# Beyond Nonexpansive Operations in Quantitative Algebraic Reasoning 

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#### Abstract

The framework of quantitative equational logic has been successfully applied to reason about algebras whose carriers are metric spaces and operations are nonexpansive. We extend this framework in two orthogonal directions: algebras endowed with generalised metric space structures, and operations being nonexpansive up to a lifting. We apply our results to the algebraic axiomatisation of the Łukaszyk-Karmowski distance on probability distributions, which has recently found application in the field of representation learning on Markov processes.


## 1 Introduction

Equational reasoning and algebraic methods are widespread in all areas of computer science, and in particular in program semantics. Indeed, initial algebra semantics and monads are cornerstones of the modern theory of functional programming and allow us to reason about inductive definitions, computational effects and specifications in a formal way (see, e.g., Moggi [1991], Rutten and Turi [1993], Hyland et al. [2006]). In elementary terms, this is due to the fact that many objects of interest in programming are free algebras of some algebraic theory, i.e., a signature $\Sigma$ together with a set of equational axioms $E$ between $\Sigma$-terms. Examples include: finite sets (free algebras of the theory of semilattices)

$$
\Sigma=\{\vee: 2\} \quad E=\left\{\begin{array}{l}
x \vee y=y \vee x, x \vee x=x \\
x \vee(y \vee z)=(x \vee y) \vee z
\end{array}\right\}
$$

finite lists (free monoids), finitely supported distributions (free convex algebras) etc. Since free algebras are (up to isomorphism) term algebras-i.e., sets of $\Sigma$-terms modulo the congruence relation $\equiv_{E}$ generated from the axioms $E$ using the deduction rules of the syntactic apparatus of equational logic-they are easy to manipulate formally in a computer.

Objects definable as free algebras, as in the framework outlined above, are sets $X$ equipped with operations of type $X^{n} \rightarrow X$. This means it is not straightforward, or even possible, to describe objects that are sets endowed with some additional structure such as, e.g., a metric $d: X^{2} \rightarrow[0,1]$. To address this limitation, in a series of recent papers (including Bacci et al. [2018b], Mardare et al. [2016, 2017], Bacci et al. [2021], Mardare et al. [2021]), the authors have proposed the notion of quantitative algebras: algebras whose carriers are metric spaces.

At the syntactic level, the apparatus of equational logic is replaced by a deductive system allowing the derivation of judgments of the form $s={ }_{\varepsilon} t$, where $s, t$ are $\Sigma$-terms and $\varepsilon \in[0,1]$, with the intended meaning that $d(s, t) \leq$ $\varepsilon$. These judgments are derived using quantitative inferences, i.e., deduction rules of the form:

$$
\left\{s_{1}={ }_{\varepsilon_{1}} t_{1}, \ldots, s_{n}=\varepsilon_{\varepsilon_{n}} t_{n}\right\} \vdash s={ }_{\varepsilon} t
$$

In particular, the deductive system includes rules such as:

$$
\begin{gathered}
\varnothing \vdash x={ }_{0} x \quad\left\{x==_{\varepsilon} y\right\} \vdash y={ }_{\varepsilon} x \\
\left\{x=\varepsilon_{\varepsilon_{1}} y, y==_{\varepsilon_{2}} z\right\} \vdash x==_{\varepsilon_{1}+\varepsilon_{2}} z
\end{gathered}
$$

corresponding to properties of metrics such as reflexivity $(d(x, x)=0)$, symme$\operatorname{try}(d(x, y)=d(y, x))$ and triangular inequality $(d(x, y) \leq d(x, y)+d(y, z))$. A quantitative theory over a signature $\Sigma$ is generated from a set of quantitative inferences, playing the role of implicational axioms, by closing under deducibility in the apparatus. Models of quantitative theories are quantitative algebras, which are metric spaces $(A, d)$ equipped with interpretations $\llbracket \mathrm{op} \rrbracket: A^{n} \rightarrow A$ of the operations such that for each op $\in \Sigma$

$$
d\left(\llbracket \mathrm{op} \rrbracket\left(a_{1}, \ldots, a_{n}\right), \llbracket \mathrm{op} \rrbracket\left(a_{1}^{\prime}, \ldots, a_{n}^{\prime}\right)\right) \leq \max \left\{d\left(a_{i}, a_{i}^{\prime}\right)\right\}_{1 \leq i \leq n}
$$

This is equivalent to requiring that $\llbracket \mathrm{op} \rrbracket:\left(A^{n}, d_{\times}\right) \rightarrow(A, d)$ is nonexpansive (also known as 1 -Lipschitz), with $d_{\times}$being the (categorical) product metric on $A^{n}$. This is reflected in the deductive system by a rule called NE:

$$
\left\{x_{i}=\varepsilon_{i} y_{i}\right\}_{1 \leq i \leq n} \vdash \operatorname{op}\left(x_{1}, \ldots, x_{n}\right)={ }_{\max \left(\varepsilon_{1}, \ldots, \varepsilon_{n}\right)} \operatorname{op}\left(y_{1}, \ldots, y_{n}\right)
$$

Consider, for example, the theory of quantitative semilattices of Mardare et al. [2016] having signature $\Sigma=\{V: 2\}$ and implicational axioms (we just write $s={ }_{\varepsilon} t$ for $\left.\varnothing \vdash s={ }_{\varepsilon} t\right)$ :

$$
x \vee y={ }_{0} y \vee x \quad x \vee x==_{0} x \quad x \vee(y \vee z)==_{0}(x \vee y) \vee z
$$

These just state the usual axioms of semilattices. Indeed, since in any metric space it holds that $d(x, y)=0$ implies $x=y$, the judgment $s={ }_{0} t$ expresses equality. From these axioms, further quantitative inferences can be obtained using the deductive apparatus, like the NE rule:

$$
\left\{x=\varepsilon_{\varepsilon_{1}} x^{\prime}, y=\varepsilon_{\varepsilon_{2}} y^{\prime}\right\} \vdash x \vee y={\max \left(\varepsilon_{1}, \varepsilon_{2}\right)} x^{\prime} \vee y^{\prime}
$$

which expresses that the interpretation of the binary operation $\vee: 2$ must be nonexpansive.

Given a quantitative theory over a signature $\Sigma$ generated by a set of implicational axioms $E$, we have a category $\operatorname{Alg}(\Sigma, E)$ consisting of quantitative algebras modelling the theory and their homomorphisms, i.e., nonexpansive maps $f:\left(A, d_{A}\right) \rightarrow\left(B, d_{B}\right)$ preserving all operations 【op】. Among the main results of Mardare et al. [2016, 2017], Bacci et al. [2018b] the following is of key importance:
Theorem 3.3 in Bacci et al. [2018b]. The free quantitative algebra generated by a metric space $(A, d)$ exists in $\operatorname{Alg}(\Sigma, E)$ and is isomorphic to the quantitative term algebra $T_{\Sigma, E}(A, d)$.

More can be said if the implicational axioms $E$ have a constrained form, where all the terms in their premises are variables: $x_{1}={ }_{\varepsilon_{1}} y_{1}, \ldots, x_{n}=\varepsilon_{\varepsilon_{n}} y_{n} \vdash$ $s={ }_{\varepsilon} t$. In this case, which covers several interesting examples (e.g., quantitative semilattices), we have a stronger result:

Theorem 4.2 in Bacci et al. [2018b]. The Eilenberg-Moore category $\mathbf{E M}\left(T_{\Sigma, E}\right)$ of the term monad $T_{\Sigma, E}$ is isomorphic to the category $\operatorname{Alg}(\Sigma, E)$.

Several interesting metric spaces can be identified with free quantitative algebras. For example the collection of non-empty finite subsets of $(A, d)$, endowed with the Hausdorff metric and interpreting $\llbracket \vee \rrbracket=\cup$ (union), can be shown (see Mardare et al. [2016]) to be isomorphic to the free quantitative semilattice generated by the metric space $(A, d)$.

### 1.1 Beyond Metric Spaces and Nonexpansive Maps

The main purpose of this paper is to extend the framework of Bacci et al. [2018b] outlined above, while maintaining its key characteristics and properties, in order to reason equationally about additional interesting mathematical objects which do not fit the constraints of the original framework.

We immediately discuss a specific example arising from recent research in the field of learning and artificial intelligence Castro et al. [2021], which will serve as a main motivation. Other examples are discussed in Section 5. In Castro et al. [2021], the authors have developed new techniques for representation learning on Markov processes based on the Łukaszyk-Karmowski (ŁK for short) distance Łukaszyk [2004]. This is a distance $d_{\mathrm{EK}}: \mathcal{D} X \times \mathcal{D} X \rightarrow[0,1]$ on finitely supported distributions on a set $X$ endowed with an arbitrary map $d: X^{2} \rightarrow[0,1]$ (i.e., $(X, d)$ is not necessarily a metric space). Even if $d$ is a metric, the $Ł K$ distance $d_{\mathrm{ŁK}}$ does not satisfy all axioms of metric spaces. Specifically the reflexivity property is in general not satisfied: $d_{\mathrm{EK}}(\varphi, \varphi) \neq 0$. However, $d_{\mathrm{EK}}$ always satisfies the symmetry and triangular inequality axioms (see Equations (1) and (4) in Section 2) and, therefore, ( $\mathcal{D} X, d_{\mathrm{EK}}$ ) is a diffuse metric space (see Castro et al. [2021] or Section 2.3 for precise definitions). If we consider the convex algebra operation $+_{p}: \mathcal{D}(X) \times \mathcal{D}(X) \rightarrow \mathcal{D}(X)$ on probability
distributions defined by

$$
(\varphi+p \psi)(x)=p \varphi(x)+(1-p) \psi(x)
$$

then it can be shown (see Lemma 5.3) that $+_{p}$ fails to be nonexpansive with respect to the ŁK distance:

$$
d_{\mathrm{EK}}\left(\varphi+{ }_{p} \varphi^{\prime}, \psi+_{p} \psi^{\prime}\right)>\max \left\{d_{\mathrm{EK}}(\varphi, \psi), d_{\mathrm{EK}}\left(\varphi^{\prime}, \psi^{\prime}\right)\right\}
$$

Thus we have an interesting mathematical object, the diffuse metric space $\left(\mathcal{D}(X), d_{\mathrm{EK}}\right)$, whose underlying set $\mathcal{D}(X)$ is the free convex algebra over the set $X$ (see, e.g., Jacobs [2010]), not fitting the framework of Bacci et al. [2018b] due to two reasons: (1) $d_{\mathrm{EK}}$ is not a metric, and (2) the algebraic (convex algebra) operation $+_{p}$ is not nonexpansive.

Our contribution is to extend the framework of Mardare et al. [2016, 2017], Bacci et al. [2018b] along two orthogonal axes in order to accomodate examples (see Section 5) such as the one just discussed.

First extension axis: our framework can be instantiated on structures $(X, d)$ where $d: X^{2} \rightarrow[0,1]$ is a generalised metric such as any of the following (see Section 2 for details): an ultrametric, metric, pseudometric, quasimetric, diffuse metric or just a fuzzy relation (i.e., $d$ unconstrained).

This first contribution is natural, yet requires some technical care. Most notably, we need to carefully distinguish in the deductive apparatus between the notions of equality $(=)$ and zero distance $\left(=_{0}\right)$. This is due to the fact that, unlike the case of metric spaces, in generalised metric spaces (e.g., pseudometric or diffuse metrics) it does not hold that $d(x, y)=0$ implies $x=y$. As a consequence, the identification of $=$ and $={ }_{0}$ is generally unsound. Our deductive apparatus, unlike that of Bacci et al. [2018b], will therefore handle both ordinary equations $(s=t)$ and quantitative equations $\left(s=_{\varepsilon} t\right)$, connected by the following congruence principle:

$$
x=y \Rightarrow\left(\left(x={ }_{\varepsilon} z \Rightarrow y={ }_{\varepsilon} z\right) \text { and }\left(z={ }_{\varepsilon} x \Rightarrow z={ }_{\varepsilon} y\right)\right) .
$$

Second extension axis: our framework can deal with quantitative algebras whose operations are not nonexpansive with respect to the categorical product. The motivating example being the diffuse metric space $\left(\mathcal{D}(X), d_{\mathrm{EK}}\right)$ with the convex combination operation $+_{p}$ discussed earlier. This is in our opinion the main conceptual and technical contribution of the paper.

To achieve this flexibility, we consider lifted signatures $\widehat{\Sigma}=\left\{\mathrm{op}_{i}: n_{i}: L_{\mathrm{op}_{i}}\right\}_{i \in I}$. Each operation op has an arity $n \in \mathbb{N}$, as for standard signatures, and is further equipped with a lifting which maps any generalised metric space $(X, d)$ to a generalised metric space $\left(X^{n}, L_{\mathrm{op}}(d)\right)$ whose underlying set is the product set $X^{n}$, subject to some technical constraints.

In this new setting, quantitative algebras for a lifted signature $\widehat{\Sigma}$ are (generalised) metric spaces $(X, d)$ in GMet where, for each op $\in \Sigma$, the interpretation
$\llbracket \mathrm{op} \rrbracket: X^{n} \rightarrow X$ is nonexpansive up to $L_{\mathrm{op}}$, namely:

$$
\llbracket \mathrm{op} \rrbracket:\left(X^{n}, L_{\mathrm{op}}(d)\right) \rightarrow(X, d) \quad \text { is nonexpansive. }
$$

At the syntactic level, our deductive apparatus replaces the NE rule of Bacci et al. [2018b] with a rule denoted by $L-N E$ (see Definition 3.11) expressing that each op : $n: L_{\mathrm{op}} \in \widehat{\Sigma}$ is nonexpansive up to $L_{\mathrm{op}}$.

The framework of Bacci et al. [2018b] can be seen as a particular case of ours by taking GMet $=$ Met and restricting all $L_{\mathrm{op}_{i}}$ to be the standard $n$-ary (categorical) product in Met: $L_{\mathrm{op}_{i}}(X, d)=\left(X^{n_{i}}, d_{\times}\right)$.

### 1.2 Outline and Main Results

After presenting some background material in Section 2, we introduce in Section 3 our new framework for quantitative reasoning based on liftings, and we prove the soundness of the associated deductive apparatus. In Section 4, we define the term monad and we recover the key results of the framework of Bacci et al. [2018b] in our new "lifted" setting. In particular we obtain proofs of the corresponding variants of Theorem 3.3 (free algebras exist and are term algebras) and Theorem $4.2\left(\mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, E}\right) \cong \operatorname{Alg}(\widehat{\Sigma}, E)\right)$ from Bacci et al. [2018b]. We give examples of applications of our new apparatus in Section 5, covering in particular the interesting case of the ŁK diffuse metric on probability distributions. Full proofs can be found in the appendix.

## 2 Background

### 2.1 Monads

We present some definitions and results regarding monads. We assume the reader is familiar with basic concepts of category theory (see, e.g., Awodey [2010]). Facts easily derivable from known results in the literature are systematically marked as "Proposition" throughout the paper.

Definition 2.1 (Monad). A monad on a category C is a triple $(M, \eta, \mu)$ comprising a functor $M: \mathbf{C} \rightarrow \mathbf{C}$ together with two natural transformations: a unit $\eta$ : $\operatorname{id}_{\mathrm{C}} \Rightarrow M$, where $\mathrm{id}_{\mathrm{C}}$ is the identity functor on $\mathbf{C}$, and a multiplication $\mu: M^{2} \Rightarrow M$, satisfying $\mu \circ \eta M=\mu \circ M \eta=\operatorname{id}_{M}$ and $\mu \circ M \mu=\mu \circ \mu M$.

A monad $M$ has an associated category of $M$-algebras.
Definition 2.2 ( $M$-algebras). Let $(M, \eta, \mu)$ be a monad on C. An algebra for $M$ (or $M$-algebra) is a pair $(A, \alpha)$ where $A \in \mathrm{C}$ is an object and $\alpha: M(A) \rightarrow A$ is a morphism such that (1) $\alpha \circ \eta_{A}=\mathrm{id}_{A}$ and (2) $\alpha \circ M \alpha=\alpha \circ \mu_{A}$ hold. An $M$-algebra morphism between two $M$-algebras $(A, \alpha)$ and $\left(A^{\prime}, \alpha^{\prime}\right)$ is a morphism $f: A \rightarrow A^{\prime}$ in $\mathbf{C}$ such that $f \circ \alpha=\alpha^{\prime} \circ M(f)$. The category of $M$-algebras and their morphisms, denoted by $\operatorname{EM}(M)$, is called the Eilenberg-Moore category for $M$.

### 2.2 Universal Algebra

We recall basic definitions and results from universal algebra, Burris and Sankappanavar [1981] is a standard reference.

Definition 2.3 (Signature). A signature is a set $\Sigma$ containing operations symbols each with an arity $n \in \mathbb{N}$. We denote op : $n \in \Sigma$ for a symbol op with arity $n$ in $\Sigma$. With some abuse of notation, we also denote with $\Sigma$ the functor $\Sigma:$ Set $\rightarrow$ Set with the following action:

$$
\Sigma(A):=\coprod_{\text {op: }: n \in \Sigma} A^{n} \quad \Sigma(f):=\coprod_{\text {op:n }} f_{\Sigma} f^{n}
$$

Definition 2.4 ( $\Sigma$-algebra). A $\Sigma$-algebra is an algebra for the functor $\Sigma$. Equivalently, it is a set $A$ equipped with a set $\llbracket \Sigma \rrbracket_{A}$ of interpretations of the operation symbols, i.e., for every op $: n \in \Sigma$ there is a function $\llbracket \mathrm{op} \rrbracket_{A}: A^{n} \rightarrow A$ in $\llbracket \Sigma \rrbracket_{A}$. We call $A$ the carrier set. A homomorphism between two $\Sigma$-algebras with carrier sets $A$ and $B$ is a function $f: A \rightarrow B$ preserving $\llbracket-\rrbracket$, i.e., satisfying $\forall$ op : $n \in \Sigma, \forall a_{1}, \ldots, a_{n}$,

$$
f\left(\llbracket \mathrm{op} \rrbracket_{A}\left(a_{1}, \ldots, a_{n}\right)\right)=\llbracket \mathrm{op} \rrbracket_{B}\left(f\left(a_{1}\right), \ldots, f\left(a_{n}\right)\right) .
$$

The category of $\Sigma$-algebras and their homomorphisms is denoted $\operatorname{Alg}(\Sigma)$.
Definition 2.5 (Term algebra). Let $\Sigma$ be a signature and $A$ be a set. We denote with $T_{\Sigma} A$ the set of terms built from $A$ using the operations in $\Sigma$, i.e., the set inductively defined as follows: $a \in T_{\Sigma} A$ for any $a \in A$, and op $\left(t_{1}, \ldots, t_{n}\right) \in$ $T_{\Sigma} A$ for any op $: n \in \Sigma$ and $t_{1}, \ldots t_{n} \in T_{\Sigma} A$. The set $T_{\Sigma} A$ has a canonical $\Sigma$ algebra structure with the interpretation of the operations op : $n \in \Sigma$, defined as:

$$
\llbracket \mathrm{op} \rrbracket\left(t_{1}, \ldots, t_{n}\right)=\mathrm{op}\left(t_{1}, \ldots, t_{n}\right) .
$$

It is called the term algebra over $A$ and denoted $T_{\Sigma} A$ (like its carrier set). We often identify elements $a \in A$ with the corresponding terms $a \in T_{\Sigma} A$.

Definition 2.6 (Term monad). The assignment $A \mapsto T_{\Sigma} A$ can be turned into a functor $T_{\Sigma}$ : Set $\rightarrow$ Set by inductively defining, for any function $f: A \rightarrow B$, the homomorphism $T_{\Sigma} f: T_{\Sigma} A \rightarrow T_{\Sigma} B$ as follows: for any $a \in A,\left(T_{\Sigma} f\right)(a)=f(a)$, and $\forall$ op $: n \in \Sigma$ and $\forall t_{1}, \ldots t_{n} \in T_{\Sigma} A$,

$$
T_{\Sigma} f\left(\operatorname{op}\left(t_{1}, \ldots, t_{n}\right)\right)=\operatorname{op}\left(T_{\Sigma} f\left(t_{1}\right), \ldots, T_{\Sigma} f\left(t_{n}\right)\right)
$$

This becomes a monad by defining the unit $\eta_{A}^{\Sigma}: A \rightarrow T_{\Sigma} A$ as mapping $a \in A$ to the term $a \in T_{\Sigma} A$, and the multiplication $\mu_{A}^{\Sigma}: T_{\Sigma}\left(T_{\Sigma} A\right) \rightarrow T_{\Sigma} A$ as mapping a term built out of terms $t\left(t_{1}, \ldots, t_{n}\right)$ to the flattened term $t\left(t_{1}, \ldots, t_{n}\right)$. We call $\left(T_{\Sigma}, \eta^{\Sigma}, \mu^{\Sigma}\right)$ the term monad for $\Sigma$.

Proposition 2.7. For any signature $\Sigma, \operatorname{Alg}(\Sigma) \cong \mathbf{E M}\left(T_{\Sigma}\right)$.

For the rest of this paper, let $X$ be a fixed countable set of variables. An interpretation of $X$ in a $\Sigma$-algebra $\mathbb{A}=(A, \llbracket \Sigma \rrbracket)$ is a map $\iota: X \rightarrow A$. The interpretation extends to arbitrary $T_{\Sigma} X$ terms by inductively defining $\llbracket-\rrbracket^{\iota}$ : $T_{\Sigma} X \rightarrow A$ as:

$$
\llbracket x \rrbracket^{\iota}=\iota(x) \text { and } \llbracket \mathrm{op}\left(t_{1}, \ldots, t_{n}\right) \rrbracket^{\iota}=\llbracket \mathrm{op} \rrbracket\left(\llbracket t_{1} \rrbracket^{\iota}, \ldots, \llbracket t_{n} \rrbracket^{\iota}\right) .
$$

In cases where $\iota: X \rightarrow T_{\Sigma} A$ is an interpretation in a term algebra, we denote $\llbracket-\rrbracket^{\iota}$ with $\iota^{*}$ to emphasize that its action is straightforward. It can be seen as a completely syntactical rewriting procedure, as $\iota^{*}$ takes a term in $T_{\Sigma} X$ and replaces all occurrences of $x$ with the term $t(x)$.
Definition 2.8 (Equations and their models). An equation over $\Sigma$ is a pair of $\Sigma$-terms over $X$, i.e., an element of $T_{\Sigma} X \times T_{\Sigma} X$ which we denote $s=t$. We say a $\Sigma$-algebra $\mathbb{A}=(A, \llbracket \Sigma \rrbracket)$ satisfies an equation $s=t$, denoted $\mathbb{A} \vDash s=t$, if for any $\iota: X \rightarrow A, \llbracket s \rrbracket^{\iota}=\llbracket t \rrbracket^{\iota}$. We write $\mathbb{A} \vDash^{\iota} s=t$ when the equality holds for a particular interpretation $\iota$. Given a set $E$ of equations over $\Sigma$, we denote by $\operatorname{Alg}(\Sigma, E)$ the full subcategory of $\operatorname{Alg}(\Sigma)$ of all algebras that satisfy all equations in $E$.

Definition 2.9. A congruence relation on $\mathbb{A}=\left(A, \llbracket \Sigma \rrbracket_{A}\right) \in \operatorname{Alg}(\Sigma)$ is an equivalence relation $R \subseteq A^{2}$ such that for every op : $n \in \Sigma$, if $\left(a_{1}, b_{1}\right) \in R, \ldots$, $\left(a_{n}, b_{n}\right) \in R$ then it holds that $\left(\llbracket \mathrm{op} \rrbracket_{A}\left(a_{1}, \ldots, a_{n}\right), \llbracket \mathrm{op} \rrbracket_{A}\left(b_{1}, \ldots, b_{n}\right)\right) \in R$. If $R$ is a congruence then the interpretation of each op $\in \Sigma$ is well-defined on the set $A / R$ of $R$-equivalence classes, by:

$$
\left.\llbracket \mathrm{op} \rrbracket_{A / R}\left(\left[a_{1}\right]_{R}, \ldots,\left[a_{n}\right]_{R}\right)=\left[\llbracket \mathrm{op} \rrbracket_{A}\left(a_{1}, \ldots, a_{n}\right)\right)\right]_{R}
$$

Then we have the algebra $\mathbb{A} / R=\left(A / R, \llbracket \Sigma \rrbracket_{A / R}\right)$.
Definition 2.10 (Term monad, with equations). Let $\Sigma$ be a signature, $E$ a set of equations over $\Sigma$, and $A$ a set. Denote with $\equiv_{E_{A}}$ the smallest congruence on the term algebra $T_{\Sigma} A$ such that $\left(T_{\Sigma} A\right) / \equiv_{E_{A}} \in \mathbf{A l g}(\Sigma, E)$, i.e., $\left(T_{\Sigma} A\right) / \equiv_{E_{A}}$ satisfies all equations in $E$. We define a variant of the term monad denoted $T_{\Sigma, E}$ that sends a set $A$ to $T_{\Sigma} A / \equiv_{E_{A}}$. Given a function $f: A \rightarrow B$, we define the function $T_{\Sigma, E} f: T_{\Sigma, E} A \rightarrow T_{\Sigma, E} B$ using the already defined $T_{\Sigma} f$ : for any $t \in T_{\Sigma} A, T_{\Sigma, E} f\left([t]_{\equiv_{E_{A}}}\right)=\left[T_{\Sigma} f(t)\right]_{\equiv_{E_{B}}}$. One can check that $T_{\Sigma, E} f$ is well-defined and makes $T_{\Sigma, E}$ into a functor. In fact, it is a monad with unit $\eta_{A}^{\Sigma, E}=a \mapsto[a]_{\equiv_{E_{A}}}$ and multiplication

$$
\mu_{A}^{\Sigma, E}=\left[t\left(\left[t_{1}\right]_{\equiv_{E_{A}}}, \ldots,\left[t_{n}\right]_{\equiv_{E_{A}}}\right)\right]_{\equiv_{E_{T_{\Sigma, E^{A}}}}} \mapsto\left[t\left(t_{1}, \ldots, t_{n}\right)\right]_{\equiv_{E_{A}}} .
$$

We call $\left(T_{\Sigma, E}, \eta^{\Sigma, E}, \mu^{\Sigma, E}\right)$ the term monad for $(\Sigma, E)$.
Proposition 2.11. For any signature $\Sigma$ and any set $E$ of equations over $\Sigma, \operatorname{Alg}(\Sigma, E) \cong$ $\operatorname{EM}\left(T_{\Sigma, E}\right)$.

A corollary of the above proposition is that the free $(\Sigma, E)$-algebra over a set $A$ is $\left(T_{\Sigma} A / \equiv_{E_{A}}, \llbracket \Sigma \rrbracket\right)$, with the canonical interpretation of operations:

$$
\llbracket \mathrm{op} \rrbracket\left(\left[t_{1}\right] \equiv_{E_{A}}, \cdots,\left[t_{n}\right] \equiv_{E_{A}}\right)=\left[\mathrm{op}\left(t_{1}, \ldots, t_{n}\right)\right]_{E_{E_{A}}}
$$

### 2.3 Generalized Metric Spaces

Definition 2.12 (FRel). A fuzzy relation on a set $A$ is a map $d: A \times A \rightarrow$ $[0,1]$. A morphism between two fuzzy relations $(A, d)$ and $(B, \Delta)$ is a map $f: A \rightarrow B$ that is nonexpansive (also referred to as 1-Lipschitz) namely, $\forall a, a^{\prime} \in$ $A, \Delta\left(f(a), f\left(a^{\prime}\right)\right) \leq d\left(a, a^{\prime}\right)$. We denote by FRel the category of fuzzy relations and nonexpansive maps.

Here is a non-exhaustive ${ }^{1}$ list of constraints on fuzzy relations that have been considered in the literature.

$$
\begin{align*}
\forall a, b \in A, & d(a, b)=d(b, a)  \tag{1}\\
\forall a \in A, & d(a, a)=0  \tag{2}\\
\forall a, b \in A, & d(a, b)=0 \Longrightarrow a=b  \tag{3}\\
\forall a, b, c \in A, & d(a, c) \leq d(a, b)+d(b, c)  \tag{4}\\
\forall a, b, c \in A, & d(a, c) \leq \max \{d(a, b), d(b, c)\} \tag{5}
\end{align*}
$$

Each has a somewhat standard name, (1) is symmetry, (2) is indiscernibility of identicals or reflexivity, (3) is identity of indiscernibles, (4) is triangle inequality, and (5) is strong triangle inequality. Restricting FRel to relations that satisfy a subset of the axioms above, we get many categories of interest whose objects were studied at least once in the literature.


For example, metrics (Met) are fuzzy relations that satisfy axioms (1)-(4), pseudometrics (PMet) satisfy (1), (2) and (4), and diffuse metrics (DMet) satisfy (1) and (4). Other examples include: quasimetrics (QMet), pseudoquasimetrics (PQMet), metametrics (MMet), semimetrics (SMet), pseudosemimetrics (PSMet), ultrametrics (UMet). Different notions of morphisms between these objects have been considered (e.g.: continuous functions, contracting maps, etc.) but, for our purposes, we will work with full subcategories of FRel and hence keep nonexpansiveness as the only condition on morphisms. This choice implies that isomorphisms of fuzzy relations are bijections that preserve distances. In the sequel, we write GMet for a category of generalized metric spaces, which

[^0]can stand for any full subcategory of FRel satisfying a fixed subset of axioms (1)-(5).

All products and coproducts exist in GMet and are easy to define. Let $\left\{\left(A_{i}, d_{i}\right) \mid i \in I\right\}$ be a non-empty family of generalized metric spaces. The product is $\left(\prod_{i \in I} A_{i}, \sup _{i \in I} d_{i}\right)$, with $\sup _{i \in I} d_{i}:\left(\prod_{i \in I} A_{i}\right) \times\left(\prod_{i \in I} A_{i}\right) \rightarrow[0,1]$ defined for $\vec{a}, \vec{b} \in \prod_{i \in I} A_{i}$ as:

$$
\left(\sup _{i \in I} d_{i}\right)(\vec{a}, \vec{b})=\sup _{i \in I} d_{i}\left(\vec{a}_{i}, \vec{b}_{i}\right) .
$$

We denote the sup-metric $\sup _{i \in I} d_{i}$ just as $d_{\times}$when the index set $I$ is clear. The coproduct is given by $\amalg_{i \in I} d_{i}:\left(\coprod_{i \in I} A_{i}\right) \times\left(\coprod_{i \in I} A_{i}\right) \rightarrow[0,1]$, defined for $a \in A_{j}$ and $b \in A_{k}$ as:

$$
\left(\amalg_{i \in I} d_{i}\right)(a, b)= \begin{cases}d_{j}(a, b) & \text { if } j=k \\ 1 & \text { otherwise }\end{cases}
$$

The empty product, i.e., the terminal object, is given by $d_{1}:\{*\} \times\{*\} \rightarrow[0,1]$, defined by

$$
d_{\mathbf{1}}(*, *)= \begin{cases}0 & \text { if constraint }(2) \text { holds in GMet } \\ 1 & \text { otherwise }\end{cases}
$$

The empty coproduct, i.e., the initial object, is the only possible fuzzy relation on the empty set (which vacuously satisfies all the axioms that must hold in GMet).

Definition 2.13 (Isometric embedding). A nonexpansive map $f:(A, d) \rightarrow$ $(B, \Delta)$ is an isometry if for any $a, a^{\prime} \in A, \Delta\left(f(a), f\left(a^{\prime}\right)\right)=d\left(a, a^{\prime}\right)$. An isometric embedding is an isometry that is injective. ${ }^{2}$ For any generalized metric space $(A, d)$ and subset $A^{\prime} \subseteq A$, the inclusion $i:\left(A^{\prime},\left.d\right|_{A^{\prime}}\right) \rightarrow(A, d)$ is an isometric embedding.

## 3 Quantitative Reasoning with Liftings

We introduce in this section our novel framework. In Subsection 3.1 we present the notion of liftings of signatures, the associated concept of quantitative $\widehat{\Sigma}$ algebras and define the classes $\operatorname{Alg}(\widehat{\Sigma}, S)$ definable by sets of implicational axioms. In Subsection 3.2 we define the syntactical deductive apparatus used to reason about equality and distance in quantitative $\widehat{\Sigma}$-algebras, and prove the soundness theorem, stating that the syntactic apparatus guarantees correct derivations.

[^1]
### 3.1 Generalized Quantitative Algebras

In what follows, a given category GMet is fixed.
Definition 3.1. Given functors $F:$ Set $\rightarrow$ Set and $L:$ GMet $\rightarrow$ GMet we say that $L$ is a lifting of $F$ (from Set to GMet) if the following diagram commute, where $U$ is the expected forgetful functor:


Hence, for any lifting $L$, on objects we have $L(A, d)=\left(F(A), d^{\prime}\right)$ for some $d^{\prime}$ which we denote with $d^{\prime}=L(d)$. We will interchangeably use both notations $L(A, d)$ and $(F(A), L(d))$.

Definition 3.2. A lifting $L$ preserves isometric embeddings if, whenever $f:(A, d) \rightarrow$ $(B, \Delta)$ is an isometric embedding then $L(f): L(A, d) \rightarrow L(B, \Delta)$ is also an isometric embedding.

Informally, this property holds when $L$ is compatible with the operation of taking subspaces. In the rest of this paper, we will be only interested in liftings that preserve isometric embeddings and often just refer to them as liftings.

Example 3.3. Take as GMet the category Met of metric spaces. Consider the functor $F=$ id. Then, for any lifting $L,(A, d) \stackrel{L}{\mapsto}(A, L(d))$, so $L(d)$ is a distance on $A$. As examples of liftings of $F$ preserving isometric embeddings, we list:

1. the identity: $L(d)\left(a, a^{\prime}\right)=d\left(a, a^{\prime}\right)$,
2. the scaling: $L(d)\left(a, a^{\prime}\right)=r \cdot d\left(a, a^{\prime}\right)$ for $r \in(0,1)$,
3. the discrete distance: $L(d)\left(a, a^{\prime}\right)=1$ if $a \neq a^{\prime}$.

Similarly, consider $F=(-)^{2}$, i.e., $F(A)=A \times A$ and $F(f)=f \times f$. In this case, $L(d)$ is a distance on $A \times A$. Examples of liftings preserving isometric embeddings include:

- the standard product distance: $L(d)\left(\left(a_{1}, a_{1}^{\prime}\right),\left(a_{2}, a_{2}^{\prime}\right)\right)=\max \left\{d\left(a_{1}, a_{2}\right), d\left(a_{1}^{\prime}, a_{2}^{\prime}\right)\right\}$,
- the discrete distance: $L(d)\left(\left(a_{1}, a_{1}^{\prime}\right),\left(a_{2}, a_{2}^{\prime}\right)\right)=1$ if $\left(a_{1}, a_{1}^{\prime}\right) \neq\left(a_{2}, a_{2}^{\prime}\right)$.

We note, as in some of the examples above, that for any GMet, if $F$ is the $n$-ary product endofunctor $(-)^{n}$ on Set, then the $n$-ary product $d_{\times}$in GMet is a lifting of $F$ preserving isometric embeddings. We refer to it as the supproduct lifting and denote it by $L_{\times}$. Accordingly, for $n=0$, the lifting $L_{\times}$maps any object to the terminal object in GMet and, for $n=1$, the lifting $L_{\times}$is the identity functor.

Definition 3.4 (Nonexpansiveness up to lifting). Let $F:$ Set $\rightarrow$ Set and $L$ a lifting of $F$. Let $(A, d)$ and $(B, \Delta)$ in GMet. We say that a function $f: F(A) \rightarrow$ $B$ is nonexpansive up to $L$ (or $L$-nonexpansive) if $f:(F(A), L(d)) \rightarrow(B, \Delta)$ is nonexpansive (i.e., it is a morphism in GMet).

Example 3.5. As in the previous example, fix GMet $=$ Met and consider $F=$ id. Consider the metric space $([0,1], d)$, the unit interval with its standard Euclidean metric (i.e. $d(x, y)=|x-y|$ ), and the map $f: F([0,1]) \rightarrow[0,1]$ defined as $f(x)=x^{2}$. If we take as lifting of $F$ the identity lifting $L$ from Example 3.3 (i.e., the lifting $L_{\times}$) the function $f$ is not $L$-nonexpansive because, e.g., $\frac{4}{10}=d\left(\frac{6}{10}, 1\right)<d\left(\left(\frac{6}{10}\right)^{2}, 1^{2}\right)=\frac{64}{100}$. By contrast, if we take as lifting $L$ the discrete lifting then $f$ is trivially $L$-nonexpansive. In fact, any function $f:[0,1] \rightarrow[0,1]$ is nonexpansive up to the discrete lifting.

We are now ready to introduce the concept of lifted signature, which extends the usual notion of signature $\Sigma$ from universal algebra.

Definition 3.6 (Lifted signature). Given a signature $\Sigma=\left\{\mathrm{op}_{i}: n_{i}\right\}_{i \in I}$, a lifting of $\Sigma$ to GMet is a choice, for each $i \in I$, of lifting $L_{\mathrm{op}_{i}}$ of the $n_{i}$-ary product $(-)^{n_{i}}$ : Set $\rightarrow$ Set. An operation symbol op with arity $n$ and associated lifting $L_{\mathrm{op}}$ is now denoted op : $n: L_{\mathrm{op}}$. We denote lifted signatures $\widehat{\Sigma}=\left\{\mathrm{op}_{i}: n_{i}:\right.$ $\left.L_{\mathrm{op}_{i}}\right\}_{i \in I}$ to clearly distinguish them from ordinary signatures.

Note that, given any signature $\Sigma$, it is possible to obtain a lifted signature $\widehat{\Sigma}$ by choosing, for each op : $n \in \Sigma$, the sup-product lifting $L_{\times}$of $(-)^{n}$.

As in the classical case, any lifted signature $\widehat{\Sigma}$ gives rise to an endofunctor on GMet (denoted $\widehat{\Sigma}$ too) with the following action:

$$
\widehat{\Sigma}(A, d):=\coprod_{\text {op: } n: L_{\mathrm{op}} \in \widehat{\Sigma}} L_{\mathrm{op}}(A, d) \quad \widehat{\Sigma}(f):=\coprod_{\text {op:}: n: L_{\mathrm{op}} \in \hat{\Sigma}} L_{\mathrm{op}}(f)
$$

Definition 3.7 (Quantitative $\widehat{\Sigma}$-algebra). A quantitative $\widehat{\Sigma}$-algebra is an algebra for the functor $\widehat{\Sigma}$. Equivalently, it is a generalised metric space $(A, d) \in \mathbf{G M e t}$ equipped with a set $\llbracket \widehat{\Sigma} \rrbracket_{A}$ of interpretations of operation symbols, as follows: every op : $n: L_{\mathrm{op}} \in \widehat{\Sigma}$ is interpreted as a map $\llbracket \mathrm{op} \rrbracket_{A}: A^{n} \rightarrow A$ which is $L_{\text {op }}-$ nonexpansive, i.e., such that

$$
\llbracket \mathrm{op} \rrbracket_{A}:\left(A^{n}, L_{\mathrm{op}}(d)\right) \rightarrow(A, d) \text { is nonexpansive. }
$$

We call $(A, d)$ the carrier space. A homomorphism between two quantitative $\widehat{\Sigma}$ algebras with carrier spaces $(A, d)$ and $(B, \Delta)$ is a nonexpansive map $f: A \rightarrow B$ preserving all operations, i.e., $\forall \mathrm{op}: n: L_{\mathrm{op}} \in \widehat{\Sigma}$ and $\forall a_{1}, \ldots, a_{n} \in A$,

$$
f\left(\llbracket \mathrm{op} \rrbracket_{A}\left(a_{1}, \ldots, a_{n}\right)\right)=\llbracket \mathrm{op} \rrbracket_{B}\left(f\left(a_{1}\right), \ldots, f\left(a_{n}\right)\right) .
$$

The category of quantitative $\widehat{\Sigma}$-algebras is denoted $\operatorname{Alg}(\widehat{\Sigma})$.

We remark that, in the particular case of GMet $=$ Met and $\widehat{\Sigma}$ being the sup-product lifting of some signature $\Sigma$, the notion of quantitative $\widehat{\Sigma}$-algebra coincides with that of quantitative algebra for the signature $\Sigma$ of the framework of Bacci et al. [2018b].

Any quantitative $\widehat{\Sigma}$-algebra yields a $\Sigma$-algebra by applying the forgetful functor to $U$ : GMet $\rightarrow$ Set because $U \llbracket$ op $\rrbracket_{A}$ has type $A^{n} \rightarrow A$ and morphisms in $\operatorname{Alg}(\widehat{\Sigma})$ are already $\Sigma$-algebra homomorphisms. We obtain the following commutative square of forgetful functors.


Definition 3.8 (Equations). Given a quantitative $\widehat{\Sigma}$-algebra $\mathbb{A}:=(A, d, \llbracket \widehat{\Sigma} \rrbracket)$ and an equation $e \in T_{\Sigma} X \times T_{\Sigma} X$, we say that $\mathbb{A}$ satisfies $e$, denoted $\mathbb{A} \vDash e$, if its underlying $\Sigma$-algebra satisfies $e$.
Definition 3.9 (Quantitative equation). A quantitative equation in the signature $\widehat{\Sigma}$ is an element $e \in T_{\Sigma} X \times T_{\Sigma} X \times[0,1]$, i.e. a triple comprising two $\Sigma$-terms $s$ and $t$ and a real number $\varepsilon \in[0,1]$. We denote it $s=_{\varepsilon} t$. We say that $\mathbb{A}:=$ $(A, d, \llbracket \widehat{\Sigma} \rrbracket)$ satisfies $s={ }_{\varepsilon} t$, denoted $\mathbb{A} \vDash s={ }_{\varepsilon} t$, if for any variable assignment $\iota: X \rightarrow A, d\left(\llbracket s \rrbracket^{\iota}, \llbracket t \rrbracket^{\iota}\right) \leq \varepsilon$. We write $A \vDash^{\iota} s={ }_{\varepsilon} t$ when the inequality holds for a particual assignment $l$.

Let $\mathcal{V}_{\Sigma} X=T_{\Sigma} X \times T_{\Sigma} X \cup T_{\Sigma} X \times T_{\Sigma} X \times[0,1]$ denote the set of equations and quantitative equations over the signature $\Sigma$ and variables $X$. We use the letter $\phi$ to range over $\mathcal{V}_{\Sigma} X$.

Following Bacci et al. [2018b], we will consider classes of $\widehat{\Sigma}$-algebras axiomatised by (quantitative) equational implications, rather than just (quantitative) equations. While this level of generality is not required in many applications, as several useful examples (see Section 5) are purely (quantitative) equational, it allows for a direct comparison of our results and those of Bacci et al. [2018b].

Definition 3.10 (Horn clauses). In the sequel, we denote $\mathcal{H}_{\Sigma}(X)=\mathcal{P}\left(\mathcal{V}_{\Sigma} X\right) \times$ $\mathcal{V}_{\Sigma} X$ the set of (possibly infinitary) Horn clauses over the signature $\Sigma$ and variables $X$. A Horn clause $H \in \mathcal{H}_{\Sigma}(X)$ is denoted $\bigwedge_{i \in I} \phi_{i} \Rightarrow \phi$ as its intended semantics is that $\phi$ holds whenever each $\phi_{i}$ holds. More formally, we say that an algebra $\mathbb{A}=\left(A, d, \llbracket \widehat{\Sigma} \rrbracket_{A}\right) \in \operatorname{Alg}(\widehat{\Sigma})$ satisfies a clause $H=\bigwedge_{i \in I} \phi_{i} \Rightarrow \phi$, denoted $\mathbb{A} \vDash H$, if for any variable assignment $\iota: X \rightarrow A, \mathbb{A} \vDash^{\iota} \phi$ whenever $\mathbb{A} \vDash^{\iota} \phi_{i}$ for every $i$. We write $\mathbb{A} \vDash^{\iota} H$ when the implication is true for a particular assignment $\iota$. We call $H=\bigwedge_{i \in I} \phi_{i} \Rightarrow \phi$ basic if each premise $\phi_{i}$ is a (quantitative) equation between variables: $\phi_{i}$ is either of the form $x=y$ or $x={ }_{\varepsilon} y$, for $x, y \in X$ and $\varepsilon \in[0,1]$.

Given a set $S \subseteq \mathcal{H}_{\Sigma}(X)$, we denote $\operatorname{Alg}(\widehat{\Sigma}, S)$ the full subcategory of $\operatorname{Alg}(\widehat{\Sigma})$ containing all quantitative $\widehat{\Sigma}$-algebras that satisfy all clauses in $S$.

### 3.2 Syntactic Apparatus for Quantitative Reasoning

Following Bacci et al. [2018b], we now introduce a logical apparatus for reasoning about quantitative $\widehat{\Sigma}$-algebras. We use the following notation to improve readability: for a set $\vdash$ of Horn clauses $\left(\vdash \subseteq \mathcal{H}_{\Sigma}(X)\right.$ ) we write $\left\{\phi_{i}\right\}_{i \in I} \vdash \phi$ to denote that the Horn clause $\bigwedge_{i \in I} \phi_{i} \Rightarrow \phi$ belongs to the set $\vdash$.

Definition 3.11. A quantitative theory over $\widehat{\Sigma}$ is a set of Horn clauses $\vdash \subseteq \mathcal{H}_{\Sigma}(X)$ such that conditions (I)-(VI) hold:
(I) $\vdash$ is closed under the following inference rules for any $\Gamma, \Gamma^{\prime} \subseteq \mathcal{V}_{\Sigma} X, \phi, \psi \in$ $\mathcal{V}_{\Sigma} X$ and substitution $\sigma: X \rightarrow T_{\Sigma} X$ :

$$
\begin{gathered}
\frac{\Gamma \vdash \phi}{\sigma^{*}(\Gamma) \vdash \sigma^{*}(\phi)} \text { Sub } \\
\frac{\forall \phi \in \Gamma^{\prime}, \Gamma \vdash \phi \quad \Gamma^{\prime} \vdash \psi}{\Gamma \vdash \psi} \text { Cut } \frac{\phi \in \Gamma}{\Gamma \vdash \phi} \mathrm{Hyp}
\end{gathered}
$$

(II) $\vdash$ contains, for any op : $n: L_{\mathrm{op}} \in \widehat{\Sigma}$ and $x, y, z \in X$, the clauses:

$$
\begin{array}{rr}
\text { (Refl) } & \varnothing \vdash x=x \\
\text { (Sym) } & x=y \vdash y=x \\
\text { (Trans) } & x=y, y=z \vdash x=z \\
\text { (App) } & \left\{x_{i}=y_{i} \mid i \in 1, \ldots, n\right\} \vdash \mathrm{op}(\vec{x})=\mathrm{op}(\vec{y}) \tag{Trans}
\end{array}
$$

(III) $\vdash$ contains, for any $x, y \in X, \varepsilon^{\prime} \geq \varepsilon, \varepsilon_{i} \in[0,1]$, the clauses:

$$
\begin{aligned}
\text { (1-bdd) } & \varnothing \vdash x \\
\text { (Max) } & ={ }_{1} y \\
\text { (Arch) } & x={ }_{\varepsilon} y \vdash x=\varepsilon_{\varepsilon^{\prime}} y \\
& \left\{x=_{\varepsilon_{i}} y \mid i \in I\right\} \vdash x==_{\inf \left\{\varepsilon_{i} \mid i \in I\right\}} y
\end{aligned}
$$

(IV) $\vdash$ contains, for any $x, y, z \in X, \varepsilon \in[0,1]$, the clauses:

$$
\begin{array}{ll}
\left(\operatorname{Comp}_{\ell}\right) & x=y, x={ }_{\varepsilon} z \vdash y={ }_{\varepsilon} z \\
\left(\mathrm{Comp}_{r}\right) & x=y, z={ }_{\varepsilon} x \vdash z={ }_{\varepsilon} y
\end{array}
$$

(V) depending on the notion of GMet used, $\vdash$ contains an appropriate subset of the following clauses for any $x, y, z \in X, \varepsilon, \varepsilon^{\prime} \in[0,1]$ :

$$
\begin{gather*}
x={ }_{\varepsilon} y \vdash y={ }_{\varepsilon} x  \tag{1}\\
\varnothing \vdash x={ }_{0} x  \tag{2}\\
x==_{0} y \vdash x=y  \tag{3}\\
x={ }_{\varepsilon} y, y={ }_{\varepsilon^{\prime}} z \vdash x={ }_{\varepsilon+\varepsilon^{\prime}} z  \tag{4}\\
x={ }_{\varepsilon} y, y={ }_{\varepsilon^{\prime}} z \vdash x={ }_{\max \left\{\varepsilon, \varepsilon^{\prime}\right\}} z \tag{5}
\end{gather*}
$$

(VI) $\vdash$ is closed under the following inference rule, for any op : $n: L_{\text {op }} \in \widehat{\Sigma}$ and for any set $\vec{x} \cup \vec{y}=\left\{x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right\}$ of up to $2 n$ variables (not necessarily distinct):

$$
\frac{(\vec{x} \cup \vec{y}, \Delta) \in \text { GMet } \quad \delta=L_{\mathrm{op}}(\Delta)(\vec{x}, \vec{y})}{\left\{w=_{\Delta(w, z)} z \mid w, z \in \vec{x} \cup \vec{y}\right\} \vdash \mathrm{op}(\vec{x})=_{\delta} \mathrm{op}(\vec{y})} L-\mathrm{NE}
$$

Condition (I) is standard and reflects the semantics of $\vdash$ as a theory of universally quantified implications. Condition (II) includes the standard axioms of equational logic, thus (I)+(II) allows to perform equational reasoning regarding equations $(s=t)$. Condition (III) poses the constraints on quantitative equations ( $s=\varepsilon t$ ) ensuring the intended semantics: $d(s, t) \leq \varepsilon$, for any fuzzy relation $d \in$ FRel. Condition (IV) adds two axioms governing the logical interplay between equality $(=)$ and the quantitative relations $=_{\varepsilon}$. It expresses the fact that equality is a congruence relation (both on the left and the right argument) for the relation $=_{\varepsilon}$, for all $\varepsilon \in[0,1]$. Condition (V) adds to the deductive system the implicational axioms defining each category of generalised metric spaces GMet. Finally, Condition (VI) expresses the property that, for any op $: n: L_{\mathrm{op}} \in \widehat{\Sigma}$, the operation op is $L_{\mathrm{op}}$-nonexpansive. The Horn clause introduced has up to $(2 n)^{2}$ premises: quantitative equations of the form $w={ }_{\Delta(w, z)} z$, where $\Delta(w, z)$ is a number in $[0,1]$, for each choice of $w, z \in \vec{x} \cup \vec{y}$. We can see these numbers as defining a fuzzy relation $\Delta: \vec{x} \cup \vec{y} \rightarrow[0,1]$. The proviso requires that $(\vec{x} \cup \vec{y}, \Delta)$ is a GMet space. This is therefore a constraint on the $(2 n)^{2}$ values $\Delta(w, z)$. If the proviso is satisfied, since $L$ is a GMet lifting, $L_{\mathrm{op}}(\vec{x} \cup \vec{y}, \Delta)$ is a GMet space too and the value in the quantitative equation in the conclusion (i.e., $\delta=L_{\mathrm{op}}(\Delta)(\vec{x}, \vec{y})$ ) is defined.
Example 3.12. In order to improve readability when displaying instances of the $L$-NE rule, we will often omit some of the $(2 n)^{2}$ premises $w=_{\Delta(w, z)} z$ when $\Delta(w, z)$ is implicitly understood from the context. For instance, consider the case GMet $=$ Met and a binary operation op :2: $L_{\times}$with $L_{\times}$the sup-product lifting. An instance of the $L-N E$ rule is:

$$
x_{1}=\varepsilon_{\varepsilon_{1}} y_{1}, x_{2}=\varepsilon_{2} y_{2} \vdash \mathrm{op}\left(x_{1}, x_{2}\right)={\max \left\{\varepsilon_{1}, \varepsilon_{2}\right\}}^{o p}\left(y_{1}, y_{2}\right)
$$

thus implicitly assuming all other premises to be of the form $w={ }_{1} z$ if $w \neq z$, and $w==_{0} z$ otherwise, for $w, z \in \vec{x} \cup \vec{y}$. The fuzzy relation on $\vec{x} \cup \vec{y}$ described by these premises is therefore:

which indeed satisfies the axioms of Met. Since we are considering the supproduct lifting, $\delta=L_{\times}(\Delta)\left(\left(x_{1}, x_{2}\right),\left(y_{1}, y_{2}\right)\right)=\max \left\{\varepsilon_{1}, \varepsilon_{2}\right\}$. Hence all provisos
of the $L-N E$ rule are satisfied, meaning that this is a valid instance of the $L-N E$ rule.

Definition 3.13. Given a set of clauses $S \subseteq \mathcal{H}_{\Sigma}(X)$, we let $\vdash_{S}$ denote the smallest quantitative theory containing $S$, and refer to it as the GMet quantitative theory axiomatised by $S$.

Our first main result is the soundness theorem, stating that if a Horn clause $H$ is derivable in the deductive apparatus from an axiom set $S$ of Horn clauses (i.e., $H$ is in the quantitative theory axiomatised by $S$ ), then indeed $H$ holds true in any $\widehat{\Sigma}$ algebra satisfying the axioms $S$.

Theorem 3.14 (Soundness). Let $\mathbb{A}=(A, d, \llbracket \widehat{\Sigma} \rrbracket) \in \mathbf{A l g}(\widehat{\Sigma}, S)$ and $H \in \vdash_{s}$. Then $\mathbb{A} \vDash H$.

Proof. We show that each rule in Definition 3.11 is valid in $\mathbb{A}$.
(I) The inference rules Sub, Cut and Hyp are valid by purely logical arguments, as the semantics of clauses $\left\{\phi_{i}\right\}_{i \in I} \vdash_{S} \phi$ are universally quantified implications: $\forall \vec{x}$. $\left(\bigwedge_{i \in I} \phi_{i} \Rightarrow \phi\right)$.
(II) The clauses Refl, Sym, Trans and App are valid because equality ( $=$ ) is an equivalence relation and is trivially compatible with all operations op $\in \widehat{\Sigma}$ (i.e., it is a congruence).
(III) The clauses 1-bdd, Max and Arch are valid because the distance $d$ has type $d: A \times A \rightarrow[0,1]$ and the interpretation of quantitative equations $x={ }_{\varepsilon} y$ is $d(\iota(x), \iota(y)) \leq \varepsilon$ for any variable assignment $\iota: X \rightarrow A$.
(IV) The rules Comp and $\mathrm{Comp}_{r}$ are valid because equality ( $=$ ) is trivially a congruence for all relations $={ }_{\varepsilon}$.
(V) The clauses corresponding to axioms in GMet are valid because $\mathbb{A}=$ $(A, d, \llbracket \widehat{\Sigma} \rrbracket)$ is in GMet, and the interpretation of the Horn clauses (universally quantified implications) coincides with the axioms of GMet as stated in Section 2.3.
(VI) The inference $L-$ NE has a proviso $((\vec{x} \cup \vec{y}, \Delta) \in \mathbf{G M e t})$ stating that the finite set $\vec{x} \cup \vec{y}$ of variables endowed with the distances $\Delta(w, z)$, for $w, z \in \vec{x} \cup \vec{y}$, is a GMet space. Assume this as hypothesis. Since op : $n: L_{\mathrm{op}} \in \widehat{\Sigma}$, we know that $L_{\mathrm{op}}$ is a lifting on GMet. Therefore the set $(\vec{x} \cup \vec{y})^{n}$ equipped with the distance $L_{\mathrm{op}}(\Delta)$ is an element of GMet. Hence, the numerical value

$$
\delta=L_{\mathrm{op}}(\Delta)\left(\left(x_{1} \ldots, x_{n}\right),\left(y_{1}, \ldots, y_{n}\right)\right)
$$

is defined. We need to prove that the Horn clause

$$
\begin{equation*}
\left\{w=_{\Delta(w, z)} z \mid w, z \in \vec{x} \cup \vec{y}\right\} \vdash \mathrm{op}(\vec{x})={ }_{\delta} \operatorname{op}(\vec{y}) \tag{7}
\end{equation*}
$$

holds in $\mathbb{A}$. Let $\iota: X \rightarrow A$ be an assignment and assume that, for all $w, z \in$ $\vec{x} \cup \vec{y}$,

$$
d\left(\llbracket w \rrbracket^{\iota}, \llbracket z \rrbracket^{\iota}\right) \leq \Delta(w, z)
$$

holds. As consequence, the map

$$
f:(\vec{x} \cup \vec{y}, \Delta) \rightarrow(A, d)=w \mapsto \llbracket w \rrbracket^{\iota}
$$

is nonexpansive. Thus, the lifting

$$
L_{\mathrm{op}}(f): L_{\mathrm{op}}(\vec{x} \cup \vec{y}, \Delta) \rightarrow L_{\mathrm{op}}(A, d)
$$

i.e.,

$$
\begin{equation*}
L_{\mathrm{op}}(f):\left((\vec{x} \cup \vec{y})^{n}, L_{\mathrm{op}}(\Delta)\right) \rightarrow\left(A^{n}, L_{\mathrm{op}}(d)\right) \tag{8}
\end{equation*}
$$

is also nonexpansive, and we have the following derivation which implies (7), i.e., the validity of the conclusion:

$$
\begin{align*}
& d\left(\llbracket \mathrm{op}(\vec{x}) \rrbracket^{l}, \llbracket \mathrm{op}(\vec{y}) \rrbracket^{\iota}\right) \\
& =d\left(\llbracket \mathrm{op} \rrbracket\left(\llbracket x_{1} \rrbracket^{l}, \ldots, \llbracket x_{n} \rrbracket^{l}\right), \llbracket \mathrm{op} \rrbracket\left(\llbracket y_{1} \rrbracket^{\iota}, \ldots, \llbracket y_{n} \rrbracket^{l}\right)\right) \\
& \leq L_{\mathrm{op}}(d)\left(\left(\llbracket x_{1} \rrbracket^{l}, \ldots, \llbracket x_{n} \rrbracket^{\iota}\right),\left(\llbracket y_{1} \rrbracket^{\iota}, \ldots, \llbracket y_{n} \rrbracket^{l}\right)\right)  \tag{A}\\
& =L_{\mathrm{op}}(d)\left(L_{\mathrm{op}}(f)(\vec{x}), L_{\mathrm{op}}(f)(\vec{y})\right)  \tag{B}\\
& \leq L_{\mathrm{op}}(\Delta)(\vec{x}, \vec{y})  \tag{C}\\
& =\delta
\end{align*}
$$

where $(A)$ applies the fact that $\llbracket \mathrm{op} \rrbracket$ is $L_{\mathrm{op}}-$ nonexpansive, $(B)$ follows as $L_{\mathrm{op}}(f)$ applies $f$ pointwise to $n$-ary tuples, and (C) uses nonexpansiveness of $L_{\mathrm{op}}(f)$ from (8).

## 4 Term Monad and Free Quantitative Algebras

Given a lifted signature $\widehat{\Sigma}$ and a set of Horn clauses $S$ axiomatising a theory $\vdash_{S}$, we describe in Subsection 4.1 the construction of the term ( $\widehat{\Sigma}, S$ )-algebra (denoted $\widehat{T}_{\widehat{\Sigma}, S}(A, d)$ ) on a given GMet space $(A, d)$. We then show (Theorem 4.5) how this yields a monad $\widehat{T}_{\widehat{\Sigma}, S}$ on GMet.

Next, in Subsection 4.2 we show two main results regarding this monad. First (Theorem 4.6), for any given $(A, d) \in \mathbf{G M e t}$, the algebra $\widehat{T}_{\widehat{\Sigma}, S}(A, d)$ is the free algebra in $\operatorname{Alg}(\widehat{\Sigma}, S)$ generated by $(A, d)$. Second (Theorem 4.7), if all Horn clauses in $S$ are basic (see Definition 3.10), then $\operatorname{Alg}(\widehat{\Sigma}, S) \cong \mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, S}\right)$.

The definition of the monad $\widehat{T}_{\widehat{\Sigma}, S}$ and the proof techniques used to establish the two theorems are inspired by those of Mardare et al. [2016], Bacci et al. [2018b]. In fact, the latter can be seen as special instances, in our framework, when GMet $=$ Met and all liftings in $\widehat{\Sigma}$ are sup-product liftings.

### 4.1 The Term Monad

Fix a quantitative theory $\vdash$ over a lifted signature $\widehat{\Sigma}$. The construction of the term monad is done via several steps. First, we consider the set of ground
terms, i.e., the set of terms without variables $\left(T_{\Sigma} \varnothing\right)$ and we define on them a congruence $\equiv_{\vdash}$ and a fuzzy relation $d_{\vdash}$ induced by the equations and quantitative equations in $\vdash$. We then show in Lemma 4.2 how these allow us to build a quantitative $\widehat{\Sigma}$-algebra over quotiented $T_{\Sigma} \varnothing$ terms.

Definition 4.1. We let $\mathrm{E}(\vdash)$ (resp. $\mathrm{QE}(\vdash)$ ) be the set of equations (resp. quantitative equations) over $T_{\Sigma} \varnothing$ that are conclusions of Horn clauses $H \in \vdash$ having no premises. Formally:

$$
\begin{aligned}
& \mathrm{E}(\vdash)=\left\{s=t \mid \varnothing \vdash s=t, \text { for } s, t \in T_{\Sigma} \varnothing\right\} \\
& \mathrm{QE}(\vdash)=\left\{s={ }_{\varepsilon} t \mid \varnothing \vdash s={ }_{\varepsilon} t, \text { for } s, t \in T_{\Sigma} \varnothing\right\}
\end{aligned}
$$

Based on these, we define the following relation and fuzzy relation over $T_{\Sigma} \varnothing$ :

$$
\begin{gathered}
\equiv \vdash \subseteq T_{\Sigma} \varnothing \times T_{\Sigma} \varnothing \quad s \equiv_{\vdash} t \Leftrightarrow(s, t) \in \mathrm{E}(\vdash) \\
d_{\vdash}: T_{\Sigma} \varnothing \times T_{\Sigma} \varnothing \rightarrow[0,1] \quad d_{\vdash}(s, t)=\inf \left\{\varepsilon \mid s={ }_{\varepsilon} t \in \mathrm{QE}(\vdash)\right\}
\end{gathered}
$$

Lemma 4.2. The following hold:

1. The relation $\equiv_{\vdash}$ is an equivalence relation on $\Sigma$-terms without variables and is compatible with all operations.
2. $\left(T_{\Sigma} \varnothing / \equiv_{\vdash}, \llbracket \Sigma \rrbracket\right)$ is the free $\left(\Sigma, \equiv_{\vdash}\right)$-algebra on the empty set, with carrier $T_{\Sigma} \varnothing / \equiv_{\vdash}$ and operations $\llbracket \Sigma \rrbracket$ :
3. The fuzzy relation $d_{\vdash}$ satisfies the following properties:
(a) $d_{\vdash}(s, t) \leq \varepsilon$ if and only if $\left(s={ }_{\varepsilon} t\right) \in \mathrm{QE}(\vdash)$
(b) $d_{\vdash}$ preserves the equivalence $\equiv_{\vdash}$, i.e., $d_{\vdash}$ is well defined on $\equiv_{\vdash}-$ equivalence classes:

$$
d_{\vdash}: T_{\Sigma} \varnothing / \equiv \vdash \times T_{\Sigma} \varnothing / \equiv \vdash \rightarrow[0,1]
$$

4. $\left(T_{\Sigma} \varnothing / \equiv_{\vdash}, d_{\vdash}\right)$ is a GMet space.
5. $\left(T_{\Sigma} \varnothing / \equiv_{\vdash}, d_{\vdash}, \llbracket \Sigma \rrbracket\right)$ is a quantitative $\widehat{\Sigma}$-algebra.

Proof. All points are enforced by the presence of certain rules and clauses in the syntactic proof system, and the fact that $\vdash$, being a theory, is closed under them. Item 1 follows by Refl, Sym, Trans and App. Item 2 follows from the characterisation of free $\Sigma$-algebras from Subsection 2.2. Item 3a follows from Max and Arch, and 3 b from $\mathrm{Comp}_{\ell}$ and Comp $r$. Item 4 follows from the axioms in (V) corresponding to GMet.

Lastly, Item 5 is enforced by the $L-N E$ rule. We discuss this case in greater detail. We need to show that, for any op : $n: L_{\mathrm{op}} \in \widehat{\Sigma}$, the interpretation $\llbracket \mathrm{op} \rrbracket$

$$
\llbracket \mathrm{op} \rrbracket\left(\left[t_{1}\right]_{\equiv_{\vdash}}, \ldots,\left[t_{n}\right]_{\equiv_{\vdash}}\right)=\left[\mathrm{op}\left(t_{1}, \ldots, t_{n}\right)\right]_{\equiv_{\vdash}}
$$

is $L_{\mathrm{op}}$-nonexpansive. This means checking that

$$
\llbracket \mathrm{op} \rrbracket:\left(\left(T_{\Sigma} \varnothing / \equiv_{\vdash}\right)^{n}, L_{\mathrm{op}}\left(d_{\vdash}\right)\right) \rightarrow\left(T_{\Sigma} \varnothing / \equiv_{\vdash}, d_{\vdash}\right)
$$

is nonexpansive, i.e., that for any $\vec{s}=\left(s_{1}, \ldots, s_{n}\right)$ and $\vec{t}=\left(t_{1}, \ldots, t_{n}\right)$ in $\left(T_{\Sigma} \varnothing\right)^{n}$,

$$
\begin{equation*}
d_{\vdash}(\mathrm{op}(\vec{s}), \mathrm{op}(\vec{t})) \leq L_{\mathrm{op}}\left(d_{\vdash}\right)(\vec{s}, \vec{t}) \tag{9}
\end{equation*}
$$

Using the Sub rule, we instantiate the $L-\mathrm{NE}$ rule with premises

$$
p==_{\Delta(p, q)} q \text { for } p, q \in\left\{s_{1}, \ldots, s_{n}\right\} \cup\left\{t_{1}, \ldots, t_{n}\right\}
$$

where $\Delta(p, q)=d_{\vdash}(p, q)$. This set of premises satisfies the proviso of the $L-\mathrm{NE}$ rule, since $d_{\vdash}$ is a GMet relation (Item 4) because all premises are in $\mathrm{QE}(\vdash)$ (Item 3a). Hence, also the quantitative equation in the conclusion of the $L-\mathrm{NE}$ rule is in $\mathrm{QE}(\vdash)$ (apply Cut):

$$
\begin{equation*}
\mathrm{op}(\vec{s})=_{L_{\mathrm{op}}(\Delta)(\vec{s}, \vec{t})} \operatorname{op}(\vec{t}) \in \mathrm{QE}(\vdash) \tag{10}
\end{equation*}
$$

Now, since we have the isometric embedding

$$
\left(\left(\left\{s_{1}, \ldots, s_{n}\right\} \cup\left\{t_{1}, \ldots, t_{n}\right\}\right), \Delta\right) \hookrightarrow\left(T_{\Sigma} \varnothing, d_{\vdash}\right)
$$

and $L_{\mathrm{op}}$ preserves isometric embeddings, this implies

$$
L_{\mathrm{op}}(\Delta)(\vec{s}, \vec{t})=L_{\mathrm{op}}\left(d_{\vdash}\right)(\vec{s}, \vec{t}),
$$

and thus by Item 3a and (10), we conclude (9) holds.
We remark that the last step of this proof uses the technical assumption that liftings $L_{\mathrm{op}}$ preserve isometric embeddings. In contrast, the proof of Theorem 3.14 (soundness) can be carried out without this hypothesis. Therefore, this technical assumption is not needed to reason syntactically about equality and distance in quantitative algebras but is required to ensure that the construction of the term algebra (à la Mardare et al. [2016]) is valid. It is also used in the proof of Theorem 4.7.

Now, given a GMet space $(A, d)$, we aim at defining a $\widehat{\Sigma}$-algebra over terms generated from $A$ (i.e., $T_{\Sigma} A$ instead of $T_{\Sigma} \varnothing$ ), taking into account the distance on $A$ given by $d$. We do so via an extension of the theory $\vdash$.
Definition 4.3 (Theory Extension). Given a GMet space ( $A, d$ ), a lifted signature $\widehat{\Sigma}$ and a theory $\vdash$ over $\widehat{\Sigma}$, we define:

- a new lifted signature

$$
\widehat{\Sigma}_{A}=\widehat{\Sigma} \cup\left\{a: 0: L_{a} \mid a \in A\right\},
$$

where we add a fresh constant $a$ (of arity 0 ) for each element $a \in A$ where $L_{a}=L_{x}$, is the 0 -ary sup-product lifting from Example 3.3. Note that we can identify $T_{\Sigma} A$ ( $\Sigma$-terms with variables in $A$ ) with $T_{\Sigma_{A}} \varnothing\left(\Sigma_{A}\right.$-terms without variables).

- a new theory $\vdash_{A}$ over $\widehat{\Sigma}_{A}$ defined as the GMet theory generated by the set of clauses

$$
\vdash \cup\left\{\varnothing \vdash a=_{d\left(a, a^{\prime}\right)} a^{\prime} \mid\left(a, a^{\prime}\right) \in A \times A\right\},
$$

i.e., all the clauses in $\vdash$ and new ones describing the distances between the new constants in $A$.

We refer to $\widehat{\Sigma}_{A}$ as the signature $\widehat{\Sigma}$ extended by the GMet space $(A, d)$. Similarly, $\vdash_{A}$ is the theory $\vdash$ extended by $(A, d)$.

In what follows, we fix an axiom set of Horn clauses $S$ and the associated $\widehat{\Sigma}$-theory $\vdash_{S}$ axiomatised by $S$. Its extension by a GMet space $(A, d)$ is the theory $\vdash_{S_{A}}$ over $\widehat{\Sigma}_{A}$ whose term algebra (as in Lemma 4.2, Item 5) is

$$
\left(T_{\Sigma_{A}} \varnothing / \equiv \equiv_{\vdash_{S_{A}}}, d_{\vdash{ }_{S_{A}}}, \llbracket \Sigma_{A} \rrbracket\right)
$$

or, identifying $T_{\Sigma_{A}} \varnothing$ with $T_{\Sigma} A$, the $\widehat{\Sigma}_{A}$-algebra

$$
\left(T_{\Sigma} A / \equiv_{\vdash_{S_{A}}}, d_{\vdash_{S_{A}}}, \llbracket \Sigma_{A} \rrbracket\right)
$$

We can turn this into a $\widehat{\Sigma}$-algebra

$$
\left(T_{\Sigma} A / \equiv \equiv_{\vdash_{S_{A}}}, d_{\vdash_{S_{A}}}, \llbracket \Sigma \rrbracket\right)
$$

by forgetting the interpretations $\llbracket a \rrbracket$ of all constants $a \in A$. Since the set of Horn clauses $S$ is fixed, we introduce the following shortcuts to ease the notation:

$$
\equiv_{A}:=\equiv_{\vdash_{S_{A}}} \quad \widehat{T}_{\widehat{\Sigma}, S} A:=T_{\Sigma} A / \equiv_{A} \quad \widehat{T}_{\widehat{\Sigma}, S} d:=d_{\vdash S_{A}}
$$

so that, for any $(A, d)$ we have a $(\widehat{\Sigma}, S)$-algebra $\left(\widehat{T}_{\widehat{\Sigma}, S} A, \widehat{T}_{\widehat{\Sigma}, S} d, \llbracket \Sigma \rrbracket\right)$. The assignment

$$
(A, d) \mapsto\left(\widehat{T}_{\widehat{\Sigma}, S} A, \widehat{T}_{\widehat{\Sigma}, S} d, \llbracket \Sigma \rrbracket\right)
$$

can be turned into a functor $\widehat{T}_{\widehat{\Sigma}, S}: \mathbf{G M e t} \rightarrow \mathbf{\operatorname { A l g }}(\widehat{\Sigma}, S)$ by defining, for each nonexpansive map $f:(A, d) \rightarrow(B, \Delta)$,

$$
\widehat{T}_{\widehat{\Sigma}, S}(f):=[t]_{\equiv_{A}} \mapsto\left[T_{\Sigma}(f)(t)\right]_{\equiv_{B}}
$$

which is equivalent to

$$
\widehat{T}_{\widehat{\Sigma}, S}(f):=\left[t\left(a_{1}, \ldots, a_{n}\right)\right]_{\equiv_{A}} \mapsto\left[t\left(f\left(a_{1}\right), \ldots, f\left(a_{n}\right)\right)\right]_{\equiv_{B}}
$$

To check that $\widehat{T}_{\widehat{\Sigma}, S}$ is indeed a functor, one needs to verify that $\widehat{T}_{\widehat{\widehat{ }}, S}(f)$ is well-defined on equivalence classes, nonexpansive, and commutes with operations in $\Sigma$, and that $\widehat{T}_{\widehat{\Sigma}, S}$ preserves composition. The following lemma implies the first two properties.

Lemma 4.4. Let $f:(A, d) \rightarrow(B, \Delta)$ be an arrow in $\mathbf{G M e t}$. For all $s, t \in T_{\Sigma} A$,

$$
\begin{gathered}
{[s]_{\equiv_{A}}=[t]_{\equiv_{A}} \Rightarrow\left[T_{\Sigma} f(s)\right]_{\equiv_{B}}=\left[T_{\Sigma} f(t)\right]_{\equiv_{B}}} \\
\widehat{T}_{\widehat{\Sigma}, S} d\left([s]_{\equiv_{A}}[t]_{\equiv_{A}}\right) \leq \varepsilon \Rightarrow \widehat{T}_{\widehat{\Sigma}, S} \Delta\left(\left[T_{\Sigma} f(s)\right]_{\equiv_{B}}\left[T_{\Sigma} f(t)\right]_{\equiv_{B}}\right) \leq \varepsilon
\end{gathered}
$$

The commutation with operations and preservation of composition follow from the fact that $T_{\Sigma}(f)$ commutes with the operations in $\Sigma$ and $T_{\Sigma}$ preserves composition.

Hence $\widehat{T}_{\widehat{\Sigma}, S}: \operatorname{GMet} \rightarrow \operatorname{Alg}(\widehat{\Sigma}, S)$ is indeed a functor. It can be turned, by application of the forgetful functor (every algebra in $\operatorname{Alg}(\widehat{\Sigma}, S)$ is a GMet space), to a functor of type

$$
\widehat{T}_{\widehat{\Sigma}, S}: \text { GMet } \rightarrow \text { GMet. }
$$

The latter can be given the structure of a monad on GMet by defining unit $\widehat{\eta}_{(A, d)}:(A, d) \rightarrow \widehat{T}_{\widehat{\Sigma}, S}(A, d)$ and multiplication $\widehat{\mu}_{(A, d)}: \widehat{T}_{\widehat{\Sigma}, S} \widehat{T}_{\widehat{\Sigma}, S}(A, d) \rightarrow \widehat{T}_{\widehat{\Sigma}, S}(A, d)$ as follows:

$$
\begin{aligned}
& \widehat{\eta}_{(A, d)}: a \stackrel{\widehat{\eta}_{(A, d)}}{\mapsto}[a]_{\equiv_{A}} \\
& \widehat{\mu}_{(A, d)}:\left[t\left(\left[t_{1}\right]_{\equiv_{A}}, \ldots,\left[t_{n}\right]_{\equiv_{A}}\right)\right]_{\equiv_{\widehat{T}_{\widehat{\Sigma}, S}}} \stackrel{\widehat{\mu}_{(A, d)}}{\mapsto}\left[t\left(t_{1}, \ldots, t_{n}\right)\right]_{\equiv_{A}}
\end{aligned}
$$

It can be verified that these maps are nonexpansive and well defined, and that they satisfy the conditions in Definition 2.1. Therefore, we can state:

Theorem 4.5. $\left(\widehat{T}_{\widehat{\Sigma}, S}, \widehat{\eta}, \widehat{\mu}\right)$ is a monad on GMet.

### 4.2 Freeness and Isomorphism Theorems

We are now ready to prove that $\widehat{T}_{\widehat{\Sigma}, S}(A, d)$ is free.
Theorem 4.6. Let $(A, d) \in \mathbf{G M e t}$ and $(B, \Delta, \llbracket \widehat{\Sigma} \rrbracket) \in \mathbf{A l g}(\widehat{\Sigma}, S)$. For any nonexpansive map $f:(A, d) \rightarrow(B, \Delta)$, there exists a unique $\widehat{\Sigma}$-algebra homomorphism $f^{*}: \widehat{T}_{\widehat{\Sigma}, S} A \rightarrow B$ such that $f^{*} \circ \widehat{\eta}_{(A, d)}=f$. We summarize the statement in (11).


Proof. Let $E=\mathrm{E}\left(\vdash_{S_{A}}\right)$ (see Definition 4.1). We organise the proof in four steps.
Step 1. By Lemma 4.2, the carrier of $\widehat{T}_{\widehat{\Sigma}, S}(A, d)$ is $T_{\Sigma_{A}, E} \varnothing$ (equivalently: $T_{\Sigma} A / \equiv_{A}$ ), i.e., the free $\left(\Sigma_{A}, E\right)$-algebra on $\varnothing$.

Step 2. The algebra $\left(B, \Delta, \llbracket \widehat{\Sigma} \rrbracket_{B}\right) \in \operatorname{Alg}(\widehat{\Sigma}, S)$ can be expanded to become an algebra over the extended signature with the aid of the nonexpansive map $f: A \rightarrow B$. Namely, we interpret the added constants in $A$ as follows:

$$
\llbracket a \rrbracket_{B}:=f(a)
$$

Since $f$ is nonexpansive, the expanded $\left(B, \Delta, \llbracket \widehat{\Sigma}_{A} \rrbracket_{B}\right)$ satisfies the additional clauses on constants:

$$
\Delta\left(\llbracket a \rrbracket_{B}, \llbracket a^{\prime} \rrbracket_{B}\right)=\Delta\left(f(a), f\left(a^{\prime}\right)\right) \leq d_{A}\left(a, a^{\prime}\right)
$$

and therefore it is a model of the extended theory $\vdash_{S_{A}}$. Hence $\left(B, \Delta, \llbracket \widehat{\Sigma}_{A} \rrbracket_{B}\right) \in$ $\operatorname{Alg}\left(\widehat{\Sigma}_{A}, S_{A}\right)$. This means that all equations in $E=\mathrm{E}\left(\vdash_{S_{A}}\right)$ are validated in $B$. This in turn means that $\left(B, \llbracket \Sigma_{A} \rrbracket_{B}\right)$ (forgetting the metric) is a $\left(\Sigma_{A}, E\right)$-algebra.

Step 3. Combining the first two steps, we obtain a unique $\left(\Sigma_{A}, E\right)$-algebra homomorphism

$$
g_{\varnothing}^{*}: T_{\Sigma_{A}, E} \varnothing \rightarrow B
$$

where $g_{\varnothing}^{*}$ is the homomorphic extension of the empty function $g_{\varnothing}: \varnothing \rightarrow$ $B$. By identifying $T_{\Sigma_{A}, E} \varnothing$ with $T_{\Sigma} A / \equiv_{A}$, we turn $g_{\varnothing}^{*}$ into a function of type $T_{\Sigma} A / \equiv_{A} \rightarrow B$, which we denote $f^{*}$. By the definition of $g_{\varnothing}^{*}$ we have $f^{*}\left([a]_{\equiv_{A}}\right)=$ $\llbracket a \rrbracket_{B}=f(a)$, which implies that $f^{*} \circ \widehat{\eta}_{(A, d)}=f$.

Step 4. We now conclude by proving that $f^{*}$ is a morphism in $\operatorname{Alg}(\widehat{\Sigma}, S)$, namely, it is a $\Sigma$-algebra homomorphism and it is nonexpansive. The former follows from Step 3 which defined $f^{*}$ as $g_{\varnothing}^{*}$, which is a $\left(\Sigma_{A}, E\right)$-algebra homomorphism and thus preserves all operations in $\Sigma$. For the latter, take arbitrary elements $[s]_{\equiv_{A}}[t]_{\equiv_{A}} \in \widehat{T}_{\widehat{\Sigma}, S}(A, d)$ and assume $\widehat{T}_{\widehat{\Sigma}, S}(d)\left([s]_{\equiv_{A},}[t]_{\equiv_{A}}\right)=\varepsilon$, which means that $\varnothing \vdash_{S_{A}} s=_{\varepsilon} t$. We need to show that in $\left(B, \Delta, \llbracket \widehat{\Sigma} \rrbracket_{B}\right) \in \mathbf{A l g}(\widehat{\Sigma}, S)$ it holds:

$$
\begin{equation*}
\Delta\left(f^{*}\left([s]_{\equiv_{A}}\right), f^{*}\left([t]_{\equiv_{A}}\right)\right) \leq \varepsilon . \tag{12}
\end{equation*}
$$

Since we already know that $\left(B, \Delta, \llbracket \widehat{\Sigma}_{A} \rrbracket_{B}\right) \in \mathbf{A l g}\left(\widehat{\Sigma}_{A}, S_{A}\right)$ (Step 2), we have that $\left(B, \Delta, \llbracket \widehat{\Sigma}_{A} \rrbracket_{B}\right) \vDash s={ }_{\varepsilon} t$, which means that

$$
\begin{equation*}
\Delta\left(\llbracket s \rrbracket_{B}, \llbracket t \rrbracket_{B}\right) \leq \varepsilon \tag{13}
\end{equation*}
$$

To conclude, it is sufficient to observe, using the definition of $g_{\varnothing}^{*}$, that:

$$
\begin{equation*}
f^{*}\left([t]_{\equiv_{A}}\right)=\llbracket t \rrbracket_{B} \tag{14}
\end{equation*}
$$

where $\llbracket-\rrbracket_{B}$ is the extended interpretation to $\Sigma_{A}$. Hence, from (13) and (14) we derive the desired inequality (12).

We now focus our attention on the case of $\widehat{\Sigma}$ theories $\vdash_{S}$ generated by a set of basic Horn clauses (Definition 3.10), that is, of the form $\bigwedge_{i=1}^{n} \phi_{i} \Rightarrow \phi$, where each $\phi_{i}$ is a (quantitative) equation $(x=y$ or $x=\varepsilon y$ ) between variables.

Theorem 4.7. Let $\widehat{\Sigma}$ be a lifted signature and $S$ a set of basic Horn clauses. Then $\operatorname{EM}\left(\widehat{T}_{\widehat{\Sigma}, S}\right) \cong \operatorname{Alg}(\widehat{\Sigma}, S)$.

Proof sketch. Let $(A, d, \alpha) \in \mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, S}\right)$, we define the interpretations $\llbracket \widehat{\Sigma} \rrbracket_{\alpha}$ as follows: for any op : $n \in \Sigma$ and $\vec{a} \in A^{n}$,

$$
\llbracket \mathrm{op} \rrbracket_{\alpha}(\vec{a})=\alpha([\operatorname{op}(\vec{a})])
$$

where $[t]$ stands for $[t]_{\equiv_{A}}$. We claim that $\left(A, d,\left[\widehat{\Sigma} \rrbracket_{\alpha}\right) \in \operatorname{Alg}(\widehat{\Sigma}, S)\right.$. First, we show $\llbracket \mathrm{op} \rrbracket_{\alpha}$ is $L_{\mathrm{op}}$-nonexpansive. Given $\vec{a}, \vec{b} \in L_{\mathrm{op}}(A, d)$, let $\Delta$ be the restriction of $d$ on $\vec{a} \cup \vec{b}$, we have

$$
\begin{aligned}
d(\alpha([\operatorname{op}(\vec{a})]), \alpha([\operatorname{op}(\vec{b})])) & \leq \widehat{T}_{\widehat{\Sigma}, S} d([\operatorname{op}(\vec{a})],[\operatorname{op}(\vec{b})]) \\
& \leq L_{\mathrm{op}}(\Delta)(\vec{a}, \vec{b}) \\
& =L_{\mathrm{op}}(d)(\vec{a}, \vec{b}) .
\end{aligned}
$$

The first inequality holds because $\alpha$ is nonexpansive, the second inequality uses the rule $L-N E$, and the equality is the fact that $L_{\mathrm{op}}$ preserves isometric embeddings.

An adaptation of the argument in the proof of Theorem 4.2 in Bacci et al. [2018b] shows $\left(A, d, \llbracket \widehat{\Sigma} \rrbracket_{\alpha}\right)$ satisfies the clauses in $S$. This defines a functor $\widehat{P}$ : $\mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, S}\right) \rightarrow \mathbf{A l g}(\widehat{\Sigma}, S)$ acting trivially on morphisms and sending $(A, d, \alpha)$ to $\left(A, d, \llbracket \widehat{\Sigma} \rrbracket_{\alpha}\right)$.

In the converse direction, let $\mathbb{A}=(A, d, \llbracket \widehat{\Sigma} \rrbracket) \in \mathbf{A l g}(\widehat{\Sigma}, S)$, we define $\widehat{\alpha}_{\mathbb{A}}$ : $\widehat{T}_{\widehat{\Sigma}, S}(A, d) \rightarrow(A, d)$ inductively as follows: for any $a \in A, \widehat{\alpha}_{\mathbb{A}}([a])=a$ and $\forall \mathrm{op}: n \in \Sigma, \forall t_{1}, \ldots, t_{n} \in T_{\Sigma} A$,

$$
\widehat{\alpha}_{\mathbb{A}}\left(\left[\operatorname{op}\left(t_{1}, \ldots, t_{n}\right)\right]\right)=\llbracket \operatorname{op} \rrbracket\left(\widehat{\alpha}_{\mathbb{A}}\left(\left[t_{1}\right]\right), \ldots, \widehat{\alpha}_{\mathbb{A}}\left(\left[t_{n}\right]\right)\right) .
$$

This defines a functor $\widehat{P}^{-1}: \operatorname{Alg}(\widehat{\Sigma}, S) \rightarrow \mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, S}\right)$. It actstrivially on morphisms and sends $\mathbb{A}=(A, d, \llbracket \widehat{\Sigma} \rrbracket)$ to $\left(A, d, \widehat{\alpha}_{\mathbb{A}}\right)$.

The functor $\widehat{P}$ and $\widehat{P}^{-1}$ are inverses and we conclude the desired isomorphism.

## 5 Examples

In Sections 3 and 4 we have introduced the new notions of lifted signatures $\widehat{\Sigma}$ and quantitative $\widehat{\Sigma}$-algebras, the deductive apparatus to reason about them, and we stated our main results: Theorem 3.14 (soundness), Theorem 4.6 (free algebras) and Theorem $4.7\left(\mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, S}\right) \cong \mathbf{A l g}(\widehat{\Sigma}, S)\right.$ for basic theories $)$. We now show the applicability of our framework.

### 5.1 Applications already studied in the literature

As already pointed out, the framework of Mardare et al. [2016, 2017], Bacci et al. [2018b] can be seen as a special case of our framework when: (1) the generalised metric space GMet considered is Met and (2) all liftings in the lifted signature $\widehat{\Sigma}$ are the sup-product lifting $L_{\times}$(see Example 3.3). For several interesting examples of applications, more can be said.

We first recall some definitions. Given a set $A$, we let $\mathcal{D}(A)$ denote the set of finitely supported probability distributions on $A$, i.e., functions $\varphi: A \rightarrow[0,1]$ such that $|\{a \mid \varphi(a)>0\}|$ is finite. For a given $a \in A$, the Dirac distribution $\delta_{a} \in \mathcal{D}(A)$ assigns 1 to $a$, and 0 to all other elements. Convex algebras are algebras for the following signature and set of axioms:

$$
\Sigma=\left\{+_{p}: 2\right\}_{p \in(0,1)} \quad E=\left\{\begin{array}{c}
x+_{p} x=x, x+p y=y+_{1-p} x \\
\left(x+_{q} y\right)+_{p} z=x+_{p q}\left(y+_{\frac{p(1-q)}{1-p q}} z\right)
\end{array}\right\}
$$

It is well-known (see, e.g., Jacobs [2010]) that $\mathcal{D}(A)$ with operations defined as:

$$
\llbracket+_{p} \rrbracket(\varphi, \psi):=a \mapsto(p \cdot \varphi(a)+(1-p) \cdot \psi(a))
$$

is (up to isomorphism) the free convex algebra on the set $A$.
The (Met) quantitative theory of convex algebras from Mardare et al. [2016], Bacci et al. [2018b] can be formalised in our framework by taking GMet = Met, lifted signature $\widehat{\Sigma}=\left\{+_{p}: 2: L_{\times}\right\}_{p \in(0,1)}$ and as generating set of Horn clauses the axioms $E$ of convex algebras together with the clause:

$$
\left\{x_{1}=\varepsilon_{1} y_{1}, x_{2}=\varepsilon_{\varepsilon_{1}} y_{2}\right\} \Rightarrow x_{1}+{ }_{p} x_{2}={ }_{p \varepsilon_{1}+(1-p) \varepsilon_{2}} y_{1}+{ }_{p} y_{2}
$$

known as "Kantorovich rule". Note that, since the inequality

$$
p \varepsilon_{1}+(1-p) \varepsilon_{2} \leq \max \left\{\varepsilon_{1}, \varepsilon_{2}\right\}
$$

holds for all $p, \varepsilon_{1}, \varepsilon_{2} \in[0,1]$, the Kantorovich rule strictly subsumes (using the Max rule) the $L-N E$ rule for $+p$, which only states (omitting some premises, cf. Example 3.12):

$$
\left\{x_{1}=\varepsilon_{\varepsilon_{1}} y_{1}, x_{2}=\varepsilon_{1} y_{2}\right\} \Rightarrow x_{1}+{ }_{p} x_{2}={ }_{\max \left\{\varepsilon_{1}, \varepsilon_{2}\right\}} y_{1}+p y_{2} .
$$

Hence, in quantitative (Met) convex algebras, the operation $\llbracket+{ }_{p} \rrbracket$ is not merely $L_{x}$-nonexpansive, as it needs to satisfy the stronger constraint of the Kantorovich rule.

Consider now, for every $p \in(0,1)$, the lifting $L_{K}^{p}$ of the binary product defined as follows:

$$
\begin{aligned}
& L_{K}^{p}:(A, d) \mapsto\left(A \times A, L_{K}^{p}(d)\right) \\
& L_{K}^{p}(d)\left(\left(a_{1}, a_{2}\right),\left(b_{1}, b_{2}\right)\right)=d_{K}\left(\llbracket+{ }_{p} \rrbracket\left(\delta_{a_{1}}, \delta_{a_{2}}\right), \llbracket+{ }_{p} \rrbracket\left(\delta_{b_{1}}, \delta_{b_{2}}\right)\right)
\end{aligned}
$$

where $d_{K}$ is the well-known Kantorovich distance over distributions $\mathcal{D}(A)$. This lifting is easily seen to preserve isometric embeddings.

Then it can be shown that the (Met) quantitative theory of convex algebras, axiomatised above, can also be presented as the theory over the lifted signature $\widehat{\Sigma}_{K}=\left\{+_{p}: 2: L_{K}^{p}\right\}_{p \in(0,1)}$, taking as generating set of Horn clauses only the set $E$ of axioms of convex algebras. In other words, we have cast the Kantorovich rule as a $L-N E$ rule, by choosing the appropriate lifting $L_{K}^{p}$ for every operation $+_{p}$. Note that the remaining clauses are just the purely equational axioms of the (Set) theory of convex algebras.

The same applies in several other interesting examples. For example, also the (Met) quantitative theory of convex semilattices of Mio and Vignudelli [2020], Mio et al. [2021] can be presented as the (Met) quantitative theory with generating clauses just the equational axioms of convex semilattices, by choosing the appropriate liftings in the lifted signature.

### 5.2 No constraints on algebraic operations

Among the variants of the framework of Bacci et al. [2018b] that have been considered in the literature, the work of Bacci et al. [2018a] is relevant in our discussion. Indeed, the authors have observed that certain fixed-point operations on metric spaces fail to be nonexpansive (up to the sup-product lifting $L_{\times}$) and, as such, cannot be cast in the framework of Mardare et al. [2016, 2017], Bacci et al. [2018b]. The solution adopted in Bacci et al. [2018a] is to drop entirely all constraints on the interpretation of the algebraic operations $\llbracket o p \rrbracket$ and allow arbitrary maps $\llbracket \mathrm{op} \rrbracket^{n}: A^{n} \rightarrow A$.

This approach can be seen as a particular instance of our framework by taking GMet $=$ Met and using lifted signatures $\widehat{\Sigma}$ where for all op : $n: L_{\mathrm{op}} \in \widehat{\Sigma}$ the lifting $L_{\mathrm{op}}$ is the "discrete" lifting defined as follows:

$$
L_{\mathrm{op}}(d)\left(\left(a_{1}, \ldots, a_{n}\right),\left(b_{1}, \ldots, b_{n}\right)\right)= \begin{cases}0 & \text { if } \forall_{i=1}^{n} . a_{i}=b_{i} \\ 1 & \text { otherwise }\end{cases}
$$

Indeed, with this choice of lifting, the $L-N E$ rule

$$
\frac{(\vec{x} \cup \vec{y}, \Delta) \in \text { GMet } \quad \delta=L_{\mathrm{op}}(\Delta)(\vec{x}, \vec{y})}{\left\{w=_{\Delta(w, z)} z \mid w, z \in \vec{x} \cup \vec{y}\right\} \vdash \operatorname{op}(\vec{x})=_{\delta} \mathrm{op}(\vec{y})} L-\mathrm{NE}
$$

is rendered useless, as it can always by substituted with instances of the clause 1-bdd, if $\vec{x} \neq \vec{y}$, or with instances of the clause $\varnothing \vdash \mathrm{op}(\vec{x})={ }_{0}$ op $(\vec{y})$ (coming from the axioms of Met) if $\vec{x}=\vec{y}$.

Therefore our free algebra and isomorphism theorems from Section 4 hold for the theory developed in Bacci et al. [2018a] and for further variants that can be conceived. Such results could not be automatically derived from the original framework of Bacci et al. [2018b] only allowing for $L_{\times}$-nonexpansive operations.

### 5.3 The Łukaszyk-Karmowski distance on probability distributions

In this subsection we develop our main example, already presented in the introduction: the axiomatisation of the Łukaszyk-Karmowski distance ( $d_{\mathrm{EK}}$ ) on probability distributions Łukaszyk [2004]. The distance $d_{\text {モK }}$ has very recently found application in the field of representation learning and it is at the core of the definition of the MICo ("matching under independent couplings") behavioural distance on Markov processes of Castro et al. [2021].

Recall that a diffuse metric space $(A, d) \in$ DMet is a set $A$ with a fuzzy relation $d: A \times A \rightarrow[0,1]$ satisfying reflexivity and triangular inequality, i.e., for all $a, b, c \in A$ :

$$
d(a, b)=d(b, a) \quad d(a, c) \leq d(a, b)+d(b, c)
$$

The notion of diffuse metric has been introduced in [Castro et al., 2021, §4.2]. The following diagrams depict some diffuse metric spaces $(A, d)$ with $A$ finite.


Definition 5.1. Let $(A, d)$ be a diffuse metric space. The Łukaszyk-Karmowski distance is the fuzzy relation $d_{\mathrm{EK}}$ on the set of finitely supported probability distributions $\mathcal{D}(A)$ defined for any $\varphi, \psi \in \mathcal{D}(A)$ as

$$
d_{\mathrm{EK}}(\varphi, \psi)=\sum_{x \in \operatorname{supp}(\varphi)} \sum_{y \in \operatorname{supp}(\psi)} \varphi(x) \cdot \psi(y) \cdot d(x, y)
$$

Proposition 5.2. For any diffuse metric space $(A, d)$, the space $\left(\mathcal{D}(A), d_{\mathrm{EK}}\right)$ is a diffuse metric space.

Recall from Subsection 5.1 that convex algebras are algebras for the signature $\Sigma=\left\{+_{p}: 2\right\}_{p \in(0,1)}$ satisfying the axioms $E$, and that the free convex algebra generated by $A$ is $\mathcal{D}(A)$. We now observe, however, that on probability distributions equipped with the Łukaszyk-Karmowski distance the operation $\llbracket+{ }_{p} \rrbracket$ generally fails to be nonexpansive (up to the sup-product lifting $L_{\times}$).
Lemma 5.3. There exists a diffuse metric space $(A, d)$ such that the following map is not nonexpansive:

$$
\llbracket+{ }_{p} \rrbracket:\left(\mathcal{D}(A), d_{\mathrm{EK}}\right) \times\left(\mathcal{D}(A), d_{\mathrm{EK}}\right) \rightarrow\left(\mathcal{D}(A), d_{\mathrm{EK}}\right)
$$

Proof. Fix the DMet space $A=\{a, b\}$ with $d(a, a)=d(b, b)=\frac{1}{2}$ and $d(a, b)=$ $d(b, a)=1$. Take the Dirac distributions $\delta_{a}, \delta_{b} \in \mathcal{D}(A)$. We have $d_{\mathrm{EK}}\left(\delta_{a}, \delta_{a}\right)=$ $d_{\text {モK }}\left(\delta_{b}, \delta_{b}\right)=\frac{1}{2}$, and

$$
d_{\mathrm{EK}}\left(\llbracket+_{\frac{1}{2}} \rrbracket\left(\delta_{a}, \delta_{b}\right), \llbracket+_{\frac{1}{2}} \rrbracket\left(\delta_{a}, \delta_{b}\right)\right)=\frac{3}{4} .
$$

Recall that $d_{\mathrm{EK}} \times d_{\mathrm{EK}}$ is the sup-product lifting of $d_{\mathrm{EK}}$. Hence, $\llbracket+{ }_{p} \rrbracket$ is not nonexpansive:

$$
\begin{aligned}
\frac{1}{2} & =\max \left\{d_{\mathrm{EK}}\left(\delta_{a}, \delta_{a}\right), d_{\mathrm{EK}}\left(\delta_{b}, \delta_{b}\right)\right\} \\
& =d_{\mathrm{EK}} \times d_{\mathrm{EK}}\left(\left(\delta_{a}, \delta_{b}\right),\left(\delta_{a}, \delta_{b}\right)\right) \\
& <d_{\mathrm{EK}}\left(\llbracket+\frac{1}{2} \rrbracket\left(\delta_{a}, \delta_{b}\right), \llbracket+_{\frac{1}{2}} \rrbracket\left(\delta_{a}, \delta_{b}\right)\right)=\frac{3}{4}
\end{aligned}
$$

We now introduce a new lifting $L_{\mathrm{EK}}^{p}$ of the binary product ensuring that $\llbracket+p \rrbracket$ is $L_{\mathrm{EK}}^{p}$-nonexpansive. For every $p \in(0,1)$, we define the DMet lifting of the binary product:

$$
\begin{aligned}
& L_{\mathrm{EK}}^{p}:(A, d) \mapsto\left(A \times A, L_{\mathrm{EK}}^{p}(d)\right) \\
& L_{\mathrm{EK}}^{p}(d)\left(\left(a_{1}, a_{2}\right),\left(b_{1}, b_{2}\right)\right)=d_{\mathrm{EK}}\left(\llbracket+_{p} \rrbracket\left(\delta_{a_{1}}, \delta_{a_{2}}\right), \llbracket+_{p} \rrbracket\left(\delta_{b_{1}}, \delta_{b_{2}}\right)\right) .
\end{aligned}
$$

Lemma 5.4. The lifting $L_{\mathrm{EK}}^{p}$ preserves isometric embeddings.
Lemma 5.5. For every DMet space $(A, d)$, the operation $\llbracket+{ }_{p} \rrbracket: \mathcal{D}(A) \times \mathcal{D}(A) \rightarrow$ $\mathcal{D}(A)$ is $L_{\mathrm{EK}}^{p}$-nonexpansive.

We can then consider the following DMet lifting of the signature $\Sigma$ of convex algebras: $\widehat{\Sigma}_{\mathrm{EK}}:=\left\{+_{p}: 2: L_{\mathrm{EK}}^{p}\right\}_{p \in(0,1)}$, and the quantitative $\widehat{\Sigma}_{\mathrm{EK}}$-theory $\vdash_{E}$ generated by the set $E$ of axioms of convex algebras. In this theory the $L-$ NE rule for $+_{p}$ takes the following form (omitting some premises, cf. Example 3.12):

$$
\left\{\begin{array}{l}
x_{1}=\varepsilon_{11} x_{1}, x_{2}=\varepsilon_{\varepsilon_{21}} x_{1} \\
x_{1}=\varepsilon_{\varepsilon_{12}} y_{2}, y_{2}=\varepsilon_{\varepsilon_{22}} y_{2}
\end{array}\right\} \vdash x_{1}+{ }_{p} x_{2}={ }_{\delta} y_{1}+{ }_{p} y_{2}
$$

with $\delta=p^{2} \varepsilon_{11}+(1-p) p \varepsilon_{21}+p(1-p) \varepsilon_{12}+(1-p)^{2} \varepsilon_{22}$.
By application of Theorem 4.6 we know that $\operatorname{Alg}\left(\widehat{\Sigma}_{\mathrm{EK}}, E\right)$ has free algebras on $(A, d)$, for every DMet space $(A, d)$, and that these are term algebras $\widehat{T}_{\widehat{\Sigma}_{\mathrm{Lk}}, S}(A, d)$ on which we can reason syntactically. The following theorem states that these term algebras are isomorphic to ( $\left.\mathcal{D}(A), d_{\mathrm{EK}}, \llbracket \Sigma \rrbracket\right)$, the collection of finitely supported probability distributions, with $Ł K$ distance and standard convex algebras operations.
Theorem 5.6. The free algebra in $\operatorname{Alg}\left(\widehat{\Sigma}_{\mathrm{EK}}, E\right)$ on a DMet space $(A, d)$ is $\left(\mathcal{D}(A), d_{\mathrm{EK}}, \llbracket \Sigma \rrbracket\right)$.
Hence we can say that the theory $\vdash_{E}$ axiomatises convex algebras $(\mathcal{D}(A), \llbracket \Sigma \rrbracket)$ with the ŁK distance.

## 6 Conclusion

We have presented an extension of the quantitative algebra framework of Bacci et al. [2018b], Mardare et al. [2016, 2017], Bacci et al. [2021], Mardare et al. [2021] allowing us to reason on generalised metric spaces and on algebraic operations
that are nonexpansive up to a lifting. This has allowed, as an illustrative example, the axiomatisation of the Łukaszyk-Karmowski distance on probability distributions.

One direction of future work is to explore if, and how, the recent results developed for the framework of Bacci et al. [2018b] can be adapted and generalised to our setting. For example, tensor product of theories (Bacci et al. [2021] and techniques to handle fixedpoints Mardare et al. [2021].

In another direction, one can look for further generalisations. For example, it would be interesting to investigate how our treatment of GMet compares with the general relational apparatus of Ford et al. [2021] and find a way to lift their more general arities. Another interesting possibility is to consider liftings of the entire signature functor

$$
\Sigma:=\coprod_{\text {op: }: n \in \Sigma} A^{n} \quad \Sigma(f):=\coprod_{\text {op:n }: n} f^{n} .
$$

rather than just liftings of each of the operations.
From a foundational standpoint, the question of what classes of monads (e.g., finitary ones) can be constructed as term monads for quantitative theories is still open.

Generally, we plan to look at more interesting examples to drive our research on all these topics.

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## 7 Appendix

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### 7.1 Background

### 7.1.1 Additional Result on Monads

We need an additional result on monads in the full proof of Theorem 4.7.
Definition 7.1 (Monad functor). Let $\left(M, \eta^{M}, \mu^{M}\right)$ be a monad on $\mathbf{C}$ and $\left(T, \eta^{T}, \mu^{T}\right)$ a monad on $\mathbf{D}$. A monad functor from $M$ to $T$ is a pair $(F, \lambda)$ comprising a functor $F_{T}: \mathbf{C} \rightarrow \mathbf{D}$ and a natural transformation $\lambda: T F \Rightarrow F M$ such that (1) $\lambda \circ \eta^{T} F=F \eta^{M}$ and (2) $\lambda \circ \mu^{T} F=F \mu^{M} \circ \lambda M \circ T \lambda$.

Proposition 7.2 (Street [1972]). Let $(F, \lambda): M \rightarrow T$ be a monad functor, then there is a functor $F-\circ \lambda: \mathbf{E M}(M) \rightarrow \mathbf{E M}(T)$ sending an $M$-algebra $\alpha: M A \rightarrow A$ to $F \alpha \circ \lambda_{A}: T(F A) \rightarrow F A$ and a morphism $f:(A, \alpha) \rightarrow\left(A^{\prime}, \alpha^{\prime}\right)$ to $F f$.
Proof. A lower level proof is drawn in Marsden [2014].

### 7.1.2 Products and Coproducts in GMet

In Section 2.3, we gave the construction of products and coproducts in the category GMet with no proof nor reference to a proof. We prove this here. We first prove the base case in FRel.

Proposition 7.3. Let $\left\{\left(A_{i}, d_{i}\right) \mid i \in I\right\}$ be non-empty family of fuzzy relations $\left\{\left(A_{i}, d_{i}\right) \mid i \in I\right\}$, the product is $\left(\prod_{i \in I} A_{i}, \sup _{i \in I} d_{i}\right)$ with the usual projections and the coproduct is $\left(\coprod_{i \in I} A_{i}, \amalg_{i \in I} d_{i}\right)$ with the usual coprojections.

Proof. Product. The projections $\pi_{i}$ are clearly nonexpansive. Let $(A, d) \xrightarrow{f_{i}}$ $\left(A_{i}, d_{i}\right)$ be a family of nonexpansive maps. The universal property of the product in Set yields a unique function $!:(A, d) \rightarrow\left(\prod_{i \in I} A_{i}, \sup _{i \in I} d_{i}\right)$ such that $\pi_{i} \circ!=f_{i}$. Now, it is enough to prove the function is nonexpansive. For any $a, b \in A$, we have

$$
\left(\sup _{i \in I} d_{i}\right)\left(!(a),!(b)=\left(\sup _{i \in I} d_{i}\right)\left(\left(f_{i}(a)\right)_{i \in I}\left(f_{i}(b)\right)_{i \in I}\right)\right.
$$

$$
\begin{aligned}
& =\sup _{i \in I} d_{i}\left(f_{i}(a), f_{i}(b)\right) \\
& \leq \sup _{i \in I} d(a, b) \\
& =d(a, b)
\end{aligned}
$$

Coproduct. The coprojections $\kappa_{i}$ are clearly nonexpansive (they are in fact isometries). Let $\left(A_{i}, d_{i}\right) \xrightarrow{f_{i}}(A, d)$ be a family of nonexpansive maps. The universal property of the coproduct in Set yields a unique function ! : $\left(\coprod_{i \in I} A_{i}, \amalg_{i \in I} d_{i}\right) \rightarrow$ $(A, d)$ such that $!\circ \kappa_{i}=f_{i}$. Now, it is enough to prove the function is nonexpansive. For any $a \in A_{j}$ and $b \in A_{k}$, if $j \neq k$,

$$
\left(\amalg_{i \in I} d_{i}\right)(a, b)=1 \geq d(!(a),!(b))
$$

If $j=k$,

$$
\left(\amalg_{i \in I} d_{i}\right)(a, b)=d_{j}(a, b) \geq d\left(f_{j}(a), f_{j}(b)\right)=d(!(a),!(b))
$$

Now, we prove that if each fuzzy relation $\left(A_{i}, d_{i}\right)$ satisfies an axiom of (1)-(5), then the product and coproduct satisfy that axiom. It follows that (co)products in GMet exist and are computed just like those in FRel.

Proposition 7.4. Fix a subset $G$ of the axioms (1)-(5) and let $\left\{\left(A_{i}, d_{i}\right) \mid i \in I\right\}$ be non-empty family of fuzzy relations. If every $\left(A_{i}, d_{i}\right)$ satisfies the axioms in $G$, then the product and the coproduct satisfy the axioms in $G$.

Proof. Product. We proceed with each axiom independently: we suppose each $\left(A_{i}, d_{i}\right)$ satisifies it and show $\left(\prod_{i \in I} A_{i}, \sup _{i \in I} d_{i}\right)$ also satisfies it.
(1) For any $\vec{a}, \vec{b} \in \prod_{i \in I}$, since $d_{i}\left(\vec{a}_{i}, \vec{b}_{i}\right)=d_{i}\left(\vec{b}_{i}, \vec{a}_{i}\right)$ for all $i \in I$, the two sets $\left\{d_{i}\left(\vec{a}_{i}, \vec{b}_{i}\right) \mid i \in I\right\}$ and $\left\{d_{i}\left(\vec{b}_{i}, \vec{a}_{i}\right)\right\}$ are equal, and so are their supremums. We conclude

$$
\left(\sup _{i \in I} d_{i}\right)(\vec{a}, \vec{b})=\left(\sup _{i \in I} d_{i}\right)(\vec{b}, \vec{a})
$$

(2) For any $\vec{a} \in \prod_{i \in I} A_{i}$, since $d_{i}\left(\vec{a}_{i}, \vec{a}_{i}\right)=0$ for all $i \in I$, the two sets $\left\{d_{i}\left(\vec{a}_{i}, \vec{a}_{i}\right) \mid i \in I\right\}$ and $\{0\}$ are equal, and so are their supremums. We conclude

$$
\left(\sup _{i \in I} d_{i}\right)(\vec{a}, \vec{a})=0
$$

(3) For any $\vec{a}, \vec{b} \in \prod_{i \in I}$, if $\left(\sup _{i \in I} d_{i}\right)(\vec{a}, \vec{b})=0$, we have

$$
\forall i \in I, d_{i}\left(\vec{a}_{i}, \vec{b}_{i}\right) \leq\left(\sup _{i \in I} d_{i}\right)(\vec{a}, \vec{b})=0
$$

which implies $\forall i \in I, \vec{a}_{i}=\vec{b}_{i}$. We conclude $\vec{a}=\vec{b}$.
(4) For any $\vec{a}, \vec{b}, \vec{c} \in \prod_{i \in I}$, we have

$$
\begin{equation*}
d_{i}\left(\vec{a}_{i}, \vec{c}_{i}\right) \leq d_{i}\left(\vec{a}_{i}, \vec{c}_{i}\right)+d_{i}\left(\vec{b}_{i}, \vec{c}_{i}\right), \tag{15}
\end{equation*}
$$

and using standard properties of the supremum, we obtain

$$
\begin{aligned}
\left(\sup _{i \in I} d_{i}\right)(\vec{a}, \vec{c}) & =\sup _{i \in I} d_{i}\left(\vec{a}_{i}, \vec{c}_{i}\right) \\
& \leq \sup _{i \in I}\left(d_{i}\left(\vec{a}_{i}, \vec{b}_{i}\right)+d_{i}\left(\vec{b}_{i}, \vec{c}_{i}\right)\right) \\
& \leq \sup _{i \in I} d_{i}\left(\vec{a}_{i}, \vec{b}_{i}\right)+\sup _{i \in I} d_{i}\left(\vec{b}_{i}, \vec{c}_{i}\right) \\
& =\left(\sup _{i \in I} d_{i}\right)(\vec{a}, \vec{b})+\left(\sup _{i \in I} d_{i}\right)(\vec{b}, \vec{c})
\end{aligned}
$$

(5) For any $\vec{a}, \vec{b}, \vec{c} \in \prod_{i \in I}$, we have

$$
\begin{equation*}
d_{i}\left(\vec{a}_{i}, \vec{c}_{i}\right) \leq \max \left\{d_{i}\left(\vec{a}_{i}, \vec{c}_{i}\right), d_{i}\left(\vec{b}_{i}, \vec{c}_{i}\right)\right\}, \tag{16}
\end{equation*}
$$

and using standard properties of the supremum, we obtain

$$
\begin{aligned}
\left(\sup _{i \in I} d_{i}\right)(\vec{a}, \vec{c}) & =\sup _{i \in I} d_{i}\left(\vec{a}_{i}, \vec{c}_{i}\right) \\
& \leq \sup _{i \in I}\left(\max \left\{d_{i}\left(\vec{a}_{i}, \vec{b}_{i}\right), d_{i}\left(\vec{b}_{i}, \vec{c}_{i}\right)\right\}\right) \\
& \leq \max \left\{\sup _{i \in I} d_{i}\left(\vec{a}_{i}, \vec{b}_{i}\right), \sup _{i \in I} d_{i}\left(\vec{b}_{i}, \vec{c}_{i}\right)\right\} \\
& =\max \left\{\left(\sup _{i \in I} d_{i}\right)(\vec{a}, \vec{b}),\left(\sup _{i \in I} d_{i}\right)(\vec{b}, \vec{c})\right\}
\end{aligned}
$$

Coproduct. We proceed with each axiom independently: we suppose each $\left(A_{i}, d_{i}\right)$ satisifies it and show $\left(\amalg_{i \in I} A_{i}, \amalg_{i \in I} d_{i}\right)$ also satisfies it.
(1) For any $a \in A_{j}$ and $b \in A_{k}$, if $j \neq k$, then

$$
\left(\amalg_{i \in I} d_{i}\right)(a, b)=1=\left(\amalg_{i \in I} d_{i}\right)(b, a),
$$

otherwise if $j=k$,

$$
\left(\amalg_{i \in I} d_{i}\right)(a, b)=d_{j}(a, b)=d_{j}(b, a)=\left(\amalg_{i \in I} d_{i}\right)(b, a) .
$$

(2) For any $a \in A_{j}$, we have

$$
\left(\amalg_{i \in I} d_{i}\right)(a, a)=d_{j}(a, a)=0 .
$$

(3) For any $a \in A_{j}$ and $b \in A_{k}$, if $j \neq k$, then

$$
\left(\amalg_{i \in I} d_{i}\right)(a, b)=1 \neq 0,
$$

otherwise if $j=k$,

$$
\left(\amalg_{i \in I} d_{i}\right)(a, b)=0 \Longrightarrow d_{j}(a, b)=0 \Longrightarrow a=b .
$$

(4) For any $a \in A_{j}$ and $b \in A_{k}, c \in A_{\ell}$, if either $j \neq k$ or $k \neq \ell$, then

$$
\left(\amalg_{i \in I} d_{i}\right)(a, c) \leq 1 \leq\left(\amalg_{i \in I} d_{i}\right)(a, b)+\left(\amalg_{i \in I} d_{i}\right)(b, c) .
$$

otherwise if $j=k$ and $k=\ell$, then $j=\ell$, thus

$$
\begin{aligned}
\left(\amalg_{i \in I} d_{i}\right)(a, c) & =d_{j}(a, c) \\
& \leq d_{j}(a, b)+d_{j}(b, c) \\
& =\left(\amalg_{i \in I} d_{i}\right)(a, b)+\left(\amalg_{i \in I} d_{i}\right)(b, c) .
\end{aligned}
$$

(5) For any $a \in A_{j}$ and $b \in A_{k}, c \in A_{\ell}$, if either $j \neq k$ or $k \neq \ell$, then

$$
\left(\amalg_{i \in I} d_{i}\right)(a, c) \leq 1 \leq \max \left\{\left(\amalg_{i \in I} d_{i}\right)(a, b),\left(\amalg_{i \in I} d_{i}\right)(b, c)\right\} .
$$

otherwise if $j=k$ and $k=\ell$, then $j=\ell$, thus

$$
\begin{aligned}
\left(\amalg_{i \in I} d_{i}\right)(a, c) & =d_{j}(a, c) \\
& \leq \max \left\{d_{j}(a, b)+d_{j}(b, c)\right\} \\
& =\max \left\{\left(\amalg_{i \in I} d_{i}\right)(a, b),\left(\amalg_{i \in I} d_{i}\right)(b, c)\right\} .
\end{aligned}
$$

### 7.2 Proofs of Section 4

### 7.2.1 Proof of Theorem 4.5

We divide the proof in multiple lemmas, the first being a technical lemma. In short, it states that if $f:(A, d) \rightarrow(B, \Delta)$ is nonexpansive, then any $(\widehat{\Sigma}, S)$ algebra with carrier $(B, \Delta)$ can be extended to a $\left(\widehat{\Sigma}_{A}, S_{A}\right)$-algebra. ${ }^{3}$

Lemma 7.5. Let $\mathbb{B}:=\left(B, \Delta,\left[\widehat{\Sigma} \rrbracket_{B}\right)\right.$ be a $(\widehat{\Sigma}, S)$-algebra. For any nonexpansive map $f:(A, d) \rightarrow(B, \Delta)$, there is a $\left(\widehat{\Sigma}_{A}, S_{A}\right)$-algebra $\left(B, \Delta, \llbracket \widehat{\Sigma}_{A} \rrbracket_{B, f}\right)$ such that for any $t \in T_{\Sigma} A=T_{\Sigma_{A}} \varnothing, \llbracket t \rrbracket_{B, f}^{\ell}=\llbracket T_{\Sigma} f(t) \rrbracket_{B}^{\mathrm{I}_{B}}$ with $\iota: \varnothing \rightarrow B$ being the only possible assignment $\left(\mathrm{id}_{B}\right.$ is omitted in the sequel).

[^2]Proof. Setting $\llbracket \mathrm{op} \rrbracket_{B, f}=\llbracket \mathrm{op} \rrbracket_{B}$ for every op $\in \Sigma$ and $\llbracket a \rrbracket_{B, f}=f(a)$ for every $a \in A$, we get all the interpretations in $\llbracket \widehat{\Sigma}_{A} \rrbracket_{B, f}$. We write $\mathbb{B}_{f}:=\left(B, \Delta, \llbracket \widehat{\Sigma}_{A} \rrbracket_{B, f}\right)$. Note that $\mathbb{B}_{f}$ still satisfies the clauses in $S$ as they do not involve the constants from $A$ and $\mathbb{B}$ satisfied them. Moreover, since $f$ is nonexpansive,

$$
\Delta\left(\llbracket a \rrbracket_{B}, \llbracket a^{\prime} \rrbracket_{B}\right)=\Delta\left(f(a), f\left(a^{\prime}\right)\right) \leq d\left(a, a^{\prime}\right),
$$

thus $\mathbb{B}_{f}$ satisfies the additional clauses on constants $\left(\varnothing \Rightarrow a=_{d\left(a, a^{\prime}\right)} a^{\prime}\right)$ that belong to $S_{A}$. We conclude $\mathbb{B}_{f} \in \operatorname{Alg}\left(\widehat{\Sigma}_{A}, S_{A}\right)$.

We proceed by induction for the last part of the lemma. If $t=a \in A$, we have $\llbracket t \rrbracket_{B, f}^{l}=f(a)=\llbracket f(a) \rrbracket_{B}$. If $t=\operatorname{op}\left(t_{1}, \ldots, t_{n}\right)$ and we assume $\llbracket t_{i} \rrbracket_{B, f}=$ $\llbracket T_{\Sigma} f\left(t_{i}\right) \rrbracket_{B}$ for each $1 \leq i \leq n$, we have

$$
\begin{aligned}
\llbracket t \rrbracket_{B, f}^{\prime} & =\llbracket \mathrm{op} \rrbracket_{B, f}\left(\llbracket \llbracket_{1} \rrbracket_{B, f}^{l}, \ldots, \llbracket t_{n} \rrbracket_{B, f}^{l}\right) \\
& =\llbracket \mathrm{op} \rrbracket_{B}\left(\llbracket t_{1} \rrbracket_{B, f}^{\prime}, \cdots, \llbracket t_{n} \rrbracket_{B, f}^{\prime}\right) \\
& =\llbracket \mathrm{op} \rrbracket_{B}\left(\llbracket T_{\Sigma} f\left(t_{1}\right) \rrbracket_{B}, \ldots, \llbracket T_{\Sigma} f\left(t_{n}\right) \rrbracket_{B}\right) \\
& =\llbracket T_{\Sigma} f(t) \rrbracket_{B} .
\end{aligned}
$$

Lemma 7.6. The map $\widehat{\eta}_{(A, d)}$ is nonexpansive.
Proof. Apply Lemma 7.5 to the term algebra $\left(\widehat{T}_{\widehat{\Sigma}, S} B, \widehat{T}_{\widehat{\Sigma}, S} \Delta, \llbracket \widehat{\Sigma} \rrbracket\right)$ and the map $f^{\prime}: A \rightarrow \widehat{T}_{\widehat{\Sigma}, S} B$ defined by $a \mapsto[f(a)]_{\equiv_{B}}$ which is nonexpansive as

$$
\widehat{T}_{\widehat{\Sigma}, S^{\prime}} \Delta\left([f(a)]_{\equiv_{B}},\left[f\left(a^{\prime}\right)\right]_{\equiv_{B}}\right) \leq \Delta\left(f(a), f\left(a^{\prime}\right)\right) \leq d\left(a, a^{\prime}\right) .
$$

We find that $\left(\widehat{T}_{\widehat{\Sigma}, S} B, \widehat{T}_{\widehat{\Sigma}, S} \Delta,[\widehat{\Sigma}]_{f}\right)$ satisfies all the clauses in $S_{A}$ and for any $t \in T_{\Sigma} A, \llbracket t \rrbracket_{f}^{\iota}=\llbracket T_{\Sigma} f(t) \rrbracket=\left[T_{\Sigma} f(t)\right]_{\equiv_{B}}$ (with $\iota: \varnothing \rightarrow \widehat{T}_{\widehat{\Sigma}, S} B$ ). We obtain the following implications (we leave the equivalences implicit) which prove the lemma.

$$
\begin{array}{ll}
{[s]=[t]} & \widehat{T}_{\widehat{\Sigma}, S} d([s],[t]) \leq \varepsilon \\
\Leftrightarrow \varnothing \vdash_{S_{A}} s=t & \Leftrightarrow \varnothing \vdash_{S_{A}} s=_{\varepsilon} t \\
\Rightarrow \llbracket s \rrbracket_{f}^{l}=\llbracket t \rrbracket_{f}^{l} & \Rightarrow \widehat{T}_{\widehat{\Sigma}, S} \Delta\left(\llbracket s \rrbracket_{f}^{l}, \llbracket \rrbracket \rrbracket_{f}^{\iota}\right) \leq \varepsilon \\
\Rightarrow\left[T_{\Sigma} f(s)\right]=\left[T_{\Sigma} f(t)\right] & \Rightarrow \widehat{T}_{\widehat{\Sigma}, S} \Delta\left(\left[T_{\Sigma} f(s)\right],\left[T_{\Sigma} f(t)\right]\right) \leq \varepsilon
\end{array}
$$

Lemma 7.7. The map $\widehat{\mu}_{(A, d)}$ is well-defined on equivalence classes and nonexpansive.
Proof. Apply Lemma 7.5 to the term algebra $\left(\widehat{T}_{\widehat{\Sigma}, S} A, \widehat{T}_{\widehat{\Sigma}, S} d, \llbracket \widehat{\Sigma} \rrbracket\right)$ and the identity id : $\widehat{T}_{\widehat{\Sigma}, S} A \rightarrow \widehat{T}_{\widehat{\Sigma}, S} A$ which is nonexpansive. We find that $\left(\widehat{T}_{\widehat{\Sigma}, S} A, \widehat{T}_{\widehat{\Sigma}, S} d, \llbracket \widehat{\Sigma} \rrbracket_{\text {id }}\right)$
satisfies all the clauses in $S_{\widehat{T}_{\widehat{\Sigma}, S} A}$ and for any $t \in T_{\Sigma}\left(\widehat{T}_{\widehat{\Sigma}} A\right), \llbracket t \rrbracket_{\mathrm{id}}^{\iota}=\llbracket T_{\Sigma} \mathrm{id}(t) \rrbracket=$ $\left[T_{\Sigma} \mathrm{id}(t)\right]_{\equiv_{A}}$ (with $\iota: \varnothing \rightarrow \widehat{T}_{\widehat{\Sigma}, S} A$ ). We obtain the following implications which prove nonexpansiveness of $\widehat{\mu}_{(a, d)}$ (well-definedness is proven similarly).

$$
\begin{aligned}
& \widehat{T}_{\widehat{\Sigma}, S}\left(\widehat{T}_{\widehat{\Sigma}, S} d\right)\left([s]_{\equiv_{\widehat{T}_{\widehat{\Sigma}, S^{A}}}}[t]_{\left.\equiv_{\widehat{T}_{\widehat{\Sigma}, S}{ }^{A}}\right) \leq \varepsilon}\right. \\
& \Leftrightarrow \varnothing \vdash{S_{\widehat{T}_{\widehat{\Sigma}, S^{A}}} s={ }_{\varepsilon} t} \\
& \Rightarrow \widehat{T}_{\widehat{\Sigma}, S} d\left(\llbracket s \rrbracket_{\mathrm{id}}^{\iota}, \llbracket t \rrbracket_{\mathrm{id}}^{\iota}\right) \leq \varepsilon \\
& \Rightarrow \widehat{T}_{\widehat{\Sigma}, S} d\left(\left[T_{\Sigma} \operatorname{id}(s)\right]_{\equiv_{A}},\left[T_{\Sigma} \operatorname{id}(t)\right]_{\equiv_{A}}\right) \leq \varepsilon
\end{aligned}
$$

The last implication holds by $\left[T_{\Sigma} \operatorname{id}(t)\right]_{\equiv_{A}}=\widehat{\mu}_{(A, d)}\left([t]_{\equiv_{\widehat{\Gamma}_{\Sigma, S} A}}\right)$.
The associativity of $\hat{\mu}$, the unitality of $\widehat{\eta}$ and the naturality of both all follow from their counterpart for $\mu^{\Sigma}$ and $\eta^{\Sigma}$. This concludes the proof of Theorem 4.5.

### 7.2.2 Proof of Theorem 4.7

Let $E=\mathrm{E}\left(\vdash_{S}\right)$. In order to construct the isomorphism $\mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, S}\right) \cong \operatorname{Alg}(\widehat{\Sigma}, S)$ we will make use of the isomorphism $P: \operatorname{EM}\left(T_{\Sigma, E}\right) \cong \operatorname{Alg}(\Sigma, E): P^{-1}$ which exists by Proposition 2.11. We will also make use of the forgetful functor $U$ : $\operatorname{Alg}(\widehat{\Sigma}, S) \rightarrow \operatorname{Alg}(\Sigma, E)$ that forgets about the generalized metric space structure and the fact some clauses in $S$ are satisfied. Our proof that $\operatorname{EM}\left(\widehat{T}_{\widehat{\Sigma}, S}\right) \cong$ $\operatorname{Alg}(\widehat{\Sigma}, S)$ is divided in three key steps.

In (Step 1) we construct a functor $F: \mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, S}\right) \rightarrow \mathbf{E M}\left(T_{\Sigma, E}\right)$ so that we have the following picture.


In (Step 2), we prove that:

1. for any $(A, d, \alpha) \in \mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, S}\right)$, there exists $\left(A, d, \llbracket \widehat{\Sigma} \rrbracket_{\alpha}\right) \in \mathbf{A l g}(\widehat{\Sigma}, S)$ such that $P F(A, d, \alpha)=U\left(A, d, \llbracket \widehat{\Sigma} \rrbracket_{\alpha}\right)$, and
2. for any $\mathbb{A}=(A, d, \llbracket \widehat{\Sigma} \rrbracket) \in \operatorname{Alg}(\widehat{\Sigma}, S)$, there exists $\left(A, d, \alpha_{\mathbb{A}}\right)$ such that $P^{-1} U(\mathbb{A})=F\left(A, d, \widehat{\alpha}_{\mathbb{A}}\right)$.

Finally, in (Step 3) we conclude $\widehat{P}$ and $\widehat{P}^{-1}$ acting trivially on morphisms and as below on objects (with the notation introduced in Step 2) define functors that are inverse to each other.

$$
\begin{aligned}
\widehat{P}: \mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, S}\right) \rightarrow \mathbf{A l g}(\widehat{\Sigma}, S) & \mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, S}\right) \leftarrow \mathbf{A l g}(\widehat{\Sigma}, S): \widehat{P}^{-1} \\
(A, d, \alpha) \mapsto\left(A, d, \llbracket \widehat{\Sigma} \rrbracket_{\alpha}\right) & \left(A, d, \widehat{\alpha}_{\mathbb{A}}\right) \leftarrow(A, d, \llbracket \widehat{\Sigma} \rrbracket)=\mathbb{A}
\end{aligned}
$$

Unrolling the definitions, it will emerge that (18) commutes.


Then, using the fact that $P$ and $P^{-1}$ are inverses and the fact that the vertical functors forget information that is not modified by $\widehat{P}$ and $\widehat{P}^{-1}$, we can infer the latter pair are inverses too. We conclude $\mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, S}\right) \cong \mathbf{A l g}(\widehat{\Sigma}, S)$.

Step 1: Construction of $F$. The functor $F$ will be constructed, by application of Proposition $7.2(F=(U-\circ \mathfrak{q}))$, by proving that $(U, \mathfrak{q})$ is a monad functor from $\widehat{T}_{\widehat{\Sigma}, S}$ to $T_{\Sigma, E}$ where $U:$ GMet $\rightarrow$ Set is the forgetful functor and the natural transformation

$$
\mathfrak{q}_{(A, d)}: T_{\Sigma, E} A \rightarrow T_{\Sigma} A / \equiv s_{A}
$$

is defined as:

$$
[t]_{E} \stackrel{\mathfrak{q}_{(A, d)}}{\mapsto}[t]_{\equiv S_{A}},
$$

where $\equiv S_{A}=\mathrm{E}\left(\vdash_{S_{A}}\right)$ is the set of equations in the theory $\vdash_{S}$ extended by $(A, d) .{ }^{4}$ To lighten the notation, we write $[t]$ as a shorthand for $[t]_{E}$, and $\langle t\rangle$ for either $[t]_{\equiv_{S_{A}}}$ or $[t]_{\equiv_{\widehat{T}_{\widehat{\Sigma}, S}(A, d)}}$ with the context making it clear which of the two is intended. Therefore the action of $\mathfrak{q}$ always looks like

$$
[t] \stackrel{\mathfrak{q}_{(A, d)}}{\mapsto}\langle t\rangle .
$$

We will show that $(U, \mathfrak{q})$ is a monad functor from $\widehat{T}_{\widehat{\Sigma}, S}$ to $T_{\Sigma, E}$. First, we show $\mathfrak{q}: T_{\Sigma, E} U \Rightarrow U \widehat{T}_{\widehat{\Sigma}, S}$ is natural, i.e.: it makes (19) commute for any $f$ :

[^3]$$
(A, d) \rightarrow(B, \Delta)
$$
\[

$$
\begin{align*}
& T_{\Sigma, E} A \xrightarrow{T_{\Sigma, E} f} T_{\Sigma, E} B  \tag{19}\\
& \mathfrak{q}_{(A, d)} \downarrow \\
& T_{\Sigma} A / \equiv S_{A} \underset{\widehat{T}_{\widehat{\Sigma}, S} f}{ } \\
& T_{\Sigma} B / \equiv S_{B}
\end{align*}
$$
\]

Starting with $[t]$ in the top left, the bottom path yields $\langle t\rangle$ then $\widehat{T}_{\widehat{\Sigma}, S} f(\langle t\rangle)=$ $\left\langle T_{\Sigma} f(t)\right\rangle$ and the top path yields $T_{\Sigma, E} f([t])=\left[T_{\Sigma} f(t)\right]$ then $\left\langle T_{\Sigma} f(t)\right\rangle$. Second, we show that $\mathfrak{q} \cdot \eta^{\Sigma, E} U=U \hat{\eta}$. At component $(A, d)$ and for any $a \in A$, we have

$$
\mathfrak{q}_{(A, d)}\left(\eta_{A}(a)\right)=\mathfrak{q}_{(A, d)}([a])=\langle a\rangle=U \hat{\eta}_{(A, d)}(a)
$$

Finally, we prove that (20) commutes as follows:


Concretely, the functor $F=(U-\circ \mathfrak{q})$, which is indeed a functor by Proposition 7.2, acts as follows on objects:

$$
\begin{array}{cc}
\mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, S}\right) & \mathbf{E M}\left(T_{\Sigma, E}\right) \\
\alpha: \widehat{T}_{\widehat{\Sigma}, S}(A, d) \rightarrow(A, d) & \mapsto
\end{array} \quad U \alpha \circ \mathfrak{q}_{(A, d)}: T_{\Sigma, E} A \rightarrow A
$$

It acts trivially on morphisms, namely if $f: A \rightarrow B$ is a $\widehat{T}_{\widehat{\Sigma}, S}$-algebra homomor$\operatorname{phism}(A, d, \alpha) \rightarrow(B, \Delta, \beta)$, then it is sent to $f: A \rightarrow B$ which is a $T_{\Sigma, E}$-algebra homomorphism $F(A, d, \alpha) \rightarrow F(B, \Delta, \beta)$.

Step 2.1: the functor $\widehat{P}: \mathbf{E M}\left(\widehat{T}_{\widehat{\Sigma}, S}\right) \rightarrow \mathbf{A l g}(\widehat{\Sigma}, S)$. Let $\alpha: \widehat{T}_{\widehat{\Sigma}, S}(A, d) \rightarrow(A, d)$ be in $\operatorname{EM}\left(\widehat{T}_{\widehat{\Sigma}, S}\right)$ and denote $\left(A, \llbracket \widehat{\Sigma} \rrbracket_{\alpha}\right)$ the $\Sigma$-algebra obtained from applying $P$ to $U \alpha \circ \mathfrak{q}_{(A, d)}$. Explicitly, for each op : $n \in \Sigma, \llbracket o p \rrbracket_{\alpha}$ sends $\left(a_{1}, \ldots, a_{n}\right)$ to $\alpha\left(\left\langle\operatorname{op}\left(a_{1}, \ldots, a_{n}\right)\right\rangle\right)$. We claim that $\mathbb{A}_{\alpha}:=\left(A, d, \llbracket \widehat{\Sigma} \rrbracket_{\alpha}\right) \in \mathbf{A} \lg (\widehat{\Sigma}, S)$. Namely, for any op : $n: L \in \widehat{\Sigma}$, $\llbracket \mathrm{op} \rrbracket_{\alpha}$ is nonexpansive with respect to the lifting $L(A, d)$ and $\mathbb{A}_{\alpha} \vDash S$.

First, we show $\llbracket \mathrm{op} \rrbracket_{\alpha}$ is nonexpansive. Given $\vec{a}, \vec{b} \in L(A, d)$, let $\Delta$ be the restriction of $d$ on $\vec{a} \cup \vec{b}$, we have

$$
d(\alpha(\langle\operatorname{op}(\vec{a})\rangle), \alpha(\langle\operatorname{op}(\vec{b})\rangle)) \leq \widehat{T}_{\widehat{\Sigma}, S} d(\langle\operatorname{op}(\vec{a})\rangle,\langle\operatorname{op}(\vec{b})\rangle)
$$

$$
\begin{aligned}
& \leq L(\Delta)(\vec{a}, \vec{b}) \\
& =L(d)(\vec{a}, \vec{b}) .
\end{aligned}
$$

The first inequality holds because $\alpha$ is nonexpansive, the second inequality uses the rule $L-\mathrm{NE}$, and the equality is the fact that $L$ preserves isometric embeddings.

Next, we show $\mathbb{A}_{\alpha}$ satisfies $S$. We can view any assignment $\iota: X \rightarrow A$ as an assignment $\iota: X \rightarrow \widehat{T}_{\widehat{\Sigma}, S} A$, and to distinguish the two extensions to arbitrary $\Sigma$-terms, we write

$$
\llbracket-\rrbracket_{\alpha}^{\iota}: T_{\Sigma} X \rightarrow A \text { and } \iota^{*}: T_{\Sigma} X \rightarrow \widehat{T}_{\widehat{\Sigma}, S} A
$$

We claim that for any basic (quantitative) equation $\phi \in \mathcal{V}_{\Sigma} X$,

$$
\mathbb{A}_{\alpha} \vDash^{l} \phi \Longrightarrow \widehat{T}_{\widehat{\Sigma}, S}(A, d) \vDash^{l} \phi
$$

Indeed, for any $x \in X$, we have $\llbracket x \rrbracket_{\alpha}^{\iota}=\iota(x)=\iota^{*}(x)$, thus if $\phi$ is quantitative, w.l.o.g. it is $x=\varepsilon y$, the following implications hold:

$$
\begin{aligned}
\mathbb{A}_{\alpha} \vDash^{\iota} \phi & \left.\left.\Leftrightarrow d\left(\llbracket x \rrbracket_{\alpha}^{\iota}\right), \llbracket y \rrbracket_{\alpha}^{l}\right)\right) \leq \varepsilon \\
& \Leftrightarrow d(\iota(x), \iota(y)) \leq \varepsilon \\
& \Rightarrow \varnothing \vdash_{S_{A}} \iota(x)={ }_{\varepsilon} \iota(y) \\
& \Leftrightarrow \widehat{T}_{\widehat{\Sigma}, S} d\left(\iota^{*}(x), \iota^{*}(y)\right) \leq \varepsilon \\
& \Leftrightarrow \widehat{T}_{\widehat{\Sigma}, S}(A, d) \vDash^{\iota} \phi .
\end{aligned}
$$

The non-invertible implication holds because $l(x)$ and $\iota(y)$ are elements of $A$, so the clause $\varnothing \Longrightarrow \iota(x)={ }_{d(\iota(x), \iota(y))} \iota(y)$ belongs to $S_{A}$. Using Max yields $\varnothing \vdash_{S_{A}} \iota(x)={ }_{\varepsilon} \iota(y)$. A very similar argument works when $\phi$ is not quantitative.

Let $\bigwedge_{i \in I} \phi_{i} \Rightarrow \phi$ be a clause in $S$ (each $\phi_{i}$ is basic), and suppose $\mathbb{A}_{\alpha} \vDash^{\ell} \phi_{i}$ for each $i \in I$. By our argument above, we also have $\widehat{T}_{\widehat{\Sigma}, S}(A, d) \vDash^{\ell} \phi_{i}$, and since $\widehat{T}_{\widehat{\Sigma}, S}(A, d) \in \operatorname{Alg}(\widehat{\Sigma}, S)$, it satisfies all clauses in $S$. We infer $\widehat{T}_{\widehat{\Sigma}, S}(A, d) \vDash^{\iota} \phi$. Now, suppose $\phi$ has the shape $s={ }_{\varepsilon} t$ (a very similar argument will work if $\phi$ is not quantitative), we have $\widehat{T}_{\widehat{\Sigma}, S} d\left(\iota^{*}(s), \iota^{*}(t)\right) \leq \varepsilon$ and since $\alpha$ is nonexpansive, we also have $d\left(\alpha\left(\iota^{*}(s)\right), \alpha\left(\iota^{*}(t)\right)\right) \leq \varepsilon$. Now, one can show by induction that $\llbracket-\rrbracket_{\alpha}^{\iota}=\alpha \circ \iota^{*}$, thus $\mathbb{A}_{\alpha} \vDash^{\iota} s={ }_{\varepsilon} t$. We conclude $\mathbb{A} \vDash S$.

This describes the action of $\widehat{P}$ on objects. On morphisms, we said the action is trivial because if $f:\left(A, d_{A}, \alpha\right) \rightarrow\left(B, d_{B}, \beta\right)$ is a homomorphism of $\widehat{T}_{\widehat{\Sigma}, S^{-}}$ algebras, then the underlying function $f: A \rightarrow B$ is nonexpansive and it is a homomorphism of $\Sigma$-algebras $\left(A, \llbracket \Sigma \rrbracket_{\alpha}\right) \rightarrow\left(B, \llbracket \Sigma \rrbracket_{\beta}\right)$. Therefore, it is also a $(\widehat{\Sigma}, S)$-algebra homomorphism $\left(A, d_{A}, \llbracket \Sigma \rrbracket_{\alpha}\right) \rightarrow\left(B, d_{B}, \llbracket \Sigma \rrbracket_{\beta}\right)$. Functoriality is easy to check.

Step 2.2: the functor $\widehat{P}: \operatorname{EM}\left(\widehat{T}_{\widehat{\Sigma}, S}\right) \rightarrow \operatorname{Alg}(\widehat{\Sigma}, S)$. Let $\mathbb{A}=(A, d, \llbracket \widehat{\Sigma} \rrbracket)$ be in $\operatorname{Alg}(\widehat{\Sigma}, S)$ and denote $\alpha_{\mathbb{A}}: T_{\Sigma, E} A \rightarrow A$ the $T_{\Sigma, E}$-algebra obtained from apply-
ing $P^{-1}$ to $U \mathbb{A}$. We claim that $\alpha_{\mathbb{A}}$ is in the image of $U-\circ \mathfrak{q}$. We first show $\alpha_{\mathbb{A}}$ is compatible with $\equiv S_{A}$ and nonexpansive with respect to $d_{S_{A}}$.

For the former, suppose that $\varnothing \vdash_{s_{A}} s=t$ with $s, t \in T_{\Sigma} A$. Setting $\llbracket a \rrbracket=a$ for every $a \in A$, we can check that $\mathbb{A}^{+}:=\left(A, d, \llbracket \widehat{\Sigma}_{A} \rrbracket\right) \in \operatorname{Alg}\left(\widehat{\Sigma}_{A}, S_{A}\right)$. Therefore, by Theorem 3.14, we have $\mathbb{A}^{+} \vDash s=t$. Thus, for the only possible assignment $\iota: \varnothing \rightarrow A$ (recall that $s, t \in T_{\Sigma} A \subseteq T_{\Sigma_{A}} \varnothing$ ), we find

$$
\alpha_{\mathbb{A}}(s)=\llbracket s \rrbracket^{\iota}=\llbracket t \rrbracket^{\iota}=\alpha_{\mathbb{A}}(t) .
$$

For the latter, we can use the same reasoning starting with the assumption $\varnothing \vdash s_{A} s={ }_{\varepsilon} t$ to obtain $d\left(\alpha_{\mathbb{A}}(s), \alpha_{\mathbb{A}}(t)\right) \leq \varepsilon$.

We now have a nonexpansive map $\widehat{\alpha}_{\mathbb{A}}: \widehat{T}_{\widehat{\Sigma}, S}(A, d) \rightarrow(A, d)$ defined by $\widehat{\alpha}_{\mathbb{A}}(\langle t\rangle)=\alpha_{\mathbb{A}}(t)$. Equivalently, it can be inductively defined: for any $a \in A$, $\widehat{\alpha}_{\mathbb{A}}(\langle a\rangle)=a$ and $\forall \mathrm{op}: n \in \Sigma, \forall t_{1}, \ldots, t_{n} \in T_{\Sigma} A$,

$$
\widehat{\alpha}_{\mathbb{A}}\left(\left\langle\operatorname{op}\left(t_{1}, \ldots, t_{n}\right)\right\rangle\right)=\llbracket \mathrm{op} \rrbracket\left(\widehat{\alpha}_{\mathbb{A}}\left(\left\langle t_{1}\right\rangle\right), \ldots, \widehat{\alpha}_{\mathbb{A}}\left(\left\langle t_{n}\right\rangle\langle )\right\rangle\right) .
$$

It remains to show it is a $\widehat{T}_{\widehat{\Sigma}, S}$-algebra. This is a direct consequence of $\alpha_{\mathbb{A}}$ being a $T_{\Sigma, E^{-}}$algebra. Indeed, for any $a \in A$, we have

$$
\begin{aligned}
\widehat{\alpha}_{\mathbb{A}}\left(\widehat{\eta}_{(A, d)}(a)\right) & =\widehat{\alpha}_{\mathbb{A}}(\langle a\rangle) \\
& =\alpha_{\mathbb{A}}(a)=a,
\end{aligned}
$$

and for any $\left\langle t\left(\left\langle t_{1}\right\rangle, \ldots,\left\langle t_{n}\right\rangle\right)\right\rangle \in$, we have

$$
\begin{aligned}
& \widehat{\alpha}_{\mathbb{A}}\left(\widehat{T}_{\widehat{\Sigma}, S}\left(\widehat{\alpha}_{\mathbb{A}}\right)\left(\left\langle t\left(\left\langle t_{1}\right\rangle, \ldots,\left\langle t_{n}\right\rangle\right)\right\rangle\right)\right) \\
& =\widehat{\alpha}_{\mathbb{A}}\left(\left\langle t\left(\alpha_{\mathbb{A}}\left(t_{1}\right), \ldots, \alpha_{\mathbb{A}}\left(t_{n}\right)\right)\right\rangle\right) \\
& =\alpha_{\mathbb{A}}\left(t\left(\alpha_{\mathbb{A}}\left(t_{1}\right), \ldots, \alpha_{\mathbb{A}}\left(t_{n}\right)\right)\right) \\
& =\alpha_{\mathbb{A}}\left(t\left(t_{1}, \ldots, t_{n}\right)\right) \\
& =\widehat{\alpha}_{\mathbb{A}}\left(\left\langle t\left(t_{1}, \ldots, t_{n}\right)\right\rangle\right) \\
& =\widehat{\alpha}_{\mathbb{A}}\left(\widehat{\mu}_{(A, d)}\left(\left\langle t\left(\left\langle t_{1}\right\rangle, \ldots,\left\langle t_{n}\right\rangle\right)\right\rangle\right)\right)
\end{aligned}
$$

This describes the action of $\widehat{P}^{-1}$ on objects. On morphisms, an argument similar the one above yield the functor $\widehat{P}^{-1}$.

### 7.3 Proofs of Section 5

In this Section, whenever $p \in(0,1)$, we denote $\bar{p}=1-p$.

### 7.3.1 Proof of Proposition 5.2

Let $\varphi, \psi \in \mathcal{D} A$, since $d_{\mathrm{EK}}(\varphi, \psi)$ is a sum of products of numbers in $[0,1]$, we find that $d_{\mathrm{EK}}$ has type $\mathcal{D} A \times \mathcal{D} A \rightarrow[0,1]$, i.e. it is a fuzzy relation. It is also
clear that $d_{\mathrm{EK}}$ 's definition does not depend on the order of the inputs, so it is symmetric (1). For the triangle inequality (4), we have the following derivation for all $\varphi, \psi, \theta \in \mathcal{D} A$, where $x, y$, and $z$ range in $\operatorname{supp}(\varphi), \operatorname{supp}(\psi)$ and $\operatorname{supp}(\theta)$ respectively.

$$
\begin{aligned}
& d_{\mathrm{EK}}(\varphi, \psi)+d_{\mathrm{EK}}(\psi, \theta) \\
& =\sum_{(x, y)} \varphi(x) \psi(y) d(x, y)+\sum_{(y, z)} \psi(y) \theta(z) d(y, z) \\
& \left.=\sum_{y} \psi(y)\left(\sum_{x} \varphi(x) d(x, y)+\sum_{z} \theta(z) d(y, z)\right)\right) \\
& \left.=\sum_{y} \psi(y)\left(\sum_{(x, z)} \varphi(x) \theta(z) d(x, y)+\sum_{(x, z)} \varphi(x) \theta(z) d(y, z)\right)\right) \\
& =\sum_{y} \psi(y)\left(\sum_{(x, z)} \varphi(x) \theta(z)(d(x, y)+d(y, z))\right) \\
& \geq \sum_{y} \psi(y)\left(\sum_{(x, z)} \varphi(x) \theta(z) d(x, z)\right) \\
& =\sum_{(x, z)} \varphi(x) \theta(z) d(x, z)=d_{\mathrm{EK}}(\varphi, \theta)
\end{aligned}
$$

### 7.3.2 Proof of Lemma 5.4

Let $(A, d)$ be a diffuse metric space, and $p \in(0,1)$. For any subset $A^{\prime} \subseteq A$ and any $a, b, a^{\prime}, b^{\prime} \in A^{\prime}$,

$$
\begin{aligned}
& L_{\mathrm{EK}}^{p}(d)\left((a, b),\left(a^{\prime}, b^{\prime}\right)\right) \\
& =d_{\mathrm{EK}}\left(\llbracket+{ }_{p} \rrbracket(a, b), \llbracket+{ }_{p} \rrbracket\left(a^{\prime}, b^{\prime}\right)\right) \\
& =d_{\mathrm{EK}}\left(p a+\bar{p} b, p a^{\prime}+\bar{p} b^{\prime}\right) \\
& =p^{2} d\left(a, a^{\prime}\right)+p \bar{p} d\left(a, b^{\prime}\right)+\bar{p} p d\left(b, a^{\prime}\right)+\bar{p}^{2} d\left(b, b^{\prime}\right) \\
& =\left.p^{2} d\right|_{A^{\prime}}\left(a, a^{\prime}\right)+\left.p \bar{p} d\right|_{A^{\prime}}\left(a, b^{\prime}\right)+\left.\bar{p} p d\right|_{A^{\prime}}\left(b, a^{\prime}\right)+\left.\bar{p}^{2} d\right|_{A^{\prime}}\left(b, b^{\prime}\right) \\
& =\left.d\right|_{A^{\prime} \mathrm{£K}}\left(p a+\bar{p} b, p a^{\prime}+\bar{p} b^{\prime}\right) \\
& =\left.d\right|_{A^{\prime} \mathrm{£K}}\left(\llbracket+p \rrbracket(a, b), \llbracket+p \rrbracket\left(a^{\prime}, b^{\prime}\right)\right) \\
& =L_{\mathrm{EK}}^{p}\left(\left.d\right|_{A^{\prime}}\right)\left((a, b),\left(a^{\prime}, b^{\prime}\right)\right) .
\end{aligned}
$$

In other words, if $i: A^{\prime} \hookrightarrow A$ is the inclusion function (without loss of generality, these are the only isometric embeddings we need to consider) $L_{£ K}^{p}(i)$ is an isometric embedding.

### 7.3.3 Proof of Lemma 5.5

For any $\varphi, \varphi^{\prime}, \psi, \psi^{\prime} \in \mathcal{D} A$, we have the following derivation where $x$ and $y$ range over the union of the support of all these distributions.

$$
\begin{aligned}
& d_{\mathrm{EK}}\left(\llbracket+p \rrbracket(\varphi, \psi), \llbracket+{ }_{p} \rrbracket\left(\varphi^{\prime}, \psi^{\prime}\right)\right) \\
& =d_{\mathrm{EK}}\left(p \varphi+\bar{p} \psi, p \varphi^{\prime}+\bar{p} \psi^{\prime}\right) \\
& =\sum_{x, y}(p \varphi(x)+\bar{p} \psi(x))\left(p \varphi^{\prime}(y)+\bar{p} \psi^{\prime}(y)\right) d(x, y) \\
& =\sum_{x, y}\left(p^{2} \varphi(x) \varphi^{\prime}(y)+p \bar{p} \varphi(x) \psi^{\prime}(y)\right. \\
& \left.\quad+\bar{p} p \psi(x) \varphi^{\prime}(y)+\bar{p}^{2} \psi(x) \psi^{\prime}(y)\right) d(x, y) \\
& =p^{2} \sum_{x, y} \varphi(x) \varphi^{\prime}(y) d(x, y)+p \bar{p} \sum_{x, y} \varphi(x) \psi^{\prime}(y) d(x, y) \\
& \quad+\bar{p} p \sum_{x, y} \psi(x) \varphi^{\prime}(y) d(x, y)+\bar{p}^{2} \sum_{x, y} \psi(x) \psi^{\prime}(y) d(x, y) \\
& =p^{2} d_{\mathrm{EK}}\left(\varphi, \varphi^{\prime}\right)+p \bar{p} d_{\mathrm{EK}}\left(\varphi, \psi^{\prime}\right) \\
& \quad+\bar{p} p d_{\mathrm{EK}}\left(\psi, \varphi^{\prime}\right)+\bar{p}^{2} d_{\mathrm{EK}}\left(\psi, \psi^{\prime}\right) \\
& = \\
& d_{\mathrm{EK} K}\left(\llbracket+p \rrbracket\left(\delta_{\varphi}, \delta_{\psi}\right), \llbracket+p \rrbracket\left(\delta_{\varphi^{\prime}}, \delta_{\psi^{\prime}}\right)\right) \\
& = \\
& L_{\mathrm{EK}}^{p}\left(d_{\mathrm{EK}}\right)\left((\varphi, \psi),\left(\varphi^{\prime}, \psi^{\prime}\right)\right) .
\end{aligned}
$$

### 7.3.4 Proof of Theorem 5.6

Let $\eta_{(A, d)}:(A, d) \rightarrow\left(\mathcal{D} A, d_{\mathrm{EK}}\right)$ be defined by $a \mapsto \delta_{a}$, we show that for any $\left(B, \Delta, \llbracket+{ }_{p} \rrbracket_{B}\right)$ and nonexpansive map $f:(A, d) \rightarrow(B, \Delta)$, there exists a unique homomorphism $f^{*}: A \rightarrow B$ in $\operatorname{Alg}\left(\widehat{\Sigma}_{\mathrm{EK}}, E\right)$ such that $f^{*} \circ \eta=f$. This is summarized in (21).


First, we show that $\eta_{(A, d)}$ is nonexpansive. Let $a, a^{\prime} \in(A, d)$, the following derivation shows $\eta_{(A, d)}^{Ł K}$ is an isometry.

$$
\begin{aligned}
d_{\mathrm{EK}}\left(\delta_{a}, \delta_{a^{\prime}}\right) & =\sum_{x \in \operatorname{supp}\left(\delta_{a}\right)} \sum_{y \in \operatorname{supp}\left(\delta_{a^{\prime}}\right)} \delta_{a}(x) \delta_{a^{\prime}}(y) d\left(a, a^{\prime}\right) \\
& =\delta_{a}(a) \delta_{a^{\prime}}\left(a^{\prime}\right) d\left(a, a^{\prime}\right) \\
& =d\left(a, a^{\prime}\right)
\end{aligned}
$$

We already know $\left(\mathcal{D} A, \llbracket+{ }_{p} \rrbracket\right)$ is a convex algebra and Lemma 5.5 tells us each $\llbracket+{ }_{p} \rrbracket: L_{\mathrm{EK}}^{p}\left(\mathcal{D} A, d_{\mathrm{EK}}\right) \rightarrow\left(\mathcal{D} A, d_{\mathrm{EK}}\right)$ is nonexpansive, thus $\left(\mathcal{D} A, d_{\mathrm{EK}}, \llbracket+{ }_{p} \rrbracket\right)$ is a convex $Ł K$ algebra. Now, since $\mathcal{D} A$ is the free convex algebra on $A$ and $a \mapsto \delta_{a}$ is the universal morphism witnessing this, we have convex algebra homomorphism $f^{*}:\left(\mathcal{D} A, \llbracket+{ }_{p} \rrbracket\right) \rightarrow\left(B, \llbracket+{ }_{p} \rrbracket_{B}\right)$ making the triangle above commute. It remains to show it is nonexpansive to conclude it is a morphism in $\operatorname{Alg}\left(\widehat{\Sigma}_{\mathrm{EK}}, E\right)$.

Briefly, $f^{*}$ sends a probability distribution $\varphi$ on $A$ to the interpretation in $B$ of a term in $T_{\Sigma_{\mathrm{EK}}} A$ corresponding to $\varphi$ where every occurence of $a$ has been replaced by $f(a)$. For instance, if $\operatorname{supp}(\varphi)=\left\{a_{1}, \ldots, a_{n}\right\}$, one could write

$$
f^{*}(\varphi)=\llbracket+_{\varphi\left(a_{1}\right)} \rrbracket_{B}\left(f\left(a_{1}\right), \llbracket+_{\frac{\varphi\left(a_{2}\right)}{1-\varphi\left(a_{1}\right)}} \rrbracket_{B}\left(f\left(a_{2}\right), \cdots\right) .\right.
$$

In particular, we have $f^{*}\left(\delta_{a}\right)=f(a)$ for any $a \in A$. Moreover, since $f^{*}$ is a homomorphism, for any $\varphi, \varphi^{\prime} \in \mathcal{D} A$ and $p \in(0,1)$, we have

$$
f^{*}\left(p \varphi+\bar{p} \varphi^{\prime}\right)=f^{*}\left(\llbracket+{ }_{p} \rrbracket\left(\varphi, \varphi^{\prime}\right)\right)=\llbracket+{ }_{p} \rrbracket_{B}\left(f^{*}(\varphi), f^{*}\left(\varphi^{\prime}\right) .\right.
$$

More details can be inferred from Jacobs [2010].
We are now ready to show $f^{*}$ is nonexpansive. We proceed by induction on the size of the support of $\varphi, \psi \in \mathcal{D} A$. For the base case, we must have $\varphi=\delta_{a}$ and $\psi=\delta_{b}$ for $a, b \in A$, then it is easy to compute

$$
\Delta\left(f^{*}\left(\delta_{a}\right), f^{*}\left(\delta_{b}\right)\right)=\Delta(f(a), f(b)) \leq d(a, b)=d_{\mathrm{EK}}\left(\delta_{a}, \delta_{b}\right)
$$

Suppose $f^{*}$ is nonexpansive on all pairs of distributions $\varphi$ and $\psi$ with $2<$ $|\operatorname{supp}(\varphi)|+|\operatorname{supp}(\psi)|<n$, and fix any $\varphi, \psi \in \mathcal{D} A$ with $|\operatorname{supp}(\varphi)|+|\operatorname{supp}(\psi)|=$ $n$. It is always possible to rewrite $\varphi=p \delta_{a}+\bar{p} \varphi^{\prime}$ and $\psi=p \delta_{b}+\bar{p} \psi^{\prime}$ such that $\left|\operatorname{supp}\left(\varphi^{\prime}\right)\right|+\left|\operatorname{supp}\left(\psi^{\prime}\right)\right|<n$ (without loss of generality, we can pick $a$ that has the smallest weight $p$ in $\varphi$ and $b$ has weight at least $p$ in $\psi$ ).

By the induction hypothesis, we have the following inequalities (recalling that $f^{*}\left(\delta_{a}\right)=f(a)$ and $\left.f^{*}\left(\delta_{b}\right)=f(b)\right)$.

$$
\begin{aligned}
\Delta(f(a), f(b)) & \leq d_{\mathrm{EK}}\left(\delta_{a}, \delta_{b}\right) \\
\Delta\left(f(a), f^{*}\left(\psi^{\prime}\right)\right) & \leq d_{\mathrm{EK}}\left(\delta_{a}, \psi^{\prime}\right) \\
\Delta\left(f^{*}\left(\varphi^{\prime}\right), f(b)\right) & \leq d_{\mathrm{EK}}\left(\varphi^{\prime}, \delta_{b}\right) \\
\Delta\left(f^{*}\left(\varphi^{\prime}\right), f^{*}\left(\psi^{\prime}\right)\right) & \leq d_{\mathrm{EK}}\left(\varphi^{\prime}, \psi^{\prime}\right)
\end{aligned}
$$

Then, we have the following derivation where $x$ and $y$ range over supp $(\varphi)$ and $\operatorname{supp}\left(\varphi^{\prime}\right)$ respectively.

$$
\begin{aligned}
& \Delta\left(f^{*}(\varphi), f^{*}(\psi)\right) \\
& =\Delta\left(\llbracket+{ }_{p} \rrbracket_{B}\left(f^{*}\left(\delta_{a}\right), f^{*}\left(\varphi^{\prime}\right)\right), \llbracket+{ }_{p} \rrbracket_{B}\left(f^{*}\left(\delta_{b}\right), f^{*}\left(\psi^{\prime}\right)\right)\right) \\
& \leq L_{\mathrm{EK}}^{p}(\Delta)\left(\left(f^{*}\left(\delta_{a}\right), f^{*}\left(\varphi^{\prime}\right)\right),\left(f^{*}\left(\delta_{b}\right), f^{*}\left(\psi^{\prime}\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= L_{\mathrm{EK}}^{p}(\Delta)\left(\left(f(a), f^{*}\left(\varphi^{\prime}\right)\right),\left(f(b), f^{*}\left(\psi^{\prime}\right)\right)\right) \\
&= \Delta_{\mathrm{EK}}\left(p f(a)+\bar{p} f^{*}\left(\varphi^{\prime}\right), p f(b)+\bar{p} f^{*}\left(\psi^{\prime}\right)\right) \\
&= p^{2} \Delta(f(a), f(b))+p \bar{p} \Delta\left(f(a), f^{*}\left(\psi^{\prime}\right)\right) \\
&+\bar{p} p \Delta\left(f^{*}\left(\varphi^{\prime}\right), f(b)\right)+\bar{p}^{2} \Delta\left(f^{*}\left(\varphi^{\prime}\right), f^{*}\left(\psi^{\prime}\right)\right) \\
& \leq p^{2} d_{\mathrm{EK}}\left(\delta_{a}, \delta_{b}\right)+p \bar{p} d_{\mathrm{EK}}\left(\delta_{a}, \psi^{\prime}\right) \\
&+\bar{p} p d_{\mathrm{EK}}\left(\varphi^{\prime}, \delta_{b}\right)+\bar{p}^{2} d_{\mathrm{EK}}\left(\varphi^{\prime}, \psi^{\prime}\right) \\
&= p^{2} \sum_{x, y} \delta_{a}(x) \delta_{b}(y) d(x, y)+p \bar{p} \sum_{x, y} \delta_{a}(x) \psi^{\prime}(y) d(x, y) \\
&+\bar{p} p \sum_{x, y} \varphi(x) \delta_{b}(y) d(x, y)+\bar{p}^{2} \sum_{x, y} \varphi(x) \psi^{\prime}(y) d(x, y) \\
&= \sum_{x, y}\left(p \delta_{a}(x)+\bar{p} \varphi^{\prime}(x)\right)\left(p \delta_{b}(y)+\bar{p} \psi^{\prime}(y)\right) d(x, y) \\
&= \sum_{x, y} \varphi(x) \psi(y) d(x, y) \\
&= d_{\mathrm{EK}}(\varphi, \psi)
\end{aligned}
$$

The first inequality holds by $L_{\mathrm{EK}}^{p}-$ nonexpansiveness of $\llbracket+_{p} \rrbracket_{B}$, the second holds by the induction hypothesis (the four inequalities written above).


[^0]:    ${ }^{1}$ A wider class of implicational constraints can be handled in our framework. We chose these five as running example since they are well known.

[^1]:    ${ }^{2}$ In the category Met of metric spaces, any isometry is injective, but this is not true for all GMet.

[^2]:    ${ }^{3}$ This result was implicitly used in the proof of Theorem 4.6.

[^3]:    ${ }^{4}$ It is a non-trivial observation that $E$ can be a strict subset of $E\left(\vdash s_{A}\right)$. An instance of this happening is in the theory of convex semilattices with black-hole (see Theorem 44 of Mio et al. [2021]).

