(a) \perp_{d} V_{X}(a), a_{A'} \mid V_{Z}(a))_{\mathcal{G}(V(a))} \text{ then, for some } f(\cdot),$$

$$p(V_{Y}(a) = v_{Y} \mid V_{Z}(a) = v_{Z}, V_{X}(a) = v_{X}) = p(V_{Y}(a) = v_{Y} \mid V_{Z}(a) = v_{Z})$$

$$= f(v_{Y}, v_{Z}, a_{A \setminus A'}).$$
(11)

Example 1 Consider the global Markov property associated with the SWIG G(a) in Figure 2(b), corresponding to the FFRCISTG model shown in Figure 2 (a). Since a is d-separated from Y(a) given M(a) in G(a),

$$p(Y(a) = y \mid M(a) = m) = f(y, m).$$
(12)

Hence this distribution is not a function of a, even though M(a) and Y(a) are minimally labeled, so $M(a) \neq M(a')$ and $Y(a) \neq Y(a')$ for $a \neq a'$. In addition, it is well known that in the FFRCISTG model corresponding to Figure 2 (a),

$$p(Y(a) = y \mid M(a) = m) = \sum_{a'} p(y \mid m, a')p(a'),$$

which is not equal to $p(y \mid m)$ under the given model. Hence, the function $f(v_Y, v_Z, a_{A \setminus A'})$ is not necessarily equal to the conditional distribution $p(V_Y(a_{A \setminus A'}) \mid V_Z(a_{A \setminus A'}))$.

^{19.} This represents an extension of the notion of d-separation in Richardson and Robins (2013). Our extension here consists only in allowing fixed vertices to appear in at most one side of a d-separation statement (not the conditioning set). The semantics for these extended d-separation statements are given in (11) below.

Remark 1 Since, by construction, all edges in $\mathcal{G}(V(a))$ are directed out of a_j , in the case where Z is the empty set, there is a d-connecting path between a_i and $V_i(a)$ if and only if a_i is an ancestor of $V_i(a)$ in $\mathcal{G}(V(a))$; as noted above, this is automatically reflected with the minimal labeling of the vertices.

In (12) we see an example where p(Y(a)|M(a)) does not depend on a, even though Y(a)and M(a) are minimally labeled. One might wonder whether it is possible to have the converse situation whereby a conditional distribution does depend on a fixed vertex that is not present in any minimal label. The next Proposition shows that this cannot arise:

Proposition 2 In a minimally labeled SWIG $\mathcal{G}(a)$, if a fixed vertex a_i is d-connected to $V_j(a_{\mathrm{an}_i^*})$ given $\{V_{k_1}(a_{\operatorname{an}_{k_1}^*}),\ldots,V_{k_p}(a_{\operatorname{an}_{k_p}^*})\}$ then either $i\in\operatorname{an}_j^*$ or $i\in\operatorname{an}_{k_s}^*$ for some s.

In words, if a fixed vertex is d-connected by a path to a random vertex given some conditioning set, then the fixed vertex either appears in the minimal label for the other endpoint, or a vertex in the conditioning set. This follows since if there is a d-connecting path on which a_i is an endpoint then, since a_i only has children in $\mathcal{G}(a)$, the path is directed out of a_i . The conclusion then follows since if the path contains no colliders then $V_i(a_{\text{an}_i})$ is a descendant of a_i ; if the path contains a collider then a_i is an ancestor of that collider, which, by definition of d-connection is itself an ancestor of a vertex in $V_{k_s}(a_{\operatorname{an}_{k_s}^*})$.

2.4 Factorization associated with the SWIG global Markov property

As noted above, the factorization (9) corresponds solely to the induced subgraph of $\mathcal{G}(a)$ on the random vertices. We now derive the factorization corresponding to the SWIG global Markov property. Consider a single term in (9):

$$p(V_{i}(a) = v_{i} \mid V_{\text{pa}_{i} \setminus A}(a) = v_{\text{pa}_{i} \setminus A})$$

$$= p(V_{i}(a, v_{\text{pa}_{i} \setminus A}) = v_{i} \mid V_{\text{pa}_{i} \setminus A}(a, v_{\text{pa}_{i} \setminus A}) = v_{\text{pa}_{i} \setminus A})$$

$$= p(V_{i}(a, v_{\text{pa}_{i} \setminus A}) = v_{i})$$

$$= p(V_{i}(a_{\text{pa}_{i} \cap A}, v_{\text{pa}_{i} \setminus A}) = v_{i})$$

$$= p(V_{i}(a_{\text{pa}_{i} \cap A}, v_{\text{pa}_{i} \setminus A}) = v_{i} \mid V_{\text{pa}_{i} \cap A}(a_{\text{pa}_{i} \cap A}, v_{\text{pa}_{i} \setminus A}) = a_{\text{pa}_{i} \cap A},$$

$$V_{\text{pa}_{i} \setminus A}(a_{\text{pa}_{i} \cap A}, v_{\text{pa}_{i} \setminus A}) = v_{\text{pa}_{i} \setminus A})$$

$$= p(V_{i} = v_{i} \mid V_{\text{pa}_{i} \cap A} = a_{\text{pa}_{i} \cap A}, V_{\text{pa}_{i} \setminus A} = v_{\text{pa}_{i} \setminus A}).$$

$$(13)$$

Here the first equality follows from consistency; the second follows from (9) for $\mathcal{G}(a, v_{\text{pa}_i \setminus A})$. The third equality follows from causal irrelevance since if we intervene on all the parents of V_i then no other variables have a causal effect on V_i . The fourth line follows from (9) for $\mathcal{G}(a_{pa_i} \cap A, v_{pa_i} \setminus A)$. The fifth line again follows from consistency. Thus we have:

Proposition 3 Under the FFRCISTG models associated with a graph \mathcal{G} we have the following identification formula:

$$p(V(a) = v) = \prod_{i=1}^{K} p(V_i(a) = v_i \mid V_{\text{pa}_i \setminus A}(a) = v_{\text{pa}_i \setminus A})$$

$$= \prod_{i=1}^{K} p(v_i \mid a_{\text{pa}_i \cap A}, v_{\text{pa}_i \setminus A}).$$
(15)

$$= \prod_{i=1}^{K} p(v_i \mid a_{\operatorname{pa}_i \cap A}, v_{\operatorname{pa}_i \setminus A}). \tag{16}$$

Thus p(V(a)) is identified if all of the conditional distributions in Equation (16) are identified.²⁰

^{20.} This may not hold in the absence of positivity; see (Robins et al., 2021, Section 3) for further discussion.

Now consider a DAG $\mathcal{G}^*(V \cup A^*)$ containing disjoint sets of vertices V and A^* , with the same set of edges as in $\mathcal{G}(a)$ under the natural correspondence: $V_i \Leftrightarrow V_i(a)$ and $A_i^* \Leftrightarrow a_i$. Then (16) corresponds syntactically to the (subset of) terms in the DAG factorization for \mathcal{G}^* associated with the variables in V. This then establishes the SWIG global Markov property via results on conditional graphs (Richardson et al., 2017).²¹

The modified factorization (16) is known as the *extended g-formula* (Richardson and Robins, 2013; Robins et al., 2004). Like the original factorization (4), Equation (16) has a term for every $V_i \in V$ not merely for every $V_i \in V \setminus A$.²² An alternative proof of the extended g-formula is given in (Richardson and Robins, 2013).

The following Proposition follows directly from Equation (16) and is included here because a generalization of this result, Proposition 5 below, plays an important role in the identification of causal effects in DAGs with hidden variables.

Proposition 4 If \mathcal{G} is a DAG with SWIG $\mathcal{G}(a)$ then for all $c_k \in \mathfrak{X}_k$

$$\begin{split} p(V(a, c_k) &= v) \\ &= p(V_{-k}(a) = v_{-k}, V_k(a) = c_k) \times \frac{p\Big(V_k(a) = v_k \ \Big| \ V_{\mathrm{pa}_k^{\mathcal{G}(a)}}(a) = v_{\mathrm{pa}_k^{\mathcal{G}(a)}}\Big)}{p\Big(V_k(a) = c_k \ \Big| \ V_{\mathrm{pa}_k^{\mathcal{G}(a)}}(a) = v_{\mathrm{pa}_k^{\mathcal{G}(a)}}\Big)}, \end{split}$$

where $V_{-k} \equiv \{V_1, \dots, V_{k-1}, V_{k+1}, \dots, V_K\}$, provided the conditional probability in the denominator is positive.

2.5 SWIG representation of the defining FFRCISTG assumptions

Consider the special case in which A = V; in the resulting graph $\mathcal{G}(V(v))$ every variable (in V) has been split and thus no pair of random variables are joined by an edge. The factorization (9) then becomes:

$$p(V(v^*) = v) = \prod_{i=1}^{K} p(V_i(v^*) = v_i) = \prod_{i=1}^{K} p(V_i(v_{pa_i}^*) = v_i),$$
(17)

and thus for a fixed $v^* \in \mathfrak{X}_V$ the one-step ahead counterfactuals $V_1(v^*_{\mathrm{pa}_1}), \ldots, V_K(v^*_{\mathrm{pa}_K})$ are independent. Note that (17) holding for all $v^* \in \mathfrak{X}_V$ is equivalent to (6) and thus defines the FFRCISTG model.

3. The Potential Outcomes Calculus And Identification

Pearl presented the three rules of do-calculus as an inference system for deriving identification results for causal inference problems. The do-calculus is stated as three identities involving (conditional) interventional distributions, with preconditions given by d-separation (or m-separation) statements on graphs derived from the causal diagram $\mathcal{G}(V)$.

^{21.} For the sole purpose of establishing the SWIG global Markov property it is sufficient to show that $p(V_i(a) = v_i \mid V_{\mathrm{pa}_i \setminus A}(a) = v_{\mathrm{pa}_i \setminus A})$ is not a function of the fixed nodes that are not in pa_i , that is $a_{A \setminus \mathrm{pa}_i}$. This is established by (13). Under the FFRCISTG, $p(V_i(a_{\mathrm{pa}_i \cap A}, v_{\mathrm{pa}_i \setminus A}))$ exists even if $p(a_{\mathrm{pa}_i \cap A}, v_{\mathrm{pa}_i \setminus A}) = 0$.

^{22.} This is because the extended g-formula includes the value a variable takes on just before it is intervened upon and set to a constant a_i .

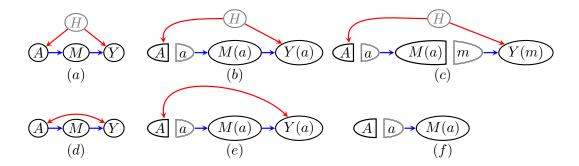


Figure 2: (a) A hidden variable causal model. (b) A SWIG corresponding to an intervention that sets A to a in the causal model represented by (a). (c) A SWIG corresponding to an intervention that sets A to a and M to m. (d) A latent projection of the DAG in (a). (e) A latent projection of the SWIG in (b). (f) A latent projection of the SWIG in (b) onto an ancestral set of vertices A, M(a) and a.

Here we reformulate and extend these three rules as a "potential outcomes calculus" or "pocalculus" for short. The rules are as follows:

1:
$$p(Y(x)|Z(x),W(x)) = p(Y(x)|W(x))$$
 if $(Y(x) \perp \!\!\! \perp \!\!\! Z(x)|W(x))_{\mathcal{G}(x)},$
2: $p(Y(x,z)|W(x,z)) = p(Y(x)|W(x),Z(x) = z)$ if $(Y(x,z) \perp \!\!\! \perp \!\!\! Z(x,z)|W(x,z))_{\mathcal{G}(x,z)},$
3: $p(Y(x,z)) = p(Y(x))$ if $(Y(x,z) \perp \!\!\! \perp \!\!\! Z(x,z)|W(x,z))_{\mathcal{G}(x,z)},$

where $\mathcal{G}(x)$ and $\mathcal{G}(x,z)$ are SWIGs describing interventions on X and $X \cup Z$. The sets Z,Y and W are assumed to be disjoint; X may overlap with the other sets, but if $Z \cap X \neq \emptyset$ then we require $x_{X \cap Z} = z_{X \cap Z}$, so that the assignments are consistent.

Rule 1 can be viewed as the part of the SWIG global Markov property pertaining to random (rather than fixed) variables.

Rule 2 can be viewed as a kind of generalized conditional ignorability rule which follows from Rule 1 and recursive substitution. Specifically by recursive substitution or minimal labelling Z(x,z)=Z(x); further,

$$p(Y(x,z) | W(x,z)) = p(Y(x,z) | W(x,z), Z(x) = z)$$

= $p(Y(x) | W(x), Z(x) = z)$

here the first equality follows by the given d-separation and Rule 1, while the second follows from consistency (or recursive substitution) since Y(x,z)=Y(x) and W(x,z)=W(x) given that Z(x)=z.

Rule 3 expresses the property of causal irrelevance that interventions only affect descendants: Note that the Rule 3 condition $(Y(x,z) \perp \!\!\! \perp z)_{\mathcal{G}(x,z)}$ is, by definition, equivalent to the fixed vertex (or vertices) z not being an ancestor of any vertex in Y(x,z) in the SWIG $\mathcal{G}(x,z)$ where the vertices in X and Z have been split.

Further, if a variable Y(x) appears in the SWIG $\mathcal{G}(x,z)$ (with minimal labeling), then there is no directed path from any fixed vertex in z to Y(x), and $Y(x) = Y(x,z)^{23}$. Thus the minimal labeling of the SWIG implicitly encodes all applications of Rule 3, in the sense that if the

^{23.} Thus under the counterfactual model, as defined by one-step ahead counterfactuals, $V_i(a_{pa_i})$ and recursive substitution (1), we have a stronger implication than Rule 3: if $(Y(x,z) \perp \!\!\! \perp z)_{\mathcal{G}(x,z)}$ then Y(x,z) = Y(x). Notwithstanding

(minimally labeled) vertex Y(x) is present in the SWIG then for any set Z, disjoint from X, p(Y(x)) = p(Y(x, z)).

As we have shown here, the po-calculus directly follows from the SWIG global Markov property, which is implied by both the FFRCISTG model (and thus the NPSEM-IE), consistency, and causal irrelevance, where the latter two hold for any NPSEM.

Rule 3, as stated here, 24 simply states that interventions only affect descendants and thus is simpler than Rule 3 in the original formulation of the do-calculus. It is proved in (Malinsky et al., 2019) that this reformulated Rule 3, in conjunction with the other two rules, is equivalent to Pearl's do-calculus, in the sense that the three rules stated here imply the original three rules. The rules stated here are more general in that we allow X to overlap with Y, Z and W, which is not possible within the framework and notation of the original do-calculus. As a consequence, as we will show below, there are additional identification results that follow from the po-calculus, but not the do-calculus. However, if we restrict the po-calculus rules to the case where X does not overlap with Y and Z then they are equivalent to the do-calculus.

It may also be noted that the po-calculus is formulated using a uniform type of graph, the SWIG, for displaying the preconditions for each rule.²⁵

Remark 2 We note that there are other types of equality between distributions that do not correspond to a single application of the po-calculus rules. For instance, in Example 1, it follows that

$$p(Y(a) | M(a)) = p(Y(a') | M(a')) \text{ for } a, a' \in \mathfrak{X}_A.$$
 (18)

This holds even though Y(a) and M(a) depend on a and $p(Y(a) | M(a)) \neq p(Y | M)$. This is a form of independence. Such constraints are captured by the full global Markov property for SWIGs: notice that a is d-separated from Y(a) given M(a) in the SWIG shown in Figure 2(b). However, the equality in (18) may be derived from three applications of the po-calculus (or the do-calculus) rules.

4. Identification In Hidden Variable Causal Models

If some variables in an NPSEM are unobserved, identification becomes more complicated, and some interventional distributions become non-identified. Identification theory in NPSEMs associated with $\mathcal{G}(V \cup H)$, where H are hidden variables is often described in terms of a special acyclic directed mixed graph (ADMG) $\mathcal{G}(V)$ obtained from $\mathcal{G}(V \cup H)$ via the *latent projection* operation (Verma and Pearl, 1990). Any two distinct hidden variable DAGs $\mathcal{G}_1(V \cup H_1)$, $\mathcal{G}_2(V \cup H_2)$ that share the latent projection $\mathcal{G}(V) = \mathcal{G}_1(V) = \mathcal{G}_2(V)$ also share all equality constraints on the observed marginal distribution (Evans, 2018), as well as non-parametric identification theory, in the sense that effects are identified in \mathcal{G}_1 if and only if they are identified in \mathcal{G}_2 , and by the same functional (Richardson et al., 2017).

In cases where p(V(a)) is identified, the functional is a kind of modified factorization associated with nested Markov models of ADMGs (Richardson et al., 2017).

4.1 Latent Projection ADMGs

Given a DAG $\mathcal{G}(V \cup H)$, where V are observed and H are hidden variables, a latent projection $\mathcal{G}(V)$ is the following ADMG with a vertex set V. An edge $A \to B$ exists in $\mathcal{G}(V)$ if there exists

this, we formulate Rule 3 in terms of the equality of distributions because we wish these rules to be logically equivalent to the original *do*-calculus and also apply to weaker causal models; see Footnote 18 and Section 6.2.

^{24.} Rule 3 in this paper is called Rule 3* by Malinsky et al. (2019).

^{25.} Whereas the original do-calculus involves three different constructions: $G_{\overline{X}}$, $G_{\overline{X}\underline{Z}}$ and $G_{\overline{X}(Z(W))}$.

^{26.} Formally we may think of P(Y(a) | M(a)) as forming a kernel q(y | m, a), which is a set of conditional distributions indexed by a. The constraint is then an independence in this kernel; (Richardson et al., 2017).

a directed path from A to B in $\mathcal{G}(V \cup H)$ with all intermediate vertices in H. Similarly, an edge $A \leftrightarrow B$ exists in $\mathcal{G}(V)$ if there exists a path without consecutive edges $\to \circ \leftarrow$ from A to B with the first edge on the path of the form $A \leftarrow$, the last edge on the path of the form $\to B$, and all intermediate vertices on the path in H. Latent projections of hidden variable DAGs may be viewed as graphical versions of marginal distributions, in the following sense. Just as conditional independences may be read off a DAG using d-separation, they may be read from an ADMG via the natural extension of d-separation to ADMGs which is called m-separation Richardson (2003).

If $p(V \cup H)$ factorizes with respect to $\mathcal{G}(V \cup H)$, then for any disjoint subsets A, B, C of V, if A is m-separated from B given C, then A is independent of B conditionally on C in the marginal distribution p(V). Since latent projections define an infinite class of hidden variable DAGs that share identification theory, identification algorithms are typically defined directly on latent projections for simplicity.

Given $A \subseteq V$ in a hidden variable DAG $\mathcal{G}(V \cup H)$, we can construct the latent projection of the SWIG $\mathcal{G}(V(a) \cup H(a))$ directly from the ADMG $\mathcal{G}(V)$, we denote the resulting ADMG (with fixed nodes) by $\mathcal{G}(V(a))$. We can extend d-separation on SWIGs constructed from DAGs to m-separation on SWIGs constructed from ADMGs, and define the SWIG global Markov property on SWIG ADMGs analogously to the SWIG global Markov property on SWIG DAGs. Similarly, we can restate po-calculus rules using m-separation on SWIG ADMGs.

As an example, the latent projection of the hidden variable DAG in Figure 2 (a) is shown in Figure 2 (d), while the latent projection of the SWIG in Figure 2 (b) is shown in Figure 2 (e).

All vertex relations defined in (3) translate without change to any SWIG $\mathcal{G}(V(a))$, except by convention $\operatorname{dis}_i^{\mathcal{G}(V(a))}$, $\operatorname{mb}_i^{\mathcal{G}(V(a))}$, and $\operatorname{pre}_i^{\mathcal{G}(V(a))}$ may only contain random vertices, in other words, they are subsets of V(a).

We will describe a complete identification algorithm in hidden variable DAG models for all distributions of the form p(Y(a)), where Y may potentially intersect A. The original formulation of the problem in (Tian and Pearl, 2002; Shpitser and Pearl, 2006a; Richardson et al., 2017) assumed $Y \cap A = \emptyset$, and yielded the *ID algorithm*.

We call our version of the algorithm the *extended ID algorithm*, by analogy with the *extended g-formula* (16). The extended ID algorithm will be formulated using SWIGs defined on latent projection ADMGs of the underlying hidden variable DAG. The algorithm will take advantage of the fact that under certain assumptions given by the causal model, a single splitting operation that defines a counterfactual distribution in a SWIG can be phrased in terms of the observed data distribution. This insight can be applied inductively to obtain results of multiple splitting operations as functionals of the observed data distribution.

The extended ID algorithm expresses the functional for p(Y(a)) as a counterfactual factorization in a certain SWIG ADMG, where terms of the factorization correspond to districts in the SWIG. It then aims to identify each term by finding a sequence of splitting operations, possibly interleaved with marginalization operations. Perhaps surprisingly, this always suffices to obtain identification whenever identification is possible.

4.2 The Identified Splitting Operation In A SWIG

A general identification algorithm for interventional distributions in hidden variable DAG models involves, as an essential step, expressing the counterfactual distribution $p(V(a, c_k))$ as a function of another counterfactual distributions p(V(a)), where one fewer variable (V_k) has been intervened on, using restrictions in $\mathcal{G}(V(a))$. Specifically, we have the following generalization of Proposition 4:

Proposition 5 Given an ADMG $\mathcal{G}(V)$ with SWIG $\mathcal{G}(V(a))$, if $V_k(a)$ is not split, so $k \notin A$, and $V_k(a)$ is such that there is no other random vertex that is both a descendant of $V_k(a)$ and in the same district as $V_k(a)$ then

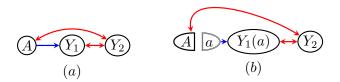


Figure 3: (a) A graph where $p(Y_1(a))$ and $p(Y_2(a))$ are identified, but Proposition 5 may not be applied. (b) A SWIG showing a splitting operation that is not identified according to Proposition 5.

for all $c_k \in \mathfrak{X}_k$:

$$\begin{split} p(V(a,c_k) &= v) \\ &= p(V_{-k}(a) = v_{-k}, V_k(a) = c_k) \times \frac{p\Big(V_k(a) = v_k \ \Big|\ V_{\mathbf{mb}_k^{\mathcal{G}(V(a))}}(a) = v_{\mathbf{mb}_k^{\mathcal{G}(V(a))}}\Big)}{p\Big(V_k(a) = c_k \ \Big|\ V_{\mathbf{mb}_k^{\mathcal{G}(V(a))}}(a) = v_{\mathbf{mb}_k^{\mathcal{G}(V(a))}}\Big)}, \end{split}$$

where $V_{-k} \equiv \{V_1, \dots, V_{k-1}, V_{k+1}, \dots, V_K\}$, provided the conditional probability in the denominator is positive.

In words, this Proposition states that if $V_k(a)$ satisfies the graphical condition in $\mathcal{G}(V(a))$ then $p(V(a, c_k))$, the joint distribution over all variables (including A and V_k) resulting from intervening to set A to a and V_k to c_k , may be obtained from p(V(a)) by evaluating at $V_k(a) = c_k$ and multiplying by a ratio of conditional densities for $V_k(a)$.

The graphical condition may be interpreted as requiring that in the world where we have already intervened on A, there is no sequence of variables between V_k and any of its causal descendants such that there is an unmeasured confounder between each pair.

There exist counterfactual distributions which are identified, but where the above Proposition does not directly apply to the observed data distribution. For example, in the graph in Figure 3 (a), $p(Y_1(a)) = p(Y_1 \mid a)$, and $p(Y_2(a)) = p(Y_2)$. Nevertheless, the preconditions to applying Proposition 5 do not apply to the original graph, meaning that the distribution represented by the SWIG in Figure 3 (b) is not equal to the functional of the observed data distribution described in the Proposition. In fact, the joint distribution associated with this SWIG is not identified at all, as was shown in Tian and Pearl (2002). Nevertheless, identification of $p(Y_2(a))$ and $p(Y_1(a))$ may be obtained by the identification algorithm we describe below.

In the next section we will apply the Proposition iteratively, in conjunction with marginalization steps, in order to obtain a complete algorithm for identifying a margin p(Y(a)).

Proof: Fix an ordering \prec' on vertex indices such that \prec' is topological in $\mathcal{G}(V(a))$ and such that no element in the district of $V_k(a)$ occurs later in the ordering than $V_k(a)$. Such a topological ordering exists because, by hypothesis, no vertex in the district of $V_k(a)$ is a descendant of $V_k(a)$. For any index j, define pre_j' to be the set of predecessor indices according to \prec' .

$$\begin{split} p(V(a,c_k) = v) &= \prod_{i \in \operatorname{pre}_k'} p(V_i(a,c_k) = v_i \,|\, V_{\operatorname{pre}_i'}(a,c_k) = v_{\operatorname{pre}_i'}) \times p(V_k(a,c_k) = v_k \,|\, V_{\operatorname{pre}_k'}(a,c_k) = v_{\operatorname{pre}_k'}) \\ &\times \prod_{j \notin \{k\} \cup \operatorname{pre}_k'} p(V_j(a,c_k) = v_j \,|\, V_{\operatorname{pre}_j'}(a,c_k) = v_{\operatorname{pre}_j'}) \\ &= \prod_{i \in \operatorname{pre}_k'} p(V_i(a) = v_i \,|\, V_{\operatorname{pre}_i'}(a) = v_{\operatorname{pre}_i'}) \times p(V_k(a) = v_k \,|\, V_{\operatorname{pre}_k'}(a) = v_{\operatorname{pre}_k'}) \\ &\times \prod_{j \notin \{k\} \cup \operatorname{pre}_k'} p(V_j(a,c_k) = v_j \,|\, V_{\operatorname{pre}_j'}(a,c_k) = v_{\operatorname{pre}_j'}) \\ &= \prod_{i \in \operatorname{pre}_k'} p(V_i(a) = v_i \,|\, V_{\operatorname{pre}_i'}(a) = v_{\operatorname{pre}_i'}) \times p(V_k(a) = v_k \,|\, V_{\operatorname{pre}_k'}(a) = v_{\operatorname{pre}_k'}) \\ &\times \prod_{j \notin \{k\} \cup \operatorname{pre}_k'} p(V_j(a) = v_j \,|\, V_{\operatorname{pre}_j'} \setminus \{k\}(a) = v_{\operatorname{pre}_j'}, V_k(a) = c_k) \\ &= \prod_{i \in \operatorname{pre}_k'} p(V_i(a) = v_i \,|\, V_{\operatorname{pre}_i'}(a) = v_{\operatorname{pre}_i'}) \times p(V_k(a) = c_k \,|\, V_{\operatorname{pre}_k'}(a) = v_{\operatorname{pre}_k'}) \\ &\times \prod_{j \notin \{k\} \cup \operatorname{pre}_k'} p(V_j(a) = v_j \,|\, V_{\operatorname{pre}_j'} \setminus \{k\}(a) = v_{\operatorname{pre}_j'}, V_k(a) = c_k) \\ &\times p(V_k(a) = v_k \,|\, V_{\operatorname{pre}_k'}(a) = v_{\operatorname{pre}_k'}) \Big/ p(V_k(a) = c_k \,|\, V_{\operatorname{pre}_k'}(a) = v_{\operatorname{pre}_k'}) \\ &= p(V_{-k}(a) = v_{-k}, V_k(a) = c_k) \\ &\times p(V_k(a) = v_k \,|\, V_{\operatorname{pre}_k'}(a) = v_{\operatorname{pre}_k'}) \Big/ p(V_k(a) = c_k \,|\, V_{\operatorname{pre}_k'}(a) = v_{\operatorname{pre}_k'}) \\ &= p(V_{-k}(a) = v_{-k}, V_k(a) = c_k) \frac{p(V_k(a) = v_k \,|\, V_{\operatorname{pre}_k'}(a) = v_{\operatorname{pre}_k'})}{p(V_k(a) = c_k \,|\, V_{\operatorname{pre}_k'}(a) = v_{\operatorname{pre}_k'})}. \end{split}$$

Here the first identity is via the chain rule of probability applied to $p(V(a,c_k))$ using the ordering \prec' , the second by Rule 3 (causal irrelevance) applied to elements indexed by $\{k\} \cup \operatorname{pre}_k'$ in $\mathcal{G}(V(a,c_k))$, the third by Rule 2 (generalized ignorability) applied to every term in the second product in $\mathcal{G}(V(a,c_k))$, and the assumption on \prec' that all elements of $\operatorname{dis}_k^{\mathcal{G}(V(a))}$ are in $V_{\operatorname{pre}_k'}(a)$, the fourth by multiplying and dividing by $p(V_k(a)=c_k \mid V_{\operatorname{pre}_k'}(a)=v_{\operatorname{pre}_k'})$, the fifth by the chain rule, the sixth by Rule 1 (m-separation) applied to $\mathcal{G}(V(a))$ and the definition of $\operatorname{mb}_k^{\mathcal{G}(V(a))}$.

4.3 The Extended ID Algorithm

There are SWIGs $\mathcal{G}(V(a))$ for which, for some variable $V_k(a)$ we are not able to apply Proposition 5, but where it may be applied to a SWIG $\mathcal{G}(Y(a))$, where Y(a) is an ancestral subset of V(a) in $\mathcal{G}(V(a))$. Here a set Y of vertices in a (SWIG) ADMG \mathcal{G}^* is said to be *ancestral* if $V_i \in Y$ implies $\operatorname{an}_i^{g^*} \subseteq Y$.

Marginal distributions p(Y(a)) obtained from p(V(a)) that correspond to ancestral sets in $\mathcal{G}(V(a))$ have the nice property that a latent projection $\mathcal{G}(Y(a))$ is always equal to an induced subgraph $(\mathcal{G}(V(a)))_{Y(a)}$ of a SWIG $\mathcal{G}(V(a))$, with $\mathcal{G}(Y(a))$ having strictly fewer vertices and

edges than $\mathcal{G}(V(a))$ if $Y(a) \subset V(a)$. For example, given the SWIG in Figure 1 (e), the latent projection onto the ancestral subset A, M(a) and a yields the SWIG shown in Figure 1 (f). We describe the precise way in which splitting and ancestral margin operations are used to obtain identification below.

Specifically, complete non-parametric identification for intervention distributions associated with the FFRCISTG model may be obtained from: (i) the district factorization in the appropriate SWIG, (ii) the identified splitting operation described in the previous section, and (iii) marginalization steps that lead to marginal distributions corresponding to ancestral sets of vertices in SWIGs. All of these steps may be justified via the po-calculus.

For any (possibly intersecting) subsets Y, A of V in a latent projection $\mathcal{G}(V)$ representing a causal DAG $\mathcal{G}(V \cup H)$, define $Y^*(a)$ to be the random ancestors of Y(a) in $\mathcal{G}(V(a))$. Clearly, if $p(Y^*(a))$ is identified, then we may recover p(Y(a)) since:

$$P(Y(a) = y) = \sum_{u \in \mathfrak{X}_{Y^* \setminus Y}} p(V_Y(a) = y, V_{Y^* \setminus Y}(a) = u).$$
(19)

Though less obvious, extensions of results in (Shpitser and Pearl, 2006a) imply that the converse also holds, so that if p(Y(a)) is identified (for all parameter values) then $p(Y^*(a))$ is identified. Consequently, in the foregoing we will assume that Y(a) is an ancestral set of (random) vertices in $\mathcal{G}(V(a))$.

If p(Y(a)) is identified then this may be obtained by breaking this joint distribution into districts in $\mathcal{G}(Y(a))$. For each such district D(a), define the set of *strict* (random) parents as $\operatorname{pas}_{D(a)}^{\mathcal{G}(Y(a))} \equiv \operatorname{pa}_{D(a)}^{\mathcal{G}(Y(a))} \setminus (D(a) \cup a)$.

First, we show that p(Y(a) = y) can be factorized into a set of terms of the form $p(D(a, v_{\text{pas}_{D(a)}^{\mathcal{G}(Y(a))}}))$, as follows.

$$p(Y(a) = v_{Y})$$

$$= \prod_{i \in Y} p(V_{i}(a) = v_{i} \mid V_{Y \cap \text{pre}_{i}}(a) = v_{Y \cap \text{pre}_{i}})$$

$$= \prod_{D \in \mathcal{D}(\mathcal{G}(Y(a)))} \prod_{i \in D} p(V_{i}(a) = v_{i} \mid V_{Y \cap \text{pre}_{i}}(a) = v_{Y \cap \text{pre}_{i}})$$

$$= \prod_{D \in \mathcal{D}(\mathcal{G}(Y(a)))} \prod_{i \in D} p(V_{i}(a, v_{\text{pas}_{D}^{\mathcal{G}(Y(a))}}) = v_{i} \mid V_{D \cap \text{pre}_{i}}(a, v_{\text{pas}_{D}^{\mathcal{G}(Y(a))}}) = v_{D \cap \text{pre}_{i}})$$

$$= \prod_{D \in \mathcal{D}(\mathcal{G}(Y(a)))} p(V_{D}(a, v_{\text{pas}_{D}^{\mathcal{G}(Y(a))}}) = v_{D}).$$
(21)

Here the first two lines follow by the chain rule of probability, term grouping, and the fact that in any ADMG, including a SWIG ADMG, the set of districts partitions the set of random vertices. The third equality follows because of the following:

$$p(V_{i}(a) = v_{i} \mid V_{Y \cap \text{pre}_{i}}(a) = v_{Y \cap \text{pre}_{i}})$$

$$= p\left(V_{i}(a) = v_{i} \mid V_{\text{mb}_{i}}^{\mathcal{G}(Y(a))} \cap \text{pre}_{i}}(a) = v_{\text{mb}_{i}}^{\mathcal{G}(Y(a))} \cap \text{pre}_{i}}\right)$$

$$= p\left(V_{i}(a) = v_{i} \mid V_{D \cap \text{pre}_{i}}(a) = v_{D \cap \text{pre}_{i}}, V_{\text{pas}_{D}^{\mathcal{G}(Y(a))} \cap \text{pre}_{i}}(a) = v_{\text{pas}_{D}^{\mathcal{G}(Y(a))} \cap \text{pre}_{i}}\right)$$

$$= p(V_{i}(a, b_{i}) \mid V_{D \cap \text{pre}_{i}}(a, b_{i}) = v_{D \cap \text{pre}_{i}})$$

$$= p\left(V_{i}(a, v_{\text{pas}_{D}^{\mathcal{G}(Y(a))}}) \mid V_{D \cap \text{pre}_{i}}(a, v_{\text{pas}_{D}^{\mathcal{G}(Y(a))}}) = v_{D \cap \text{pre}_{i}}\right)$$

$$(22)$$

where $b_i = v_{\operatorname{pas}_D^{\mathcal{G}(Y(a))} \cap \operatorname{pre}_i}$, and $D = \operatorname{dis}_i^{\mathcal{G}(Y(a))}$. Here the first equality follows from Rule 1;²⁷ the second follows from the definition of the Markov blanket of $V_i(a)$ in $\mathcal{G}(Y(a))$; the third follows from Rule 2 since $V_i(a,b_i)$ is m-separated from $B_i(a,b_i) \equiv V_{\operatorname{pas}_D^{\mathcal{G}(Y(a))} \cap \operatorname{pre}_i}(a,b_i)$ in $\mathcal{G}(a,b_i)$; the fourth is an application of Rule 3 since vertices in $V_{\operatorname{pas}_D^{\mathcal{G}(Y(a))} \setminus \operatorname{pre}_i}$ are ordered after V_i and hence are not ancestors of V_i in \mathcal{G} , and thus also in $\mathcal{G}(a,b_i)$.

Next, we consider whether each term of the form $p(V_D(a, v_{\operatorname{pas}_D^{\mathcal{G}(Y(a))}}))$ is identified from p(V) by inductively applying the identified splitting operation in Proposition 5 to every element V_j in $A \cup (V \setminus D)$ in a sequence such that the precondition of Proposition 5 is satisfied at every step, and marginalizing $V_j(a)$ at every step whenever $V_j \notin D$. (Hence V_j will be split unless $V_j \in D \setminus A$.) p(Y(a)) is identified if for every district $D \in \mathcal{D}(\mathcal{G}(Y(a)))$, the corresponding term $p(V_D(a, v_{\operatorname{pas}_D^{\mathcal{G}(Y(a))}}))$ is identified in this way. In fact, the above method of identification is sufficient and necessary for identification of p(Y(a)). See the Appendix for details.

For the special case where $Y \cap A = \emptyset$ the resulting identified functionals were first described as an algorithm in Tian and Pearl (2002), and proven to be complete in Huang and Valtorta (2006); Shpitser and Pearl (2006a). In both versions of the algorithm, the identifiable terms corresponding to districts D(a) in $\mathcal{G}(Y(a))$ form parts of the *nested Markov factorization* of an ADMG, and the algorithm may thus be viewed as giving a modified nested factorization of an ADMG, just as the extended g-formula is a modified DAG factorization. For more details, see Richardson et al. (2017).

4.4 Identification Of Conditional Interventional Distributions

Targets of inference in causal inference are often functions of conditional counterfactual distributions p(Y(a)|Z(a)) rather than marginal distributions p(Y(a)). Such targets arise, for instance, when effects within certain subgroups are of interest, or when investigating relationships between primary and secondary outcomes. A straightforward modification of the above algorithm yields identification in such settings.

Fix Y,Z,A where Y,Z are disjoint, but may both intersect A. Fix the largest subset $W\subseteq Z$, with $Z'=Z\setminus W$, such that Z'(a,z') is m-separated from Y(a,z') given W(a,z') in $\mathcal{G}(V(a,z'))$. Then, by rule 2, $p(Y(a)\mid W(a),Z'(a)=z')=p(Y(a,z')\mid W(a,z'))$. Next, let A' be a maximal subset of $Z\cap A$ such that $A'(a,z')\bot Y(a,z')\mid \{W(a,z'):W\in Z\setminus (Z'\cup A')\}$. Then $p(Y(a)\mid Z(a))$ is identified if $p(Y(a,z'),\{W(a,z'):W\in Z\setminus (Z'\cup A')\})$ is identified. In fact, we have:

$$p(Y(a) \mid Z(a)) = \left. \frac{p(Y(a,z'), \{W(a,z') : W \in Z \setminus (Z' \cup A')\})}{p(\{W(a,z') : W \in Z \setminus (Z' \cup A')\})} \right|_{Z \setminus A' = z_{Z \setminus A'}}.$$

As we show in the Appendix, this condition is also necessary since if $p(Y(a, z'), \{W(a, z') : W \in Z \setminus (Z' \cup A')\})$ is not identified, $p(Y(a) \mid Z(a))$ is also not identified.

4.5 Representing context specific independence using SWIGs

We now discuss an extension of SWIGs due to Dahabreh et al. (2019) and Sarvet et al. (2020) that demonstrates that SWIGs have greater expressive power than standard causal DAGs because of their ability to represent interventional context specific conditional independence.

Consider the causal DAG shown in Figure 5(a) where A, M, Y are observed and U, R, S are unobserved. The latent projection is given in Figure 5(a*). The SWIG resulting from a joint intervention setting A to a and M to m is shown in Figure 5(b); the latent projection of this SWIG

^{27.} Note that the Markov blanket of i in the subgraph of $\mathcal{G}(Y(a))$ restricted to predecessors of i is, in general, a strict subset of the predecessors of i in the Markov blanket of i in $\mathcal{G}(Y(a))$. Consequently, the conditioning set is in the terms of (22) may not be minimal.



Figure 4: A simple DAG containing a treatment A, an intermediate M and a response Y.

is shown in Figure 5(b*). The distribution of Y(a, m) is not identified owing to the presence of the edges $M \to Y \leftrightarrow M$ (also called a bow arc).

However, suppose that, additional context specific subject matter knowledge²⁸ implies that the following counterfactual independences hold:

$$U \perp \perp R(a = 0, m);$$
 $U \perp \perp M(a = 1).$

As a consequence, the edges $U \to R(0,m)$ in $\mathcal{G}(0,m)$ and $U \to M(1)$ in $\mathcal{G}(1,m)$ may be removed, leading to the SWIGs shown in Figure 5(c) and (d), with corresponding latent projections shown in Figure 5(c*) and (d*).

Applying d-separation to the latent projections in Figure $5(c^*)$ and (d^*) we see that 29

$$Y(a,m) \perp \perp M(a), A \quad \text{for } a = 0, 1. \tag{24}$$

Consequently,

$$P(Y | A = a, M = m) = P(Y(a, m)),$$
 (25)

so that the joint effect of A and M on Y is identified for both a=0 and a=1.

Given solely the DAG in Figure 5 (a), with the latent projection in Figure 5 (a*), the equality (25) would not be expected since it does not follow from existing methods such as the do-calculus, the ID algorithm or the back-door criterion Pearl (2009), though see the recent work Tikka et al. (2019). However, (25) has a structural explanation in terms of the SWIGs corresponding to different treatment values of A.

Since, in addition to (24), we also have $M(a) \perp \!\!\! \perp A$ for a=0,1, it follows that the distribution of the counterfactuals $\{A,M(a),Y(a,m)$ for all $a,m\}$ obeys the FFRCISTG model associated with the graph shown in Figure 4 in which there are no bi-directed edges. However, interestingly, the distribution of these counterfactuals does not obey the NPSEM-IE associated with Figure 4, though the distribution does obey the NPSEM-IE (hence also the FFRCISTG) associated with Figure 5 (a^*) .

5. Conclusion

We wholeheartedly applaud Judea Pearl for his development and advocacy of graphical approaches to causal modeling. His approach represents a fundamental advance leading to many new insights

^{28.} See the ivermectin study described in the companion paper (Robins et al., 2021).

^{29.} Recall that when testing d-separation in SWIGs, fixed nodes such as a=0 in Figure 5(c*) and a=1 in Figure 5(d*) always block paths on which they occur as non-endpoint vertices.

^{30.} See (Robins et al., 2021, Section 2.6).

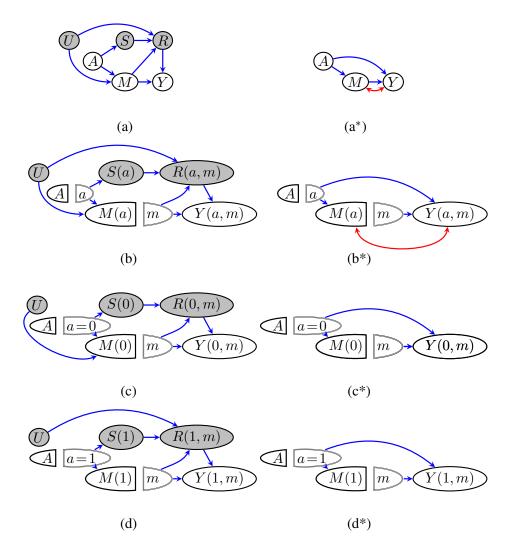


Figure 5: (a) A DAG $\mathcal G$ representing two studies of river blindness, described in (Robins et al., 2021, Section 2.6). (b) The SWIG $\mathcal G(a,m)$ resulting from $\mathcal G$; (c) and (d) show SWIGs $\mathcal G(a=0,m)$ and $\mathcal G(a=1,m)$ that incorporate additional context specific causal information. (a*), (b*), (c*), (d*) show the corresponding latent projections.

and methods, including complete identification theory for causal queries of all types, and extensions of d-separation to complex questions in causal modeling and missing data.

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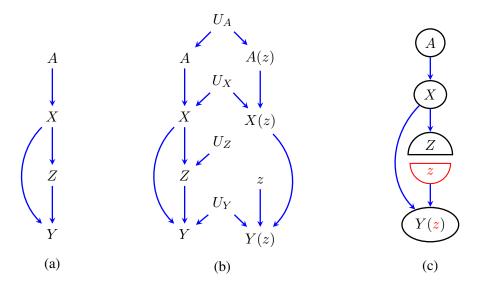


Figure 6: (a) A DAG \mathcal{G} . (b) The twin-network arising from intervening to set Z to z. (c) The SWIG $\mathcal{G}(z)$.

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6. Appendix

6.1 Incompleteness of d-separation in Twin Networks due to deterministic relations.

Twin networks (Balke and Pearl, 1994) are an alternative way to combine graphs and counterfactuals which allow some of the counterfactual independence relations implied by the NPSEM-IE to be read off via d-separation; see also (Shpitser and Pearl, 2008), (Pearl, 2009, Section 7.1.4). However, d-separation is not complete for Twin Networks (Richardson and Robins, 2013) since, as a consequence of consistency, certain variables in a twin network may be deterministically related. Consequently, it is possible for there to be a d-connecting path in a Twin Network and yet the corresponding conditional independence holds for all distributions in the model.

To see a simple example, consider the DAG shown in Figure 6(a), with the Twin Network and SWIG associated with intervening to set Z to z, shown in (b) and (c) respectively. Note that A and Y(z) are d-connected given X in the twin-network by two different d-connecting paths, ³¹ However, in spite of this $A \perp \!\!\!\perp Y(z) \mid X$ under the associated NPSEM-IE because X(z) = X, and A and Y(z) are d-separated given X(z) in the twin-network. The SWIG $\mathcal{G}(z)$ shown in Figure 6(c) makes manifest that A is d-separated from Y(z) given X, hence $A \perp \!\!\!\perp Y(z) \mid X$ under the FFRCISTG, hence also under the NPSEM-IE.

In addition, it may also be inferred from the SWIG that $A(z) \perp\!\!\!\perp Y(z) \mid X; A \perp\!\!\!\perp Y(z) \mid X(z)$ and $A(z) \perp\!\!\!\perp Y(z) \mid X(z)$ hold under the FFRCISTG (and hence also the NPSEM-IE). This is because it follows from causal irrelevance that, given a SWIG $\mathcal{G}(a)$, if a label a_i is present on some random node (equivalently if the SWIG contains a fixed node a_i), then a_i may always be added to the label of any random node on which it is not already present. Consequently we are free to add z to the label for X and A in $\mathcal{G}(z)$, from which these independences follow. Note that in the Twin Network, although A and A(z) are d-separated from Y(z) given X(z), the path $A(z) \to X(z) \to Y(z)$ d-connects A(z) and Y(z) given X, hence we cannot read off $A(z) \perp\!\!\!\!\perp Y(z) \mid X$ from the Twin Network.

Shpitser and Pearl (2008) provide an algorithm for merging nodes in a Twin Network, under a particular instantiation of the variables. This algorithm is conjectured to be complete for checking equality of the probability of counterfactual events. A conditional independence statement corresponds to a (potentially exponential) set of equalities between probabilities of events. Thus, if the conjecture holds, then the algorithm of Shpitser and Pearl (2008) provides a way to check counterfactual conditional independence implied by an NPSEM-IE. Though this approach is more involved, as noted earlier, in footnote 7, it addresses a harder problem than SWIGs since it is determining all independencies implied by an NPSEM-IE model which also includes "cross-world" independencies.

6.2 Weaker causal models to which the po-calculus also applies

In Section 3, we chose to express rule 3 of the po-calculus on the distribution level: p(Y(x,z)) = p(Y(x)), although the equality holds on the individual level: Y(x,z) = Y(x) under the FFR-CISTG, see also footnotes 18 and 23. We chose to do so for several reasons. First, this form is closer in spirit to Pearl's original formulation of the *do*-calculus.

Second, the weaker equality is expressible in the language of interventions, say via the do operator: $p(Y \mid do(x,z)) = p(Y \mid do(x))$. This allows us to apply this rule, and other rules of po-calculus to causal models that are not counterfactual, but which allow discussion of interventional distributions, such as the *agnostic causal model* of Spirtes et al. (2001) which is *defined* by the relationship between the observed data distribution and interventional distributions given by the extended g-formula (16) re-expressed via the *do* operator. Indeed, the FFRCISTG and the NPSEM-IE imply all distribution-level interventional statements that hold under the agnostic causal model,

^{31.} Precisely: $A \leftarrow U_A \rightarrow A(z) \rightarrow X(z) \rightarrow Y(z)$ and $A \rightarrow X \leftarrow U_X \rightarrow X(z) \rightarrow Y(z)$.

and these are the only statements that are relevant for the purposes of identification of interventional quantities expressible by the *do* operator. Note that the distribution level equality has a graphical representation via *population SWIGs* in which missing edges correspond to the absence of population level direct effects, whereas the individual level counterfactuals are not necessarily the same. See also (Richardson and Robins, 2013, Section 7).

6.3 Completeness Proofs

Here we describe a number of completeness results referred to in the main body of the paper. Before doing so, we state necessary preliminaries. Given an acyclic directed mixed graph (ADMG) $\mathcal{G}(V)$ and a set $S \subseteq V$, an induced subgraph $\mathcal{G}(V)_S$ is defined to be a graph containing vertices S, and all edges in $\mathcal{G}(V)$ between elements in S.

Given an acyclic directed mixed graph (ADMG) $\mathcal{G}(V)$, we define a set $W \subseteq V$ to be fixable if $W = \emptyset$, or $W = \{W_1, W_2, \ldots\}$ and there exists a set of ADMGs $\mathcal{G}_0(V)$, $\mathcal{G}_1(V \setminus \{W_1\})$, $\mathcal{G}_2(V \setminus \{W_1, W_2\}), \ldots, \mathcal{G}_k(V \setminus W)$, such that

- $\mathcal{G}_0(V) = \mathcal{G}(V)$.
- For every i = 0, ..., k-1, W_{i+1} has no element $V_j \in V \setminus \{W_1, ..., W_i, W_{i+1}\}$ with a directed path from W_{i+1} to V_j and a path consisting exclusively of bidirected edges from W_{i+1} to V_j in G_i .
- For every $i=1,\ldots,k$, $\mathcal{G}_i(V\setminus\{W_1,\ldots,W_i\})$ is obtained from $\mathcal{G}_{i-1}(V\setminus\{W_1,\ldots,W_{i-1}\})$ by removing W_i and all edges adjacent to W_i .

If $W \subseteq V$ is fixable, the set $S \equiv V \setminus W$ is said to be *reachable*. A set S reachable in $\mathcal{G}(V)$ is said to be *intrinsic* if the vertices in $\mathcal{G}(V)_S$ form a bidirected connected set. Note the relationship between reachable sets and the precondition for Proposition 5.

We have the following result.

Theorem 1 Fix possibly intersecting sets Y, A such that Y(a) is ancestral in the SWIG $\mathcal{G}(V(a))$. Then

$$p(Y(a) = v_Y) = \prod_{D \in \mathcal{D}(\mathcal{G}(Y(a)))} p(V_D(a, v_{\text{pas}_D^{\mathcal{G}(A)}}) = v_D),$$

and p(Y(a)) is not identified if there exists $D \in \mathcal{D}(\mathcal{G}(Y(a)))$ such that no inductive sequence of applications of Proposition 5 exists where every element $V_j \in A \cup (V \setminus D)$ is split such that the precondition of Proposition 5 is satisfied at every step, and $V_j(a)$ is marginalized from the resulting SWIG whenever $V_j \notin D$.

Proof: Assume such a set D exists. Assume D is not a reachable set in $\mathcal{G}(V)$. Then the results in Richardson et al. (2017) imply that there exists a hedge for p(Y(a)) and that p(Y(a)) is not identified Shpitser and Pearl (2006a).

Assume D is a reachable set, but some element $A_i \in D$ cannot be split by applying Proposition 5. This implies there exists a set of vertices W_1, \ldots, W_k in D that are bidirected connected, and W_k is a child of A_i in $\mathcal{G}(V)$. Since W_1, \ldots, W_k , being elements of D, are in the set of ancestors of Y in $\mathcal{G}(V(a))$, the sets $\{A_i\}$, and $\{A_i, W_1, \ldots, W_k\}$ form a hedge for p(Y(a)), so p(Y(a)) is not identifiable.

Theorem 2 Fix subsets Y, Z, A of V, in some ADMG $\mathcal{G}(V)$, where Y, Z are disjoint, but may both intersect A. Fix the largest subset $W \subseteq Z$, with $Z' = Z \setminus W$, such that Z'(a,z') is m-separated from Y(a,z') given W(a,z') in $\mathcal{G}(V(a,z'))$, and let A' be a maximal subset of $Z \cap A$ such that A'(a,z') is m-separated from Y(a,z') given $\{W(a,z'): W \in Z \setminus (Z' \cup A')\}$. Then $p(Y(a) \mid Z(a))$ is identified if $p(Y(a,z'), \{W(a,z'): W \in Z \setminus (Z' \cup A')\})$ is identified. If identification holds, we have:

$$p(Y(a) \mid Z(a)) = \left. \frac{p(Y(a,z'), \{W(a,z') : W \in Z \setminus (Z' \cup A')\})}{p(\{W(a,z') : W \in Z \setminus (Z' \cup A')\})} \right|_{Z \backslash A' = z_{Z \backslash A'}}.$$

Proof: If the stated assumptions hold, and $p(Y(a, z'), \{W(a, z') : W \in Z \setminus (Z' \cup A')\})$ is identified, the conclusion follows by definition of conditioning.

Assume $p(Y(a,z'),\{W(a,z'):W\in Z\setminus (Z'\cup A')\})$ is not identified. It suffices to consider the case where $p(\{W(a,z'):W\in Z\setminus (Z'\cup A')\})$ is not identified. The proof then follow the proof structure for analogous results in Shpitser and Pearl (2006b); Malinsky et al. (2019), with the fact that $A\cap Y$ is potentially not an empty set not influencing the structure of the proof.

Non-identification of $p(\{W(a,z'):W\in Z\setminus (Z'\cup A')\})$ implies the existence of a hedge, and the preconditions of the theorem imply the existence of an m-connecting (given W) path from an element in W in the hedge to some element in Y. Non-identification is established by induction on the structure of this path. Specifically, fix an element L on the path such that the inductive hypothesis that p(L(a,z')|W'(a,z')) is not identified holds, where W' is the subset of W involved in the hedge, or in the m-connecting path from the hedge to L. Thus, there exist two elements of the causal model that disagree on this distribution, but agree on the observed data distribution. The induction then establishes that the distribution p(L'(a,z')|W''(a,z')), where L' is the next element on the m-connecting path, and W'' are all elements of W that are either in the hedge, or witness m-connection of the path from the hedge to L', is also not identified. This is established by extending existing two elements with an appropriate distribution that yields a one to one mapping from distributions p(L(a,z')|W'(a,z')) to distributions p(L'(a,z')|W''(a,z')).