

Effects of Stubbornness on Opinion Dynamics

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ABSTRACT

As an important factor governing opinion dynamics, stubbornness strongly affects various aspects of opinion formation. However, a systematically theoretical study about the influences of heterogeneous stubbornness on opinion dynamics is still lacking. In this paper, we study a popular opinion model in the presence of inhomogeneous stubbornness. We show analytically that heterogeneous stubbornness has a great impact on convergence time, expressed opinion of every node, and the overall expressed opinion. We provide an explanation of the expressed opinion in terms of stubbornness-dependent spanning diverging forests. We propose quantitative indicators to quantify some social concepts, including conflict, disagreement, and polarization by incorporating heterogeneous stubbornness, and develop a nearly linear time algorithm to approximate these quantities, which has a proved theoretical guarantee for the error of each quantity. To demonstrate the performance of our algorithm, we perform extensive experiments on a large set of real networks, which indicate that our algorithm is both efficient and effective, scalable to large networks with millions of nodes.

CCS CONCEPTS

• **Human-centered computing** → **Social network analysis**; • **Networks** → **Network dynamics**; • **Information systems** → **Social networks**.

KEYWORDS

Opinion dynamics, social network, multi-agent system, polarization, disagreement, conflict, Laplacian solver

1 INTRODUCTION

As a research subject of computational social science [27], opinion dynamics has received considerable attention from the scientific community [7], especially in recent years due to the explosive growth of social media and online social networks [32]. An important factor affecting opinion dynamics is the attributes of individuals, for example, stubbornness that measures the extent to which an individual sticks to his own initial opinion/belief or is willing to change his opinion in reaction to its neighbors' opinions. Thus, stubbornness indicates people's susceptibility to persuasion [1, 2]. In social psychology, a rich body of empirical work [6, 17, 20, 47,

48] focused on this key element, unveiling some far-reaching implications of stubbornness in diverse aspects, including product marketing, public health campaigns, and political candidates. Some recent work studied persuasive technologies to change an individual's attitude or stubbornness [21, 28, 29], showing that individual's stubbornness are heterogeneous.

Perhaps the most important step for opinion dynamics study is to establish a mathematical model by capturing various fundamental factors affecting opinion formation. During the past decades, a large body of literature has been devoted to the modeling of opinion dynamics [41], and several classic models have been developed, providing a deep understanding of opinion diffusion and formulation. Among these models, Friedkin-Johnsen (FJ) model [23] is one of the most popular models. As a significant extension of the DeGroot model [16], the FJ model grasps some intricate social behaviors by considering French's "theory of social power" [22]. Very recently, by modifying the FJ model a study was initiated in [1, 2] to explore how to change individuals' stubbornness to optimize the overall opinion. However, there is still no theoretical studies that systematically uncover the influences of heterogeneous stubbornness on opinion dynamics, including convergence speed, overall opinion, and some resulting social phenomena such as polarization, disagreement, and conflict.

In order to fill the aforementioned gap, in this paper, we study the FJ model for opinion dynamics on an undirected weighted graph, aiming at unveiling the role played by heterogeneous stubbornness in opinion dynamics. The main contributions of this paper are summarized below.

- We give an explanation of the expressed opinion in terms of spanning diverging forests of a corresponding defined directed graph.
- We show that heterogeneous stubbornness has a strong effect on various aspects of opinion dynamics, including convergence velocity, expressed opinion of every node, and the overall expressed opinion.
- We make an extension of some relevant quantities for the FJ model with uniform stubbornness to the case with heterogeneous stubbornness, including conflict, disagreement, and polarization by incorporating heterogeneous stubbornness. We then develop an approximation algorithm to evaluate these quantities, which has a nearly linear time complexity and a theoretically guaranteed error.

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- We perform extensive experiments on large datasets, which demonstrate that our algorithm achieves both good efficiency and effectiveness, being scalable to large networks with millions of nodes.

2 PRELIMINARIES

In this section, we briefly introduce some concepts about directed and undirected graphs, their related matrices, as well as some useful notations.

2.1 Digraph and Related Concepts

Let $\Gamma = (V(\Gamma), E(\Gamma), w)$ denote a weighted directed graph (network) abbreviated as digraph with node set $V(\Gamma)$, arc (edge) set $E(\Gamma)$ and edge weight function $w : E(\Gamma) \rightarrow \mathbb{R}_+$. Suppose $n = |V(\Gamma)|$ and $V(\Gamma) = \{v_1, v_2, \dots, v_n\}$. In what follows, v_i and i are used interchangeably to denote node v_i in the case without inducing any confusion. Let $i \rightarrow j$ denote an arc in $E(\Gamma)$ pointing to j from i . In Γ , a loop is an arc having identical end nodes, and an isolated node is a node having no arcs pointing to or coming from it. For two nodes v_1 and v_k in digraph Γ , a path from v_1 to v_k is an alternating sequence of nodes and arcs $v_1, e_1, v_2, \dots, e_{k-1}, v_k$, in which nodes are distinct and every arc e_i is $v_i \rightarrow v_{i+1}$. A circuit is a path plus an arc from the ending node to the starting node. Digraph Γ is called (strongly) connected if there exists at least a path for any pair of nodes. Γ is called weakly connected if it is connected when one replaces any arc $i \rightarrow j$ by two arcs $i \rightarrow j$ and $j \rightarrow i$ in opposite directions. A directed tree is a weakly connected digraph with no loops and cycles. An isolated node is considered as a tree. A directed forest is a particular digraph that is a disjoint union of directed trees.

For a weighted digraph Γ , many connection properties are encoded in its weighted adjacency matrix $A = (w_{ij})_{n \times n}$, whose entry w_{ij} at row i and column j represents the weight of arc $i \rightarrow j$. For each arc $i \rightarrow j \in E$ in Γ , its weight w_{ij} is strictly nonnegative, representing the influence of j on i . If there is no arcs from i to j , $w_{ij} = 0$. For a node i , its weighted in-degree d_i^+ is defined by $d_i^+ = \sum_{j=1}^n w_{ji}$, and its weighted out-degree d_i^- is $d_i^- = \sum_{j=1}^n w_{ij}$. Let Θ_i be the set of nodes to which node i points. Then, d_i^- is also equal to $\sum_{j \in \Theta_i} w_{ij}$. In the sequel, we use d_i to represent the out-degree d_i^- . The diagonal degree matrix of Γ is $D = \text{diag}(d_1, d_2, \dots, d_n)$, with the i -th diagonal entry being d_i . The Laplacian matrix of Γ is $L = (l_{ij})_{n \times n} = D - A$. Let $\mathbf{1}$ and $\mathbf{0}$ be the two n -dimensional vectors with all entries being ones and zeros, respectively. Let \mathbf{O} represent the zero matrix of appropriate dimension. By definition, the sum of all entries in each row of L equals 0, implying $L\mathbf{1} = \mathbf{0}$. Let I be the identity matrix. Then matrix $I + L$ is invertible, with each entry of matrix $(I + L)^{-1}$ being positive [11, 12].

2.2 Undirected Graph and Its Laplacian Matrix

In a weighted digraph Γ , if for any arc $i \rightarrow j$ with weight w_{ij} , the arc $j \rightarrow i$ exists and has weight w_{ji} equal to w_{ij} , then Γ is called an undirected weighted graph, denoted by $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), w)$. Arcs $j \rightarrow i$ and $i \rightarrow j$ together form an undirected edge in \mathcal{G} . Let w_{\max} and w_{\min} denote, respectively, the maximum and minimum weight among all edges in $E(\mathcal{G})$ of graph \mathcal{G} . By definition, for an undirected graph \mathcal{G} , $w_{ij} = w_{ji}$ holds for any pair of nodes i and j , and $d_i^+ = d_i^-$ holds for any node i . Moreover, both the weighted

adjacency matrix A and Laplacian matrix L of \mathcal{G} are symmetric, with the latter obeying $L\mathbf{1} = \mathbf{0}$.

For an undirected weighted graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), w)$ with n nodes and m edges, there is an alternative construction for its Laplacian matrix L . Let $B \in \mathbb{R}^{|E(\mathcal{G})| \times |V(\mathcal{G})|}$ represent the signed edge-node incidence matrix of \mathcal{G} . It is an $m \times n$ matrix, whose entry b_{ev} with $e \in E(\mathcal{G})$ and $v \in V(\mathcal{G})$ is defined as follows: $b_{ev} = 1$ (resp. $b_{ev} = -1$) if node v is the head (resp. tail) of edge e , and $b_{ev} = 0$ otherwise. Let \mathbf{e}_i denote the i -th standard basis vector. In matrix B , the row vector \mathbf{b}_e corresponding to e linking two nodes i and j can be written as $\mathbf{b}_e = \mathbf{e}_i - \mathbf{e}_j$. We use W to denote the $m \times m$ diagonal weight matrix $\text{diag}(w_1, w_2, \dots, w_m)$, the e -th diagonal entry of which is equal to the weight of edge w_e . Then the Laplacian matrix L of \mathcal{G} can also be written as $L = B^T W B = \sum_{e \in E(\mathcal{G})} w_e \mathbf{b}_e \mathbf{b}_e^T$, implying that L is symmetric and positive semidefinite.

The positive semidefiniteness of Laplacian matrix L for graph \mathcal{G} means that all eigenvalues of L are non-negative. Moreover, when graph \mathcal{G} is connected, its Laplacian matrix L has a unique zero eigenvalue. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \lambda_n = 0$ be the n eigenvalues of L for connected \mathcal{G} . Then, $\lambda_{\max} = \lambda_1 \leq n w_{\max}$ [45], and $\lambda_{\min} = \lambda_{n-1} \geq w_{\min}/n^2$ [34]. Moreover, it has been shown that for any diagonal matrix $K = \text{diag}(k_1, k_2, \dots, k_n)$ with $k_i > 0$ ($i = 1, 2, \dots, n$), matrix $K + L$ is invertible, and every entry of matrix $(K + L)^{-1}$ is positive [33].

2.3 Some Useful Notations

In this subsection, we introduce some notations and their properties, which are useful for our description and proofs. For two non-negative scalars $a \geq 0$ and $b \geq 0$, a is called an ϵ -approximation of b and denoted by $a \approx_\epsilon b$, if $(1 - \epsilon)a \leq b \leq (1 + \epsilon)a$ for $0 \leq \epsilon \leq 1/2$. The ϵ -approximation has the following basic properties. For nonnegative scalars a, b, c , and d , if $a \approx_\epsilon b$ and $c \approx_\epsilon d$, then $a + c \approx_\epsilon b + d$. For two matrices X and Y , if $Y - X$ is positive semidefinite we write $X \leq Y$. In other words, for every real vector \mathbf{x} , the relation $\mathbf{x}^T X \mathbf{x} \leq \mathbf{x}^T Y \mathbf{x}$ holds. For a diagonal matrix $K = \text{diag}(k_1, k_2, \dots, k_n)$, we use k_{\max} and k_{\min} to denote, respectively, the maximum and minimum values of its diagonal entries. Then, for the Laplacian matrix L of a connected undirected graph \mathcal{G} , the following expressions hold true: $K \leq L + K$, $L \leq L + K$, $L + K \leq (k_{\max} + n w_{\max})I$, and $\frac{1}{k_{\max} + n w_{\max}} L \leq I$.

3 FJ MODEL WITH HETEROGENEOUS STUBBORNNESS

In this section, we study the FJ model [23] of opinion dynamics on an undirected weighted graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ with heterogeneous stubborn agents. We first introduce the FJ model on a connected digraph Γ with homogeneous stubbornness.

3.1 FJ Model on a Digraph with Unit Stubbornness

The FJ model is a classic model for opinion formulation [23]. For the FJ model on a digraph $\Gamma = (V(\Gamma), E(\Gamma), w)$, each node $i \in V$ has two different opinions, internal (or innate) opinion s_i and expressed opinion z_i . Moreover, every individual displays some stubbornness k_i against changing its opinion. The internal opinion s_i is assumed to remain constant in the interval $[-1, 1]$, while the

expressed opinion $z_i(t+1)$ at time $t+1$ evolves as a weighted average of its internal opinion s_i and the expressed opinions of its neighbors at time t as

$$z_i(t+1) = \frac{s_i + \sum_{j \in \Theta_i} w_{ij} z_j(t)}{1 + \sum_{j \in \Theta_i} w_{ij}}. \quad (1)$$

Note that during the above updating process of expressed opinions, the stubbornness k_i of every node i is 1, which is a common assumption in the literature. After evolution of enough time, the expressed opinions of nodes converge. Let $\mathbf{z} = (z_1, z_2, \dots, z_n)^\top$ be the vector of expressed opinions, where z_i is the expressed opinion of node i at equilibrium. It was shown [8] that the unique equilibrium opinion vector \mathbf{z} is

$$\mathbf{z} = (\mathbf{I} + \mathbf{L})^{-1} \mathbf{s}. \quad (2)$$

In [36], $\Omega = (\mathbf{I} + \mathbf{L})^{-1} = (\omega_{ij})_{n \times n}$ is called the fundamental matrix of the FJ opinion dynamics model. It has been proved that matrix Ω is row stochastic satisfying $\Omega \mathbf{1} = \mathbf{1}$ [11, 12]. Then, (2) indicates that for any node $i \in V$, its expressed opinion z_i at equilibrium is $z_i = \sum_{j=1}^n \omega_{ij} s_j$, which is a convex combination of the internal opinions of all nodes including i itself. Since for $i \neq j$, $\omega_{ii} > \omega_{ij}$ holds [11, 12], z_i has a bias towards the internal opinion s_i . That is, at equilibrium any node is inclined to keep its internal opinion of its own rather than others.

3.2 FJ Model on a Graph with Heterogeneous Stubbornness

Many research works [21, 28, 29] show that the stubbornness of individuals are heterogeneous. Here we study FJ model on an undirected weighted graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), \mathbf{w})$ with inhomogeneous stubbornness, with an aim to explore the influences of inhomogeneous stubbornness on various aspects of opinion dynamics.

3.2.1 Evolution Rule. Different from the (1), for the FJ model on graph \mathcal{G} , each node i has different inclination to his initial opinion. At time $t+1$, the expressed opinion of node i updates in the following way:

$$z_i(t+1) = \frac{k_i s_i + \sum_{j \in \Theta_i} w_{ij} z_j(t)}{k_i + \sum_{j \in \Theta_i} w_{ij}}, \quad (3)$$

where the stubbornness coefficient $k_i \geq 0$ measures the stubbornness of node i to his initial opinion. We consider the case that the stubbornness coefficients of all nodes are greater than 0 obeying $k_1 k_2 \dots k_n \neq 0$. In what follows, we call $\mathbf{K} = \text{diag}(k_1, k_2, \dots, k_n)$ the stubbornness matrix of \mathcal{G} .

3.2.2 Existence and Expression of Equilibrium Opinions. We now determine the expressed opinions at equilibrium. We first express (3) in matrix form as

$$\mathbf{z}(t+1) = \mathbf{Q} \mathbf{A} \mathbf{z}(t) + \mathbf{Q} \mathbf{K} \mathbf{s}, \quad (4)$$

where $\mathbf{Q} = \text{diag}(q_1, q_2, \dots, q_n)$ is an $n \times n$ diagonal matrix with the i -th diagonal entry q_i being $q_i = 1/(k_i + \sum_{j \in \Theta_i} w_{ij})$.

By iterating (4), the expressed opinion vector $\mathbf{z}(t)$ at time t is obtained to be

$$\mathbf{z}(t) = (\mathbf{Q} \mathbf{A})^t \mathbf{z}(0) + \sum_{s=0}^{t-1} (\mathbf{Q} \mathbf{A})^s \mathbf{Q} \mathbf{K} \mathbf{z}(0). \quad (5)$$

Note that the sum of the i -th row for matrix $\mathbf{Q} \mathbf{A}$ is $\frac{\sum_{j \in \Theta_i} w_{ij}}{k_i + \sum_{j \in \Theta_i} w_{ij}} < 1$ for any $i \in V$, implying that matrix $\mathbf{Q} \mathbf{A}$ is row sub-stochastic. According to Perron-Forbenius theorem [35], the absolute values of all eigenvalues of matrix $\mathbf{Q} \mathbf{A}$ are less than 1. Thus, we have $\lim_{t \rightarrow \infty} (\mathbf{Q} \mathbf{A})^t = \mathbf{O}$. Then, we can conclude the existence of vector \mathbf{z} for equilibrium expressed opinions, which reads

$$\mathbf{z} = \lim_{t \rightarrow \infty} \mathbf{z}(t) = (\mathbf{L} + \mathbf{K})^{-1} \mathbf{K} \mathbf{z}(0) = (\mathbf{L} + \mathbf{K})^{-1} \mathbf{K} \mathbf{s}. \quad (6)$$

Let $\Phi = (\phi_{ij})_{n \times n} \triangleq (\mathbf{L} + \mathbf{K})^{-1} \mathbf{K}$ be the fundamental matrix of this opinion model. Due to $\mathbf{L} \mathbf{1} = \mathbf{0}$, relation $(\mathbf{L} + \mathbf{K}) \mathbf{1} = \mathbf{K} \mathbf{1}$ holds. On the other hand, matrix $\mathbf{L} + \mathbf{K}$ is invertible, then $(\mathbf{L} + \mathbf{K})^{-1} \mathbf{K} \mathbf{1} = \mathbf{1}$, implying that $\Phi = (\mathbf{L} + \mathbf{K})^{-1} \mathbf{K}$ is row-stochastic. Equation (6) shows that, for each node $i \in V$ its expressed opinion z_i is $z_i = \sum_{j=1}^n \phi_{ij} s_j$, which is a convex combination of the internal opinions for all nodes in V . Since Φ is determined by network structure, edge weighs, and the levels of stubbornness of nodes, its is the same with the equilibrium expressed opinion for every node when the internal opinions are fixed.

3.2.3 Properties of Opinions. Using the properties of the fundamental matrix $(\mathbf{L} + \mathbf{K})^{-1} \mathbf{K}$, we give some relations between internal opinion vector \mathbf{s} and expressed opinion vector \mathbf{z} .

PROPERTY 1. For a graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), \mathbf{w})$, if the weighted sum of internal opinions $\mathbf{1}^\top \mathbf{K} \mathbf{s}$ is 0, then the weighted sum of expressed opinions $\mathbf{1}^\top \mathbf{K} \mathbf{z}$ in equilibrium is also 0.

PROPERTY 2. For opinion dynamics on a graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), \mathbf{w})$, if the internal opinion vector \mathbf{s} is changed to $\mathbf{s}' = \mathbf{s} + c \mathbf{1}$ with c being a constant, then the corresponding equilibrium opinion vector is changed from \mathbf{z} to $\mathbf{z}' = \mathbf{z} + c \mathbf{1}$.

Thus, without loss of generality, in the sequel we assume that the internal opinions obey $\mathbf{1}^\top \mathbf{K} \mathbf{s} = 0$, which means $\mathbf{1}^\top \mathbf{K} \mathbf{z} = 0$. Otherwise, if $\mathbf{1}^\top \mathbf{K} \mathbf{s} \neq 0$, we change \mathbf{s} to $\mathbf{s}' = \mathbf{s} - \frac{\mathbf{1}^\top \mathbf{K} \mathbf{s}}{n} \mathbf{1}$. By Property 2, the expressed opinion vector \mathbf{z} is changed to $\mathbf{z}' = \mathbf{z} - \frac{\mathbf{1}^\top \mathbf{K} \mathbf{s}}{n} \mathbf{1}$.

3.3 Interpretation in Terms of Spanning Diverging Forests

Here we provide an explanation for the expressed opinions $\mathbf{z} = (\mathbf{L} + \mathbf{K})^{-1} \mathbf{K} \mathbf{s}$. For this purpose, we map $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), \mathbf{w})$ with stubbornness matrix \mathbf{K} into a weighted digraph \mathcal{G}' . The node set V' of digraph \mathcal{G}' is the same as the node set V of graph \mathcal{G} . For each edge e with weight $w_{ij} = w_{ji}$ in \mathcal{G} linking two end nodes i and j , we create two arcs $i \rightarrow j$ and $j \rightarrow i$ in \mathcal{G}' , the weights of which are w_{ij}/k_i and w_{ji}/k_j , respectively. Recall that the fundamental matrix Φ for opinion dynamics on graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), \mathbf{w})$ with stubbornness matrix \mathbf{K} can be written as $\Phi = (\mathbf{L} + \mathbf{K})^{-1} \mathbf{K} = (\mathbf{I} + \mathbf{K}^{-1} \mathbf{L})^{-1}$. Since $\mathbf{K}^{-1} \mathbf{L}$ is the Laplacian matrix of digraph \mathcal{G}' , according to (2), it is easy to verify that Φ is in fact the fundamental matrix of opinion dynamics on \mathcal{G}' , when the stubbornness of all nodes is 1. Below, we give an interpretation of expressed opinions $\mathbf{z} = (\mathbf{L} + \mathbf{K})^{-1} \mathbf{K} \mathbf{s}$ in terms of spanning converging forests of digraph \mathcal{G}' .

For a digraph \mathcal{G}' with node set V' and arc set E' , a subdigraph of \mathcal{G}' is a digraph, whose node set and arc set are subsets of V' and E' , respectively, but the node set V' must comprise the head and tail

of each arc in set E' . A spanning subdigraph of \mathcal{G}' is a subdigraph with node set V' . A spanning diverging tree or an out-tree is a weakly connected digraph in which one node, called the root, has in-degree 0 and the remaining nodes have in-degree 1. An isolated node is considered as a diverging tree with the root being itself. A spanning diverging forest of \mathcal{G}' is a spanning subdigraph of \mathcal{G}' , where every weakly connected component is a diverging tree. A spanning diverging forest is also called a spanning out-forest [3, 10].

For a subdigraph $\bar{\mathcal{G}}$ of digraph \mathcal{G}' , its weight $\varepsilon(\bar{\mathcal{G}})$ is defined as the product of the weights of all arcs in $\bar{\mathcal{G}}$. If $\bar{\mathcal{G}}$ has no arc, its weight $\varepsilon(\bar{\mathcal{G}})$ is set to be 1. For a nonempty set S of subdigraphs, define its weight $\varepsilon(S)$ as $\varepsilon(S) = \sum_{\bar{\mathcal{G}} \in S} \varepsilon(\bar{\mathcal{G}})$. If S is empty, $\varepsilon(S)$ is defined to be zero [11, 12]. For a digraph \mathcal{G}' , let Y be the set of all its out-forests, and Y_{ij} the set of out-forests with nodes i and j in the same out-tree rooted at node i . In [11, 12], the diverging forest matrix of a digraph \mathcal{G}' is proposed and defined as $\Phi = (\phi_{ij})_{n \times n} = (I + K^{-1}L)^{-1}$, where the ij -th entry ϕ_{ij} is $\phi_{ij} = \varepsilon(Y_{ij})/\varepsilon(Y)$. Obviously, the diverging forest matrix Φ is exactly the fundamental matrix of opinion dynamics on \mathcal{G}' , when the stubbornness matrix is an identity matrix.

In this way, we have provided an interpretation of the equilibrium expressed opinion vector z given in (6) for a undirected graph \mathcal{G} , in terms of the diverging forest matrix of a corresponding digraph \mathcal{G}' , where edge weights contain the information of node stubbornness. The perspective of our interpretation is novel, which is different from previous ones, such as absorbing random walks [26], game theory [8], and electrical networks [25].

4 EFFECTS OF HETEROGENEOUS STUBBORNNESS ON OPINION DYNAMICS

In this section, we give insight on how the stubbornness affects the convergence velocity to equilibrium opinions, overall opinion, and the expressed opinion of every node.

4.1 Convergence Time

Equation (6) characterizes the equilibrium expressed opinions. This equilibrium behavior is relevant only if it converges in a reasonable time [19]. Below study the impact of the level of stubbornness on the convergence time. We first give a definition of error vector for expressed opinion vector $z(t)$ at time t . For brevity, in the sequel, we use $\mathcal{G} = (V, E, w)$ to represent $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), w)$.

DEFINITION 4.1. For opinion dynamics on a graph $\mathcal{G} = (V, E, w)$, the error vector $\mathbf{e}(t)$ of expressed opinion vector $\mathbf{z}(t)$ at time t is the difference of $\mathbf{z}(t)$ and the equilibrium opinion vector \mathbf{z} : $\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{z}$.

Based on (5), $\mathbf{e}(t)$ can be expressed as

$$\mathbf{e}(t) = (QA)^t \left(\mathbf{z}(0) - \sum_{s=0}^{\infty} (QA)^s QK\mathbf{z}(0) \right),$$

which leads to the following recursive relation governing $\mathbf{e}(t)$ and $\mathbf{e}(t+1)$:

$$\mathbf{e}(t+1) = QA\mathbf{e}(t). \quad (7)$$

In order to give a better analysis of the convergence time, we introduce a new error vector $\mathbf{f}(t)$ defined as $\mathbf{f}(t) = Q^{-\frac{1}{2}}\mathbf{e}(t)$, the i -th

entry of which is $f_i(t) = \frac{e_i(t)}{\sqrt{d_i + k_i}}$. Using (7), we obtain

$$\mathbf{f}(t+1) = Q^{\frac{1}{2}}AQ^{\frac{1}{2}}\mathbf{f}(t). \quad (8)$$

We now define the convergence time of opinion dynamics on graph $\mathcal{G} = (V, E, w)$ and give an upper bound for it.

DEFINITION 4.2. For opinion dynamics on a graph $\mathcal{G} = (V, E, w)$, its convergence time $t(\varepsilon)$ is defined as

$$t(\varepsilon) = \inf \{t \geq 0, |\mathbf{f}(t)| \leq \varepsilon\}. \quad (9)$$

Let ρ_{\max} be the spectral radius of matrix QA . Since QA is sub-stochastic, ρ_{\max} is always less than 1.

PROPOSITION 4.3. For opinion dynamics on a graph $\mathcal{G} = (V, E, w)$, its convergence time $t(\varepsilon)$ is not greater than $\log_{\rho_{\max}} \varepsilon - \log_{\rho_{\max}} |\mathbf{f}(0)|$.

Proof. Since $Q^{\frac{1}{2}}AQ^{\frac{1}{2}} = Q^{-\frac{1}{2}}(QA)Q^{\frac{1}{2}}$, $Q^{\frac{1}{2}}AQ^{\frac{1}{2}}$ and QA has the same set of eigenvalues. Let $1 > \rho_{\max} = |\rho_1| \geq |\rho_2| \geq \dots \geq |\rho_n|$ be the eigenvalues of the symmetric matrix $Q^{\frac{1}{2}}AQ^{\frac{1}{2}}$, and let $\xi_1, \xi_2, \dots, \xi_n$ be their corresponding orthogonal eigenvectors. Then $\mathbf{f}(t)$ can be recast as

$$\mathbf{f}(t) = \sum_{i=1}^n \frac{\xi_i^\top \mathbf{f}(t)}{\xi_i^\top \xi_i} \xi_i. \quad (10)$$

Based on (8) and (10), we obtain the recursive relation between $\mathbf{f}(t+1)$ and $\mathbf{f}(t)$ as

$$\mathbf{f}(t+1) = Q^{\frac{1}{2}}AQ^{\frac{1}{2}}\mathbf{f}(t) = \sum_{i=1}^n \frac{\xi_i^\top \mathbf{f}(t)}{\xi_i^\top \xi_i} \rho_i \xi_i. \quad (11)$$

Then, we have

$$\begin{aligned} |\mathbf{f}(t+1)|^2 &= \sum_{i=1}^n \left(\frac{\xi_i^\top \mathbf{f}(t)}{\xi_i^\top \xi_i} \right)^2 \rho_i^2 \xi_i^2 \leq \rho_{\max}^2 \sum_{i=1}^n \left(\frac{\xi_i^\top \mathbf{f}(t)}{\xi_i^\top \xi_i} \right)^2 \xi_i^2 \\ &= \rho_{\max}^2 |\mathbf{f}(t)|^2. \end{aligned}$$

Thus, $|\mathbf{f}(t+1)| \leq \rho_{\max} |\mathbf{f}(t)|$ and $|\mathbf{f}(t)| \leq \rho_{\max}^t |\mathbf{f}(0)|$, which means that $|\mathbf{f}(t)|$ decreases exponentially with $|\mathbf{f}(0)|$. When $|\mathbf{f}(t)|$ drops below ε with $0 \leq \varepsilon \leq |\mathbf{f}(0)|$, the convergence time is not great than $\log_{\rho_{\max}} \varepsilon - \log_{\rho_{\max}} |\mathbf{f}(0)|$. \square

Proposition 4.3 indicates that the convergence time of opinion dynamics on graph $\mathcal{G} = (V, E, w)$ is related to the leading eigenvalue ρ_{\max} of matrix QA , with smaller ρ_{\max} corresponding to faster convergence.

The next theorem shows the change of the largest eigenvalue ρ_{\max} for matrix QA with respect to the stubbornness k_i of node i .

THEOREM 4.4. For opinion dynamics on an undirected graph \mathcal{G} with stubbornness matrix K , the largest eigenvalue ρ_{\max} of matrix QA is a decreasing function of stubbornness k_i of node i , if the stubbornness of all other nodes is fixed.

Proof. Let \mathbf{u} be the unit eigenvector of matrix QA corresponding to the largest eigenvalue ρ_{\max} . If we increase the stubbornness k_i of node i to k'_i , while fixing the values of the stubbornness of other nodes, we use ρ'_{\max} and \mathbf{v} to represent, respectively, the largest eigenvalue and its corresponding unit eigenvector for the resulting matrix $Q'A$. By Perron-Frobenius theorem [35], all elements in \mathbf{u} and \mathbf{v} are positive. Then, we have

$$\rho_{\max} = \mathbf{u}^\top QA\mathbf{u} \geq \mathbf{v}^\top QA\mathbf{v} > \mathbf{v}^\top Q'A\mathbf{v} = \rho'_{\max},$$

which finishes the proof. \square

Thus, increasing the stubbornness value of a node can accelerate the convergence of opinion dynamics. On the contrary, reducing the level of stubbornness of a node may deteriorate the convergence speed.

4.2 Overall Expressed Opinion

We continue to study the influence of inhomogeneous stubbornness on the overall opinion, that is, the sum of expressed opinions of all nodes, which has received considerable recent attention [2, 26, 53].

It was shown in [26] that for the FJ model of opinion dynamics on an undirected graph with each node having a unit stubbornness, the overall opinion is equal to the total internal opinion, that is $\sum_{i=1}^n s_i = \sum_{i=1}^n z_i$, although the equilibrium expressed opinion for an individual node may differ from its internal opinion. Moreover, this conservation law does not rely on the network structure and the distribution of edge weights. This conservation law can also be understood as follows. Equation (6) shows when $K = I$, the fundamental matrix is reduced to $\Phi = (I + L)^{-1}$, which is symmetric and doubly stochastic [37] obeying $\sum_{j=1}^n \phi_{ij} = 1$ for $i = 1, 2, \dots, n$, and $\sum_{i=1}^n \phi_{ij} = 1$ for $j = 1, 2, \dots, n$. Then, $\sum_{i=1}^n z_i = \sum_{i=1}^n \sum_{j=1}^n \phi_{ij} s_j = \sum_{j=1}^n \sum_{i=1}^n \phi_{ij} s_j = \sum_{i=1}^n s_i$. Actually, for any $K = kI$ with k being a positive constant, $\sum_{i=1}^n s_i = \sum_{i=1}^n z_i$ always holds.

In the case that the stubbornness matrix K is not a scalar matrix, the opinion dynamics on undirected graph \mathcal{G} is equivalent to opinion dynamics on a digraph \mathcal{G}' with stubbornness matrix being I . Note that for any pair of arcs $i \rightarrow j$ and $j \rightarrow i$ in digraph \mathcal{G}' , their weights are usually unequal, with the exception of $k_i = k_j$. Thus, the Laplacian matrix $K^{-1}L$ of digraph \mathcal{G}' is asymmetric, it is the same with the diverging forest matrix $\Omega = (K^{-1}L + I)^{-1}$. This asymmetry is stemmed from the heterogeneous stubbornness, which is ubiquitous in social systems. Then matrix $\Omega = (K^{-1}L + I)^{-1}$ might not be column stochastic, although it is always row stochastic. This reciprocity may lead to the disappearance of conservation of total opinions, namely, $\sum_{i=1}^n z_i \neq \sum_{i=1}^n s_i$. It is not difficult to verify that heterogeneous stubbornness can not only increase but also decrease the overall opinion, exerting a substantial influence on the overall opinion.

4.3 Equilibrium Expressed Opinions of Nodes

As shown in (6), the expressed opinion z_i of a node $i \in V$ is $z_i = \sum_{j=1}^n \phi_{ij} s_j$, where ϕ_{ij} is the ij -entry of the fundamental matrix Φ . Since Φ depends on the stubbornness matrix, it is expected that ϕ_{ij} is related to the level of stubbornness of every node. Next we study the influence of node stubbornness on the entries of matrix Φ .

THEOREM 4.5. *For opinion dynamics on an undirected graph $\mathcal{G} = (V, E, w)$ with Laplacian matrix L and stubbornness matrix K , if the stubbornness k_v of node $v \in V$ is decreased while the stubbornness of all other nodes in V is fixed, then for matrix $\Phi = (L + K)^{-1}K$, the elements in column v decrease, while elements not in v column increase.*

Proof. Let $E_{v,v}$ be the $n \times n$ matrix, which has only one nonzero entry 1 at row v and column v , while other entries are zeros. For matrix $K = \text{diag}(k_1, k_2, \dots, k_n)$, if k_v is decreased while other k_i ($i \neq v$) keeps unchanged, then $H \triangleq K^{-1} = \text{diag}(1/k_1, 1/k_2, \dots, 1/k_n)$ is changed to $H' = H + dE_{v,v}$ with $d > 0$.

Define $X = \Phi = (HL + I)^{-1}$ and $Y = (H'L + I)^{-1}$, and $\Delta X = Y - X$. Then, we have $X(HL + I) = (X + \Delta X)((H^{-1} + dE_{v,v})L + I)$. Expanding the above equation gives $-\Delta X(H'L + I) = dXE_{v,v}L$, which means $\Delta X = -dXE_{v,v}LY$. Considering $Y = X + \Delta X$, we further have

$$\Delta X = -d(dXE_{v,v}L + I)^{-1}XE_{v,v}LX.$$

Define vector $x = Xe_v$ and vector $y = Le_v$. According to Sherman-Morrison formula [38], we have

$$(dXE_{v,v}L + I)^{-1} = (I + dxy^T)^{-1} = I - \frac{dxy^T}{1 + dy^Tx}.$$

Then, the i -th element of vector ΔXe_v at the v -th column of matrix ΔX is

$$e_i^T \Delta Xe_v = -\frac{de_i^T xy^T x}{1 + dy^Tx}.$$

Since the elements in $X = \Phi$ are positive, $e_i^T x > 0$ always holds. For $y^T x$, we have

$$y^T x = e_v^T LX e_v = \sum_{j=1}^n l_{vj} \phi_{jv} > \sum_{j=1}^n l_{vj} \phi_{vv} = 0. \quad (12)$$

Thus the elements in the v -th column of ΔX are less than 0, implying that the elements in the v -th column of Φ will decrease if the stubbornness k_v is reduced.

For the j -th column of the matrix Φ , $j \neq v$, using a similar approach one can obtain

$$e_i^T \Delta Xe_j = -\frac{de_i^T X e_v e_v^T L X e_j}{1 + dy^T x}. \quad (13)$$

It is clear that $y^T x > 0$ and $e_i^T X e_v > 0$. Then we only need to evaluate the elements in LX that can be recast as

$$LX = L(L + K)^{-1}K = K - K(L + K)^{-1}K. \quad (14)$$

Since K is a diagonal matrix and $(L + K)^{-1}$ is a positive matrix, all the non-diagonal elements of matrix LX are less than 0, which means $e_v^T LX e_j < 0$ for any $j \neq v$. Thus, for every $j \neq v$, all the elements in the j -th column of matrix ΔX are greater than 0, implying that all the elements in the j -th column of matrix Φ will increase if we reduce the stubbornness k_v . \square

Therefore, for opinion dynamics on an undirected graph $\mathcal{G} = (V, E, w)$, if we increase the stubbornness k_v of node $v \in V$, while fixing the stubbornness of all other nodes, all nodes (including node v) will place more weight on the initial opinion of node v to form their equilibrium expressed opinions.

5 METRICS FOR RELATED PHENOMENA

Under the influence of stubbornness for individuals, equilibrium opinions in the FJ model do not reach consensus, leading to some social phenomena, such as conflict, disagreement, polarization. For the case that nodes have unit stubbornness, These phenomena have

been quantified and studied in prior works [13, 14, 39, 50, 51]. However, for the case that node stubbornness is heterogeneous, related measures and algorithms for these quantities are still less studied.

In this section, we extend previous quantitative measures for social phenomena to the FJ opinion model on undirected graphs described in (3), which incorporates heterogeneous stubbornness of nodes. Moreover, we express these quantities in terms of quadratic forms and provide a conservation law governing them.

5.1 Definitions for Metrics

In the FJ model, the internal opinions and expressed opinions for individuals are often different, the extent of which is measured by internal conflict defined below.

DEFINITION 5.1. *For opinion dynamics on an undirected graph $\mathcal{G} = (V, E, w)$ with stubborn matrix K , its internal conflict $C(\mathcal{G})$ is the weighted sum of squares of the differences between internal and expressed opinions over all nodes in V , with the weights being their corresponding stubbornness:*

$$C(\mathcal{G}) = \sum_{i \in V} k_i (z_i - s_i)^2. \quad (15)$$

Note that (15) reduces to the expression for internal conflict in [13], when $k_i = 1$ for all $i \in V$.

DEFINITION 5.2. [14, 39] *For opinion dynamics on an undirected graph $\mathcal{G} = (V, E, w)$ with stubborn matrix K , the disagreement between two nodes i and j is defined as $w_{ij}(z_i - z_j)^2$. The disagreement of the whole graph \mathcal{G} is the sum of the squared differences between all pairs of nodes, given by*

$$D(\mathcal{G}) = \sum_{(i,j) \in E, i < j} w_{ij}(z_i - z_j)^2. \quad (16)$$

Disagreement $D(\mathcal{G})$ is also called the external conflict of graph \mathcal{G} [13].

In the FJ model, the expressed opinion of a node often deviates from the (weighted) average $\mathbf{1}^T K \mathbf{z} = 0$ of the expressed opinions for all nodes. We use polarization to measure how expressed opinions deviate from their weighted average in the equilibrium.

DEFINITION 5.3. *For opinion dynamics on an undirected graph $\mathcal{G} = (V, E, w)$ with stubborn matrix K , the polarization $P(\mathcal{G})$ is defined to be:*

$$P(\mathcal{G}) = \sum_{i \in V} k_i z_i^2. \quad (17)$$

Equation (17) is reduced (3) in [39], when $k_i = 1$ for all $i \in V$.

There is a trade-off between disagreement and polarization, the sum of which is called polarization-disagreement index [39].

DEFINITION 5.4. *For opinion dynamics on an undirected graph $\mathcal{G} = (V, E, w)$ with stubborn matrix K , the polarization-disagreement index $I_{pd}(\mathcal{G})$ is the sum of the polarization $P(\mathcal{G})$ and disagreement $D(\mathcal{G})$: $I_{pd}(\mathcal{G}) = P(\mathcal{G}) + D(\mathcal{G})$.*

It is easy to derive that the polarization-disagreement index $I_{pd}(\mathcal{G})$ equals $\sum_{i=1}^n k_i s_i z_i$, which reduces to the result in [39] when $K = I$.

5.2 Expressions in Terms of Quadratic Forms

The above-defined quantities for social phenomena can be expressed by quadric forms of related matrices.

PROPOSITION 5.5. *For opinion dynamics on an undirected graph $\mathcal{G} = (V, E, w)$ with Laplacian matrix L , stubborn matrix K and internal opinion vector \mathbf{s} , the quantities $C(\mathcal{G})$, $D(\mathcal{G})$, $P(\mathcal{G})$, and $I_{pd}(\mathcal{G})$ can be expressed, in terms of quadratic forms as:*

$$C(\mathcal{G}) = \mathbf{s}^T K(L + K)^{-1} L K^{-1} L(L + K)^{-1} K \mathbf{s}, \quad (18)$$

$$D(\mathcal{G}) = \mathbf{s}^T K(L + K)^{-1} L(L + K)^{-1} K \mathbf{s}, \quad (19)$$

$$P(\mathcal{G}) = \mathbf{s}^T K(L + K)^{-1} K(L + K)^{-1} K \mathbf{s}, \quad (20)$$

$$I_{pd}(\mathcal{G}) = \mathbf{s}^T K(L + K)^{-1} K \mathbf{s}. \quad (21)$$

The quantities concerned obey the following conservation law:

$$C(\mathcal{G}) + 2D(\mathcal{G}) + P(\mathcal{G}) = \sum_{i \in V} k_i s_i^2, \quad (22)$$

which extends the result in [39] when $K = I$.

6 ALGORITHM FOR RELATED QUANTITIES

In this section, we address the problem of fast calculation for those quantities defined in last section, including $C(\mathcal{G})$, $D(\mathcal{G})$, $P(\mathcal{G})$, and $I_{pd}(\mathcal{G})$. Proposition 5.5 shows that exactly computing the quantities involves inverting matrix $L + K$, which takes $O(n^3)$ time and is computationally unacceptable for large graphs. In order to tackle the computational barrier, a fast approximation algorithm is developed, which has a nearly linear time complexity with respect to the number of edges in \mathcal{G} . We first explicitly express the concerned quantities as ℓ_2 norms of vectors.

LEMMA 6.1. *Let $W^{1/2}$ be a diagonal matrix defined as $W^{1/2} = \text{diag}(\sqrt{w_1}, \sqrt{w_2}, \dots, \sqrt{w_m})$. For opinion dynamics on an undirected graph $\mathcal{G} = (V, E, w)$ with Laplacian matrix L , stubborn matrix K and internal opinion vector \mathbf{s} , the quantities $C(\mathcal{G})$, $D(\mathcal{G})$, $P(\mathcal{G})$, and $I_{pd}(\mathcal{G})$ can be expressed, respectively, in terms of ℓ_2 norms as:*

$$C(\mathcal{G}) = \mathbf{z}^T L K^{-1} L \mathbf{z} = \|K^{-1/2} L(L + K)^{-1} \mathbf{s}\|^2, \quad (23)$$

$$D(\mathcal{G}) = \mathbf{z}^T L \mathbf{z} = \|W^{1/2} B(L + K)^{-1} \mathbf{s}\|^2, \quad (24)$$

$$P(\mathcal{G}) = \mathbf{z}^T K \mathbf{z} = \|K^{1/2} (L + K)^{-1} \mathbf{s}\|^2, \quad (25)$$

$$I_{pd}(\mathcal{G}) = \|W^{1/2} B(L + K)^{-1} \mathbf{s}\|^2 + \|K^{-1/2} (L + K)^{-1} \mathbf{s}\|^2. \quad (26)$$

Since directly calculating the ℓ_2 norms still requires inverting matrix $L + K$, below we resort to the linear system solvers [31] that can efficiently avoid the inverse operation, thus significantly reducing the computational complexity.

LEMMA 6.2. [31] *There is a nearly linear time solver $\mathbf{y} = \text{SOLVE}(T, \mathbf{x}, \delta)$, which takes an $n \times n$ positive semi-definite matrix T with m non-zero entries, a column vector \mathbf{x} , and an accuracy parameter δ , and returns a column vector \mathbf{y} satisfying $\|\mathbf{y} - T^\dagger \mathbf{x}\|_T \leq \delta \|T^\dagger \mathbf{x}\|_T$, where $\|\mathbf{v}\|_T = \sqrt{\mathbf{v}^T T \mathbf{v}}$ and T^\dagger is the pseudo-inverse of matrix T . The expected time for performing this solver is $O\left(m \log^3 n \log\left(\frac{1}{\delta}\right)\right)$.*

We now exploit Lemma 6.2 to obtain ϵ -approximations for the concerned quantities.

LEMMA 6.3. *For opinion dynamics on an undirected weighted graph $\mathcal{G} = (V, E, w)$ with each edge weight in the interval $[w_{\min}, w_{\max}]$, Laplacian matrix L , incident matrix B , diagonal edge weight matrix W , parameter $\epsilon \in (0, \frac{1}{2})$, internal opinion vector $\mathbf{s} = (s_1, s_2, \dots, s_n)^T$,*

and stubbornness matrix K obeying relation $\mathbf{1}^\top K \mathbf{s} = 0$, let $\mathbf{q} = \text{SOLVE}(\mathbf{L} + \mathbf{K}, \mathbf{s}, \delta)$, then the following relations hold:

$$(1 - \epsilon) \|\mathbf{K}^{1/2}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|^2 \leq \|\mathbf{K}^{1/2} \mathbf{q}\|^2 \leq (1 + \epsilon) \|\mathbf{K}^{1/2}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|^2, \quad (27)$$

if δ satisfies $\delta \leq \delta_1 = \frac{\epsilon}{3\sqrt{k_{\min}^{-1} k_{\max}^{-1} (k_{\max} + n w_{\max})}}$;

$$(1 - \epsilon) \|\mathbf{W}^{1/2} \mathbf{B}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|^2 \leq \|\mathbf{W}^{1/2} \mathbf{B} \mathbf{q}\|^2 \leq (1 + \epsilon) \|\mathbf{W}^{1/2} \mathbf{B}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|^2, \quad (28)$$

if δ satisfies $\delta \leq \delta_2 = \frac{\epsilon k_{\min} \|\mathbf{s}\|}{3n(k_{\max} + n w_{\max})} \sqrt{\frac{w_{\min}}{n(k_{\max} + n w_{\max})}}$;

$$(1 - \epsilon) \|\mathbf{K}^{-1/2} \mathbf{L}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|^2 \leq \|\mathbf{K}^{-1/2} \mathbf{L} \mathbf{q}\|^2 \leq (1 + \epsilon) \|\mathbf{K}^{-1/2} \mathbf{L}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|^2, \quad (29)$$

if δ satisfies

$$\delta \leq \delta_3 = \frac{\epsilon w_{\min} k_{\min} \sqrt{k_{\min}} \|\mathbf{s}\|}{3w_{\max} n^3 (k_{\max} + n w_{\max}) \sqrt{n k_{\max} (k_{\max} + n w_{\max})}}.$$

Proof. By Lemma 6.2, we have

$$\|\mathbf{y} - (\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|_{\mathbf{L} + \mathbf{K}}^2 \leq \delta^2 \|(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|_{\mathbf{L} + \mathbf{K}}^2.$$

The term on the left-hand side (lhs) is bounded as

$$\begin{aligned} \|\mathbf{q} - (\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|_{\mathbf{L} + \mathbf{K}}^2 &\geq k_{\max}^{-1} \|\mathbf{K}^{1/2} \mathbf{q} - \mathbf{K}^{1/2}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|^2 \\ &\geq k_{\max}^{-1} \left| \|\mathbf{K}^{1/2} \mathbf{q}\| - \|\mathbf{K}^{1/2}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\| \right|^2, \end{aligned}$$

while the term on the right-hand side (rhs) is bounded by

$$\|(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|_{\mathbf{L} + \mathbf{K}}^2 \leq k_{\min}^{-1} (k_{\max} + n w_{\max}) \|\mathbf{K}^{1/2}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|^2.$$

Combining the above results leads to

$$\begin{aligned} &\left| \|\mathbf{K}^{1/2} \mathbf{q}\| - \|\mathbf{K}^{1/2}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\| \right|^2 \\ &\leq \delta^2 k_{\min}^{-1} k_{\max}^{-1} (k_{\max} + n w_{\max}) \|\mathbf{K}^{1/2}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|^2, \end{aligned}$$

which implies

$$\begin{aligned} &\frac{\left| \|\mathbf{K}^{1/2} \mathbf{q}\| - \|\mathbf{K}^{1/2}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\| \right|}{\|\mathbf{K}^{1/2}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|} \\ &\leq \sqrt{\delta^2 k_{\min}^{-1} k_{\max}^{-1} (k_{\max} + n w_{\max})} \leq \frac{\epsilon}{3} \end{aligned}$$

and

$$(1 - \frac{\epsilon}{3})^2 \|\mathbf{K}^{1/2}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|^2 \leq \|\mathbf{K}^{1/2} \mathbf{q}\|^2 \leq (1 + \frac{\epsilon}{3})^2 \|\mathbf{K}^{1/2}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|^2. \quad \text{Our code is available at } \text{https://anonymous.4open.science/r/heter_stubbornness}.$$

Using $0 < \epsilon < \frac{1}{2}$, one obtains

$$(1 - \epsilon) \|\mathbf{K}^{1/2}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|^2 \leq \|\mathbf{K}^{1/2} \mathbf{q}\|^2 \leq (1 + \epsilon) \|\mathbf{K}^{1/2}(\mathbf{L} + \mathbf{K})^{-1} \mathbf{s}\|^2,$$

which finishes the proof. \square

The proofs for (28) and (29) are similar to that of (27), and are thus omitted. On the basis of Lemmas 6.2 and 6.3, we design a fast algorithm APPROXIM to estimate $C(\mathcal{G})$, $D(\mathcal{G})$, $P(\mathcal{G})$, and $I_{\text{pd}}(\mathcal{G})$ for opinion dynamics on an undirected weighted graph \mathcal{G} . In Algorithm 1 we present the pseudocode of APPROXIM, where δ is less than or equal to the minimum of $\delta_1, \delta_2, \delta_3$ defined in Lemma 6.3. Theorem 6.4 summarizes the performance of algorithm APPROXIM.

Algorithm 1: APPROXIM ($\mathcal{G}, \mathbf{s}, \epsilon$)

Input : \mathcal{G} : a graph with edge weight in $[w_{\min}, w_{\max}]$
 \mathbf{s} : initial opinion vector with $\mathbf{1}^\top K \mathbf{s} = 0$
 ϵ : the error parameter in $(0, \frac{1}{2})$

Output : $\{\tilde{D}(\mathcal{G}), \tilde{P}(\mathcal{G}), \tilde{C}(\mathcal{G}), \tilde{I}_{\text{pd}}(\mathcal{G})\}$

```

1  $\delta = \frac{\epsilon w_{\min} k_{\min} \sqrt{k_{\min}} \|\mathbf{s}\|}{3 w_{\max} n^3 (k_{\max} + n w_{\max}) \sqrt{n k_{\max} (k_{\max} + n w_{\max})}}$ 
2  $\mathbf{q} = \text{SOLVE}(\mathbf{L} + \mathbf{K}, \mathbf{s}, \delta)$ 
3  $\tilde{C}(\mathcal{G}) = \|\mathbf{K}^{-1/2} \mathbf{L} \mathbf{q}\|^2$ 
4  $\tilde{D}(\mathcal{G}) = \|\mathbf{W}^{1/2} \mathbf{B} \mathbf{q}\|^2$ 
5  $\tilde{P}(\mathcal{G}) = \|\mathbf{K}^{1/2} \mathbf{q}\|^2$ 
6  $\tilde{I}_{\text{pd}}(\mathcal{G}) = \tilde{P}(\mathcal{G}) + \tilde{D}(\mathcal{G})$ 
7 return  $\{\tilde{C}(\mathcal{G}), \tilde{D}(\mathcal{G}), \tilde{P}(\mathcal{G}), \tilde{I}_{\text{pd}}(\mathcal{G})\}$ 

```

THEOREM 6.4. For opinion dynamics on an undirected weighted graph $\mathcal{G} = (V, E, w)$ having n nodes, m edges with weight of each edge being in the interval $[w_{\min}, w_{\max}]$, an error parameter $\epsilon \in (0, \frac{1}{2})$, and the internal opinion vector \mathbf{s} , the algorithm APPROXIM ($\mathcal{G}, \mathbf{s}, \epsilon$) runs in expected time $O\left(m \log^4 n \log\left(\frac{r}{\epsilon}\right)\right)$ where $r = \frac{w_{\max}}{w_{\min}}$. APPROXIM returns $\tilde{C}(\mathcal{G})$, $\tilde{D}(\mathcal{G})$, $\tilde{P}(\mathcal{G})$, $\tilde{I}_{\text{pd}}(\mathcal{G})$ for the ϵ -approximation of internal conflict $C(\mathcal{G})$, disagreement $D(\mathcal{G})$, polarization $P(\mathcal{G})$ and polarization-disagreement index $I_{\text{pd}}(\mathcal{G})$, satisfying $\tilde{C}(\mathcal{G}) \approx_\epsilon C(\mathcal{G})$, $\tilde{D}(\mathcal{G}) \approx_\epsilon D(\mathcal{G})$, $\tilde{P}(\mathcal{G}) \approx_\epsilon P(\mathcal{G})$, and $\tilde{I}_{\text{pd}}(\mathcal{G}) \approx_\epsilon I_{\text{pd}}(\mathcal{G})$.

7 EXPERIMENTS

In this section, we evaluate the efficiency and effectivity of our approximation algorithm APPROXIM. To achieve this goal, we implement this algorithm on a large set of real networks, and compare the accuracy and running time of APPROXIM with an exact algorithm EXACT, which computes relevant quantities via directly inverting the matrix $\mathbf{L} + \mathbf{K}$, performing matrix product of related matrices, and calculating related ℓ_2 norms.

Machine and Repeatability. All our experiments were run on a Linux box with 4-core 4.2GHz Intel i7-7700K CPU and 32GB of main memory. In our experiments, the error parameter ϵ is set to be 10^{-6} . Both approximation algorithm APPROXIM and exact algorithm EXACT are programmed with *Julia* v1.5.1 using a single thread. The solver SOLVE used in this paper is based on the method in [31], whose *Julia* language implementation is open and accessible on the website <https://github.com/danspielman/Laplacians.jl>.

Our code is available at https://anonymous.4open.science/r/heter_stubbornness.

Datasets. The real network datasets used in our experiments are publicly available in the Koblenz Network Collection [30] and Network Repository [43]. The statistics for basic information of these datasets are listed in the first three columns of Table 1.

Distributions of Innate opinions. We use four different distributions of the innate opinions in our experiments, including: uniform distribution, power-law distribution, normal distribution, and exponential distribution. The innate opinion s_i for each node i is in interval $[-1, 1]$.

Efficiency and Scalability. In Table 1, we report the running time of algorithms APPROXIM and EXACT on different networks in order to compare their computation complexity. Note that for

Table 1: Statistics of real datasets and comparison of running time (seconds, s) between EXACT and APPROXIM.

Network	n	m	Running time (s) for algorithms EXACT and APPROXIM							
			Uniform		Power-law		Normal		Exponential	
			EXACT	APPROXIM	EXACT	APPROXIM	EXACT	APPROXIM	EXACT	APPROXIM
GrQc	4158	13422	1.44	2.42	1.41	2.40	1.45	2.40	1.43	2.40
USgrid	4941	6594	2.24	2.41	2.22	2.36	2.25	2.35	2.23	2.35
Erdos992	5094	7515	2.37	2.42	2.37	2.40	2.46	2.40	2.51	2.41
Advogato	5167	39432	2.51	2.44	2.55	2.42	2.52	2.42	2.56	2.43
Bcspwr10	5300	8271	2.71	2.37	2.73	2.60	2.72	2.41	2.69	2.40
Reality	6809	7680	6.01	2.37	6.55	2.36	5.98	2.38	5.95	2.36
PagesGovernment	7057	89429	6.75	2.42	6.84	2.42	7.07	2.41	8.04	2.34
WikiElec	7115	100753	7.45	2.43	6.93	2.43	8.01	2.42	7.18	2.43
Dmela	7393	25569	8.87	2.42	8.51	2.42	9.23	2.41	8.65	2.41
HepPh	11204	117619	30.21	2.37	28.49	2.35	29.15	2.36	28.25	2.35
Anybeat	12645	49132	40.94	2.31	41.39	2.31	41.74	2.30	41.49	2.34
PagesCompany	14113	52126	56.67	2.32	57.07	2.31	59.83	2.43	59.20	2.33
AstroPh	17903	196972	122.64	2.44	119.47	2.44	114.35	2.44	118.19	2.43
CondMat	21363	91286	205.04	2.36	202.79	2.35	203.85	2.37	202.91	2.35
Gplus	23628	39194	276.76	2.63	275.54	2.63	279.40	2.79	280.99	2.69
GemsecRO	41773	125826	1584.76	3.05	1573.48	3.29	1577.72	3.05	1599.97	3.09
GemsecHU	47538	222887	2448.57	4.59	2537.67	4.65	2561.06	4.43	2480.51	4.44
WikiTalk	92117	360767	-	3.06	-	3.16	-	3.11	-	3.13
Buzznet	101163	2763066	-	5.70	-	6.01	-	5.99	-	6.02
LiveMocha	104103	2193083	-	6.33	-	6.09	-	6.09	-	6.04
Douban	154908	327162	-	3.18	-	3.02	-	3.15	-	3.17
Gowalla	196591	950327	-	4.07	-	4.02	-	4.07	-	4.06
Academia	200169	1022441	-	4.27	-	4.34	-	4.35	-	4.44
GooglePlus	211186	1141650	-	4.17	-	4.08	-	4.07	-	4.37
Pwtk	217891	5653221	-	8.16	-	8.33	-	8.24	-	8.06
Citeseer	227320	814134	-	3.74	-	3.61	-	3.66	-	3.71
MathSciNet	332689	820644	-	4.32	-	4.24	-	4.14	-	4.36
TwitterFollows	404719	713319	-	3.64	-	3.67	-	3.67	-	3.63
Flickr	513969	3190452	-	7.42	-	7.99	-	7.68	-	7.84
Delicious	536108	1365961	-	4.92	-	5.09	-	5.06	-	5.00
FourSquare	639014	3214986	-	6.78	-	6.70	-	6.78	-	6.63
Digg	770799	5907132	-	12.59	-	12.61	-	12.51	-	12.25
IMDB	896305	3782447	-	13.21	-	13.99	-	13.17	-	13.59
Ldoor	909537	20770807	-	23.37	-	23.31	-	23.26	-	23.24
RoadNetPA	1087562	1541514	-	6.65	-	7.00	-	6.47	-	6.68
YoutubeSnap	1134890	2987624	-	9.11	-	8.56	-	9.85	-	8.95
Lastfm	1191805	4519330	-	11.81	-	11.50	-	11.16	-	11.58
Pokec	1632803	22301964	-	95.00	-	96.11	-	94.52	-	95.13
RoadNetCA	1957027	2760388	-	10.26	-	10.88	-	10.57	-	10.17
Flixster	2523386	7918801	-	19.99	-	20.27	-	20.75	-	19.96
Patent	3774768	16518947	-	80.63	-	78.45	-	77.77	-	80.85
LiveJournal	4033137	27933062	-	108.10	-	97.97	-	101.74	-	104.98
ItalyOsm	6686493	7013978	-	22.18	-	23.62	-	23.08	-	22.28

large networks we cannot compute related quantities using algorithm EXACT, due to the high cost of memory and time, but we can perform algorithm APPROXIM. For each distribution of the innate opinions, we record the running times of APPROXIM and EXACT in different networks. Table 1 shows that for all networks considered, the running time of APPROXIM is smaller than that of EXACT. For moderately large networks with more than ten thousand nodes, APPROXIM is orders of magnitude faster than EXACT. In particular, APPROXIM is scalable to large network with more than one million nodes.

Accuracy. In Table 2, we present the accuracy results for algorithm APPROXIM. For each of the four distributions of innate opinions, we compare the results of APPROXIM with those of EXACT for some networks in Table 1. For each relevant quantity ρ , we apply the mean relative error $\sigma = |\rho - \tilde{\rho}|/\rho$ of $\tilde{\rho}$ obtained by APPROXIM as an estimation of ρ . Table 2 reports the mean relative errors of the four estimated quantities, $\tilde{C}(\mathcal{G})$, $\tilde{D}(\mathcal{G})$, $\tilde{P}(\mathcal{G})$, and $\tilde{I}_{pd}(\mathcal{G})$, for many real networks. We can see that for all quantities concerned,

the results of APPROXIM are very close to those associated with EXACT, since their actual relative errors are negligible, which are all less than 10^{-8} .

8 RELATED WORK

In this section, we briefly review some prior work that lies close to ours.

Stubbornness. As an important individual attribute, stubbornness is a fundamental element affecting diverse aspects of opinion dynamics. For an individual, stubbornness indicates the degree to which the individual sticks to its own internal opinion or willingness to conform to its neighbors' opinions. In fact, stubbornness is equivalent to people's susceptibility to persuasion introduced in [1, 2]. There is a rich body of work [6, 17, 20, 47, 48] in politics and social psychology, studying empirically this key factor, which show that stubbornness can be applied to a large variety of practical scenarios, such as product marketing, public health campaigns, and political candidates. Some recent work also studied how to change an individual's attitude or stubbornness [21, 28, 29], and

showed that heterogeneity of individual's stubbornness is ubiquitous. It is thus of practical and theoretical interest to analyze the influences of heterogeneous stubbornness of agents on different aspects of opinion dynamics.

Opinion dynamics model. To better understand the formation of opinion dynamics, several relevant models have been presented, among which, FJ model is a popular one. It is frequently adopted to study various aspects of opinion dynamics, and thus has been extensively studied. A sufficient condition for stability of the FJ model was obtained in [42], and the equilibrium expressed opinion was derived in [8, 15]. In the FJ model, the total expressed opinion is equal to the total internal opinion [26, 50]. Some interpretations of the FJ model were provided in [8] and [25]. Most of the previous work only considered homogeneous stubbornness. In [1, 2], the impact of stubbornness on optimizing the overall opinion dynamics were analyzed by introducing a resistance parameter to modify the FJ model. However, it is still not well understood how heterogeneous stubbornness affects other respects of opinion dynamics, such as convergence speed and overall opinion. Furthermore, some prior interpretations for expressed opinion are not applicable to the case with inhomogeneous stubbornness, for example, electrical networks [25]. Our interpretation for spanning diverging forest is novel, which differs greatly from previous ones.

Quantification of social phenomena. In recent years, social media and online social networks have experienced explosive growth, which is accompanied by diverse social phenomena, including polarization, disagreement, and conflict. Actually, these phenomena have taken place in human societies for millenia, and have been a recent hot subject of study in different disciplines, especially social science. Thus far, various measures have been developed to quantify these phenomena, such as disagreement [14, 39], polarization [14, 36, 39], conflict [13], and controversy [13]. Most of these measures are based on the FJ opinion dynamics model [23] with uniform stubbornness, which do not apply to the case in the presence of different stubbornness. To make up for the deficiency, we

extend these quantities by incorporating heterogeneous stubbornness, so that they can measure the corresponding social phenomena. The issue of how to quantify these phenomena has received increasing amount of attention. Since direct computation of these indicators involves the operations of matrix inversion and multiplication, making it computationally infeasible for large-scale graphs with millions of nodes, we present a nearly linear time algorithm to estimate all these quantities.

Impacts of reciprocity on dynamics. Note that the FJ model on an undirected network with non-zero heterogeneous stubbornness is equivalent to traditional FJ model on a corresponding weighted directed graph with uniform stubbornness and asymmetric weights between edges linking a pair of nodes. The tendency of node pairs to form mutual asymmetric connection in directed networks is also called link reciprocity, which is a common characteristic of many realistic networks [4, 24, 46, 49], such as the World Wide Web [5], e-mail networks [18, 40], and world trade web [44]. It has been shown the ubiquitous link reciprocity strongly affects dynamical processes running in binary networks, for example, spread of computer viruses [40] or information [54], percolation [9], and random walks [52]. By contrast, the influence of reciprocity on opinion dynamics has attracted much less attention, although it is suggested that reciprocity could play a crucial role in various aspects of this dynamics.

9 CONCLUSION

Stubbornness of individuals is a key factor affecting various aspects opinion dynamics. In this paper, we provided a solid theoretical analysis of a classic model for opinion dynamics on undirected graphs in the presence of heterogeneous stubbornness for different nodes. We demonstrated that heterogeneous stubbornness strongly affects almost all aspects of opinion dynamics, such as the expressed opinion, convergence velocity to equilibrium, and overall opinion. For example, when a node increase its stubbornness, the expressed opinions of nodes place more weight on the

Table 2: Relative error for estimated $C(\mathcal{G})$, $D(\mathcal{G})$, $P(\mathcal{G})$, $I_{pd}(\mathcal{G})$ for four internal distributions with input parameter $\epsilon = 10^{-6}$.

Network	Relative error of four estimated quantities for four internal opinion distributions															
	Uniform Distribution				Power-law distribution				Normal distribution				Exponential distribution			
	$C(\mathcal{G})$	$D(\mathcal{G})$	$P(\mathcal{G})$	$I_{pd}(\mathcal{G})$	$C(\mathcal{G})$	$D(\mathcal{G})$	$P(\mathcal{G})$	$I_{pd}(\mathcal{G})$	$C(\mathcal{G})$	$D(\mathcal{G})$	$P(\mathcal{G})$	$I_{pd}(\mathcal{G})$	$C(\mathcal{G})$	$D(\mathcal{G})$	$P(\mathcal{G})$	$I_{pd}(\mathcal{G})$
GrQc	2.69E-09	1.79E-10	1.23E-10	1.04E-14	1.94E-09	6.12E-10	5.42E-10	8.51E-16	6.00E-09	2.39E-09	1.30E-09	3.03E-15	2.42E-08	4.51E-11	1.84E-11	2.32E-14
USgrid	3.61E-08	1.05E-08	4.56E-09	2.89E-14	7.72E-10	3.02E-09	4.90E-09	3.78E-15	3.48E-08	1.27E-08	5.74E-09	2.71E-14	3.19E-09	4.43E-09	3.53E-09	1.24E-14
Erdos992	2.45E-08	8.04E-10	4.30E-10	4.71E-15	2.23E-10	3.12E-10	5.09E-10	4.94E-16	5.01E-09	9.26E-10	5.87E-10	7.81E-15	9.41E-09	7.38E-10	4.17E-10	1.66E-15
Advogato	1.78E-08	6.70E-09	3.21E-09	4.58E-15	4.66E-09	2.03E-09	1.70E-09	2.76E-16	2.09E-09	2.01E-10	1.60E-10	2.02E-15	9.51E-10	5.41E-10	4.59E-10	1.91E-16
Bcspwr10	2.56E-08	1.43E-08	6.28E-09	2.01E-14	8.02E-09	3.19E-09	1.58E-09	4.45E-14	3.04E-09	7.10E-10	4.17E-10	1.19E-14	1.07E-08	2.02E-09	1.18E-09	7.24E-15
Reality	7.43E-10	6.97E-10	3.32E-10	4.26E-16	9.62E-11	8.92E-10	1.05E-09	1.70E-15	4.95E-09	2.31E-11	1.24E-11	1.69E-16	1.75E-09	5.53E-10	2.95E-10	3.49E-16
PagesGovernment	3.01E-07	3.45E-08	7.91E-10	5.79E-15	1.66E-08	6.34E-10	1.28E-09	4.03E-15	2.60E-08	1.01E-08	1.07E-08	1.14E-14	4.58E-09	2.21E-09	1.72E-09	1.51E-15
WikiElec	5.37E-11	4.91E-11	2.48E-11	3.92E-16	4.98E-09	2.46E-10	4.58E-10	3.87E-16	2.67E-08	4.58E-09	2.59E-09	6.57E-16	9.24E-09	5.78E-11	6.80E-11	1.81E-15
Dmela	4.56E-11	2.62E-10	1.70E-10	9.74E-16	3.67E-09	6.35E-10	1.07E-09	6.87E-16	5.88E-09	3.76E-09	2.36E-09	3.61E-15	6.75E-08	2.99E-09	1.65E-09	9.77E-16
HepPh	1.25E-09	6.00E-09	3.61E-09	1.25E-14	4.49E-09	8.54E-10	2.18E-09	1.46E-15	1.36E-08	6.47E-09	5.06E-09	8.41E-15	7.73E-08	4.77E-08	9.19E-09	1.82E-14
Anybeat	1.38E-08	1.46E-09	9.03E-10	1.25E-16	1.02E-09	3.85E-11	3.20E-10	1.44E-15	2.83E-08	3.37E-09	1.91E-09	1.39E-14	2.50E-08	1.40E-09	8.55E-10	2.63E-15
PagesCompany	5.93E-10	1.91E-09	1.70E-09	5.05E-15	2.85E-08	7.04E-09	8.21E-09	4.37E-15	4.12E-09	2.57E-10	2.92E-10	6.40E-15	8.06E-09	6.91E-09	4.65E-09	1.66E-14
AstroPh	3.44E-09	1.08E-09	1.36E-09	4.58E-16	2.68E-08	3.50E-08	3.99E-09	7.67E-15	5.04E-09	3.18E-09	2.36E-09	8.49E-16	6.12E-09	8.24E-10	1.08E-09	1.89E-14
CondMat	1.58E-09	2.00E-09	1.67E-09	8.07E-15	4.60E-09	2.89E-09	4.69E-09	2.18E-15	8.52E-09	1.93E-09	2.17E-09	3.60E-15	8.17E-09	3.16E-10	2.34E-10	1.56E-14
Gplus	2.32E-09	3.88E-10	2.51E-10	6.70E-16	3.33E-10	1.88E-11	5.01E-11	7.62E-16	6.17E-10	6.78E-11	5.00E-11	3.23E-16	1.28E-09	1.30E-10	1.11E-10	2.26E-16
GemsecRO	9.75E-09	4.75E-09	4.12E-09	2.53E-15	4.50E-09	1.25E-09	2.05E-09	7.98E-16	4.30E-09	3.52E-09	3.05E-09	2.93E-15	4.45E-08	4.03E-08	1.18E-08	7.01E-14
GemsecHU	9.90E-09	2.03E-09	3.10E-09	1.19E-14	2.20E-09	1.40E-09	1.32E-08	2.29E-14	5.15E-09	2.98E-09	3.33E-09	1.45E-14	1.08E-08	4.40E-09	5.48E-09	3.23E-15

internal opinions of this node, showing that a more stubborn individual has more steady social power. Again for instance, when the stubbornness are different among nodes, the sum of expressed opinions is often not equal to the total initial opinion, indicating the difference between the evolution of social power with heterogeneous and homogeneous stubbornness. We provided an interpretation of the expressed opinion in terms of spanning diverging forests of a stubbornness-dependent digraph associated with the original undirected graph.

In addition, we extended previously proposed quantitative indicators for social phenomena in the opinion dynamics model with identical stubbornness to the case when stubbornness are heterogeneous. The considered social concepts include conflict, disagreement, and polarization, all of which incorporate heterogeneous stubbornness. Since direct computation of these quantities involves matrix inverse, which is computationally challenging for large graphs. We devised an approximation algorithm to estimate these quantities concerned. Our algorithm has a theoretical guarantee for accuracy and an almost linear computation time complexity with respect to the number of edges. Finally, we executed experiments on many real-world networks, demonstrating the high efficiency and good effectiveness of our algorithm, which is applicable to large graphs with millions of nodes.

REFERENCES

- [1] Rediet Abebe, T-H HUBERT Chan, Jon Kleinberg, Zhibin Liang, David Parkes, Mauro Sozio, and Charalampos E Tsourakakis. 2021. Opinion Dynamics Optimization by Varying Susceptibility to Persuasion via Non-Convex Local Search. *ACM Transactions on Knowledge Discovery from Data* 16, 2 (2021), 1–34.
- [2] Rediet Abebe, Jon Kleinberg, David Parkes, and Charalampos E Tsourakakis. 2018. Opinion dynamics with varying susceptibility to persuasion. In *Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*. ACM, 1089–1098.
- [3] Rafiq Pashaevich Agaev and P Yu Chebotarev. 2001. Spanning forests of a digraph and their applications. *Automation and Remote Control* 62, 3 (2001), 443–466.
- [4] Leman Akoglu, Pedro O. S. Vaz de Melo, and Christos Faloutsos. 2012. Quantifying reciprocity in large weighted communication networks. *Lect. Notes Comput. Sci.* 7302 (2012), 85–96.
- [5] Réka Albert, Hawoong Jeong, and Albert-László Barabási. 1999. Internet: Diameter of the World-Wide Web. *Nature* 401, 6749 (1999), 130–131.
- [6] Joseph P Allen, Maryfrances R Porter, and F Christy McFarland. 2006. Leaders and followers in adolescent close friendships: Susceptibility to peer influence as a predictor of risky behavior, friendship instability, and depression. *Development and Psychopathology* 18, 1 (2006), 155–172.
- [7] Brian DO Anderson, Fabrizio Dabbene, Anton V Proskurnikov, Chiara Ravazzi, and Mengbin Ye. 2020. Dynamical networks of social influence: Modern trends and perspectives. *IFAC-PapersOnLine* 53, 2 (2020), 17616–17627.
- [8] David Bindel, Jon Kleinberg, and Sigal Oren. 2015. How bad is forming your own opinion? *Games and Economic Behavior* 92 (2015), 248–265.
- [9] Marián Boguñá and M Ángeles Serrano. 2005. Generalized percolation in random directed networks. *Phys. Rev. E* 72, 1 (2005), 016106.
- [10] Pavel Chebotarev and Rafiq Agaev. 2002. Forest matrices around the Laplacian matrix. *Linear Algebra Appl.* 356, 1–3 (2002), 253–274.
- [11] P. Yu Chebotarev and E. V. Shamis. 1997. The matrix-forest theorem and measuring relations in small social groups. *Automation and Remote Control* 58, 9 (1997), 1505–1514.
- [12] P. Yu Chebotarev and E. V. Shamis. 1998. On proximity measures for graph vertices. *Automation and Remote Control* 59, 10 (1998), 1443–1459.
- [13] Xi Chen, Jefrey Lijffijt, and Tijl De Bie. 2018. Quantifying and minimizing risk of conflict in social networks. In *Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*. ACM, 1197–1205.
- [14] Pranav Dandekar, Ashish Goel, and David T Lee. 2013. Biased assimilation, homophily, and the dynamics of polarization. *Proceedings of the National Academy of Sciences* 110, 15 (2013), 5791–5796.
- [15] Abhimanyu Das, Sreenivas Gollapudi, Rina Panigrahy, and Mahyar Salek. 2013. Debiasing social wisdom. In *Proceedings of the 19th ACM SIGKDD international conference on Knowledge discovery and data mining*. ACM, 500–508.
- [16] Morris H DeGroot. 1974. Reaching a consensus. *J. Amer. Statist. Assoc.* 69, 345 (1974), 118–121.
- [17] TE Dielman, Deborah D Kloska, Sharon L Leech, John E Schulenberg, and Jean T Shope. 1992. Susceptibility to peer pressure as an explanatory variable for the differential effectiveness of an alcohol misuse prevention program in elementary schools. *Journal of School Health* 62, 6 (1992), 233–237.
- [18] Holger Ebel, Lutz-Ingo Mielsch, and Stefan Bornholdt. 2002. Scale-free topology of e-mail networks. *Phys. Rev. E* 66, 3 (2002), 035103.
- [19] Glenn Ellison. 1993. Learning, local interaction, and coordination. *Econometrica: Journal of the Econometric Society* (1993), 1047–1071.
- [20] William N Evans, Wallace E Oates, and Robert M Schwab. 1992. Measuring peer group effects: A study of teenage behavior. *Journal of Political Economy* 100, 5 (1992), 966–991.
- [21] Brian J Fogg. 2002. Persuasive technology: Using computers to change what we think and do. *Ubiquity* 2002, December (2002), 2.
- [22] John RP French Jr. 1956. A formal theory of social power. *Psychological review* 63, 3 (1956), 181.
- [23] Noah E Friedkin and Eugene C Johnsen. 1990. Social influence and opinions. *Journal of Mathematical Sociology* 15, 3–4 (1990), 193–206.
- [24] Diego Garlaschelli and Maria I Loffredo. 2004. Patterns of link reciprocity in directed networks. *Phys. Rev. Lett.* 93, 26 (2004), 268701.
- [25] Javad Ghaderi and Rayadurgam Srikant. 2014. Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate. *Automatica* 50, 12 (2014), 3209–3215.
- [26] Aristides Gionis, Evimaria Terzi, and Panayiotis Tsaparas. 2013. Opinion maximization in social networks. In *Proceedings of the 2013 SIAM International Conference on Data Mining*. SIAM, 387–395.
- [27] Jake M Hofman, Duncan J Watts, Susan Athey, Filiz Garip, Thomas L Griffiths, Jon Kleinberg, Helen Margetts, Sendhil Mullainathan, Matthew J Salganik, Simine Vazire, et al. 2021. Integrating explanation and prediction in computational social science. *Nature* 595 (2021), 181–188.
- [28] Wijnand IJsselstein, Yvonne De Kort, Cees Midden, Berry Eggen, and Elise Van Den Hoven. 2006. Persuasive technology for human well-being: setting the scene. In *International conference on persuasive technology*. Springer, 1–5.
- [29] Maurits Kaptein, Panos Markopoulos, Boris de Ruyter, and Emile Aarts. 2009. Can you be persuaded? Individual differences in susceptibility to persuasion. In *IFIP conference on human-computer interaction*. Springer, 115–118.
- [30] Jérôme Kunegis. 2013. KONECT: The Koblenz Network Collection. In *Proceedings of the 22nd International Conference on World Wide Web (Rio de Janeiro, Brazil)*. ACM, 1343–1350.
- [31] Rasmus Kyng and Sushant Sachdeva. 2016. Approximate Gaussian elimination for Laplacians - fast, sparse, and simple. In *Proceedings of IEEE 57th Annual Symposium on Foundations of Computer Science*. IEEE, 573–582.
- [32] Heidi Ledford. 2020. How Facebook, Twitter and other data troves are revolutionizing social science. *Nature* 582, 7812 (2020), 328–330.
- [33] Huan Li, Richard Peng, Liren Shan, Yuhao Yi, and Zhongzhi Zhang. 2019. Current flow group closeness centrality for complex networks. (2019), 961–971.
- [34] Huan Li and Aaron Schild. 2018. Spectral subspace Sparsification. In *Proceedings of 2018 IEEE 59th Annual Symposium on Foundations of Computer Science*. IEEE, 385–396.
- [35] Charles R MacCluer. 2000. The many proofs and applications of Perron’s theorem. *SIAM Rev.* 42, 3 (2000), 487–498.
- [36] Antonis Matakos, Evimaria Terzi, and Panayiotis Tsaparas. 2017. Measuring and moderating opinion polarization in social networks. *Data Mining and Knowledge Discovery* 31, 5 (2017), 1480–1505.
- [37] Russell Merris. 1997. Doubly stochastic graph matrices. *Publikacije Elektrotehničkog fakulteta. Serija Matematika* 8 (1997), 64–71.
- [38] Carl D Meyer, Jr. 1973. Generalized inversion of modified matrices. *SIAM J. Appl. Math.* 24, 3 (1973), 315–323.
- [39] Cameron Musco, Christopher Musco, and Charalampos E Tsourakakis. 2018. Minimizing polarization and disagreement in social networks. In *Proceedings of the 2018 World Wide Web Conference*. 369–378.
- [40] Mark EJ Newman, Stephanie Forrest, and Justin Balthrop. 2002. Email networks and the spread of computer viruses. *Phys. Rev. E* 66, 3 (2002), 035101.
- [41] Anton V Proskurnikov and Roberto Tempo. 2017. A tutorial on modeling and analysis of dynamic social networks. Part I. *Annual Reviews in Control* 43 (2017), 65–79.
- [42] Chiara Ravazzi, Paolo Frasca, Roberto Tempo, and Hideaki Ishii. 2015. Ergodic randomized algorithms and dynamics over networks. *IEEE Transactions on Control of Network Systems* 1, 2 (2015), 78–87.
- [43] Ryan Rossi and Nesreen Ahmed. 2015. The network data repository with interactive graph analytics and visualization. In *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence*. AAAI, 4292–4293.
- [44] Ma Ángeles Serrano and Marián Boguñá. 2003. Topology of the world trade web. *Phys. Rev. E* 68, 1 (2003), 015101.
- [45] Daniel A Spielman and Nikhil Srivastava. 2011. Graph sparsification by effective resistances. *SIAM J. Comput.* 40, 6 (2011), 1913–1926.

- [46] Tiziano Squartini, Francesco Picciolo, Franco Ruzzenenti, and Diego Garlaschelli. 2013. Reciprocity of weighted networks. *Sci. Rep.* 3 (2013), 2729.
- [47] Laurence Steinberg and Kathryn C Monahan. 2007. Age differences in resistance to peer influence. *Developmental psychology* 43, 6 (2007), 1531.
- [48] Julie A Wall, Thomas G Power, and Consuelo Arbona. 1993. Susceptibility to antisocial peer pressure and its relation to acculturation in Mexican-American adolescents. *Journal of Adolescent Research* 8, 4 (1993), 403–418.
- [49] Cheng Wang, Omar Lizardo, David Hachen, Anthony Strathman, Zoltán Toroczkai, and Nitesh V Chawla. 2013. A dyadic reciprocity index for repeated interaction networks. *Netw. Sci.* 1, 01 (2013), 31–48.
- [50] Wanyue Xu, Qi Bao, and Zhongzhi Zhang. 2021. Fast evaluation for relevant quantities of opinion dynamics. In *Proceedings of The Web Conference*. ACM, 2037–2045.
- [51] Yuhao Yi and Stacy Patterson. 2020. Disagreement and polarization in two-party social networks. *IFAC-PapersOnLine* 53, 2 (2020), 2568–2575.
- [52] Zhongzhi Zhang, Huan Li, and Yibin Sheng. 2014. Effects of reciprocity on random walks in weighted networks. *Scientific reports* 4, 1 (2014), 1–7.
- [53] Xiaotian Zhou and Zhongzhi Zhang. 2021. Maximizing influence of leaders in social networks. In *Proceedings of the 27th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*. ACM, 2400–2408.
- [54] Yu Xiao Zhu, Xiao-Guang Zhang, Gui-Quan Sun, Ming Tang, Tao Zhou, and Zi Ke Zhang. 2014. Influence of reciprocal links in social networks. *PLoS ONE* 9, 7 (2014), e103007.