

On the Parameterized Complexity of Learning First-Order Logic

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Abstract

We analyse the complexity of learning first-order queries in a model-theoretic framework for supervised learning introduced by (Grohe and Turán, TOCS 2004). Previous research on the complexity of learning in this framework focussed on the question of when learning is possible in time sublinear in the background structure.

Here we study the parameterized complexity of the learning problem. We have two main results. The first is a hardness result, showing that learning first-order queries is at least as hard as the corresponding model-checking problem, which implies that on general structures it is hard for the parameterized complexity class AW[*]. Our second main contribution is a fixed-parameter tractable agnostic PAC learning algorithm for first-order queries over sparse relational data (more precisely, over nowhere dense background structures).

1 Introduction

We study the complexity of learning first-order queries from examples. This is a Boolean classification problem where hypotheses are specified by formulas of first-order logic over finite structures. In Boolean classification problems, the goal is to learn an unknown Boolean function $f^* : \mathcal{X} \rightarrow \{0, 1\}$, the *target function*, defined on an *instance space* \mathcal{X} , from a sequence $(x_1, \lambda_1), \dots, (x_m, \lambda_m) \in \mathcal{X} \times \{0, 1\}$ of *labelled examples*, where (in the simplest setting) for all i the label λ_i is the target value $f^*(x_i)$. Given the labelled examples, the learner computes a hypothesis $h : \mathcal{X} \rightarrow \{0, 1\}$ that is supposed to be close to the target function f^* . To make this problem feasible at all, one usually assumes that the hypothesis is from some restricted *hypothesis class* \mathcal{H} . If the target function is also from this class, we call the learning problem *realisable*, but we do not insist on this. We mainly frame our results in Valiant’s *probably approximately correct (PAC) learning* model [38] and the more general *agnostic PAC learning* model [25]. In the latter, we do not require a fixed target function and instead just assume that there is an unknown probability distribution \mathcal{D} on $\mathcal{X} \times \{0, 1\}$. Then the learner’s goal is to find a hypothesis h in the hypothesis class that minimises the probability $\Pr_{(x, \lambda) \sim \mathcal{D}} (h(x) \neq \lambda)$ of being wrong on a random (unseen) example. This probability is known as the *generalisation error* (or *risk*) of the hypothesis. It is known that, information theoretically, PAC learning and agnostic PAC learning are possible if and only if the hypothesis class \mathcal{H} has bounded VC dimension [5, 41]. It is also known that if this is the case, then we can cast the algorithmic problem as a minimisation problem known as *empirical risk minimisation*: find the hypothesis $h \in \mathcal{H}$ that minimises the *training error* (a.k.a. *empirical risk*), that is, the fraction of falsely classified training examples. An approximate minimisation with an additive error ε is sufficient for an (agnostic) PAC learning algorithm.

After this very brief review of some necessary definitions and facts from algorithmic learning theory (for more background, see Section 3 and [27, 32]), let us now describe the



logical setting from [23, 22] that we consider here. In this setting, we want to learn a first-order query from positive and negative examples—tuples that are in the query answer and tuples that are not—over a relational database instance D , called the *background structure* in [23, 22]. Formally, our hypotheses are specified by first-order formulas $\varphi(\bar{x}; \bar{y})$ together with tuples \bar{w} of elements of D . We can think of these elements as constants used in the query. Listing them explicitly as “parameters” of the query, rather than implicitly as part of the formula φ , allows for a cleaner complexity analysis. The formula $\varphi(\bar{x}; \bar{y})$ together with the parameter tuple \bar{w} specifies a Boolean function $h_{\varphi, \bar{w}}$ defined by $h_{\varphi, \bar{w}}(\bar{v}) = 1$ if $D \models \varphi(\bar{v}; \bar{w})$ and $h_{\varphi, \bar{w}}(\bar{v}) = 0$ otherwise.

Previous work in this logical learning framework focussed on learning algorithms running in time sublinear in the database D (see the “Related Work” section below). Here, we analyse the parameterized complexity.

Our Contributions The algorithmic problem we study is the empirical risk minimisation problem FO-ERM in the logical framework: we are given a database instance D and a sequence Λ of training examples of the form (\bar{v}, λ) , where \bar{v} is a k -tuple of elements of D and $\lambda \in \{0, 1\}$, and the goal is find a hypothesis of the form $h_{\varphi, \bar{w}}$ for a first-order formula $\varphi(\bar{x}; \bar{y})$ and a tuple \bar{w} that minimises the fraction of examples (\bar{v}, λ) from Λ where $h_{\varphi, \bar{w}}(\bar{v}) \neq \lambda$. We restrict the hypothesis space by giving a bound q on the quantifier rank of φ and a bound ℓ on the length of \bar{w} . Furthermore, we allow for an additive error $\varepsilon > 0$ in the minimisation. Provided the sequence Λ of training examples is sufficiently long (logarithmic in the size of D or linear in the VC dimension of the hypothesis space), this empirical risk minimisation problem is equivalent to PAC learning; see the discussion in Section 3. In addition to the additive approximation error ε , we also allow for a relaxation in the quantifier rank and parameter number of the output hypothesis. We only require them to be bounded in terms of functions L, Q of k, ℓ, q (where we always assume $L \geq \ell$ and $Q \geq q$). The resulting version of the problem is denoted by (L, Q) -FO-ERM. Note that this relaxation strengthens our hardness result (Theorem 1) and weakens the tractability result (Theorem 2).

In our parameterized complexity analysis, we take $k, \ell, q, \frac{1}{\varepsilon}$ as parameters; the input size depends on the size n of D and the size m of Λ . Note that FO-ERM and (L, Q) -FO-ERM trivially have polynomial-time data complexity, or in terms of parameterized complexity theory, belong to the class XP. The main question we study here is whether they are *fixed-parameter tractable*, that is, solvable in time $f(k, \ell, q, 1/\varepsilon) \cdot (n + m)^{O(1)}$ for some function f .

Our first result is a hardness result, which states that our learning problem is at least as hard as the model-checking problem for first-order logic under suitable parameterized reductions. Since the model-checking problem is known to be complete for the parameterized complexity class AW[*] (see [14]), we can state this result as follows.

► **Theorem 1.** *(L, Q) -FO-ERM is hard for the parameterized complexity class AW[*] under parameterized Turing reductions (for all L, Q).*

While the theorem may be unsurprising, its proof is surprisingly hard. Using a combinatorial trick based on Ramsey’s Theorem, we can compute an equivalence relation with a bounded number of classes that refines first-order equivalence on the vertices of a graph. We can do this using a polynomial number of oracle calls to (L, Q) -FO-ERM. With this equivalence relation, we can implement a recursive fixed-parameter tractable model-checking algorithm.

In view of the hardness theorem, we look at restricted classes of database instances where the model-checking problem is fixed-parameter tractable. Without loss of generality, we

restrict our attention to vertex-coloured graphs, because arbitrary relational structures can easily be encoded as graphs.

It is relatively easy to see that for classes with a fixed-parameter tractable model-checking problem satisfying mild closure conditions, the *unary* version of the learning problem (i.e. where $k = 1$) is reducible to the model-checking problem and hence fixed-parameter tractable as well. Unfortunately, the simple reduction argument breaks down for $k \geq 2$. Therefore, we consider specific graph classes. One of the most general family of graph classes with a tractable first-order model-checking problem is the family of *effectively nowhere dense graph classes*. Our second main result states that on those the ERM problem is fixed-parameter tractable.

► **Theorem 2.** *For all nowhere dense graph classes \mathcal{C} , there are functions L, Q such that the restriction of (L, Q) -FO-ERM to input graphs from \mathcal{C} is fixed-parameter tractable.*

The proof of this theorem heavily depends on the characterisation of nowhere dense classes in terms of the so-called *splitter game* [20]. Interestingly, the parameters \bar{w} in the hypothesis $h_{\varphi, \bar{w}}$ computed by our algorithm are essentially the vertices the splitter chooses in her winning strategy for the game.

Related Work The model-theoretic learning framework we study here was introduced in [23], where the authors proved information-theoretic learnability results for both first-order and monadic second-order logic obtained by restricting the background structures, for example, to be planar or of bounded tree width. The algorithmic question was first studied in [22], where it was proved that on graphs of maximum degree d , empirical risk minimisation for first-order definable hypotheses is possible in time polynomial in d and the number m of labelled examples the algorithm receives as input, independently of the size of the background structure A . This was generalised to first-order logic with counting [39] and with weight aggregation [40]. All these results are mainly interesting in structures of small, say, polylogarithmic degree, because there they yield learning algorithms running in time sublinear in the size of the background structure. It was shown in [21, 39] that sublinear-time learning is no longer possible if the degree is unrestricted. To address this issue, in [21] it was proposed to introduce a preprocessing phase where (before seeing any labelled examples) the background structure is converted to some data structure that supports sublinear-time learning later. This model was applied to monadic second-order logic on strings [21] and trees [19].

The framework is related to, but different from the framework of *inductive logic programming* (e.g. [10, 29, 30]), which may be viewed as the classical logic-learning framework.

In the database literature, there are various approaches to learning queries [1, 4, 6, 7, 34, 24, 26, 33, 42]. Many of these are concerned with active learning scenarios (most recently [34]), whereas we are in a statistical learning setting. Furthermore, most of these results are concerned with conjunctive queries (or queries outside of the relational database model), whereas our focus is on full first-order logic. A related problem that has also received some attention in the database literature is learning schema mappings [3, 18, 35, 37].

Related logical learning frameworks have also been studied in formal verification (e.g. [9, 16, 13, 31, 43]).

2 Preliminaries

We use \mathbb{N} and $\mathbb{N}_{\geq 1}$ to denote the non-negative and positive integers and use the shorthand $[m] = \{1, \dots, m\}$. All variables in this paper denote integers except for ε, δ and their

variations such as ε^* and δ_0 .

For the ease of presentation, we will state and prove all our results for (coloured) graphs instead of relational structures (or relational databases). All results can easily be extended to arbitrary relational structures, either directly, or indirectly by coding relational structures as graphs.

We consider undirected and simple graphs that may be vertex-coloured. Formally, we view a graph as a relational structure $G = (V(G), E(G), P_1(G), \dots, P_\ell(G))$ of some relational vocabulary $\tau = \{E, P_1, \dots, P_\ell\}$ where E is binary and the P_j are unary. We always assume the edge relation $E(G)$ to be symmetric and irreflexive. If we want to specify the vocabulary explicitly, we call such a graph a τ -coloured graph. For $\tau' \supseteq \tau$, a τ' -expansion of a τ -coloured graph G is a τ' -coloured graph G' such that $V(G') = V(G)$, $E(G') = E(G)$, and $P(G') = P(G)$ for all $P \in \tau$. The *order* of a graph is the number of its vertices $|V(G)|$. A graph $G' \subseteq G$ is a subgraph of G if $V(G') \subseteq V(G)$, $E(G') \subseteq E(G)$, and $P(G') \subseteq P(G)$ for all $P \in \tau$. Given a set $S \subseteq V(G)$, the *induced subgraph* $G[S]$ is defined as the graph with nodes S and all relations restricted to the nodes in S . The *distance* $\text{dist}(u, v)$ between two nodes $u, v \in V(G)$ is the length of the shortest path between u and v . The distance between a node $u \in V(G)$ and a vector $\bar{v} \in (V(G))^k$ is defined as $\text{dist}(u, \bar{v}) = \min_{v \in \bar{v}} (\text{dist}(u, v))$. Similarly, we define the distance between two vectors $\bar{u} \in (V(G))^k$ and $\bar{v} \in (V(G))^\ell$ as $\text{dist}(\bar{u}, \bar{v}) = \min_{u \in \bar{u}, v \in \bar{v}} (\text{dist}(u, v))$. The r -neighbourhood of a vector $\bar{u} \in (V(G))^k$ is the set $N_r^G(\bar{u}) = \{v \in V(G) \mid \text{dist}(v, \bar{u}) \leq r\}$ of nodes up to a distance of r from \bar{u} . The r -neighbourhood of a set $S \subseteq V(G)$ is $N_r^G(S) = \{u \in V(G) \mid s \in S, \text{dist}(u, s) \leq r\}$. The graph $\mathcal{N}_r^G(\bar{u}) = G[N_r^G(\bar{u})]$ induced from the neighbourhood of \bar{u} is called the (induced) r -neighbourhood graph. The following lemma is an easy consequence of the so-called Vitali Covering Lemma.

► **Lemma 3.** *Let G be a graph, $X \subseteq V(G)$, and $r \geq 1$. Then there is a set $Z \subseteq X$ and an $R = 3^i r$, where $0 \leq i \leq |X| - 1$, such that*

- (i) $N_R^G(z) \cap N_R^G(z') = \emptyset$ for all distinct $z, z' \in Z$ and
- (ii) $N_r^G(X) \subseteq N_R^G(Z)$.

Proof. For $i \geq 0$, let $R_i := 3^i r$. We inductively construct a sequence $Z_0 \supset Z_1 \supset \dots \supset Z_p$ of subsets of X such that for every i we have $N_r^G(X) \subseteq N_{R_i}^G(Z_i)$ and for p we additionally have $N_{R_p}^G(z) \cap N_{R_p}^G(z') = \emptyset$ for all distinct $z, z' \in Z_p$. Then $p \leq |X| - 1$ and Z_p is the desired set.

Let $Z_0 := X$. Now suppose that Z_i is defined. If $N_{R_i}^G(z) \cap N_{R_i}^G(z') = \emptyset$ for all distinct $z, z' \in Z_i$, we set $p := i$ and stop the construction. Otherwise, let $Z_{i+1} \subseteq Z_i$ be inclusion-wise maximal such that $N_{R_i}^G(z) \cap N_{R_i}^G(z') = \emptyset$ for all distinct $z, z' \in Z_{i+1}$. Clearly $Z_{i+1} \subset Z_i$, as otherwise we would have stopped the construction. Then for all $y \in Z_i$ there is a $z \in Z_{i+1}$ such that $N_{R_i}^G(y) \cap N_{R_i}^G(z) \neq \emptyset$, which implies that $N_{R_i}^G(y) \subseteq N_{3R_i}^G(z) = N_{R_{i+1}}^G(z)$. Thus, $N_r^G(X) \subseteq N_{R_i}^G(Z_i) \subseteq N_{R_{i+1}}^G(Z_{i+1})$.

In the worst case, all pairs Z_i, Z_{i+1} differ by exactly one element and we actually need $p = |X| - 1$ which implies that $|Z_p| = 1$. This case can be reached when each x_i is at position $3^{i-1}r$ on a path and $x_1 \in Z_i$ for all i . ◀

Nowhere Dense Graphs Let G be a graph and $r, s > 0$. The (r, s) -splitter game on G is played by two players called *Connector* and *Splitter*. The game is played in a sequence of at most s rounds. We let $G_0 := G$. In round $i + 1$ of the game, Connector picks a vertex $v_{i+1} \in V(G_i)$, and Splitter answers by picking $w_{i+1} \in N_r^{G_i}(v_{i+1})$. We let $G_{i+1} := G_i[N_r^{G_i}(v_{i+1}) \setminus \{w_{i+1}\}]$. Splitter *wins* if $G_{i+1} = \emptyset$; otherwise the play continues.

► **Fact 4** ([20]). *A class \mathcal{C} of graphs is nowhere dense if for every $r > 0$ there is an $s > 0$ such that for every $G \in \mathcal{C}$, Splitter has a winning strategy for the (r, s) -splitter game on G .*

In a modified version of the game, in each round Connector not only picks a vertex v , but also a new radius $r' \leq r$, and the game continues in $N_{r'}(v)$. Clearly, reducing the radius does not help Connector, and the fact remains true for the same s . Nevertheless, it will be convenient for us later to work with this modified version of the game.

A graph class is *effectively nowhere dense* if it is nowhere dense and s is given by a computable function $f(r)$. In this paper, we consider only effectively nowhere dense graph classes.

Types Let us fix a vocabulary τ (of coloured graphs) in the following and assume that all graphs are τ -coloured. $\text{FO}[\tau]$ denotes the set of all formulas of first-order logic of vocabulary τ , and $\text{FO}[\tau, q]$ denotes the subset of $\text{FO}[\tau]$ consisting of all formulas of quantifier rank up to q .

The q -type of a tuple $\bar{v} = (v_1, \dots, v_k) \in (V(G))^k$ of arity k is the set $\text{tp}_q(G, \bar{v})$ of all $\text{FO}[\tau, q]$ -formulas $\varphi(\bar{x})$ such that $G \models \varphi(\bar{v})$. Moreover, for $q, r \geq 0$, the *local (q, r) -type* of \bar{v} is the set $\text{ltp}_{q,r}(G, \bar{v}) := \text{tp}_q(\mathcal{N}_r^G(\bar{v}), \bar{v})$. A k -variable q -type (of vocabulary τ) is a set θ of $\text{FO}[\tau, q]$ -formulas whose free variables are among x_1, \dots, x_k . Since, up to logical equivalence, there are only finitely many $\text{FO}[\tau, q]$ -formulas whose free variables are among x_1, \dots, x_k , we can view types as finite sets of formulas. Formally, we syntactically define a normal form such that for all τ, q, k , there are only finitely many $\text{FO}[\tau, q]$ -formulas in normal form with free variables among x_1, \dots, x_k . Furthermore, there is an algorithm that transforms every formula into an equivalent formula in normal form without increasing the quantifier rank. Then, we view types as sets of formulas in normal form. We denote the set of all k -variable q -types of vocabulary τ by $\text{Tp}[\tau, k, q]$.

It is easy to see that for every $\text{FO}[\tau, q]$ -formula $\varphi(x_1, \dots, x_k)$, there is a set $\Phi \subseteq \text{Tp}[\tau, k, q]$ such that for all τ -coloured graphs G and all $\bar{v} \in (V(G))^k$,

$$G \models \varphi(\bar{v}) \iff \text{tp}_q(G, \bar{v}) \in \Phi.$$

The following fact is a consequence of Gaifman's Theorem [15].

► **Fact 5.** *For all $q \geq 0$, there is an $r := r(q) \in 2^{\mathcal{O}(q)}$ such that for all $k \geq 1$, all vocabularies τ , all τ -coloured graphs G , and all $\bar{v}, \bar{v}' \in (V(G))^k$, if $\text{ltp}_{q,r}(G, \bar{v}) = \text{ltp}_{q,r}(G, \bar{v}')$, then $\text{tp}_q(G, \bar{v}) = \text{tp}_q(G, \bar{v}')$.*

It is important for us to note that this r can be chosen to depend only on q , independent of the vocabulary τ . From Fact 5, we directly get the following corollary.

► **Corollary 6.** *Let $\varphi(x_1, \dots, x_k)$ be an $\text{FO}[\tau, q]$ -formula, and let $r := r(q)$ be chosen according to Fact 5. Then for every graph G , there is a set Φ of k -variable q -types such that for all $\bar{v} \in (V(G))^k$,*

$$G \models \varphi(\bar{v}) \iff \text{ltp}_{q,r}(G, \bar{v}) \in \Phi.$$

Parameterized Complexity A *parameterized problem* (over a finite alphabet Σ) is a pair (L, κ) , where $L \subseteq \Sigma^*$ and $\kappa: \Sigma^* \rightarrow \mathbb{N}$ is a polynomial-time computable function, called *parameterization* of Σ^* . If the parameterization κ is clear from the context, we can omit it. We say that (L, κ) is *fixed-parameter tractable* or $(L, \kappa) \in \text{FPT}$ if there is an algorithm \mathcal{L} , a computable function $f: \mathbb{N} \rightarrow \mathbb{N}$, and a polynomial p such that \mathcal{L} decides L and runs

in time $f(\kappa(x)) \cdot p(|x|)$ on input $x \in \Sigma^*$. The class FPT essentially captures all tractable parameterized problems.

An *fpt Turing reduction* from (L, κ) to (L', κ') is an fpt algorithm with oracle access to (L', κ') solving (L, κ) in such a way that on input x , for all oracle queries x' , it holds that $\kappa'(x') \leq g(\kappa(x))$ for some computable function g . Clearly, if there is an fpt Turing reduction from (L, κ) to (L', κ') and $(L', \kappa') \in \text{FPT}$ then $(L, \kappa) \in \text{FPT}$.

For additional background, we refer the reader to [11, 14].

3 Learning Problems

Recall the supervised learning scenario we described in the introduction. We are given a sequence $(\bar{v}_1, \lambda_1), \dots, (\bar{v}_m, \lambda_m) \in V(G)^k \times \{0, 1\}$ of labelled examples over some graph G , which we call the *background graph*. In the query learning scenario, the graph represents a database instance. We assume that the (\bar{v}_i, λ_i) are drawn independently from some unknown “data generating” probability distribution \mathcal{D} on $V(G)^k \times \{0, 1\}$. Note that we make no assumptions about the distribution \mathcal{D} . In particular, we do not assume that for every \bar{v} there is exactly one (or at most one) $\lambda \in \{0, 1\}$ such that $\mathcal{D}(\{(\bar{v}, \lambda)\}) > 0$. That is, there is not necessarily an (unknown) target function that assigns a “true” value $\lambda \in \{0, 1\}$ to every \bar{v} . Of course there could be such a function, then essentially \mathcal{D} would be a distribution on $V(G)^k$, but our setting is more general and admits uncertainty in the data. This setting is sometimes referred to as *agnostic learning*. The special case where an unknown target function exists and is actually one of the hypotheses is called the *realisable* case; we shall see in Section 4 that our hardness result even applies in this special case.

Our hypotheses are first-order queries, formally represented as Boolean functions $h_{\varphi, \bar{w}} : V(G)^k \rightarrow \{0, 1\}$ for some first-order formula $\varphi(\bar{x}; \bar{y})$ and tuple $\bar{w} \in V(G)^\ell$. Our goal is to generate a hypothesis $h_{\varphi, \bar{w}}$ that generalises well, that is, minimises the generalisation error

$$\Pr_{(\bar{v}, \lambda) \sim \mathcal{D}} (h_{\varphi, \bar{w}}(\bar{v}) = \lambda).$$

It is well-known that, at least from a theoretical perspective, it suffices to minimise the *empirical risk* or *training error*, that is, the fraction of training examples that are classified wrong. This paradigm is known as *empirical risk minimisation (ERM)*. The two main results relevant for us here are the following. The *Uniform Convergence Theorem* (see [32, Chapter 4]) assumes that we have a finite hypothesis class \mathcal{H} . It states that for all $\varepsilon, \delta > 0$, it suffices to see $\mathcal{O}(\log |\mathcal{H}|)$ labelled examples to guarantee that with probability at least $1 - \delta$, the training error of a hypothesis $h \in \mathcal{H}$ deviates from the generalisation error by at most ε . This implies that the generalisation error of a hypothesis $h \in \mathcal{H}$ minimising the empirical risk deviates from the generalisation error of the best possible hypothesis $h^* \in \mathcal{H}$ by at most 2ε . In other words: for finite hypothesis classes, ERM is *probably approximately correct*, or a *PAC learning algorithm*. The second result, sometimes called the *Fundamental Theorem of PAC Learning* (see [32, Chapter 6]), lifts this to infinite hypothesis classes \mathcal{H} , as long as their *VC dimension* $\text{VC}(\mathcal{H})$ is finite. It states that uniform convergence already holds if we see $\mathcal{O}(\text{VC}(\mathcal{H}))$ examples, and hence ERM is probably approximately correct in this more general setting. It will not be necessary here to understand VC dimension. To tie things together, just note that for a finite hypothesis class \mathcal{H} , the VC dimension is at most $\log |\mathcal{H}|$.

The uniform convergence results relating the training error and the generalisation error not only show that an ERM algorithm yields a PAC learning algorithm, but conversely they also allow us to obtain a (randomised) ERM algorithm from a PAC learning algorithm, simply by applying the PAC learning algorithm with a data generating distribution that is

uniform on the training examples (and zero everywhere else). Thus, ERM and PAC learning are essentially equivalent, and therefore we focus on ERM from now on.

The hypothesis classes we are interested in here are classes $\mathcal{H}_{k,\ell,q}(G)$, where G is a graph and $k, \ell, q \in \mathbb{N}$, consisting of all $h_{\varphi, \bar{w}}$ for first-order formulas $\varphi(\bar{x}; \bar{y})$ of quantifier rank q with $|\bar{x}| = k$, $|\bar{y}| = \ell$, and tuples $\bar{w} \in (V(G))^\ell$. Since up to logical equivalence there are only finitely many first-order formulas of quantifier rank q with $(k + \ell)$ free variables, these hypothesis classes are finite sets of size $f(k, \ell, q) \cdot n^\ell$, where n is the order of G . Thus, an ERM-learning algorithm that sees $\mathcal{O}(\log n)$ training examples is a PAC learning algorithm. Furthermore, for graphs G from a nowhere dense class \mathcal{C} of graphs, the VC dimension of the hypothesis classes $\mathcal{H}_{k,\ell,q}(G)$ is uniformly bounded by a constant $d = d(\mathcal{C}, k, \ell, q)$, and therefore an ERM-learning algorithm only needs to see $\mathcal{O}(d)$ (independent of n) examples to be a PAC learning algorithm.

To conclude: the algorithmic problem we need to solve is empirical risk minimisation. We always denote the background graph by G and its vocabulary by τ . We assume the input sequence of training examples is

$$\Lambda = \left((\bar{v}_1, \lambda_1), \dots, (\bar{v}_m, \lambda_m) \right) \in \left(V(G)^k \times \{0, 1\} \right)^m.$$

The *training error* of a hypothesis h is

$$\text{err}_\Lambda(h) := \frac{1}{m} |\{i \in [m] \mid h(\bar{v}_i) \neq \lambda_i\}|.$$

For hypotheses $h_{\varphi, \bar{w}}$, we write $\text{err}_\Lambda(\varphi, \bar{w})$ instead of $\text{err}_\Lambda(h_{\varphi, \bar{w}})$. Here is the precise statement of the ERM problem.

FO-ERM
Input Graph G , training sequence $\Lambda \in (V(G)^k \times \{0, 1\})^m$, and $k, \ell, q \in \mathbb{N}$, $\varepsilon > 0$.
Parameter $k + \ell + q + |\tau| + \frac{1}{\varepsilon}$
Problem Return a hypothesis $h_{\varphi, \bar{w}} \in \mathcal{H}_{k,\ell,q}(G)$ such that

$$\text{err}_\Lambda(\varphi, \bar{w}) \leq \varepsilon^* + \varepsilon,$$

where $\varepsilon^* := \min\{\text{err}_\Lambda(h) \mid h \in \mathcal{H}_{k,\ell,q}(G)\}$.

Note that we only require a hypothesis approximately minimising the risk, because such an approximation is sufficient for PAC learning.

Unfortunately, in our positive learnability result for nowhere dense graphs, we cannot guarantee to match the quantifier rank and parameter number of the optimal hypothesis, but may return a hypothesis with larger quantifier rank and parameter number. To make this precise, we introduce the following relaxation of the problem. Let $L, Q: \mathbb{N}^3 \rightarrow \mathbb{N}$ be functions with $L(k, \ell, q) \geq \ell$ and $Q(k, \ell, q) \geq q$ for all $k, \ell, q \in \mathbb{N}$.

(L, Q) -FO-ERM
Input Graph G , training sequence $\Lambda \in (V(G)^k \times \{0, 1\})^m$, and $k, \ell^*, q^* \in \mathbb{N}$, $\varepsilon > 0$.
Parameter $k + \ell^* + q^* + |\tau| + \frac{1}{\varepsilon}$
Problem Return a hypothesis $h_{\varphi, \bar{w}} \in \mathcal{H}_{k,\ell,q}(G)$ for some $q \leq Q(k, \ell^*, q^*)$ and $\ell \leq L(k, \ell^*, q^*)$ such that

$$\text{err}_\Lambda(\varphi, \bar{w}) \leq \varepsilon^* + \varepsilon,$$

where $\varepsilon^* := \min\{\text{err}_\Lambda(h) \mid h \in \mathcal{H}_{k,\ell^*,q^*}(G)\}$.

4 The Hardness of Learning

In this section, we shall prove that the ERM problem is hard; at least as hard as the following *model-checking problem* for first-order logic.

FO-MC
Input Graph G , FO-sentence φ
Parameter $|\varphi|$
Problem Decide whether $G \models \varphi$ holds.

As FO-MC is hard for the parameterized complexity class AW[*] [12], this will prove Theorem 1.

► **Lemma 7.** *Let $L, Q: \mathbb{N}^3 \rightarrow \mathbb{N}$ with $L(k, \ell, q) \geq \ell$ and $Q(k, \ell, q) \geq q$ for all k, ℓ, q . Then FO-MC is fpt Turing reducible to (L, Q) -FO-ERM.*

Proof. We show that there is an fpt algorithm with access to a (L, Q) -FO-ERM oracle that solves FO-MC. To turn this into an fpt reduction, we need to be careful that the algorithm makes only oracle calls with parameter $k + \ell^* + q^* + |\tau| + \frac{1}{\varepsilon} \leq g(p)$, where p is the length of the input formula of the model-checking problem and g is some computable function.

We first give the proof under the assumption $L(1, 0, q) = 0$ for all q .

Let G be a τ -colored graph of order n , and let φ be an FO-sentence of length p . Let q be the quantifier rank of φ . Of course we have $q \leq p$, and we may also assume that $|\tau| \leq p$, because we can ignore relation symbols not occurring in φ .

Without loss of generality, we may assume $\varphi = \exists x \psi(x)$ for some FO-formula $\psi(x)$ of quantifier rank at most $q - 1$.

Using $\binom{n}{2}$ calls to (L, Q) -FO-ERM, we aim to partition the nodes $V(G)$ into a bounded number of classes in such a way that nodes in the same class have the same $(q-1)$ -type. The number of classes of the partition will be bounded in terms of the parameter p .

▷ **Claim 8.** Let $u, v \in V(G)$ with $\text{tp}_{q-1}(G, u) \neq \text{tp}_{q-1}(G, v)$. Then (L, Q) -FO-ERM on input $\Lambda = ((u, 0), (v, 1))$ with parameters $k = 1$, $\ell^* = 0$, $q^* = q - 1$ and $\varepsilon = 1/4$ returns a formula $\varphi'(x)$ of quantifier rank at most $Q(1, 0, q - 1)$ such that $G \not\models \varphi'(u)$ and $G \models \varphi'(v)$.

Proof. Let $\varphi^*(x) = \bigwedge_{\gamma \in \text{tp}_{q-1}(G, v)} \gamma(x)$, which is an FO-formula of quantifier rank $q - 1$. Then, $G \not\models \varphi^*(u)$ and $G \models \varphi^*(v)$. Thus, $\text{err}_\Lambda(\varphi^*) = 0$. Since $\varepsilon = \frac{1}{4}$, (L, Q) -FO-ERM has to return a formula $\varphi'(x)$ with $\text{err}_\Lambda(\varphi') \leq \frac{1}{4}$. Note that here we use $L(k, 0, q) = 0$ to make sure that the formula φ' does not have additional parameters. Furthermore, by the definition of the training error, $\text{err}_\Lambda(\varphi') \in \{0, \frac{1}{2}, 1\}$. Hence, $\text{err}_\Lambda(\varphi') = 0$ and $G \not\models \varphi'(u)$, $G \models \varphi'(v)$. ◁

The difficulty of the proof is that we do not have a “converse” of Claim 1. If $\text{tp}_{q-1}(G, u) = \text{tp}_{q-1}(G, v)$, then the (L, Q) -FO-ERM-oracle still returns a formula $\varphi'(x)$, but we know nothing about this formula, and we have no way of recognising that we are in this case.

As discussed in the preliminaries, by introducing a suitable normal form, we can always assume that the set $\text{FO}[\tau, Q(1, 0, q - 1)]$ of FO-formulas of vocabulary τ and quantifier rank at most $Q(1, 0, q - 1)$ is finite. Let s be an upper bound for the size of this set. Note that s can be bounded in terms of p .

Our next goal is to compute a “bounded” set $T \subseteq V(G)$ of representatives of the $(q - 1)$ -types, that is, we want to guarantee the following:

- (i) for every $v \in V(G)$ there is a $t \in T$ such that $\text{tp}_{q-1}(G, v) = \text{tp}_{q-1}(G, t)$;
- (ii) $|T| \leq h(p)$ for some computable function h .

To choose the function h in (ii), we recall Ramsey's Theorem (see, for example, [8]): *there is a computable function $R : \mathbb{N}^3 \rightarrow \mathbb{N}$ such that for all k, ℓ, m , every set $S = \{s_1, \dots, s_r\}$ of size $r > R(k, \ell, m)$, and every mapping $C : \binom{S}{k} \rightarrow [\ell]$, there is a subset $M \subseteq S$ of size $|M| = m$ that is monochromatic, that is, the restriction of C to $\binom{M}{k}$ is constant.* We let $h(p) := R(2, s, 3)$.

We fix an arbitrary linear order $<$ on $V(G)$. For each unordered pair $\{u, v\} \in \binom{V(G)}{2}$, say with $u < v$, we compute an FO-formula $\gamma_{u,v} = \gamma_{u,v}(x)$ of quantifier rank at most $Q(1, 0, q-1)$ by calling (L, Q) -FO-ERM on input G and $\Lambda = ((u, 0), (v, 1))$ with parameters $k = 1$, $\ell^* = 0$, $q^* = q - 1$ and $\varepsilon = 1/4$. By Claim 8, if $\text{tp}_{q-1}(G, u) \neq \text{tp}_{q-1}(G, v)$, we have $G \not\models \gamma_{u,v}(u)$ and $G \models \gamma_{u,v}(v)$.

▷ **Claim 9.** Let $S \subseteq V(G)$ of size $|S| > h(p)$. Then we can find three vertices $v_1, v_2, v_3 \in S$ such that either $\text{tp}_{q-1}(G, v_1) = \text{tp}_{q-1}(G, v_2)$ or $\text{tp}_{q-1}(G, v_2) = \text{tp}_{q-1}(G, v_3)$.

Proof. By Ramsey's Theorem, there is a monochromatic subset of size 3, that is, three vertices v_1, v_2, v_3 such that $\gamma_{v_1, v_2} = \gamma_{v_1, v_3} = \gamma_{v_2, v_3} =: \gamma$. We shall prove that these vertices satisfy the assertion of the claim.

Suppose $\text{tp}_{q-1}(G, v_1) \neq \text{tp}_{q-1}(G, v_2)$; if they are equal, there is nothing to prove. Then $G \not\models \gamma(v_1)$ and $G \models \gamma(v_2)$ by Claim 1. If in addition we had $\text{tp}_{q-1}(G, v_2) \neq \text{tp}_{q-1}(G, v_3)$, this would imply $G \not\models \gamma(v_2)$ (and $G \models \gamma(v_3)$), which is a contradiction. Thus, $\text{tp}_{q-1}(G, v_2) = \text{tp}_{q-1}(G, v_3)$.¹ ◁

We can use the claim to iteratively construct a set T satisfying (i) and (ii). We construct a sequence $T_0 \supset T_1 \supset \dots$ of subsets of $V(G)$ that satisfy (i) and stop as soon as we arrive at a set $T := T_i$ satisfying (ii). Let $T_0 := V(G)$. If $|T_i| > h(p)$, we find v_1, v_2, v_3 according to Claim 2 and let $T_{i+1} := T_i \setminus \{v_2\}$.

Recall that we assumed the input formula φ of the model-checking problem to be of the shape $\exists x \psi(x)$. Since $\psi(x)$ is a formula of quantifier rank at most $q - 1$, for all u, v with $\text{tp}_{q-1}(G, u) = \text{tp}_{q-1}(G, v)$ it holds that $G \models \psi(u) \iff G \models \psi(v)$. Thus by (i), $G \models \varphi$ if and only if there is a $t \in T$ such that $G \models \psi(t)$. This means that to find out if $G \models \varphi$, we only need to check if $G \models \psi(t)$ for $t \in T$. We can do this by recursively calling our algorithm $|T|$ times.

The only small difficulty is that the input formula to the model-checking problem needs to be a sentence, that is, a formula without free variables. But we can easily resolve this by constructing, for each $t \in T$, a sentence ψ_t and an expansion G_t of G such that $G \models \psi(t) \iff G_t \models \psi_t$ and then making the recursive call to G_t, ψ_t . We do this by adding two fresh unary relation symbols P_t, Q_t , which are interpreted by the sets $P_t(G) := \{t\}$ and $Q_t(G) := \{u \mid u \in N(t)\}$ in G_t . To obtain ψ_t from $\psi(x)$, we replace all atoms $x = y$, $y = x$ by $P_t(y)$ and all atoms $E(x, y)$, $E(y, x)$ by $Q_t(y)$, assuming without loss of generality that there are no atoms $E(x, x)$ and $x = x$.

The recursion tree has depth at most q and the degree is bounded by a function of p . Thus, the size of the recursion tree is bounded in terms of p . Each recursive call requires only fpt time, so overall we have an fpt algorithm.

This completes the proof of the lemma under the assumption that $L(1, 0, q) = 0$ for all q .

¹ Note that we do not need the full strength of Ramsey's Theorem here, because we never use $\gamma_{v_1, v_3} = \gamma$. It is not hard to prove the weaker statement that we need directly, but we found it most convenient to just apply Ramsey's Theorem.

The only place where we used the assumption $L(1, 0, q) = 0$ is in the proof of Claim 1. So let us focus on this claim. Again, let $u, v \in V(G)$ such that $\text{tp}_{q-1}(G, u) \neq \text{tp}_{q-1}(G, v)$. It is our goal to find a formula $\varphi'(x)$ (without additional free variables) such that $G \not\models \varphi'(u)$ and $G \models \varphi'(v)$. The quantifier rank of this formula needs to be bounded in terms of p , but there is no need for it to be at most $Q(1, 0, q - 1)$.

Instead of using types, here we use local types to distinguish u and v . By Fact 5, we have $\text{ltp}_{q-1, r}(G, u) \neq \text{ltp}_{q-1, r}(G, v)$ for some $r = r(q - 1) \in 2^{\mathcal{O}(q)}$. We can use this to construct an FO-formula $\varphi^*(x)$ such that $G \not\models \varphi^*(u)$, $G \models \varphi^*(v)$ and the formula φ^* is r -local, which means that for all graphs H and all vertices $w \in V(H)$ we have $H \models \varphi^*(w)$ if and only if $\mathcal{N}_r^H(w) \models \varphi^*(w)$. To achieve this, we simply need to restrict all quantifiers to vertices of distance at most r from x . This implies that the quantifier rank q^* of the formula $\varphi^*(x)$ will be in $\mathcal{O}(\max\{q, \log r\}) = \mathcal{O}(q)$.

Let $\ell := \max\{1, L(k, 0, q^*)\}$, and let \widehat{G} be the union of 2ℓ disjoint copies $G^{(1)}, \dots, G^{(2\ell)}$ of G . For every vertex $w \in V(G)$, let $w^{(i)}$ be the copy of w in $G^{(i)}$. Let $\widehat{\Lambda} := ((u^{(i)}, 0), (v^{(i)}, 1) \mid i \in [2\ell])$. Observe that for all $i \in [2\ell]$, it holds that $\widehat{G} \not\models \varphi^*(u^{(i)})$ and $\widehat{G} \models \varphi^*(v^{(i)})$, because $\mathcal{N}_r^{\widehat{G}}(u^{(i)}) = \mathcal{N}_r^G(u)$ and $\mathcal{N}_r^{\widehat{G}}(v^{(i)}) = \mathcal{N}_r^G(v)$. Thus $\text{err}_{\widehat{\Lambda}}(\varphi^*) = 0$.

We apply (L, Q) -FO-ERM on input $\widehat{G}, \widehat{\Lambda}$ with parameters $k = 1$, $\ell^* = 0$, q^* and $\varepsilon = 1/8$ and obtain a formula $\varphi(x; \bar{y})$ with $\ell' = |\bar{y}| \leq \ell$ and a tuple $\bar{w} = (w_1, \dots, w_{\ell'}) \in V(\widehat{G})^{\ell'}$ such that $\text{err}_{\widehat{\Lambda}}(\varphi, \bar{w}) \leq \frac{1}{8}$. Let us call an index $i \in [2\ell]$ *covered* if there is parameter $w_i \in V(G^{(i)})$. Moreover, we call i *wrong* if either $\widehat{G} \models \varphi(u^{(i)}; \bar{w})$ or $\widehat{G} \not\models \varphi(v^{(i)}; \bar{w})$. Note that at most $\ell' \leq \ell$ indices are covered and at most $|\Lambda|/8 = \ell/2$ are wrong. Thus, as $\ell \geq 1$, there is an index i° that is neither covered nor wrong. In the following, we let $u^\circ := u^{(i^\circ)}$ and $v^\circ := v^{(i^\circ)}$.

We expand \widehat{G} by unary relations P_i to a graph \widehat{G}' with $P_i(\widehat{G}') = \{w_i\}$. We can easily construct a formula $\varphi'(x)$ such that for all $\widehat{v} \in V(\widehat{G}) = V(\widehat{G}')$ we have $\widehat{G} \models \varphi(\widehat{v}; \bar{w}) \iff \widehat{G}' \models \varphi'(\widehat{v})$. In particular, we have $\widehat{G}' \not\models \varphi'(u^\circ)$ and $\widehat{G}' \models \varphi'(v^\circ)$.

From $\varphi'(x)$ we can construct an r' -local formula $\varphi''(x)$, for some r' bounded in terms of the quantifier rank of φ' , such that for all $\widehat{v} \in V(\widehat{G}')$ we have $\widehat{G}' \models \varphi'(\widehat{v}) \iff \widehat{G}' \models \varphi''(\widehat{v})$. Again, $\widehat{G}' \not\models \varphi''(u^\circ)$ and $\widehat{G}' \models \varphi''(v^\circ)$.

Now let $\varphi'''(x)$ be the formula (of the original vocabulary τ) obtained from $\varphi''(x)$ by replacing each atomic subformula $P_i(z)$ by FALSE (or $\neg z = z$). Then $\widehat{G} \not\models \varphi'''(u^\circ)$ and $\widehat{G} \models \varphi'''(v^\circ)$, simply because there is no w_i in the r' -neighbourhood of u° or v° . Since $\varphi'''(x)$ is r' -local and $\mathcal{N}_{r'}^{\widehat{G}}(u^\circ) = \mathcal{N}_{r'}^G(u)$ and $\mathcal{N}_{r'}^{\widehat{G}}(v^\circ) = \mathcal{N}_{r'}^G(v)$, it follows that $G \not\models \varphi'''(u)$ and $G \models \varphi'''(v)$. Thus $\varphi'''(x)$ has the desired property.

Using this generalisation of Claim 1, we can complete the proof as in the case $L(1, 0, q) = 0$. \blacktriangleleft

► **Remark 10.** The proof only uses that the (L, Q) -FO-ERM-oracle delivers correct answers in the case that $\varepsilon^* = 0$, that is, there is a hypothesis consistent with the training data. This means that the hardness result even holds for the realisable case of the learning problem.

5 Fixed-parameter Tractable Learning

For a class \mathcal{C} of graphs and a parameterized problem M , with M being FO-ERM, (L, Q) -FO-ERM, or FO-MC, we say that M is *fixed-parameter tractable on \mathcal{C}* if the restriction of M to input graphs from \mathcal{C} is fixed-parameter tractable.

Let us say that a class \mathcal{C} of (coloured) graphs is closed under *colour expansions* if for all τ -coloured graphs G , all $\tau' \subseteq \tau$, and all τ' -expansions G' of G , if $G \in \mathcal{C}$ then $G' \in \mathcal{C}$. It can easily be checked that the reduction from FO-MC to (Q, L) -FO-ERM that we gave in the

previous section can be restricted to all graph classes that are closed under colour expansions and disjoint unions. As (Q, L) -FO-ERM trivially reduces to FO-ERM, the result also applies to FO-ERM.

In the following two lemmas, we give converse reductions from FO-ERM to FO-MC, but only under additional restrictions on the parameters k and ℓ . Again, since (Q, L) -FO-ERM reduces to FO-ERM, the results also apply to (Q, L) -FO-ERM.

► **Proposition 11.** *Let \mathcal{C} be a class of graphs closed under colour expansions such that FO-MC is fixed-parameter tractable on \mathcal{C} .*

Then, for every constant $\ell^ \in \mathbb{N}$, the restriction of FO-ERM to inputs with $\ell = \ell^*$ is fixed-parameter tractable on \mathcal{C} .*

For the proof of Proposition 11, we use that a simple brute-force test of all $|V(G)|^{\ell^*}$ possible parameter configurations can be done within the given time bound. A formal proof is given in the appendix.

Recall that we call a learning problem *realisable* if there is a target function, and this target function is contained in the hypothesis class. In the realisable version of the ERM problem, we assume that the training error ε^* of the best possible hypothesis is 0. Note that formally, the restriction of FO-ERM to the realisable case is a *promise problem* where an algorithm is only required to give a correct answer if the input satisfies a certain assumption, the “promise”, in our case $\varepsilon^* = 0$. On inputs that do not fulfil the promise, the algorithm can do anything. The following proposition considers the realisable version of FO-ERM restricted to the 1-dimensional case $k = 1$.

► **Proposition 12.** *Let \mathcal{C} be a class of graphs closed under colour expansions such that FO-MC is fixed-parameter tractable on \mathcal{C} .*

Then the following restriction of FO-ERM is fixed-parameter tractable.

Input Graph $G \in \mathcal{C}$, training sequence $\Lambda \in (V(G)^k \times \{0, 1\})^m$, and $\ell, q \in \mathbb{N}$, $\varepsilon > 0$.
Parameter $\ell + q + |\tau| + \frac{1}{\varepsilon}$
Assumption There is a $h \in \mathcal{H}_{1, \ell, q}(G)$ with $\text{err}_\Lambda(h) = 0$.
Problem Return a hypothesis $h_{\varphi, \bar{w}} \in \mathcal{H}_{1, \ell, q}(G)$ such that

$$\text{err}_\Lambda(\varphi, \bar{w}) \leq \varepsilon.$$

Here, techniques similar to those used in [21, 19] can be applied to find a consistent parameter setting. Again, a formal proof is given in the appendix.

The following result is a precise version of Theorem 2.

► **Theorem 13.** *For every nowhere dense class \mathcal{C} of graphs, there are functions $L, Q: \mathbb{N}^3 \rightarrow \mathbb{N}$ such that the problem (L, Q) -FO-ERM is fixed-parameter tractable on \mathcal{C} .*

The remainder of this section is devoted to the proof of the theorem. In the following, we let \mathcal{C} be an effectively nowhere dense graph class, and we fix $k \geq 1$ and $\ell^*, q^* \geq 0$. We choose $r = r(q^*)$ according to Fact 5 and let $R := 3^{\ell^* - 1} \cdot ((k + 2)(2r + 1))$. The specific choice of R will be justified in the proof of Lemma 16 in the appendix. Let $G \in \mathcal{C}$ and let $s \geq 1$ such that Splitter has a winning strategy for the (R, s) -splitter game on G . The value s can be computed since \mathcal{C} is effectively nowhere dense. We let $V := V(G)$, $E := E(G)$, $n := |V|$, $\ell := L(k, \ell^*, q^*) := \ell^* \cdot s$ and $q := Q(k, \ell^*, q^*) := q^* + \log R$. Furthermore, let $\Lambda \in (V(G)^k \times \{0, 1\})^m$ be a training sequence and let $\Lambda^+ = \{\bar{v} \in V^k \mid (v, 1) \in \Lambda\}$ and $\Lambda^- = \{\bar{v} \in V^k \mid (v, 0) \in \Lambda\}$ be the sets of positive and negative examples. Let $\varepsilon^* \geq 0$ such

that there is an FO-formula $\varphi^*(\bar{x}; \bar{y})$ of quantifier rank at most q^* with $k + \ell^*$ free variables and a tuple $\bar{w}^* = (w_1^*, \dots, w_{\ell^*}^*) \in V^{\ell^*}$ such that $\text{err}_\Lambda(\varphi^*, \bar{w}^*) \leq \varepsilon^*$.

Let $\varepsilon > 0$. Whenever we speak of an *fpt algorithm* in the following, we mean an algorithm running in time $f(k, \ell^*, q^*, s, 1/\varepsilon) \cdot (n + m)^{\mathcal{O}(1)}$. Our goal is to compute a query (φ, \bar{w}) , our *hypothesis*, consisting of an FO-formula $\varphi(\bar{x}; \bar{y})$ of quantifier rank at most q with $k + \ell$ free variables and a tuple $\bar{w} \in V^\ell$ such that $\text{err}_\Lambda(\varphi, \bar{w}) \leq \varepsilon^* + \varepsilon$, using an fpt algorithm. This means that the hypothesis correctly classifies all but $m \cdot (\varepsilon^* + \varepsilon)$ examples from Λ .

A vector $\bar{v} \in V^{\ell'}$ is called ε' -*discriminating* if there is a formula ψ of quantifier rank at most q such that $\text{err}_\Lambda(\psi, \bar{v}) \leq \varepsilon'$. We then say that \bar{v} *discriminates* all but (at most) $\varepsilon' \cdot m$ examples. If we have an $(\varepsilon^* + \varepsilon)$ -discriminating ℓ -tuple \bar{w} , then we can find the formula φ by an fpt algorithm that simply steps through all possible formulas since model checking on nowhere dense graphs is in FPT and there are only finitely many formulas to check. Hence, in the following, we describe a procedure to find such an $(\varepsilon^* + \varepsilon)$ -discriminating ℓ -tuple.

A *conflict* in Λ is a pair $(\bar{v}^+, \bar{v}^-) \in \Lambda^+ \times \Lambda^-$ with $\text{ltp}_{q^*, r}(G, \bar{v}^+) = \text{ltp}_{q^*, r}(G, \bar{v}^-)$. The *type* of a conflict (\bar{v}^+, \bar{v}^-) is the local (q^*, r) -type $\text{ltp}_{q^*, r}(G, \bar{v}^+) = \text{ltp}_{q^*, r}(G, \bar{v}^-)$. Let Ξ be the set of all conflicts. To *resolve* a conflict $(\bar{v}^+, \bar{v}^-) \in \Xi$, we need to find parameters \bar{w} such that $\text{ltp}_{q^*, r}(G, \bar{v}^+ \bar{w}) \neq \text{ltp}_{q^*, r}(G, \bar{v}^- \bar{w})$. Only parameters in the $(2r + 1)$ -neighbourhood of $\bar{v}^+ \cup \bar{v}^-$ have an effect on the local type. We say that we *attend* to the conflict (\bar{v}^+, \bar{v}^-) if we choose at least one parameter in $N_{2r+1}^G(\bar{v}^+ \bar{v}^-)$. Note that attending to a conflict is a necessary, but not a sufficient condition for resolving the conflict. If we do not attend to a conflict, we *ignore* it. An example $\bar{v} \in \Lambda$ is *critical* if it is involved in some conflict, that is, $\bar{v} \in \Lambda^+$ and there is a $\bar{v}^- \in \Lambda^-$ such that $(\bar{v}, \bar{v}^-) \in \Xi$, or $\bar{v} \in \Lambda^-$ and there is a $\bar{v}^+ \in \Lambda^+$ such that $(\bar{v}^+, \bar{v}) \in \Xi$. Let Γ be the set of all critical tuples. For every $w \in V$, let

$$\Gamma(w) := \{\bar{v} \in \Gamma \mid w \in N_{2r+1}^G(\bar{v})\}$$

Note that if $\bar{v} \in \Gamma(w)$, then w attends to all conflicts \bar{v} is involved in.

The algorithm \mathcal{L} now uses s steps, corresponding to moves in the splitter game, to find a tuple that discriminates all but $m(\varepsilon^* + \varepsilon)$ examples. For this, we define a sequence of graphs G^0, \dots, G^s and training sequences $\Lambda^0, \dots, \Lambda^s$ with $\Lambda^i \in (V(G^i)^k \times \{0, 1\})^{m^i}$. Let $G^0 = G$ and $\Lambda^0 = \Lambda$. We will define the graphs G^i for $i > 0$ and their training sequences Λ^i later. Every tuple that is not critical can be classified correctly without additional parameters by adding its local type explicitly to the hypothesis. Thus, in every step, we consider only examples that have been involved in a conflict in all previous steps.

We now consider the i -th step of the algorithm. Let G^i be the current graph and Γ^i be the set of critical examples in the i -th step. In the following two lemmas, we strategically limit the search space for the parameters such that in each step, the error increases only by $\frac{\varepsilon}{s}$. The next lemma shows that there is a small set X of neighbourhood centres such that most of the conflicts can be attended by nodes from $N_{4r+2}^{G^i}(X)$.

► **Lemma 14.** *There is a set $X \subseteq V(G^i)$ of size $|X| \leq \frac{k\ell^*s}{\varepsilon}$ such that*

$$\frac{|\Gamma^i(u)|}{|\Gamma|} < \frac{\varepsilon}{\ell^* \cdot s} \quad \text{for all } u \in V(G^i) \setminus N_{4r+2}^{G^i}(X),$$

where $\Gamma^i(u)$ is the set of critical tuples in G^i that are affected by u and Γ is the set of critical tuples in the original graph G . Furthermore, there is an fpt algorithm computing such a set.

Proof. We inductively define a sequence $x_1, \dots, x_p \in V(G^i)$ as follows. For all $j \geq 1$, we choose $x_j \in V(G^i)$ such that $\text{dist}(x_j, x_{j'}) > 4r + 2$ for all $j' < j$ and, subject to this condition, $|\Gamma^i(x_j)|$ is maximum. If no such x_j exists, we let $p = j - 1$ and stop the construction. We

note that for every critical tuple $\bar{v} \in \Gamma$, there are at most k of the x_j such that $\bar{v} \in \Gamma^i(x_j)$. This holds as for every entry v of $\bar{v} \in (V(G^i))^k$, there is at most one of the x_j in the $(2r+1)$ -neighbourhood of v by the construction of X . Therefore,

$$\sum_{j=1}^p |\Gamma^i(x_j)| \leq k |\Gamma|.$$

This means that there are at most $k\ell^*s/\varepsilon$ of the x_j with $|\Gamma^i(x_j)| \geq \frac{\varepsilon}{\ell^*s} |\Gamma|$. As the x_j are sorted by decreasing $|\Gamma^i(x_j)|$, we have $|\Gamma^i(x_j)| < \frac{\varepsilon}{\ell^*s} |\Gamma|$ for all $j > \frac{k\ell^*s}{\varepsilon}$. We let $X = \{x_1, \dots, x_{\min\{p, k\ell^*s/\varepsilon\}}\}$. ◀

In the following, we fix a set X according to the lemma. We show in the next lemma that there is always a tuple of parameters consisting only of vertices in the neighbourhood of X that can induce a small error.

► **Lemma 15.** *If there is a tuple $\bar{v} \in (V(G^i))^{\ell^*}$ that discriminates all but $m \cdot \varepsilon'$ examples in G^i , then there is a tuple $\bar{w} \in (N_{4r+2}^{G^i}(X))^{\ell^*}$ that discriminates all but $m(\varepsilon' + \varepsilon/s)$ examples in G^i .*

In the proof, \bar{w} is computed from \bar{v} by dropping all elements that are far from X . By the choice of X , this introduces only a small error.

Proof. We split the entries v_j of \bar{v} in two sets W^*, U^* , where W^* contains all v_j that are contained in $N_{4r+2}^{G^i}(X)$ and U^* contains the remaining entries of \bar{v} . By the choice of X , each $u \in U^*$ can only discriminate up to

$$|\Gamma^i(u)| \leq \frac{m |\Gamma^i(u)|}{|\Gamma|} \leq \frac{m\varepsilon}{\ell^* \cdot s}$$

examples, where $\Gamma^i(u)$ is the set of critical tuples in G^i that are affected by u and Γ is the set of critical examples in the original graph G . Thus, the elements of U^* can discriminate at most $m\frac{\varepsilon}{s}$ examples in G^i . Since \bar{v} discriminates all but $m \cdot \varepsilon'$ examples, the elements of W^* must discriminate all but $m \cdot (\varepsilon' + \frac{\varepsilon}{s})$ examples. We let \bar{w} be an ℓ^* -tuple formed by the elements of W^* , for example choosing $w_j = v_j$ for all $v_j \in W^*$, and w_j is set to a fixed element in W^* for all other entries. Then, $\bar{w} \in (N_{4r+2}^{G^i}(X))^{\ell^*}$ and \bar{w} discriminates all but $m(\varepsilon' + \varepsilon/s)$ examples in G^i . ◀

We can now describe Step i and the way it is embedded in the overall algorithm. We know that in G^0 , the vector \bar{w}^* discriminates all but $m\varepsilon^*$ examples by the definition of \bar{w}^* and ε^* . In all later graphs G^i , there is a vector that discriminates all but $m \cdot (\varepsilon^* + \frac{i\varepsilon}{s})$ of the examples Λ^i which will be guaranteed by the construction of G^i and Λ^i in Lemma 16. By Lemma 14, there is also a tuple close to X that discriminates all but $m \cdot (\varepsilon^* + \frac{(i+1)\varepsilon}{s})$ examples. Thus, by further restricting the set of possible parameters, we increase the error in each step. In Step i we will choose ℓ^* parameters. Together with the parameters we choose in all following steps, this will result in the wanted tuple that discriminates all but (at most) $m(\varepsilon^* + \varepsilon)$ examples. Our next goal is to find such a tuple $\bar{w} \in (N_{4r+2}^{G^i}(X))^{\ell^*}$ that (together with the parameters from the following steps) discriminates all but $m(\varepsilon^* + \varepsilon)$ examples in G^i .

Clearly, for each such tuple of arity ℓ^* , there is a subset $Y \subseteq X$ of size $|Y| \leq \ell^*$ such that $\bar{w} \in (N_{4r+2}^{G^i}(Y))^{\ell^*}$. We non-deterministically guess such a set $Y = \{y_1, \dots, y_{\ell^*}\}$ and keep it fixed in the following. Simulating this non-deterministic guess by a deterministic algorithm adds a multiplicative cost of $|X|^{\ell^*} \leq (\frac{ks\ell^*}{\varepsilon})^{\ell^*}$, which is allowed in an fpt algorithm.

When searching for parameters in $N_{4r+2}^{G^i}(Y)$, we can attend only to conflicts with at least one element in $N_{6r+3}^{G^i}(Y)$ as all other conflicts cannot be attended by parameters from $N_{4r+2}^{G^i}(Y)$. We apply Lemma 3 and obtain a set $Z \subseteq Y$ and an $R' = 3^j((k+2)(2r+1))$, where $0 \leq j \leq |Y| - 1 \leq \ell^* - 1$, such that

$$N_{R'}^{G^i}(z) \cap N_{R'}^{G^i}(z') = \emptyset$$

for all distinct $z, z' \in Z$ and

$$N_{(k+2)(2r+1)}^{G^i}(Y) \subseteq N_{R'}^{G^i}(Z).$$

Note that $R' \leq R$, where R is the radius from the (R, s) -splitter game defined in the very beginning of the proof. We will see in the proof of Lemma 16 why we need $((k+2)(2r+1))$ -neighbourhoods. Suppose that $Z = \{z_1, \dots, z_{\ell''}\}$, where $\ell'' \leq \ell' \leq \ell^*$. For every $j \in [\ell'']$, let w_j be Splitter's answer if Connector picks z_j together with the radius R' in the (modified) (R, s) -splitter game on G . Note that we consider only possible picks $z_j \in Z$ and not arbitrary choices. The reason why Splitter's answers to the z_j suffice is that if we can identify every node in $N_{R'}^G(Z)$, then we can solve (almost) all conflicts consisting of vectors of vertices from that set. Essentially, Splitter guarantees that after removing a certain set of points (her answers in the splitter game), every node in the neighbourhood can be identified in at most $s - (i+1)$ steps if she had a winning strategy for the current graph in $s - i$ steps (which is guaranteed by the construction of G^i).

We then choose the vertices $\hat{w}^i = (w_1, \dots, w_{\ell''})$ as parameters in step i , where $w_j := w_{\ell''}$ for all $j \in \{\ell'' + 1, \dots, \ell^*\}$. With those parameters, we can now define the next graph G^{i+1} and the next set of examples Λ^{i+1} .

► **Lemma 16.** *There is a graph G^{i+1} and a training sequence $\Lambda^{i+1} \in (V(G^{i+1})^k \times \{0, 1\})^{m_{i+1}}$ for some $m_{i+1} \leq m_i$ such that the following holds.*

- (1) $V(G^{i+1}) = N_{R'}^{G^i}(Z) \uplus U \uplus U^*$, where U is a set of fresh isolated vertices in G^{i+1} and U^* is the set of all isolated vertices in G^i . $|U|$ depends only on k, ℓ^*, q^* but not on m or n .
- (2) $E(G^{i+1}) \subseteq E(G^i)$.
- (3) Splitter has a winning strategy for the $((R, s - (i+1))$ -splitter game on G^{i+1} .
- (4) If there is a tuple \bar{w}^i that discriminates all but $m \cdot \varepsilon'$ examples in G^i , then there is a tuple \bar{w}^{i+1} that discriminates all but $m(\varepsilon' + \frac{\varepsilon}{s})$ examples from Λ^{i+1} in G^{i+1} .
- (5) If there is a tuple \bar{w}^{i+1} from $V(G^{i+1})$ that distinguishes all but c examples from Λ^{i+1} , then the tuple $\hat{w}^i \bar{u}^{i+1}$ distinguishes all but c examples from Λ^i where \bar{u}^{i+1} is derived from \bar{w}^{i+1} by dropping all entries not in $V(G^i)$.

Furthermore, there is an fpt algorithm computing G^{i+1} and Λ^{i+1} .

The formal proof of the lemma is given in the appendix. From the lemma itself and its proof, we observe the following.

► **Remark 17.** For the graph G^{i+1} the following holds:

1. The isolated vertices in G^{i+1} will not be suitable parameters, as any conflict they resolve could also be resolved without the use of additional parameters. Thus, we can assume that all nodes from \bar{w}^i also appear in G^i and we have that $\bar{u}^i = \bar{w}^i$ in Statement (5).
2. Every conflict in the examples Λ^{i+1} in the graph G^{i+1} corresponds to a conflict between examples from Λ^i in the graph G^i . The number of conflicts can thus only decrease.
3. G^{i+1} is nowhere dense because Splitter has a winning strategy.
4. Λ^{i+1} contains exactly the critical examples from Λ^i .

5. Examples in Λ^{i+1} are projections of examples in Λ^i into $V(G^{i+1})$, essentially changing all those nodes that are not included in $N_{R'}^{G^i}(Z)$ to isolated vertices that represent types. The type each isolated vertex represents is encoded by its colour.

Note that only the last two items in the remark do not directly follow from the lemma (and are details of its proof). With the construction of the graph G^{i+1} and the corresponding set of examples Λ^{i+1} , we now have all the ingredients to prove Theorem 13.

Proof of Theorem 13. We start the algorithm by computing the number of steps s in the splitter game. We set $G^0 = G$ and $\Lambda^0 = \Lambda$, as described above. For $i = 0$ to s , we perform the following steps.

We use Lemma 14 to compute a set X . By Lemma 15, we know that there exists a tuple in the neighbourhood of X that discriminates all but $m(\varepsilon^* + \frac{(i+1)\varepsilon}{s})$ examples. This step is crucial as only in the neighbourhood of X , parameters will have a high impact and we thus limit our search for parameters to this area. Over all s steps, this additional error sums up to ε . Now, we non-deterministically guess a subset $Y \subseteq X$ of size at most ℓ^* . Unrolling this non-deterministic guess adds a factor that depends only on the parameters of the problem and can thus be performed by an fpt algorithm. Next, we apply Lemma 3 and obtain Z . By saving Splitter's answers in the splitter game to the picks $z_j \in Z$ in the parameters w_j , we obtain the vector \hat{w}^i . With this vector \hat{w}^i , we can now apply Lemma 16 and compute the next graph G^{i+1} and the next training sequence Λ^{i+1} to proceed to Step $i + 1$.

In Step s , we know that the tuple \hat{w}^s discriminates all but $m(\varepsilon^* + \varepsilon)$ examples from Λ^s in G^s . By concatenating all \hat{w}^i , we obtain a tuple \bar{w} of length $\ell = \ell^* \cdot s$ that discriminates all but $m(\varepsilon^* + \varepsilon)$ examples in the original graph G . We finish the computation by testing all possible formulas of quantifier rank q with $k + \ell$ free variables and are guaranteed to find at least one φ with $\text{err}_\Lambda(\varphi, \bar{w}) \leq \varepsilon^* + \varepsilon$.

Overall, the problem (L, Q) -FO-ERM is in FPT since there is an fpt algorithm for model checking first-order formulas on nowhere dense graphs and the bound on the number of such formulas we need to check depends only on k, ℓ , and q . All intermediate steps can also be performed by fpt algorithms. ◀

6 Conclusion

We have studied the parameterized complexity of learning first-order queries from examples. The specific problem we focussed on is *empirical risk minimisation*: find a query consistent with a set of positive or negative examples, or if no such query exists, minimise the error. While the main reason for looking at this problem is that it is essentially equivalent to PAC learning, we believe that the problem is natural and interesting in its own right, en par with other algorithmic problems typically studied for queries: evaluation (or model-checking), enumeration, or counting. The relevance of the problem goes beyond mere “learning queries by example”. In database systems, a similar problem arises for example in schema mapping [36]; the problem of finding formal specifications consistent with examples arises in other fields as well, for example in formal verification (e.g. [28, 17]).

Technically, the ERM problem poses new challenges. In this work, they became most obvious in the hardness proof, which required novel ideas quite distinct to those used in similar results. It is remarkable that the hardness result even holds for an approximate version of the problem, because hardness of approximation results are rare in the world of parameterized complexity theory.

Nowhere dense classes seem like a natural limit for the tractability of the learning problem, at least for graph classes that are closed under taking subgraphs. Adler and Adler [2] proved that nowhere dense classes are the largest family of classes closed under taking subgraphs for which first-order queries have bounded VC dimension and hence for which we have a uniform information theoretic PAC learning result. Furthermore, Grohe, Kreutzer and Siebertz [20] showed that nowhere dense classes are the largest family of classes closed under taking subgraphs for which first-order model-checking is fixed-parameter tractable. And indeed, our proof shows that the learning problem can be tightly linked with the game characterisation of nowhere-denseness. However, the result holds only for the approximation version (L, Q) -FO-ERM of the problem where we are allowed to increase the hyperparameters ℓ and q . It remains a challenging open problem whether FO-ERM is also fixed-parameter tractable on nowhere dense classes.

Our hardness result yields a reduction from the model-checking problem FO-MC to FO-ERM and (L, Q) -FO-ERM. This reduction relativises to graph classes satisfying mild closure conditions. It remains open whether the problems are actually equivalent, that is, whether there is a reduction from FO-ERM or (L, Q) -FO-ERM to FO-MC on all classes satisfying reasonable closure conditions.

While we know that first-order definable concepts admit no sublinear-time learning algorithms on nowhere dense graph classes, it might be possible to obtain such algorithms after a polynomial-time preprocessing phase (similar to the results of [21, 19] for monadic second-order logic on strings and trees).

Finally, it would be interesting to extend our results to richer logics that implement more features of commonly used relational database systems such as the extensions of first-order logic with counting or weight aggregation. Similarly, it would be interesting to prove that concepts definable in monadic second-order logic over structures of bounded tree width or bounded clique width can be learned by fixed-parameter tractable algorithms.

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A Appendix

A.1 Proof of Proposition 11

► **Proposition 11.** *Let \mathcal{C} be a class of graphs closed under colour expansions such that FO-MC is fixed-parameter tractable on \mathcal{C} .*

Then, for every constant $\ell^ \in \mathbb{N}$, the restriction of FO-ERM to inputs with $\ell = \ell^*$ is fixed-parameter tractable on \mathcal{C} .*

■ **Algorithm 1** FO-ERM learning algorithm for constant ℓ

Require: Training sequence $\Lambda \in (V(G)^k \times \{0, 1\})^m$

- 1: $min_errors \leftarrow m + 1$
- 2: **for all** $\varphi' \in \Phi'$ **do**
- 3: **for all** $\bar{w} \in (V(G))^\ell$ **do**
- 4: $errors \leftarrow 0$
- 5: **for all** $(\bar{v}, \lambda) \in \Lambda$ **do**
- 6: Let G' be the graph with $V(G') = V(G)$, $R_i(G') = \{v_i\}$ for $i \leq k$, $S_j(G') = \{w_j\}$ for $j \leq \ell$, and $R(G') = R(G)$ for other relations
- 7: **if** $(\lambda = 0$ and $G' \models \varphi')$ or $(\lambda = 1$ and $G' \not\models \varphi')$ **then**
- 8: $errors \leftarrow errors + 1$
- 9: **if** $errors < min_errors$ **then**
- 10: $\varphi'_{min} \leftarrow \varphi'$, $\bar{w}_{min} \leftarrow \bar{w}$
- 11: **return** $(\varphi_{min}(\bar{x}; \bar{y}), \bar{w}_{min})$, where φ_{min} is obtained from φ'_{min} by replacing all occurrences of $R_i x$ with $x = x_i$ and all occurrences of $S_i x$ are with $x = y_i$

Proof. We start the proof of the learnability for the setting that ℓ is constant by adding $k + \ell$ unary relations (colours) to τ : $\tau' := \tau \uplus \{R_1, \dots, R_k, S_1, \dots, S_\ell\}$. Then we have a finite set $\Phi' \subseteq \text{FO}[\tau', q]$ of formulas in Gaifman normal form (here: sentences in Gaifman normal form) with quantifier rank at most q . The algorithm for finding a hypothesis is given in pseudocode in Algorithm 1. We search for a sentence that minimises the training error in the graph G' over the extended alphabet τ' . This sentence is then translated back into a formula with $k + \ell$ free variables by substituting all occurrences of the additional relations. The translated formula is then returned by the algorithm, solving the problem FO-ERM and its relaxation (L, Q) -FO-ERM.

Let $n = |V(G)|$. Using the assumption that $k, \ell \leq n$, Algorithm 1 runs in time

$$|\Phi'| \cdot n^\ell \cdot m \cdot (k + \ell + n + g(q + |\tau'|) \cdot n^c) \in \mathcal{O}(f(k + q + |\tau|) \cdot m \cdot n^{\ell+c+1}).$$

for some functions f and g . In this running time, the first three factors correspond to the for-loops of the algorithm and the bracketed part comes from model checking. ◀

A.2 Proof of Proposition 12

► **Proposition 12.** *Let \mathcal{C} be a class of graphs closed under colour expansions such that FO-MC is fixed-parameter tractable on \mathcal{C} .*

Then the following restriction of FO-ERM is fixed-parameter tractable.

Input Graph $G \in \mathcal{C}$, training sequence $\Lambda \in (V(G)^k \times \{0, 1\})^m$, and $\ell, q \in \mathbb{N}$, $\varepsilon > 0$.
Parameter $\ell + q + |\tau| + \frac{1}{\varepsilon}$
Assumption There is a $h \in \mathcal{H}_{1, \ell, q}(G)$ with $\text{err}_\Lambda(h) = 0$.
Problem Return a hypothesis $h_{\varphi, \bar{w}} \in \mathcal{H}_{1, \ell, q}(G)$ such that

$$\text{err}_\Lambda(\varphi, \bar{w}) \leq \varepsilon.$$

■ **Algorithm 2** FO-ERM learning algorithm for $k = 1$

Require: Training sequence $\Lambda \in (V(G)^k \times \{0, 1\})^m$

- 1: **for all** $\varphi \in \Phi'$ **do**
- 2: $\text{consistent} \leftarrow \text{true}$
- 3: **for** $i = 1$ to ℓ **do**
- 4: $\varphi_i(x, y_{i+1}, \dots, y_\ell) := \exists y_1 \dots \exists y_i \left(\bigwedge_{j=1}^i S_j y_j \right. \\ \left. \wedge \varphi(x, y_1, \dots, y_\ell) \right)$
- 5: $\text{found} \leftarrow \text{false}$
- 6: **for all** $u \in V(G)$ **do**
- 7: Let G' be the graph with $V(G') = V(G)$, $S_j(G') = \{w_j\}$ for all $j < i$, $S_i(G') = \{u\}$,
 $P_+(G') = \Lambda^+$, $P_-(G') = \Lambda^-$, and $R(G') = R(G)$ for other relations
- 8: **if** $G' \models \exists y_{i+1} \dots \exists y_\ell \forall x \left((P_+ x \rightarrow \varphi_i(x, y_{i+1}, \dots, y_\ell)) \wedge \right. \\ \left. (P_- x \rightarrow \neg \varphi_i(x, y_{i+1}, \dots, y_\ell)) \right)$ **then**
- 9: $w_i \leftarrow u$
- 10: $\text{found} \leftarrow \text{true}$
- 11: **break**
- 12: **if not found then**
- 13: $\text{consistent} \leftarrow \text{false}$
- 14: **break**
- 15: **if consistent then**
- 16: **return** $(\varphi(x; \bar{y}), \bar{w})$
- 17: **reject**

Proof. Let $\tau' := \tau \uplus \{S_1, \dots, S_\ell, P_+, P_-\}$. Again, similar to the setting in Proposition 11, we have a finite set $\Phi' \subseteq \text{FO}[\tau', q]$ of formulas in Gaifman normal form with quantifier rank at most q with $\ell + 1$ free variables. The algorithm uses a relation P_+ for all positive and a relation P_- for all negative examples. Since $k = 1$, those relations are both unary and the resulting graph is still contained in \mathcal{C} . Using those unary relations for the examples, it is possible to test whether a given prefix of parameters can be extended to a consistent one. The algorithm thus starts by testing for every node $u \in V(G)$ whether it can be extended to a consistent parameter setting. If it finds such a node u , it fixes $w_1 = u$ and proceeds with the search for w_2 . Let $n = |V(G)|$. After at most $\ell \cdot n$ model-checking steps, the algorithm has discovered a consistent parameter setting if there is one. As in Proposition 11, we use model checking for sentences only and expand the background structure to contain additional unary relations (colours) for the free variables of the formula. In our case, the free variables of the formulas we evaluate are exactly the parameter prefixes. The algorithm is given in pseudocode in Algorithm 2 and runs in time

$$|\Phi'| \cdot \ell \cdot n \cdot (n + m + \ell + g(q + \ell + |\tau'|)) \cdot n^c \in \mathcal{O}(f(\ell + q + |\tau|) \cdot m \cdot n^{c+1}).$$

for some functions f and g . In this running time, the first three factors correspond to the for-loops of the algorithm and the bracketed part comes from model checking. ◀

A.3 Proof of Lemma 16

► **Lemma 16.** *There is a graph G^{i+1} and a training sequence $\Lambda^{i+1} \in (V(G^{i+1})^k \times \{0, 1\})^{m_{i+1}}$ for some $m_{i+1} \leq m_i$ such that the following holds.*

- (1) $V(G^{i+1}) = N_{R'}^{G^i}(Z) \uplus U \uplus U^*$, where U is a set of fresh isolated vertices in G^{i+1} and U^* is the set of all isolated vertices in G^i . $|U|$ depends only on k, ℓ^*, q^* but not on m or n .
- (2) $E(G^{i+1}) \subseteq E(G^i)$.
- (3) Splitter has a winning strategy for the $((R, s - (i + 1))$ -splitter game on G^{i+1} .
- (4) If there is a tuple \bar{w}^i that discriminates all but $m \cdot \varepsilon'$ examples in G^i , then there is a tuple \bar{w}^{i+1} that discriminates all but $m(\varepsilon' + \frac{\varepsilon}{s})$ examples from Λ^{i+1} in G^{i+1} .
- (5) If there is a tuple \bar{w}^{i+1} from $V(G^{i+1})$ that distinguishes all but c examples from Λ^{i+1} , then the tuple $\hat{w}^i \bar{u}^{i+1}$ distinguishes all but c examples from Λ^i where \bar{u}^{i+1} is derived from \bar{w}^{i+1} by dropping all entries not in $V(G^i)$.

Furthermore, there is an fpt algorithm computing G^{i+1} and Λ^{i+1} .

Proof of Lemma 16. We let G^{i+1} be the graph obtained from the induced R' -neighbourhood structure $\mathcal{N}_{R'}^{G^i}(Z)$ as follows. We start with the construction; the roles of each of the steps will become clear while proving the statements of the lemma.

1. Expand the graph by fresh colours $D_{j,d}$ for $j \in [\ell']$ and $d \in \{0, \dots, (k+2)(2r+1)\}$. We let $D_{j,d}(G^{i+1})$ be the set of all v such that $\text{dist}_{G^i}(v, y_j) = d$.
2. Expand the graph by fresh colours C_j for $j \in [\ell'']$. Let $C_j(G^{i+1}) := N_1^{G^i}(w_j)$.
3. Delete all edges incident with the w_j . Moreover, we add fresh colours B_j for $j \in [\ell'']$ and let $B_j(G^{i+1}) := \{w_j\}$.
4. For each non-empty set $I \subset [k]$ and each $|I|$ -variable q^* -type $\theta \in \text{Tp}[\tau, |I|, q^*]$, we add an isolated vertex $t_{I,\theta}$ and a fresh colour $A_{I,\theta}$ and let $A_{I,\theta}(G^{i+1}) = \{t_{I,\theta}\}$.

Thus, structurally, G^{i+1} consists of the neighbourhoods $G_j^{i+1} := G^i[N_{R'}^{G^i}(z_j) \setminus \{w_j\}]$ for $j \in [\ell'']$ and isolated vertices $w_1, \dots, w_{\ell''}$ and $t_{I,\theta}$ for all non-empty $I \subset [k]$ and $\theta \in \text{Tp}[\tau, |I|, q^*]$. Hence, Statements (1) and (2) of Lemma 16 hold by the construction of G^{i+1} . Note that the neighbourhoods G_j^{i+1} and $G_{j'}^{i+1}$ are disconnected for each $j \neq j'$ by the construction of Z .

Splitter's winning strategy on G^i is still valid on G^{i+1} , but one of the steps of the splitter game (removing a vertex Spoiler chose and continuing in the neighbourhood of a vertex Connector chose) has already been performed in each of the neighbourhoods G_j^{i+1} in the construction of G^{i+1} . Hence, Statement (3) follows.

Next, we define the training sequence $\Lambda^{i+1} \in (V(G^{i+1})^k \times \{0, 1\})^{m_{i+1}}$ for the next round. For this step, we will need the isolated vertices introduced in Step 4 in the construction of G^{i+1} . We only need to consider examples in Γ^i , which are involved in a conflict; all other examples can be correctly classified without the use of parameters. For each tuple $\bar{v} = (v_1, \dots, v_k) \in \Gamma_i$ with $\bar{v} \cap N_{6r+3}^{G^i}(Y) \neq \emptyset$, we define a tuple $\bar{v}' = (v'_1, \dots, v'_k) \in V(G^{i+1})$ for which we add (\bar{v}', λ) to Λ^{i+1} if $(\bar{v}, \lambda) \in \Lambda^i$. For that, we consider the graph $H_{\bar{v}}$ with vertex set $V(H_{\bar{v}}) := [k]$ and edges $\{a, b\}$ for all $a, b \in [k]$ with $1 \leq \text{dist}_{G^i}(v_a, v_b) \leq 2r+1$. Let I_1, \dots, I_p be the vertex sets of the connected components of $H_{\bar{v}}$. For each component I_j , we proceed as follows: if there is some $j_0 \in I_j$ such that $v_{j_0} \in N_{6r+3}^{G^i}(Y)$, we let $v'_a := v_a$ for all $a \in I_j$. Note that for all $a \in I_j$ we have $\text{dist}_{G^i}(v_{j_0}, v_a) \leq (k-1)(2r+1)$ and hence, $\text{dist}_{G^i}(Y, v_a) \leq 6r+3 + (k-1)(2r+1) = (k+2)(2r+1)$. Thus, $v'_a \in N_{(k+2)(2r+1)}^{G^i}(Y) \subseteq N_{R'}^{G^i}(Z) \subseteq V(G^{i+1})$. This is the point that determines the choice of R of the splitter game

in the very beginning of the proof. Otherwise, if $v_a \notin N_{6r+3}^{G^i}(Y)$ for all $a \in I_j$, we consider the restriction $\bar{v}|_{I_j}$ of \bar{v} to the indices in I_j and let $\theta := \text{ltp}_{q^*,r}(G^i, \bar{v}|_{I_j})$. In this case, we let $v'_a := t_{I_j, \theta}$ for all $a \in I_j$. Note that for some examples this means that they only consist of isolated vertices and thus will never be correctly classified by our algorithm. Observe that two tuples \bar{v}'_1, \bar{v}'_2 that appear in examples in Λ^{i+1} can only have the same type in G^{i+1} if their counterparts \bar{v}_1, \bar{v}_2 from Λ^i have the same type in G^i . Hence, we do not create any new conflicts by the construction.

Let $A = \mathcal{N}_R^{G^i}(Z)$ and A' the modified version of this neighbourhood in G^{i+1} . It is easy to interpret A in G^{i+1} using the information encoded in the fresh colours added in Step 1-3 of the construction of G^{i+1} . Using the parameters $w_1, \dots, w_{\ell''}$, we can also interpret the modified neighbourhood A' in G^i . We use this connection to prove Statement (4) and (5). Note that for encoding the distance information, the increased quantifier rank q might be necessary which is why we chose $q = q^* + \log R$.

Assume there is a tuple $\bar{w}^i \in V(G^i)^{\ell^*}$ that discriminates all but $m\varepsilon'$ examples in G^i . By Lemma 15, there is a tuple $\bar{u}^i \in (N_{4r+2}^{G^i}(Z))^{\ell^*}$ that discriminates all but $m(\varepsilon' + \frac{\varepsilon}{s})$ examples in G^i . Then \bar{u}^i also discriminates all but $m(\varepsilon' + \frac{\varepsilon}{s})$ examples in G^{i+1} due to the way we projected the examples and the fact that we can emulate A in G^{i+1} . This shows Statement (4).

If we can distinguish \bar{v}'_1, \bar{v}'_2 in G^{i+1} using parameters \bar{w}^{i+1} , then we can distinguish \bar{v}_1, \bar{v}_2 in G^i using \bar{w}^{i+1} and the chosen parameters $w_1, \dots, w_{\ell''}$. This holds as we can interpret A' in G^i . Thus, if \bar{w}^{i+1} discriminates all but c examples in Λ^{i+1} , then $\hat{w}^i \bar{w}^{i+1}$ discriminates all but c examples in Λ^i . This shows Statement (5).

To finish the proof, we need to show that all steps can be performed by an fpt algorithm. In the first three steps, the number of new colours only depends on the parameters k, ℓ^* , and q^* . In step 4, the number of new colours only depends on k, q^* and the size of the signature of G^i . Since the number of computed graphs is bounded by s (and therefore in terms of q^*, k , and ℓ^*), the total number of fresh colours only depends on the parameters. Furthermore, the computations are linear in the size of G^i . For the projection of the examples, the computation of the types can be performed by an fpt algorithm because G^i is nowhere dense. All other computations for the projection can be done by an fpt algorithm as well. \blacktriangleleft