# Simplifying Termination Proofs for Rewrite Systems by Preprocessing*



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## ABSTRACT

We prove some new results that simplify termination proofs for non-overlapping term rewriting systems. The first one is a refined modularity result (for not necessarily disjoint systems). The second, more important one, gives conditions under which the simplification of right-hand sides (using rules of the original system) is a sound preprocessing step, in the sense that termination of the original system is equivalent to termination of the simplified system, and that the equational theory is preserved. The proofs are based on some powerful structural properties known for non-overlapping systems. Finally, we show how to (partially) extend these results, in particular, to the case of conditional rewrite systems where we additionally treat simplification of conditions of rules. The presented results provide the theoretical basis for sound (and automatic) preprocessing steps when proving termination of (possibly conditional) non-overlapping rewrite systems and equational programs defined by such systems.

## 1. INTRODUCTION AND OVERVIEW

In this paper we concentrate on termination (and confluence) properties of non-overlapping term rewriting systems (TRS's for short). Using some powerful structural properties that are well-known for this class of TRS's we show how to simplify termination proofs in different settings. First, in Section 3, we consider a known modularity result for non-overlapping TRS's from [7] and show how to generalize it in various ways, by exploiting the structural properties mentioned above. Then, in the main Section 4, we investigate under which conditions a natural preprocessing step, where right-hand sides of the TRS under consideration are simplified (using rules of the original system), is sound w.r.t. proving termination. In other words, the question is, given a TRS $\mathcal{R}$, under which conditions is termination of the simplified version $\mathcal{R}'$ equivalent to termination of $\mathcal{R}$. This question is practically relevant, because eliminating redundancy (here: reducible right-hand sides) may considerably simplify the task of termination. Of course, the problem with such a preprocessing step is that in general it is not sound. There exist non-terminating TRS's where simplification of right-hand sides yields a terminating system. However, it turns out that for non-overlapping TRS's, there exist easily checkable syntactical criteria which indeed guarantee soundness. Again, this analysis heavily relies on the structural properties mentioned above. Finally, in Section 5 we investigate possible generalizations and extensions of the previously presented results. In 5.1 this is done by weakening the no-overlap requirement, and in 5.2 we consider conditional TRS's (CTRS's for short). In 5.2.1 we investigate the case of non-overlapping join CTRS's and discuss additional complications.For the slightly more special and practically relevant case of orthogonal normal (join) CTRS's, we are able in 5.2.2 to partially extend these results as well as to cover additionally arbitrary simplification of the conditions of conditional rules. In all cases, the corresponding generalization of the underlying structural properties for non-overlapping TRS's to the new setting is a technical key to prove these results. For CTRS's, however, there are — as usual — various rather subtle complications that have to be dealt with.

## 2. PRELIMINARIES

We assume familiarity with the basic no(ta)tions, terminology and theory of term rewriting (cf. e.g. [8], [20], [4]). The set of terms over some given signature $\mathcal{F}$ and some (disjoint) denumerable set $\mathcal{V}$ of variables is denoted by $\mathcal{T}(\mathcal{F},\mathcal{V})$. For substitutions we use postfix notation. The 'empty' root position is denoted by $\epsilon$. For the set of all variables occurring in a term $s$ we write $Var(s)$. The subterm of $s$ at some position $p \in Pos(s)$ is denoted by $s|_p$. The result of replacing in $s$ the subterm at position $p$ by $t$ is denoted by $s[t]_p$.

For rewrite rules $l \to r$ of a term rewriting system $\mathcal{R}^{\mathcal{F}} = \mathcal{R}$ (over $\mathcal{T}(\mathcal{F},\mathcal{V})$) we require that $l$ is not a variable, and that all variables of $r$ occur in $l$ (this excludes only degenerate cases). For left- and right-hand side of a rewrite rule we also use the abbreviations lhs and rhs, respectively. For reduction steps with the rewrite relation $\to_{\mathcal{R}} = \to$ induced by $\mathcal{R}$ we sometimes add additional information as in $s \to_{p,\sigma,l\to r} t$ with the obvious meaning. A rewrite rule $l \to r$ is said to be *non-erasing* if $Var(l) = Var(r)$. A TRS is *non-erasing*

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(denoted by NE) if all its rules are. For the set of *normal forms* of a TRS $\mathcal{R}$ we write NF($\mathcal{R}$) or simply NF when $\mathcal{R}$ is clear from the context. The *innermost* reduction (or rewrite) relation $\to_i$ (induced by $\mathcal{R}$) is given by: $s \to_i t$ if $s = s[l\sigma]_p \to_{l\to r} s[r\sigma]_p = t$ for some $l \to r \in \mathcal{R}$, some position $p$ in $s$ and some substitution $\sigma$ such that no proper subterm of $l\sigma$ is reducible.

A TRS is *non-overlapping* if it has no critical pairs. It is *weakly non-overlapping* if all its critical pairs are trivial, i.e., of the form $(s, s)$. A non-overlapping *left-linear* TRS is *orthogonal*. A TRS $\mathcal{R}$ is an *overlay system* if critical overlaps between rules of $\mathcal{R}$ occur only at the root position. For the properties *termination* (equivalently: *strong normalization*), *weak termination* (equivalently: *weak normalization*), *confluence*, *local confluence* and *uniform confluence* (i.e., $\leftarrow\circ\rightarrow \subseteq (\mathrm{id}\cup\rightarrow\circ\leftarrow)$, with id being the identity relation) we also use the abbreviations SN, WN, CR, WCR and WCR[1], respectively. *Innermost termination*, i.e., termination (or strong normalization) of the innermost reduction relation $\to_i$, is also abbreviated by SIN, *weak innermost termination* (or *weak innermost normalization*) by WIN. Furthermore, we also use local versions ('below' some term) of these properties like SN($s$) (meaning that there is no infinite derivation issuing from $s$) and CR($s$) (i.e., if $s \to^* t_1$ and $s \to^* t_2$, then there exists $t$ with $t_1 \to^* t$ and $t_2 \to^* t$). To ease readability, for a non-terminating term $t$, i.e., with $\neg$SN($t$) we also write $\infty(t)$ or $\infty_{\mathcal{R}}(t)$ (to indicate the corresponding rewrite relation involved).

## 3. COMBINED NON-OVERLAPPING SYSTEMS

In contrast to orthogonal TRS's, non-overlapping (possibly non-left-linear) ones are in general not confluent (cf. [18], [19]). In fact, very few results are known about confluence (and normal form) properties of non-terminating, non-left-linear, non-overlapping TRS's. However, concerning termination properties (and combinations of termination and confluence properties) the situation is much better.

To start with, let us present the structural properties of non-overlapping TRS's that we referred to above.

THEOREM 1. *([13, cf. Theorem 3.13, Lemma 3.11], [15, Lemma 3.2.8, Corollary 3.2.9, Theorem 3.2.11]) Let $\mathcal{R}^{\mathcal{F}}$ be a TRS and $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. If $\mathcal{R}$ is non-overlapping then the following properties hold:*

*(1) SIN($t$) $\iff$ SN($t$).*

*(2) WIN($t$) $\iff$ SIN($t$).*

*(3) If $s \to_i t$ and $\neg\infty(t)$, then $\neg\infty(s)$.*

*(4) If $s \to_{p,\sigma,l\to r} t$ with $l \to r \in \mathcal{R}$, $\infty(s)$ and $\neg\infty(t)$, then $s|_p = l\sigma$ contains some proper subterm $x\sigma$ with $x \in Var(l) \setminus Var(r)$ and $\infty(x\sigma)$.*

*(5) WIN($t$) $\Longrightarrow$ [SN($t$) $\wedge$ CR($t$)].*

*(6) NE $\Longrightarrow$ [WN($t$) $\iff$ SN($t$)].*

One easy consequence of Theorem 1, more precisely of (6) above, is the following termination property. To state it, we first need an additional definition.

*Definition 1.* Given two TRS $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ (with $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1\cup\mathcal{F}_2}$), we say that $\mathcal{R}_1$ preserves $\mathcal{R}_2$-normal forms if, whenever $s \to_{\mathcal{R}_1} t$ for $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ with $s \in \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}})$, then $t \in \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}})$.

For sufficient conditions guaranteeing preservation of normal forms and related questions we refer to [24], [5], [11].

THEOREM 2. *(cf. [7, Theorem 23, p. 101]) Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be TRS's and $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1\cup\mathcal{F}_2}$ be their (not necessarily disjoint) union. Suppose the following:*

*(1) $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating.*

*(2) $\mathcal{R}_1$ preserves $\mathcal{R}_2$-normal forms.*

*(3) $\mathcal{R}$ is non-overlapping.*

*(4) $\mathcal{R}$ is non-erasing.*

*Then $\mathcal{R}$ is terminating.*

PROOF. (See [7] for the idea.) For any term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, we can obtain by (1) an $\mathcal{R}$-normal form by first normalizing $s$ to some $s' \in \mathrm{NF}(\mathcal{R}_2)$ using $\mathcal{R}_2$, and then normalizing $s'$ to some $s'' \in \mathrm{NF}(\mathcal{R}_1)$ using $\mathcal{R}_1$. By (2), $s''$ is an $\mathcal{R}$-normal form. And by (3), (4) and Theorem 1(6) this yields termination of $s$ w.r.t. $\mathcal{R}$. $\square$

In [7, page 101], N. Dershowitz mentions that, though the non-erasing property is needed for the equivalence of weak and strong termination, i.e., for Theorem 1(6) above, "an example of non-termination for non-overlapping (normal form) preserving systems is lacking". Actually, this lack is not really surprising since such a counterexample cannot exist as we show next.

THEOREM 3. *(Theorem 2 generalized) Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be TRS's and $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1\cup\mathcal{F}_2}$ be their (not necessarily disjoint) union. Suppose the following:*

*(1) $\mathcal{R}_1$ and $\mathcal{R}_2$ are weakly innermost terminating.*

*(2) $\mathcal{R}_1$ preserves $\mathcal{R}_2$-normal forms.*

*(3) $\mathcal{R}$ is non-overlapping.*

*Then $\mathcal{R}$ is terminating.*

PROOF. Essentially the proof works by refined repeated applications of the structural properties of Theorem 1.

For a proof by contradiction suppose there exists $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ with $\infty_{\mathcal{R}}(s)$. By (1) (and the preservation of WIN under signature extension) we can reduce $s$ by $\mathcal{R}_2$-innermost $\mathcal{R}_2$-steps to an $\mathcal{R}_2$-normal form $s'$, and furthermore $s'$ by $\mathcal{R}_1$-innermost $\mathcal{R}_1$-steps to an $\mathcal{R}_1$-normal form $s''$. In the resulting derivation

$$D : s \to^*_{\mathcal{R}_2} s' \to^*_{\mathcal{R}_1} s''$$

we obviously have $\infty_{\mathcal{R}}(s)$ (by assumption) and $\neg\infty_{\mathcal{R}_1}(s'')$. Since $s'$ is in normal form w.r.t. $\mathcal{R}_2$, we conclude by (2), that every term in the sub-derivation

$$D_2 : s' \to^*_{\mathcal{R}_1} s''$$

is in normal form w.r.t. $\mathcal{R}_2$, hence $s''$ is in $\mathcal{R}$-normal form. Thus $\neg\infty_{\mathcal{R}}(s'')$ holds. Now, all steps in $D_2$ are $\mathcal{R}_1$-innermost

$\mathcal{R}_1$-steps and all terms in $D_2$ are in $\mathcal{R}_2$-normal form. Hence, all these steps are $\mathcal{R}$-innermost $\mathcal{R}_1$ (and $\mathcal{R}$-)steps. But by (3) and using Theorem 1(3) this implies $\neg\infty_{\mathcal{R}}(s')$, i.e., $D_2$ doesn't contain a step which is critical (w.r.t. termination). The same argument would apply to the sub-derivation

$$D_1 : s \to^*_{\mathcal{R}_2} s'$$

provided the $\mathcal{R}_2$-innermost $\mathcal{R}_2$-steps there were also $\mathcal{R}_1$-innermost (and consequently $\mathcal{R}$-innermost). However, this need not be the case, because an $\mathcal{R}_1$-innermost $\mathcal{R}_1$-redex of some term in $D_1$ may well contain $\mathcal{R}_2$-redexes as proper subterms! Instead we can argue as follows: Due to $\infty_{\mathcal{R}}(s)$ and $\neg\infty_{\mathcal{R}}(s')$, there must exist some first ($\mathcal{R}_2$-innermost) critical step in $D_1$, i.e., $D_1$ is of the form

$$D_1 : s \to^*_{\mathcal{R}_2} u \to_{\mathcal{R}_2} v \to^*_{\mathcal{R}_2} s'$$

with $\infty_{\mathcal{R}}(u)$ and $\neg\infty_{\mathcal{R}}(v)$. But by Theorem 1(4) this is only possible if in the $\mathcal{R}_2$-innermost step $u \to_{\mathcal{R}_2} v$, let's say using rule $l_2 \to r_2 \in \mathcal{R}_2$ at position $p$, the contracted redex $u|_p = l\sigma$ contains a proper subterm $x\sigma$ with $x \in Var(l) \setminus Var(r)$ that is non-terminating (w.r.t. $\mathcal{R}$) and erased in this step. Since $u \to_{\mathcal{R}_2} v$ is $\mathcal{R}_2$-innermost, the non-terminating proper subterm $t := x\sigma$ of $u|_p = l\sigma$ is in $\mathcal{R}_2$-normal form. But now, by (1) (more precisely, weak innermost termination of $\mathcal{R}_1$) and (2), and using the same argument as for $D_2$ above, this would imply the existence of an $\mathcal{R}_1$-innermost and also $\mathcal{R}$-innermost normalizing derivation $t \to^*_{\mathcal{R}_1} t'$ with $t'$ in $\mathcal{R}$-normal form, hence with $\neg\infty_{\mathcal{R}}(t')$ and $\neg\infty_{\mathcal{R}}(t)$ by Theorem 1 (2) and (1)

But this contradicts the non-termination of $t$. Consequently, there can be no critical step in $D_1$, too, which finishes the overall proof. $\square$

*Remark 1.* Interestingly, and in contrast to most of the results on non-overlapping TRS's in [15], it is unclear[1] whether the strengthened localized version of Theorem 3 (where precondition (1) and the conclusion only refer to a particular term $s$) also holds or not. However, replacing (1) by

(1′) $\mathcal{R}_1$ is weakly innermost terminating, and $s$ is weakly innermost $\mathcal{R}_2$-terminating, and the conclusion by "$s$ is $\mathcal{R}$-terminating"

leads to a valid generalization.

Let us finally give a simple example where Theorem 3 is applicable, but not the weaker version, Theorem 2.

*Example 1.* Consider the two TRS's

$$\mathcal{R}_1 = \{f(a,b,x) \to f(x,x,g(x))\}$$

and

$$\mathcal{R}_2 = \{h(x,x) \to g(a)\}$$

over non-disjoint signatures. Clearly, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-overlapping as well as the combined system $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$. Furthermore both systems are (easily shown to be) weakly innermost terminating. And, moreover, $\mathcal{R}_2$ preserves $\mathcal{R}_1$-normal forms by [24, Theorem 7] (which provides a decidable criterion for preservation of normal forms) since $\mathcal{R}_1$ is left-linear and for all $({\mathcal{R}_2}^{-1}, \mathcal{R}_1)$-critical pairs[2] $(s_1, t_1)$, i.e., with corresponding critical peak $s_1 \to_{\mathcal{R}_2} t_1 \to_{\mathcal{R}_1} t_2$, the term $s_1$ is $\mathcal{R}_1$-reducible. Here the only such critical peak is $f(a,b,h(x,x)) \to_{\mathcal{R}_2} f(a,b,g(a)) \to_{\mathcal{R}_1} f(g(a),g(a),g(g(a)))$. Clearly $f(a,b,h(x,x))$ is $\mathcal{R}_1$-reducible. Hence, by Theorem 3 $\mathcal{R}$ is terminating, whereas Theorem 2 is not applicable, since $\mathcal{R}_2$ is erasing.

Yet, it should be noted that preservation of normal forms is a rather restrictive property which in many cases does not hold. Hence, the practical benefits from Theorem 3 seem to be limited. Much more practically relevant is another type of termination results derived from the mentioned structural properties that we are going to present next.

## 4. SIMPLIFYING TERMINATION PROOFS FOR NON-OVERLAPPING SYSTEMS

When trying to prove termination of a TRS $\mathcal{R}$, one tempting preprocessing step is to simplify the right-hand sides of $\mathcal{R}$-rules (using $\mathcal{R}$-rules), i.e., to consider

$$\mathcal{R}' = \{l \to r' \mid l \to r \in \mathcal{R}, r \to^*_{\mathcal{R}} r'\}$$

instead of $\mathcal{R}$, because proving termination of $\mathcal{R}'$ might be considerably simpler than proving termination of $\mathcal{R}$.

Clearly, due to $\to_{\mathcal{R}'} \subseteq \to^+_{\mathcal{R}}$, termination of $\mathcal{R}$ implies termination of $\mathcal{R}'$. But how about the (interesting) reverse implication: Does termination of $\mathcal{R}'$ imply termination of $\mathcal{R}$?

Let us consider some examples.

*Example 2.* Addition over the natural numbers with $p$(redecessor) and $s$(uccessor) might be (completely, though a bit awkwardly) specified by the TRS

$$\mathcal{R} = \left\{ \begin{array}{c} p(0) \to 0 \\ p(s(x)) \to x \\ add(0,y) \to y \\ add(s(x),y) \to s(add(p(s(x)),y)) \end{array} \right\}$$

Proving termination of $\mathcal{R}$ is not entirely trivial, since for instance no simplification ordering is applicable (because the last rule is self-embedding). However, the simplified version

$$\mathcal{R}' = \left\{ \begin{array}{c} p(0) \to 0 \\ p(s(x)) \to x \\ add(0,y) \to y \\ add(s(x),y) \to s(add(x,y)) \end{array} \right\}$$

is easily proved to be terminating, e.g. via an appropriate recursive path ordering.

*Example 3.* Another, more interesting, example from an industrial context, due to Arts & Giesl ([2]), roughly is as follows: To verify (some aspects of) correctness of the Erlang[3] implementation (w.r.t. to some given specification) of some process in a network that receives and sends messages, a specification $S$ in Erlang is translated into an oriented

[1] A closer inspection of the proof shows that the weakened assumptions $\mathrm{WIN}(s,\mathcal{R}_1)$, $\mathrm{WIN}(s,\mathcal{R}_2)$ do not suffice to make the correspondingly modified proof go through (since, using the notations from above the properties $\mathrm{WIN}(s',\mathcal{R}_1)$ and $\mathrm{WIN}(t,\mathcal{R}_1)$ are needed, too). In fact, I would conjecture that it is possible to find counterexamples for this case.

[2] See [24] for a precise definition of $({\mathcal{R}_2}^{-1}, \mathcal{R}_1)$-critical pairs.

[3] Erlang is a functional programming language developed by Ericsson Telecom that is, among others, particularly suited for concurrent processes, networking, scheduling, etc., cf. `http://www.erlang.org/`.

non-overlapping conditional TRS (CTRS) $\mathcal{R}_S$, such that *left-right decreasingness*, a kind of strengthened termination property of $\mathcal{R}_S$ (under reasonable assumptions) implies the above correctness property (cf. [2] for more details). Instead of dealing with left-right decreasingness we focus here on the problem of proving ordinary termination of $\mathcal{R}_S$. This is non-trivial (for instance, simplification orderings are not applicable). In [2] the (left-right decreasingness) problem is treated by further transforming the CTRS $\mathcal{R}_S$ into an unconditional TRS $\mathcal{R}_S^u$ (using a variant of standard techniques for this purpose), such that the transformation preserves termination. Finally, termination of the resulting system $\mathcal{R}_S^u$ is tackled via the *dependency pair* approach[4] of Arts & Giesl, and — with some effort — solved using some newly developed refinements of their basic method.

What is interesting about this example, besides stemming from a "real world application", is that proving termination of the CTRS $\mathcal{R}_S$ would be considerably simpler after a preprocessing step that simplifies the right-hand sides and some condition in the conditional rules involved. However, it is unclear whether termination of the simplified system would also imply termination of the original CTRS $\mathcal{R}_S$ (cf. [2] for details).

The mentioned CTRS $\mathcal{R}_S$ consists of the two conditional rules[5]

$$\begin{aligned}&\mathrm{proc}(\mathit{store}, m) \to \mathrm{proc}(\mathrm{app}(\mathrm{map\_f}(\mathrm{self}, \mathrm{nil}), \mathrm{sndsplit}(m, \mathit{store})), m)\\ &\qquad \Leftarrow \mathrm{leq}(m, \mathrm{length}(\mathit{store})) \to^* \mathrm{true},\\ &\qquad\quad \mathrm{empty}(\mathrm{fstsplit}(m, \mathit{store})) \to^* \mathrm{false}\end{aligned} \tag{1}$$

$$\begin{aligned}&\mathrm{proc}(\mathit{store}, m) \to \mathrm{proc}(\mathrm{sndsplit}(m, \mathrm{app}(\mathrm{map\_f}(\mathrm{self}, \mathrm{nil}), \mathit{store})), m)\\ &\qquad \Leftarrow \mathrm{leq}(m, \mathrm{length}(\mathit{store})) \to^* \mathrm{false},\\ &\qquad\quad \mathrm{empty}(\mathrm{fstsplit}(m, \mathrm{app}(\mathrm{map\_f}(\mathrm{self}, \mathrm{nil}), \mathit{store}))) \to^* \mathrm{false}\end{aligned} \tag{2}$$

and the following unconditional ones (apart from some additional library functions):

$$\begin{aligned}\mathrm{length}(\mathrm{nil}) &\to 0\\ \mathrm{length}(\mathrm{cons}(h, t)) &\to \mathrm{s}(\mathrm{length}(t))\\ \mathrm{fstsplit}(0, x) &\to \mathrm{nil}\\ \mathrm{fstsplit}(\mathrm{s}(n), \mathrm{nil}) &\to \mathrm{nil}\\ \mathrm{fstsplit}(\mathrm{s}(n), \mathrm{cons}(h, t)) &\to \mathrm{cons}(h, \mathrm{fstsplit}(n, t))\\ \mathrm{sndsplit}(0, x) &\to x\\ \mathrm{sndsplit}(\mathrm{s}(n), \mathrm{nil}) &\to \mathrm{nil}\\ \mathrm{sndsplit}(\mathrm{s}(n), \mathrm{cons}(h, t)) &\to \mathrm{sndsplit}(n, t)\\ \mathrm{app}(\mathrm{nil}, x) &\to x\\ \mathrm{app}(\mathrm{cons}(h, t), x) &\to \mathrm{cons}(h, \mathrm{app}(t, x))\\ \mathrm{map\_f}(\mathit{pid}, \mathrm{nil}) &\to \mathrm{nil}\\ \mathrm{map\_f}(\mathit{pid}, \mathrm{cons}(h, t)) &\to \mathrm{app}(\mathrm{f}(\mathit{pid}, h), \mathrm{map\_f}(\mathit{pid}, t))\\ \mathrm{empty}(\mathrm{nil}) &\to \mathrm{true}\\ \mathrm{empty}(\mathrm{cons}(h, t)) &\to \mathrm{false}\\ \mathrm{leq}(0, n) &\to \mathrm{true}\\ \mathrm{leq}(\mathrm{s}(m), 0) &\to \mathrm{false}\\ \mathrm{leq}(\mathrm{s}(m), \mathrm{s}(n)) &\to \mathrm{leq}(m, n)\end{aligned}$$

The rhs of (1) can be simplified as follows:

$$\begin{aligned}&\mathrm{proc}(\mathrm{app}(\mathrm{map\_f}(\mathrm{self}, \mathrm{nil}), \mathrm{sndsplit}(m, \mathit{store})), m)\\ \to\ &\mathrm{proc}(\mathrm{app}(\mathrm{nil}, \mathrm{sndsplit}(m, \mathit{store})), m)\\ \to\ &\mathrm{proc}(\mathrm{sndsplit}(m, \mathit{store}), m)\end{aligned}$$

using the rules $\mathrm{map\_f}(\mathit{pid}, \mathrm{nil}) \to \mathrm{nil}$ and $\mathrm{app}(\mathrm{nil}, x) \to x$. Similarly, for the rhs of (2) we obtain

$$\begin{aligned}&\mathrm{proc}(\mathrm{sndsplit}(m, \mathrm{app}(\mathrm{map\_f}(\mathrm{self}, \mathrm{nil}), \mathit{store})), m)\\ \to\ &\mathrm{proc}(\mathrm{sndsplit}(m, \mathrm{app}(\mathrm{nil}, \mathit{store})), m)\\ \to\ &\mathrm{proc}(\mathrm{sndsplit}(m, \mathit{store}), m)\end{aligned}$$

using the same rules. Furthermore the second condition of (2) can also be rewritten, again using the same rules:

$$\begin{aligned}&\mathrm{empty}(\mathrm{fstsplit}(m, \mathrm{app}(\mathrm{map\_f}(\mathrm{self}, \mathrm{nil}), \mathit{store})))\\ \to\ &\mathrm{empty}(\mathrm{fstsplit}(m, \mathrm{app}(\mathrm{nil}, \mathit{store})))\\ \to\ &\mathrm{empty}(\mathrm{fstsplit}(m, \mathit{store}))\end{aligned}$$

This finally yields the two simplified conditional rules

$$\begin{aligned}&\mathrm{proc}(\mathit{store}, m) \to \mathrm{proc}(\mathrm{sndsplit}(m, \mathit{store})), m)\\ &\quad \Leftarrow \mathrm{leq}(m, \mathrm{length}(\mathit{store})) \to^* \mathrm{true},\\ &\qquad \mathrm{empty}(\mathrm{fstsplit}(m, \mathit{store})) \to^* \mathrm{false}\end{aligned} \tag{1'}$$

and

$$\begin{aligned}&\mathrm{proc}(\mathit{store}, m) \to \mathrm{proc}(\mathrm{sndsplit}(m, \mathit{store})), m)\\ &\quad \Leftarrow \mathrm{leq}(m, \mathrm{length}(\mathit{store})) \to^* \mathrm{false},\\ &\qquad \mathrm{empty}(\mathrm{fstsplit}(m, \mathit{store})) \to^* \mathrm{false}\end{aligned} \tag{2'}$$

Proving termination of the resulting system is significantly easier than for the original system (but still non-trivial). In fact, the only problematic rules are the two simplified conditional rules. Intuitively, these rules terminate because the precondition $\mathrm{empty}(\mathrm{fstsplit}(m, \mathit{store})) \to^* \mathrm{false}$ implies $|\mathrm{sndsplit}(m, \mathit{store})| < |\mathit{store}|$ (where $|l|$ denotes the length of the list $l$). Using this semantic argument the overall termination proof for the resulting CTRS $\mathcal{R}'_S$ is not very difficult. The remaining problem, however, is to justify that such a termination proof for $\mathcal{R}'_S$ also guarantees termination for the original system $\mathcal{R}_S$. We will come back to this example in 5.2, after having developed enough theory, and show there that it is indeed possible to justify the above simplifications.

Let us next consider examples where termination of $\mathcal{R}'$ does not imply termination of $\mathcal{R}$.

*Example 4.* A very simple counterexample is the non-terminating TRS

$$\mathcal{R} = \left\{ \begin{array}{c} a \to f(a) \\ f(a) \to b \end{array} \right\}$$

where (a one-step) simplification of the rhs of the first rule yields the terminating system

$$\mathcal{R}' = \left\{ \begin{array}{c} a \to b \\ f(a) \to b \end{array} \right\}$$

Hence, without imposing additional conditions on the shape of $\mathcal{R}$ and / or on the simplification steps allowed in $r \to^*_{\mathcal{R}} r'$, preservation of termination is not guaranteed.

Intuitively, in Example 4 non-termination of $\mathcal{R}$ is eliminated since the (non-terminating) redex $a$ in the rhs $f(a)$ is destroyed by applying the overlapping second rule. But even if the rules are non-overlapping, non-termination may be eliminated in some simplification step of some rhs.

*Example 5.* (cf. e.g. [2]) For the non-terminating TRS

$$\mathcal{R} = \left\{ \begin{array}{c} a \to f(a) \\ f(x) \to b \end{array} \right\}$$

[4]This approach by Arts & Giesl originates in [1], and subsequently was further generalized, extended and refined (cf. [3] for a recent survey).
[5]The function symbol "process" in [2] is abbreviated here by "proc".

simplification of the rhs of the first rule yields

$$f(a) \to_{\sigma, f(x)\to b} b$$

with $x\sigma = a$ such that the resulting system

$$\mathcal{R}' = \left\{ \begin{array}{c} a \to b \\ f(x) \to b \end{array} \right\}$$

is obviously terminating. According to Theorem 1(4) this is only possible because in the above step the non-terminating redex $a = x\sigma$ is erased by applying the rule $f(x) \to b$ (with $x$ occurring in the lhs, but not in the rhs).

The observation in this example, that for non-overlapping TRS's simplification steps on rhs's can only be critical (w.r.t. termination) if a non-terminating proper subterm is eliminated by applying some erasing rule (according to Theorem 1(4)), can easily be generalized as follows.

Let $\mathcal{R}$, $\mathcal{R}'$ be TRS's satisfying

(*) $\mathcal{R}'$ is obtained from $\mathcal{R}$ by replacing some rule $l \to r \in \mathcal{R}$ by a rule $l \to r'$ with $r \to^*_{\mathcal{R}} r'$ where all $\mathcal{R}$-rules applied in $r \to^*_{\mathcal{R}} r'$ are non-erasing.

THEOREM 4. *Suppose $\mathcal{R}$ is non-overlapping, and $\mathcal{R}$, $\mathcal{R}'$ satisfy (*). Then $\mathcal{R}$ is terminating iff $\mathcal{R}'$ is terminating.*

PROOF. First we observe that $\mathcal{R}'$ is obviously non-overlapping, too, by construction, and that the normal forms of $\mathcal{R}$ and $\mathcal{R}'$ coincide, i.e., $\mathrm{NF}(\mathcal{R}) = \mathrm{NF}(\mathcal{R}')$. For a proof by contradiction (of the non-trivial implication) suppose $\mathcal{R}'$ is terminating, but $\mathcal{R}$ is not. Thus there must exist some term $s$ with $\infty_{\mathcal{R}}(s)$ and some normalizing ($\mathcal{R}'$-)derivation such that

$$s =: s_0 \to_{\mathcal{R}'} s_1 \to_{\mathcal{R}'} \ldots \to_{\mathcal{R}'} s_n \in \mathrm{NF}(\mathcal{R}) = \mathrm{NF}(\mathcal{R}') .$$

By $s_n \in \mathrm{NF}(\mathcal{R})$ we know $\neg\infty_{\mathcal{R}}(s_n)$. Hence some step in the expanded ($\mathcal{R}$-)derivation, where every single step $s_j \to_{l\to r' \in \mathcal{R}'} s_{j+1}$ with $l \to r \in \mathcal{R}$ is expanded into a sequence of $\mathcal{R}$-steps $s_j \to_{l \to r \in \mathcal{R}} s'_j \to^*_{\mathcal{R}} s_{j+1}$, the latter part $s'_j \to^*_{\mathcal{R}} s_{j+1}$ of which corresponds to the sequence of $\mathcal{R}$-steps in $r \to^*_{\mathcal{R}} r'$, must be critical (w.r.t. termination of $\mathcal{R}$). But this is impossible due to assumption (*) by Theorem 1(4). $\square$

This result suffices to justify the simplification in the introductory Example 2 that makes the termination proof considerably easier.

Next we show how this result can be refined and extended. A first observation is that the termination statement in Theorem 4 can be *localized* in the following sense: For any term $t$, $t$ is $\mathcal{R}$-terminating iff $t$ is $\mathcal{R}'$-terminating. This is obvious by the assumptions and Theorem 1(4). Secondly, we can also relax the condition (*) a little bit, by allowing the application of erasing $\mathcal{R}$-rules under certain conditions when simplifying $r$ (with $l \to r \in \mathcal{R}$) into $r'$ (with $l \to r' \in \mathcal{R}'$). To see what we need, consider a normalizing ($\mathcal{R}'$-)derivation

$$D : u_0 \to_{\mathcal{R}'} u_1 \to_{\mathcal{R}'} \ldots \to_{\mathcal{R}'} u_n \in \mathrm{NF}(\mathcal{R}) = \mathrm{NF}(\mathcal{R}') .$$

which w.l.o.g. may be assumed to be $\mathcal{R}'$-innermost, and suppose that $\infty_{\mathcal{R}}(u_0)$ holds. As above let us expand every single step $u_j \to_{p,\sigma,l\to r' \in \mathcal{R}'} u_{j+1}$ $(0 \leq j \leq n)$ (where $\sigma$ is irreducible since the step is $\mathcal{R}'$-innermost) in $D$ into[6] $u_j \to_{p,\sigma,l\to r\in\mathcal{R}} u'_j \to^*_{\geq p,\mathcal{R}} u_{j+1}$ where $u'_j \to^*_{\geq p,\mathcal{R}} u_{j+1}$ corresponds to $r \to^*_{\mathcal{R}} r'$, and let $D'$ denote the resulting derivation. The step $u_j \to_{p,\sigma,l\to r\in\mathcal{R}} u'_j$ cannot be critical by Theorem 1(3), because it is innermost (by assumption). Now $E : u_j \to_{p,\sigma,l\to r\in\mathcal{R}} u'_j \to^*_{\geq p,\mathcal{R}} u_{j+1}$ at position $p$ is just an instance of the derivation from $l$ via $r$ to $r'$,

$$l \to r =: v_0 \to_{q_1,\tau_1,l_1\to r_1\in\mathcal{R}} v_1 \to_{q_2,\tau_2,l_2\to r_2\in\mathcal{R}} v_2 \cdots v_{k-1} \to_{q_k,\tau_k,l_k\to r_k\in\mathcal{R}} v_k := r',$$

i.e.:

$$\begin{aligned} E: u_j|p \; = \; & l\sigma \to_{\epsilon,\sigma,l\to r} r\sigma = u'_j|_p = \\ & v_0\sigma \to_{q_1,\tau_1\sigma,l_1\to r_1\in\mathcal{R}} v_1\sigma \to_{q_2,\tau_2\sigma,l_2\to r_2\in\mathcal{R}} \cdots \\ & v_{k-1}\sigma \to_{q_k,\tau_k\sigma,l_k\to r_k\in\mathcal{R}} v_k\sigma = r'\sigma = u_{j+1}|_p \, . \end{aligned}$$

Since we have $\infty_{\mathcal{R}}(u_0)$ and $\neg\infty_{\mathcal{R}}(u_n)$, this implies that for some $j$ with $0 \leq j \leq n$ there exists some (first) $\mathcal{R}$-step in $u'_j \to^*_{\geq p,\mathcal{R}} u_{j+1}$ which is critical (w.r.t. termination). Hence, the corresponding step in $E$,

$$v_i\sigma \to_{q_i,\tau_i\sigma,l_i\to r_i\in\mathcal{R}} v_{i+1}\sigma$$

where $0 \leq i \leq k-1$, is also critical. But, by Theorem 1(4) this is only possible if the latter step is non-innermost and if there exists some $x \in Var(l_i) \setminus Var(r_i)$ with $\infty_{\mathcal{R}}(x\tau_i\sigma)$. Summarizing these observations, we see that in $D$ no step can be critical (hence contradicting $\infty_{\mathcal{R}}(u_0)$) provided that the following condition holds:

(+) For every $j$, $1 \leq j \leq n$, if $u_j \to_{p,\sigma,l\to r\in\mathcal{R}} u'_j \to^*_{\geq p,\mathcal{R}} u_{j+1}$ via

$$\begin{aligned} u_j|p \; = \; & l\sigma \to r\sigma = u'_j|_p = \\ & v_0\sigma \to_{q_1,\tau_1\sigma,l_1\to r_1\in\mathcal{R}} v_1\sigma \to_{q_2,\tau_2\sigma,l_1\to r_2\in\mathcal{R}} \cdots \\ & v_{k-1}\sigma \to_{q_k,\tau_k\sigma,l_k\to r_k\in\mathcal{R}} v_k\sigma = r'\sigma = u_{j+1}|_p \end{aligned}$$

and if for all $m$, $0 \leq m \leq k-1$, the step

$$v_m\sigma \to_{q_m,\tau_m\sigma,l_{m+1}\to r_{m+1}\in\mathcal{R}} v_{m+1}\sigma$$

is such that, for all $x \in Var(l_{m+1})$,

(1) $x \in Var(r_{m+1})$ holds, or

(2) $x\tau_m\sigma$ is ($\mathcal{R}$-)terminating.

Part (1) can easily be checked, but (2) cannot be effectively verified (and is undecidable in general). However, we can give sufficient syntactical (and decidable) conditions for (2), namely by requiring $x\tau_m\sigma$ to be even irreducible: By assumption, $\sigma$ is irreducible. Furthermore we require that $\tau_m$ is irreducible. Nevertheless, this does not yet guarantee that $x\tau_m\sigma$ is irreducible. To obtain this, we additionally require that

(2′) no non-variable subterm of $x\tau_m$ unifies with some (renamed) lhs of $\mathcal{R}$.

Indeed, (2′) and irreducibility of $\sigma$ guarantee that $x\tau_m\sigma$ is irreducible, too. In the special case that $x\tau_m$ is an irreducible ground term, (2′) (and (2)) are automatically and trivially satisfied. The resulting weakening of condition (*) now looks as follows:

Let $\mathcal{R}$, $\mathcal{R}'$ be TRS's such that

(**) $\mathcal{R}'$ is obtained from $\mathcal{R}$ by replacing some rule $l \to r \in \mathcal{R}$ by a rule $l \to r'$ with $D : r \to^*_{\mathcal{R}} r'$ where for every rule instance $\hat{l}\tau \to \hat{r}\tau$ (with $\hat{l} \to \hat{r} \in \mathcal{R}$) used in $D$ and for every variable $x \in Var(\hat{l})$, we have that

[6] When writing $s \to_{\geq p} t$ or $s \to_{\geq p,\mathcal{R}} t$, the notation is to indicate that the ($\mathcal{R}$-)step takes place at some position $q \geq p$ (of $s$).

– $x \in Var(\hat{r})$ holds, or

– no non-variable subterm of $x\tau$ is unifiable with some lhs of $\mathcal{R}$.

In other words, we may simplify $r$ not only using non-erasing $\mathcal{R}$-rules, but also using erasing rules $\hat{l} \to \hat{r}$ with $x\tau$ irreducible for $x \in Var(\hat{l}) \backslash Var(\hat{r})$, provided that we can exclude a priori that any further instantiation by some irreducible substitution $\sigma$ makes some proper subterm of $x\tau\sigma$ reducible.

Thus we have proved the following generalization of Theorem 4.

THEOREM 5. *Suppose $\mathcal{R}$ is non-overlapping, and $\mathcal{R}$, $\mathcal{R}'$ satisfy (**). Then, for any term $t$, $t$ is $\mathcal{R}$-terminating iff $t$ is $\mathcal{R}'$-terminating.*

Observe that in Example 5 condition (**) is violated, because when simplifying $\mathcal{R} = \{a \to f(a), f(x) \to b\}$ into $\mathcal{R}' = \{a \to b, f(x) \to b\}$ via $f(a) \to_{\epsilon,\tau,f(x')\to b} b$ with $x'\tau = a$, the substitution $\tau$ is reducible. To see that irreducibility of $\tau$ is not sufficient, consider the following counterexample.

*Example 6.* The non-terminating non-overlapping TRS

$$\mathcal{R} = \left\{ \begin{array}{c} h(x,a) \to f(h(a,x)) \\ f(x) \to b \\ h(a,b) \to c \end{array} \right\}$$

simplifies via $f(h(a,x)) \to_{\tau,f(x')\to b} c$ with $x'\tau = h(a,x)$ into

$$\mathcal{R}' = \left\{ \begin{array}{c} h(x,a) \to b \\ f(x) \to b \\ h(a,b) \to c \end{array} \right\}$$

which is terminating. Here (**) is violated because $x'\tau = h(a,x)$ unifies with the lhs $h(x,a)$ of the first rule.

Finally, let us give a simple example where Theorem 5 is applicable, but not Theorem 4.

*Example 7.* The non-overlapping TRS

$$\mathcal{R} = \left\{ \begin{array}{c} f(a,b,x) \to f(x,x,x) \\ f(x,x,x) \to c \end{array} \right\}$$

simplifies via $f(x,x,x) \to_{\tau,f(x',x',x')\to c} c$ with $x'\tau = x$ into

$$\mathcal{R}' = \left\{ \begin{array}{c} f(a,b,x) \to c \\ f(x,x,x) \to c \end{array} \right\}$$

which is trivially terminating. Here the applied rule is erasing, hence Theorem 4 is not applicable. But (**) is still satisfied which proves termination of $\mathcal{R}$ by using Theorem 5.[7] The point here is that, independently of the termination proof method being used (like precedence-based syntactical reduction orders or dependency pairs), Theorem 5 allows for a preprocessing step that is sound in the sense that it guarantees equivalence of termination.

[7]Of course, termination of $\mathcal{R}$ can also be proved by many other known methods (this kind of TRS's, including in particular the rule $f(a,b,x) \to f(x,x,x)$, has been well investigated due to Toyama's famous counterexample to modularity of termination ([26]) that includes this rule, too).

Of course, when simplifying systems as above in order to facilitate termination proofs, one would expect (and require) also logical soundness, i.e., preservation of the equational theory under such transformations. If a TRS $\mathcal{R}$ is transformed into $\mathcal{R}'$ by simplifying rhs's of $\mathcal{R}$ (using $\mathcal{R}$-rules), this clearly implies $\to_{\mathcal{R}'} \subseteq \to^*_{\mathcal{R}}$. Yet, the other inclusion, $\to_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{R}'}$ is non-trivial, and does not hold in general as the following simple example demonstrates.

*Example 8.* One-step simplification of the rhs of the only rule of the TRS $\mathcal{R} = \{a \to f(a)\}$ yields $\mathcal{R}' = \{a \to f(f(a))\}$. Obviously, $a \leftrightarrow^*_{\mathcal{R}'} f(a)$ does not hold. Hence we have $\leftrightarrow^*_{\mathcal{R}'} \subsetneq \to^*_{\mathcal{R}}$.

We observe that in the counterexample above $\mathcal{R}'$ (as well as $\mathcal{R}$) is not weakly terminating. This is essential as shown in the next result (which is related to [16, Lemma 4.1]).

LEMMA 1. *Let $\mathcal{R}$ be a TRS and suppose $\mathcal{R}'$ is obtained from $\mathcal{R}$ by arbitrarily simplifying some rhs's of $\mathcal{R}$ (using $\mathcal{R}$-rules).[8] Then the following properties hold:*

*(1) $\to_{\mathcal{R}'} \subseteq \to^+_{\mathcal{R}}$.*

*(2) A term is $\mathcal{R}$-irreducible iff it is $\mathcal{R}'$-irreducible, i.e., $NF(\mathcal{R}) = NF(\mathcal{R}')$.*

*(3) If $\mathcal{R}$ is confluent and $\mathcal{R}'$ weakly terminating, then $\mathcal{R}'$ is also confluent, and $\mathcal{R}$, $\mathcal{R}'$ are logically equivalent.*

PROOF. (1) and (2) are obvious by the assumptions on $\mathcal{R}$ and $\mathcal{R}'$. Concerning (3), we first prove confluence of $\mathcal{R}'$. Suppose $t \to^*_{\mathcal{R}'} t_1$ and $t \to^*_{\mathcal{R}'} t_2$. By weak termination of $\mathcal{R}'$ there exist $\widehat{t_1}, \widehat{t_2} \in \mathrm{NF}(\mathcal{R}')$ with $t_1 \to^*_{\mathcal{R}'} \widehat{t_1}$, $t_2 \to^*_{\mathcal{R}'} \widehat{t_2}$. By (1), (2) and confluence of $\mathcal{R}$ we get that $\widehat{t_1}$ and $\widehat{t_2}$ coincide, hence $\widehat{t_1} = \widehat{t_2}$ is a common $\mathcal{R}'$-reduct of $t_1$ and $t_2$ as desired. For logical equivalence of $\mathcal{R}$ and $\mathcal{R}'$, by (1) it suffices to show $\to_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{R}'}$, or, equivalently: $l \leftrightarrow^*_{\mathcal{R}'} r$ for all $l \to r \in \mathcal{R}$. By weak termination of $\mathcal{R}'$ both $l$ and $r$ have $\mathcal{R}'$-normal forms, say $\hat{l}$ and $\hat{r}$, respectively: $l \to^*_{\mathcal{R}'} \hat{l}$, $r \to^*_{\mathcal{R}'} \hat{r}$. Now (1), (2) and confluence of $\mathcal{R}$ yield $\hat{l} = \hat{r}$, hence $l \leftrightarrow^*_{\mathcal{R}'} r$. $\square$

Lemma 1 implies in particular that in the cases we are interested in, namely in Theorems 4 and 5, the simplified TRS $\mathcal{R}'$ is logically equivalent to the original system $\mathcal{R}$, provided $\mathcal{R}'$ is terminating. Note that confluence (of $\mathcal{R}$) is a consequence of non-overlappingness and termination (of $\mathcal{R}$).

## 5. GENERALIZATIONS

In this section we shall investigate two kinds of generalizations. Firstly, we ask whether Theorems 3 and 5 can be generalized by weakening the required non-overlapping assumption. Secondly, and more importantly, we discuss to what extent these results also hold for conditional TRS's (CTRS's).

More details and missing proofs of some needed auxiliary lemmas can be found in [15] and, partially, also in [14].

[8]More precisely, $\mathcal{R}'$ contains for every $l \to r \in \mathcal{R}$ one(!) rule $l \to r'$ where $r \to^*_{\mathcal{R}} r'$, and nothing else.

## 5.1 Weakening the No-Overlap Requirement

Regarding Theorem 1, inspection of its proof(s) reveals where the non-overlapping property is really used. The proof of Theorem 1(2), namely: WIN($t$) $\iff$ SIN($t$), is only based on uniform confluence of innermost reduction, i.e., WCR$^1(\to_i)$, which is needed to apply the following result on *abstract reduction systems* (which are just binary relations $\to$ on some set $A$), due to Newman.

LEMMA 2 ([23]). *Let $\mathcal{A} = (A, \to)$ be an abstract reduction system, and $a \in A$. Suppose WN($a$) and WCR$^1(\mathcal{G}(a))$ hold.*[9] *Then SN($a$) holds, too.*[10]

With $\to_i$ as $\to$ in Lemma 2, WCR$^1(\mathcal{G}(a), \to_i)$ and WN$(a, \to_i)$ yield SN$(a, \to_i)$ as desired. Now it is easy to see that WCR$^1(\mathcal{G}(a), \to_i)$ does not only hold for non-overlapping TRS's, but also for the slightly more general class of systems satisfying the following *critical peak* condition CPC$'$ depending on $\mathcal{R}$ (cf. [15]):[11]

CPC$'$: For every critical peak $t_1 \leftarrow_p s \to_\epsilon t_2$ we have:

(1) If $p = \epsilon$ and both steps are innermost, then $t_1 = t_2$, and

(2) if $p > \epsilon$ and the *inside step* $t_1 \leftarrow_p s$ is innermost, then $t_1 = t_2$.

Observe that, in particular, weakly non-overlapping TRS's satisfy CPC$'$.

CPC$'$ also implies the following property: Whenever we have a reduction step $s \to_{p,\sigma,l\to r} t$ that is not innermost, then either all innermost redexes of $l\sigma$ are within the "variable parts" of $l\sigma$, or else there exists another reduction step from $s$ to $t$ that is innermost, i.e., $s \to_i t$.[12] This means in particular that when considering infinite derivations issuing from some term $t$ (in TRS's satisfying CPC$'$), one may w.l.o.g. assume that every step $t_k \to_{p,\sigma,l\to r} t_{k+1}$ therein is either innermost, or else all innermost redexes properly below $l\sigma$ are "within $\sigma$".

These two consequences of CPC$'$ are the essential ingredients needed for proving

THEOREM 6. *(generalized version of Theorem 1, cf. Theorem 3.4.33 in [15]) Let $\mathcal{R}^{\mathcal{F}}$ be a TRS and $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. If $\mathcal{R}$ satisfies the critical peak condition*[13] *CPC$'$ then all properties (1)-(6) of Theorem 1 hold here, too.*

PROOF. See [15]. $\Box$

Using this generalized result we are now also able to generalize the soundness of the preprocessing approach expressed by Theorem 5 from non-overlapping systems to systems satisfying CPC$'$.

THEOREM 7 (GENERALIZED VERSION OF THEOREM 5). *Suppose $\mathcal{R}$ satisfies CPC$'$, and (**) holds for $\mathcal{R}$, $\mathcal{R}'$. Then, for any term $t$, $t$ is $\mathcal{R}$-terminating iff $t$ is $\mathcal{R}'$-terminating.*

PROOF. Analogous to the reasoning that allowed us to generalize Theorem 4 to Theorem 5. All properties needed for this reasoning are also satisfied for TRS's with CPC$'$, because of Theorem 6 above. $\Box$

Similarly, and for the same reasons, Theorem 3 can be generalized from non-overlapping TRS's to TRS's satisfying CPC$'$.

THEOREM 8 (GENERALIZED VERSION OF THEOREM 3). *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be TRS's and $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$ be their (not necessarily disjoint) union. Suppose the following:*

*(1) $\mathcal{R}_1$ and $\mathcal{R}_2$ are weakly innermost terminating.*

*(2) $\mathcal{R}_1$ preserves $\mathcal{R}_2$-normal forms.*

*(3) $\mathcal{R}$ satisfies CPC$'$.*

*Then $\mathcal{R}$ is terminating.*

PROOF. The proof is analogous to the proof of Theorem 3 relying on Theorem 6 instead of Theorem 1. $\Box$

Finally, we observe Lemma 1 also holds for the current setting. This shows that if $\mathcal{R}'$ is terminating according to Theorem 5, then the original system $\mathcal{R}$ is terminating and logically equivalent to the simplified version $\mathcal{R}'$. Note that here confluence of $\mathcal{R}$ (which is needed to apply Lemma 1(3)) follows from CPC$'$ and termination of $\mathcal{R}$ (cf. [15, Theorem 16]).

## 5.2 Conditional Term Rewriting Systems

Finally, we shall investigate whether the presented results also extend to the case of conditional rewrite systems. Here we shall concentrate on those results that allow for simplification of termination proofs by preprocessing (simplification of rhs's). Furthermore we shall study to what extent simplification of conditions (besides simplifications of rhs's) is also possible such that (non-)termination and logical equivalence are preserved.

In 5.2.1 we consider CTRS's under a *join semantics*, i.e., where equality in the conditions is recursively interpreted as joinability (w.r.t. the rewriting relation being defined). Extra variables in conditions are allowed, but not in rhs's. As a practically important special case of join CTRS's we will also consider *normal (join)* CTRS's in 5.2.2.

Conditional rules are of form

$$l \to r \Longleftarrow c$$

where $c$ is a conjunction of condition literals $s_i = t_i$, $1 \leq i \leq n$, written just as a list $s_1 = t_n, \ldots, s_n = t_n$. If equality in the conditions is to be related to (defined simultaneously with) the reduction relation induced by a set of such rules, one has to be more precise about which operational semantics is meant. Common cases are *semi-equational* semantics, where equality is interpreted as convertibility ($\leftrightarrow^*$), and *join* semantics, where equality is interpreted as joinability ($\downarrow = \to^* \circ {}^*\!\leftarrow$). A special case of join systems, where a rule as above is denoted by $l \to r \Longleftarrow s_1 \downarrow t_n, \ldots, s_n \downarrow t_n$, are

[9] $\mathcal{G}(a)$ is the *reduction graph* of $a$ generated by $\to$.

[10] By *Newman's Lemma* this further implies CR($a$).

[11] Actually, there also exist properties guaranteeing WCR$^1(\to_i)$ that are slightly more general than CPC$'$, cf. [14, 15].

[12] This is a special case of a property called *avoidability of innermost-critical reduction steps* and denoted by AICR (cf. [14, 15]).

[13] To preserve compatibility with the notations from [15], we use here CPC$'$, which is a specialized version of a more general critical peak condition called CPC (cf. [15, Def.3.2.29]).

*normal (join)* CTRS's, where all rhs's $t_i$ of condition literals are required to be ground terms that are $\mathcal{R}_u$-irreducible. $\mathcal{R}_u$ is the unconditional version of a join CTRS $\mathcal{R}$, i.e., $\mathcal{R}_u := \{l \to r \mid l \to r \Longleftarrow c \in \mathcal{R}\}$. Since in a normal CTRS, by definition joinability of conditions can only be from left to right, we denote rules of normal CTRS also by $l \to r \Longleftarrow s_1 \to^* t_n, \ldots, s_n \to^* t_n$. The reduction relation induced by some (join) CTRS $\mathcal{R}$ is recursively defined as follows:

$$\mathcal{R}_0 := \emptyset,$$
$$\mathcal{R}_{n+1} := \mathcal{R}_n \cup \{(l\sigma, r\sigma) \mid l \to r \Longleftarrow s_1{\downarrow}t_1, \ldots, s_n{\downarrow}t_n \in \mathcal{R}, \; s_i\sigma{\downarrow}_{\mathcal{R}_n} t_i\sigma \text{ for all } 1 \leq i \leq n\},$$
$$\to_{\mathcal{R}} := \bigcup_{i \geq 0} \to_{\mathcal{R}_i}.$$

Observe that $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$ for all $n \geq 0$. The **depth** of rewriting step(s) $s \to^*_{\mathcal{R}} t$ is the minimal $n$ with $s \to^*_{\mathcal{R}_n} t$. $\mathcal{R}$ is said to be *shallow-confluent* if whenever $s \to^*_{\mathcal{R}_m} t_1$ and $s \to^*_{\mathcal{R}_n} t_2$ then there exists a term $u$ with $t_1 \to^*_{\mathcal{R}_n} u$ and $t_2 \to^*_{\mathcal{R}_m} u$ (for all $m, n \geq 0$). $\mathcal{R}$ is *level-confluent* if every TRS $\mathcal{R}_n$ ($n \geq 0$) is confluent. A *(conditional) critical pair* between two rules $l_i \to r_i \Longleftarrow c_i$, $i = 1, 2$ (having disjoint variables), is defined as in the unconditional case but adding the conjunction of the corresponding conditions, i.e., by unifying some non-variable subterm of $l_1$ with $l_2$, let's say at position $p$ with mgu $\sigma$, yielding $l_1\sigma[r_2\sigma]_p = r_1\sigma \Longleftarrow c$ with $c = c_1\sigma, c_2\sigma$. A condition $c : s_1{\downarrow}t_1, \ldots, s_n{\downarrow}t_n$ is said to be *infeasible* or *unsolvable* if there is no substitution $\sigma$ satisfying $c : s_i\sigma{\downarrow}_{\mathcal{R}} t_i\sigma$ for all $i, 1 \leq i \leq n$. A (conditional) critical pair $s = t \Longleftarrow c$ is *infeasible* if its condition $c$ is infeasible.

A CTRS $\mathcal{R}$ is *non-overlapping*, *left-linear*, *orthogonal*, *non-erasing* if $\mathcal{R}_u$ is non-overlapping, left-linear, orthogonal, non-erasing, respectively.

In the remainder of this section, we always assume that for any conditional rule $l \to r \Longleftarrow c$, extra variables occur at most in $c$ (i.e., $Var(r) \subseteq Var(l)$)

For further basic terminology and results about CTRS's we refer to e.g. [6], [10], [9], [20].

In general, conditional rewriting is known to be much more complicated and intricate than unconditional rewriting (cf. e.g. [6], [10], [22], [17]).

### 5.2.1 Join CTRS's

In this subsection we always assume, if not explicitly otherwise stated, CTRS's to be join ones.

Many properties of TRS's do not hold for CTRS's (or only under additional assumptions). However, it is also well-known that special classes of CTRS's like non-overlapping ones or conditional *overlay* systems behave more nicely. And, in fact, this is also the case for (at least some of) the theorems presented in the previous sections. Again the structural properties expressed by Theorem 1 are essential ones for making certain proofs in the conditional case go through.

The crucial structure Theorem 1 does indeed extend to the conditional case.

THEOREM 9 (THEOREM 1 EXTENDED TO CTRS'S). *Let $\mathcal{R}^{\mathcal{F}}$ be a CTRS and $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. If $\mathcal{R}$ is non-overlapping then the properties (1)-(3) and (5)-(6) of Theorem 1 hold here, too. (4) holds also, but in its adapted (to conditional rules) version:*

*(4) If $s \to_{p,\sigma,l\to r\Longleftarrow c} t$ with $l \to r \Longleftarrow c \in \mathcal{R}$, $\infty(s)$ and $\neg\infty(t)$, then $s|_p = l\sigma$ contains some proper subterm $x\sigma$ with $x \in Var(l) \setminus Var(r)$ and $\infty(x\sigma)$.*

PROOF. See [15] for a detailed proof. Actually, in contrast to the unconditional case, this proof makes use of another crucial result for CTRS, namely, that for conditional overlay systems with joinable critical pairs, innermost termination of some term $t$ implies termination (and confluence) of $t$ ([15, Theorem 3.6.10]). □

For trying to extend (the global version of) Theorem 5 to CTRS's we first need to adapt condition (**) to the conditional setting:

Let $\mathcal{R}$, $\mathcal{R}'$ be CTRS's such that

(#) $\mathcal{R}'$ is obtained from $\mathcal{R}$ by replacing some rule $l \to r \Longleftarrow c \in \mathcal{R}$ by a rule $l \to r' \Longleftarrow c$ with $D : r \to^*_{\mathcal{R}} r'$ where for every rule instance $\hat{l}\tau \to \hat{r}\tau \Longleftarrow \hat{c}\tau$ (with $\hat{l} \to \hat{r} \Longleftarrow \hat{c} \in \mathcal{R}$) used in $D$ and for every variable $x \in Var(\hat{l})$, we have that

- $x \in Var(\hat{r})$ holds, or
- no non-variable subterm of $x\tau$ is unifiable with some lhs of $\mathcal{R}$.

Note that in general reducibility of terms (here: of rhs's of $\mathcal{R}$) is undecidable, even for terminating CTRS's. However, it is clear that decidable incomplete versions of reducibility checks can be used, too. One such incomplete version is to look just for reducibility w.r.t. some unconditional rule. Moreover, the variable conditions in (#) are easily decidable because they are purely syntactic and ignore the conditions.

Now, in analogy to the unconditional case, we would like to prove the following:

- If $\mathcal{R}$ is a non-overlapping CTRS, $\mathcal{R}'$ is terminating and $\mathcal{R}$, $\mathcal{R}'$ satisfy (#), then $\mathcal{R}$ is terminating and logically equivalent to $\mathcal{R}'$.

Unfortunately, and this is quite typical in conditional rewriting, the proof from the unconditional case does not extend to the conditional setting in an obvious way. There are several major problems. First of all, one might tacitly expect the analogue of the basic (and, in the unconditional case, easy to prove) Lemma 1 for the conditional case to hold, too.

Suppose $\mathcal{R}$ is a CTRS and $\mathcal{R}'$ is obtained from $\mathcal{R}$ by arbitrarily simplifying some rhs's of $\mathcal{R}$ (using $\mathcal{R}$-rules), more precisely:

> $\mathcal{R}'$ contains for every $l \to r \Longleftarrow c \in \mathcal{R}$ one(!) rule $l \to r' \Longleftarrow c$ where $r \to^*_{\mathcal{R}} r'$, and nothing else.

Then the statements corresponding to Lemma 1,

(1) $\to_{\mathcal{R}'} \subseteq \to^+_{\mathcal{R}}$.

(2) A term is $\mathcal{R}$-irreducible iff it is $\mathcal{R}'$-irreducible.

(3) If $\mathcal{R}$ is confluent and $\mathcal{R}'$ weakly terminating, then $\mathcal{R}'$ is also confluent, and $\mathcal{R}$, $\mathcal{R}'$ are logically equivalent.

are no longer trivial. In fact, (1) still holds (as can be shown via proving

$$\to_{\mathcal{R}'_n} \subseteq \to^+_{\mathcal{R}}$$

by induction on $n$). However, the validity of the crucial property (2), i.e., the preservation of normal forms, on which the proof of (3) essentially relies, becomes unclear. The reason is that in contrast to unconditional rewrite systems, simplifying some rhs may have global(!) effects on reducibility. Observe moreover that in the proof of Theorem 4 (and of generalized versions thereof) the preservation of normal forms was also essential!

Actually, for proof-technical reasons we will need confluence of the considered CTRS's $\mathcal{R}$ without assuming its termination, in order to establish the preservation of normal forms when considering $\mathcal{R}'$ instead of $\mathcal{R}$. In the unconditional case it is well-known that any orthogonal TRS is confluent. However, orthogonal join CTRS's need not be confluent (cf. [6]).

*Example 9.* The orthogonal join CTRS

$$\mathcal{R} = \left\{ \begin{array}{ccc} b \to f(b) & & \\ f(x) \to a & \Longleftarrow & f(x) \downarrow x \end{array} \right\}$$

is not (weakly) confluent. We have $f(b) \to a$ and $f(f(b)) \to a$, hence a local divergence

$$f(a) \leftarrow f(f(b)) \to a$$

where both $a$ and $f(a)$ are distinct normal forms.

Imposing additionally that conditions are normal however suffices for recovering confluence.

### 5.2.2 Extensions for Normal Join CTRS's

THEOREM 10 (CF. [6]). *Any orthogonal normal CTRS $\mathcal{R}$ is confluent.*

In fact, as remarked in [12] (cf. also [25]) the proof in [6] even yields shallow- and hence level-confluence of $\mathcal{R}$. For this practically important subclass of join CTRS's we will now investigate the preservation of normal forms under a restricted version of the mentioned preprocessing operation for rhs's. More precisely, instead of

(A) $\mathcal{R}'$ contains for every $l \to r \Longleftarrow c \in \mathcal{R}$ one(!) rule $l \to r' \Longleftarrow c$ where $r \to^*_{\mathcal{R}} r'$, and nothing else

we require

(B) $\mathcal{R}'$ contains for every $l \to r \Longleftarrow c \in \mathcal{R}$ one(!) rule $l \to r' \Longleftarrow c$ where $r \to^*_{\mathcal{R}_1} r'$, and nothing else.

In other words, in essence simplification of rhs's of conditional (normal) rules is only allowed using unconditional rules.

THEOREM 11. *Let $\mathcal{R}$ be an orthogonal normal CTRS such that $\mathcal{R}$, $\mathcal{R}'$ satisfy (B) and $\mathcal{R}'$ is terminating. Then the following properties hold:*

*(1) $\mathcal{R}'$ is also an orthogonal normal CTRS.*

*(2) $\to_{\mathcal{R}'_k} \subseteq \to^+_{\mathcal{R}_k}$ for all $k \geq 0$.*

*(3) $\to_{\mathcal{R}'} \subseteq \to^+_{\mathcal{R}}$.*

*(4) $NF(\mathcal{R}'_k) = NF(\mathcal{R}_k)$ for all $k \geq 0$.*

*(5) $NF(\mathcal{R}') = NF(\mathcal{R})$.*

*(6) $\mathcal{R}$ and $\mathcal{R}'$ are logically equivalent.*

PROOF. (1) is obvious by the assumptions (and also holds with (A) instead of (B)).

Next we prove (2) by induction on $k$. The case $k = 0$ is trivial due $\mathcal{R}'_0 = \mathcal{R}_0 = \emptyset$. In the induction step consider a reduction $s = s[l\sigma] \to_{\mathcal{R}'_{k+1}} s[r'\sigma] = t$ using some rule $l \to r' \Longleftarrow c \in \mathcal{R}'$ with $l \to r \Longleftarrow c \in \mathcal{R}$ and $r \to^*_{\mathcal{R}_1} r'$, because of (B). For every condition $u \to^* v$ in $c$ we must have $u\sigma \to^*_{\mathcal{R}'_k} v\sigma = v$ (by normality), hence by induction $u\sigma \to^*_{\mathcal{R}_k} v\sigma = v$. Consequently, $l \to r \Longleftarrow c$ is also applicable to $s$ yielding $s = s[l\sigma] \to_{\mathcal{R}_{k+1}} s[r\sigma] \to^*_{\mathcal{R}_1} s[r'\sigma] = t$. Due to $1 \leq k+1$ and $\mathcal{R}_m \subseteq \mathcal{R}_n$ for all $m \leq n$ this implies $s \to^+_{\mathcal{R}_{k+1}} t$ as desired.

(3) is a trivial consequence of (2).

For proving (4), we first observe that $\mathrm{NF}(\mathcal{R}_k) \subseteq \mathrm{NF}(\mathcal{R}'_k)$ holds for all $k \geq 0$ by (2). Hence it suffices to show $\mathrm{NF}(\mathcal{R}'_k) \subseteq \mathrm{NF}(\mathcal{R}_k)$ for all $k \geq 0$. Again we proceed by induction on $k$. The base case $k = 0$ is trivial due to $\mathcal{R}'_0 = \mathcal{R}_0 = \emptyset$. In the induction step suppose for a proof by contradiction that there exists $s \in \mathrm{NF}(\mathcal{R}'_{k+1}) \setminus \mathrm{NF}(\mathcal{R}_{k+1})$. Hence there is a reduction $s \to_{\mathcal{R}_{k+1}} t$ of the form $s = s[l\sigma] \to_{\mathcal{R}_{k+1}} s[r\sigma] = t$ with $l \to r \Longleftarrow c \in \mathcal{R}$ such that for every condition $u \to^* v$ in $c$ we have $u\sigma \to^*_{\mathcal{R}_k} v$. Now, termination of $\mathcal{R}'$ clearly implies termination of $\mathcal{R}'_k$ for all $k \geq 0$. Hence we may reduce (in $\mathcal{R}'_k$) $u\sigma$ to some $\mathcal{R}'_k$-normal form $\bar{u}$, which by induction hypothesis is also an $\mathcal{R}_k$-normal form. By (2) this implies $u\sigma \to^*_{\mathcal{R}_k} \bar{u}$. Hence we have $u\sigma \to^*_{\mathcal{R}_k} v$ and $u\sigma \to^*_{\mathcal{R}_k} \bar{u}$ with both $v$ and $\bar{u}$ being $\mathcal{R}_k$-normal forms (note that $v$ is a ground term that is even $\mathcal{R}_u$-irreducible by normality of $\mathcal{R}$). But now level-confluence of $\mathcal{R}$ implies $v = \bar{u}$ and consequently $u\sigma \to^*_{\mathcal{R}'_k} v$. Since the above reasoning holds for all conditions $u \to^* v$ in $c$, we conclude that $s$ is $\mathcal{R}'_{k+1}$-reducible. But this is a contradiction to $s \in \mathrm{NF}(\mathcal{R}'_{k+1})$. Hence we are done.

The preservation property (5) now is an obvious consequence of (4).

Finally, for proving (6), by (3) it suffices to show that whenever $s \to_{\mathcal{R}} t$ then there exists some $u$ with $s \to^*_{\mathcal{R}'} u$, $t \to^*_{\mathcal{R}'} u$. By termination of $\mathcal{R}'$ we can ($\mathcal{R}'$-)reduce $s$ and $t$ to $\mathcal{R}'$-normal forms $\bar{s}$ and $\bar{t}$. By (5), $\bar{s}$ and $\bar{t}$ are also $\mathcal{R}$-normal forms. And from (3) we get $s \to^*_{\mathcal{R}} \bar{s}$ and $s \to_{\mathcal{R}} t \to^*_{\mathcal{R}} \bar{t}$, respectively. Since $\mathcal{R}$ is an orthogonal normal CTRS, it is confluent by Theorem 10. This implies $\bar{s} = \bar{t}$. Choosing $u = \bar{s} = \bar{t}$ we are done. □

Next we consider a version of preprocessing by simplifying rhs's that is compatible with Theorem 11 above.

Let $\mathcal{R}$, $\mathcal{R}'$ be CTRS's such that

(##) $\mathcal{R}'$ is obtained from $\mathcal{R}$ by replacing some rule $l \to r \Longleftarrow c \in \mathcal{R}$ by a rule $l \to r' \Longleftarrow c$ with $D : r \to^*_{\mathcal{R}_1} r'$ where for every rule instance $\hat{l}\tau \to \hat{r}\tau \Longleftarrow \hat{c}\tau$ (with $\hat{l} \to \hat{r} \Longleftarrow \hat{c} \in \mathcal{R}$) used in $D$ and for every variable $x \in Var(\hat{l})$, we have that

- $x \in Var(\hat{r})$ holds, or
- no non-variable subterm of $x\tau$ is unifiable with some lhs of $\mathcal{R}$.

Now we can prove an extended version of Theorem 5 (more precisely of its global version), where, however, simplification of rhs's of conditional rules can only be performed by using unconditional rules (of the original system).

THEOREM 12. *Suppose $\mathcal{R}$ is an orthogonal normal join CTRS, $\mathcal{R}'$ is terminating and $\mathcal{R}$, $\mathcal{R}'$ satisfy (##). Then $\mathcal{R}$ is terminating and logically equivalent to $\mathcal{R}'$.*

PROOF. The proof that $\mathcal{R}$ is terminating is analogous to the one of Theorem 4, taking into account the subsequent refinements. Instead of Theorem 1 (for non-overlapping TRS's) now Theorem 9 (for non-overlapping CTRS's which covers the special case of orthogonal normal CTRS's) is used. The crucial preservation of normal forms which was trivial in the unconditional case is now guaranteed by Theorem 11(5). Similarly, Theorem 11(3) replaces the corresponding trivial observation in the unconditional case. And finally, logical equivalence of $\mathcal{R}$ and $\mathcal{R}'$ is guaranteed by Theorem 11(6). □

Unfortunately, we have not been able to allow simplification of rhs's by arbitrary conditional rewriting (yet obeying the non-erasingness restrictions). It remains open whether the more liberal and natural condition (#) would also suffice for proving the preservation of non-termination of $\mathcal{R}$ and the logical equivalence of $\mathcal{R}$ and $\mathcal{R}'$. Technically, the main problem here seems to be how to guarantee the preservation of normal forms, $\mathrm{NF}(\mathcal{R}) = \mathrm{NF}(\mathcal{R}')$.

However, Theorem 12 above is sufficient to justify the simplification steps of rhs's in Example 3 as presented in Section 4. What remains to be done, in particular for justifying our reasoning in Example 3, is to take into account the simplification of conditions in conditional rules. This is tackled next.

Suppose $\mathcal{R}$ is an orthogonal normal CTRS and $\mathcal{R}'$ is obtained from $\mathcal{R}$ by arbitrarily simplifying some conditions of $\mathcal{R}$ (using $\mathcal{R}$-rules), more precisely:

(cs) $\mathcal{R}'$ contains for every

$$l \to r \Longleftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n \in \mathcal{R}$$

one(!) rule

$$l \to r \Longleftarrow s_1' \to^* t_1, \ldots, s_n' \to^* t_n \in \mathcal{R},$$

with $s_i \to_{\mathcal{R}}^* s_i'$ $(1 \leq i \leq n)$, and nothing else. Moreover, suppose that all simplications of conditions for all modified conditional rules of $\mathcal{R}$ are $\to_{\mathcal{R}_m}^*$-reductions, for some fixed $m$.[14]

THEOREM 13. *Let $\mathcal{R}$ be an orthogonal normal join CTRS, and suppose that $\mathcal{R}'$ is obtained from $\mathcal{R}$ by simplifying some conditions according to (cs) above. Then the following properties hold:*

*(a)* $\mathcal{R}_k' \subseteq \mathcal{R}_{m+k}$ *for all* $k \geq 0$.

*(b)* $\mathcal{R}_k \subseteq \mathcal{R}_k'$ *for all* $k \geq 0$.

*(c)* $\to_{\mathcal{R}} = \to_{\mathcal{R}'}$.

PROOF. We prove (a) by induction on $k$. The base case $k = 0$ is trivial due to $\mathcal{R}_0' = \emptyset$. For the induction step consider an $\mathcal{R}_{k+1}'$-reduction $s = s[l\sigma] \to_{\mathcal{R}_{k+1}'} s[r\sigma] = t$ using an $\mathcal{R}'$-rule $l \to r \Longleftarrow s_1' \to^* t_1, \ldots, s_n' \to^* t_n$ that has been obtained from an $\mathcal{R}$-rule $l \to r \Longleftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$ via $s_i \to_{\mathcal{R}_m}^* s_i'$ $(1 \leq i \leq n)$. Thus we have $s_i'\sigma \to_{\mathcal{R}_k'}^* t_i$ and $s_i\sigma \to_{\mathcal{R}_m}^* s_i'\sigma$ for all $1 \leq i \leq n$. By induction hypothesis we get $s_i'\sigma \to_{\mathcal{R}_{k+m}}^* t_i$, hence $s_i\sigma \to_{\mathcal{R}_m}^* s_i'\sigma \to_{\mathcal{R}_{m+k}}^* t_i$ and $s_i\sigma \to_{\mathcal{R}_{m+k}}^* t_i$ for all $1 \leq i \leq n$. But this implies $s = s[l\sigma] \to_{\mathcal{R}_{m+k+1}} s[r\sigma] = t$ as desired.

Next we show (b), also by induction on $k$. The base case $k = 0$ is again trivial. In the induction step consider an $\mathcal{R}_{k+1}$-reduction $s = s[l\sigma] \to_{\mathcal{R}_{k+1}} s[r\sigma] = t$ using an $\mathcal{R}$-rule $l \to r \Longleftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$, i.e., with $s_i\sigma \to_{\mathcal{R}_k}^* t_i$ (for all $1 \leq i \leq n$). The corresponding $\mathcal{R}'$-rule is $l \to r \Longleftarrow s_1' \to^* t_1, \ldots, s_n' \to^* t_n$ with $s_i \to_{\mathcal{R}_m}^* s_i'$, hence also $s_i\sigma \to_{\mathcal{R}_m}^* s_i'\sigma$ (for all $1 \leq i \leq n$). Now, $t$ is a ground $\mathcal{R}_u$-normal form, hence in particular also $\mathcal{R}$- and $\mathcal{R}_n$-irreducible. Together with shallow-confluence of $\mathcal{R}$ (note that $\mathcal{R}$ is orthogonal and normal) this gives $s_i'\sigma \to_{\mathcal{R}_k}^* t$ (for all $1 \leq i \leq n$). By induction hypothesis this implies $s_i'\sigma \to_{\mathcal{R}_k'}^* t$ (for all $1 \leq i \leq n$). But this means that the $\mathcal{R}'$-rule $l \to r \Longleftarrow s_1' \to^* t_1, \ldots, s_n' \to^* t_n$ is also applicable to $s$ yielding the same result: $s = s[l\sigma] \to_{\mathcal{R}_{k+1}'} s[r\sigma] = t$. Hence we are done.

Now (c) is an easy consequence of (a) and (b) as follows: $\to_{\mathcal{R}} = \bigcup_{i\geq 0} \to_{\mathcal{R}_i} \subseteq \bigcup_{i\geq 0} \to_{\mathcal{R}_i'} = \to_{\mathcal{R}'} \subseteq \bigcup_{i\geq 0} \to_{\mathcal{R}_{m+i}} = \bigcup_{i\geq 0} \to_{\mathcal{R}_i} = \to_{\mathcal{R}}$, hence $\to_{\mathcal{R}} = \to_{\mathcal{R}'}$. The first inclusion above is by (b), the second one by (a), and the last one is due to $\bigcup_{0\leq j\leq m} \to_{\mathcal{R}_j} = \to_{\mathcal{R}_m}$. □

COROLLARY 1. *Let $\mathcal{R}$ be an orthogonal normal join CTRS, and suppose that $\mathcal{R}'$ is obtained from $\mathcal{R}$ by simplifying some conditions according to (cs) above. Then $\mathcal{R}'$ is terminating iff $\mathcal{R}$ is terminating, and the equational theories of $\mathcal{R}$ and $\mathcal{R}'$ coincide.*

This means that simplification of conditions as preprocessing step (for orthogonal normal CTRS's) is harmless (it even preserves the induced reduction relation).

Finally, to be able to justify the simplifications in Example 3, we have to allow for left-linear normal join CTRS's $\mathcal{R}$ that are not (syntactically) non-overlapping (according to our Definition via $\mathcal{R}_u$), but still have essentially the same properties. In particular, one would like to allow infeasible critical pairs because they do not give rise to any critical peaks (since, by infeasibility, such one-step divergences do not really occur). This motivates the following definition.

*Definition 2.* A normal join CTRS is said to *semantically non-overlapping* if all its critical pairs are infeasible.

All results presented for non-overlapping (normal) CTRS's do indeed also hold for semantically non-overlapping (normal) CTRS's. However, in general it is undecidable whether a CTRS is semantically non-overlapping, since infeasibility is undecidable. Hence, in practice such a generalization is only useful if decidable criteria for infeasibility are provided. One such criterion deals with the practically very important special case of *definition by (mutually exclusive) case analysis*, in the following sense.

*Definition 3.* Let $\mathcal{R}$ be a normal CTRS. A conjunction $c$ of normal conditions $s_1 \to^* t_1, \ldots, s_n \to^* t_n$ (i.e., with $t_k$ $\mathcal{R}_u$-irreducible) is said to be *complementary* if there exist $i, j$ with $1 \leq i, j \leq n$ such that $s_i = s_j$ and $t_i \neq t_j$ (this implies in particular $i \neq j$).

[14]This is e.g. automatically guaranteed if $\mathcal{R}$ is finite.

Clearly, this notion of complementarity of conditions is easily decidable, since it is purely syntactical.

LEMMA 3. *Let $\mathcal{R}$ be a left-linear normal CTRS such that for every conditional critical pair $s_1 = s_2 \Longleftarrow c$ of $\mathcal{R}$ the condition $c$ is complementary. Then all these critical pairs are infeasible (hence $\mathcal{R}$ is semantically non-overlapping) and $\mathcal{R}$ is (shallow) confluent.*

PROOF. Again basically by using the proof technique of [6] to establish shallow-(level-)confluence of $\mathcal{R}$. The infeasibility of the critical pairs then is a by-product of the overall confluence proof. $\square$

Actually, our notion of *complementarity* above can be seen a special case of a more general complementarity notion considered in [21] within the extended framework of positive/negative CTRS's where also extended versions of decidable infeasibility checks are discussed.

*Example 10.* (Example 3 revisited) The presented results (for left-linear, semantically non-overlapping, normal CTRS's) now suffice for justifying all simplifications we had performed in this example in order to significantly facilitate the termination proof of the original system. $\mathcal{R}_S$ is left-linear, and semantically non-overlapping, because the condition of the only (conditional) critical pair is complementary. Furthermore, we observe there that for the simplification of rhs's we had used not only the non-erasing rule $\mathrm{app}(\mathrm{nil}, x) \to x$, but also the erasing one $\mathrm{map\_f}(pid, \mathrm{nil}) \to \mathrm{nil}$, where, however, the variable *pid* was substituted by the $\mathcal{R}_{S_u}$-irreducible ground term self. Both rules used here for simplification of rhs's are unconditional (as we required in (##)). And moreover, we had also performed simplification of a condition.

Actually, as already mentioned the approach pursued in [2] for treating Example 3 differs from our presentation here in the sense that there, instead of termination of $\mathcal{R}_S$, *left-to-right decreasingness* is the goal that is tried to be achieved (cf. [2] for more details). It remains open whether preprocessing techniques along the lines of the approach presented here can also be conceived and used for simplifying (left-right) decreasingness proofs.

# 6. CONCLUSIONS

We have shown how powerful structural properties for non-overlapping TRS's give rise to refined and new results on termination (proofs) of such systems. Moreover, we have investigated how to obtain generalized versions of these results by weakening the non-overlapping requirement, and for the case of conditional rewriting. These generalizations heavily depend on the correspondingly generalized structural properties. What remained open is whether for the case of orthogonal normal CTRS's simplification of rhs's using conditional rules can also be handled appropriately.

From a practical point of view we think that, in particular, Theorem 5 and the conditional extensions of its global version, Theorems 12, 13 and Corollary 1 (together with their generalization for left-linear, semantically non-overlapping normal CTRS's), may be very useful in practice, because the corresponding transformations can always be savely applied as a preprocessing step to simplify termination proofs. Natural cases of rewrite systems / programs where such preprocessing steps may be quite useful are e.g.

- unfolding of defined function calls in rhs's (i.e., the body of the definition),
- automatically generated systems, and
- non-optimized rewrite systems arising from direct modelling of problems or original (declarative) specifications.

An obvious question (which is of natural interest in a programming context where *let-* and *where*-constructs are admissible) concerning the latter results for CTRS's is whether they can also be extended to some class of CTRS's allowing for extra variables not only in conditions, but also in rhs's, cf. e.g. [25], [27]. This remains to be investigated.

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