Nondeterminism and Guarded Commands^{*}

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1 Introduction

The purpose of this chapter is to review Dijkstra's contribution to nondeterminism by discussing the relevance and impact of his guarded commands language proposal. To properly appreciate it we explain first the role of nondeterminism in computer science at the time his original article Dijkstra [1975] appeared.

The notion of computability is central to computer science. It was studied first in mathematical logic in the thirties of the last century. Several formalisms that aimed at capturing this notion were then proposed and proved equivalent in their expressive power: μ -recursive functions, lambda calculus, and Turing machines, to mention the main ones.

Alan Turing alluded to nondeterminism in his original article on his machines, writing

For some purposes we might use machines (choice machines or *c*-machines) whose motion is only partially determined by the configuration. [...] When such a machine reaches one of these ambiguous configurations, it cannot go on until some arbitrary choice has been made by an external operator. [Turing, 1937, page 232]

However, subsequently he limited his exposition to deterministic machines and it seems that the above option has not been pursued for quite some time. As pointed out in Armoni and Ben-Ari [2009], an interesting account of nondeterminism, in the classic book by Martin Davis, Turing machines were used with a restriction that

no Turing machine will ever be confronted with two different instructions at the same time [Davis, 1958, page 5].

Another early book by Hermes [1965] introduced the theory of computability with deterministic Turing machines as equivalent to the μ -recursive functions.

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Further, as pointed out in Spaan et al. [1989], another helpful study of nondeterminism, in a standard comprehensive introduction to the recursion theory by Hartley Rogers Jr., Turing machines were explicitly assumed to be deterministic:

[...] Finally, the device is to be constructed that it behaves according to a finite list of deterministic rules [...] [Rogers Jr., 1987, page 13]

It seems that a systematic study of addition of nondeterminism to formalisms concerned with computability is due to computer scientists. What follows is a short exposition of such formalisms. Then we discuss Dijkstra's contribution and its relevance. We conclude by providing a brief account of other approaches to nondeterminism that followed.

Literature on nondeterminism in computer science is really extensive. The word 'nondeterminism' yields 421 hits in the DBLP database, while 'nondeterministic' results in 1141 matches. Our intention was not to provide a survey of the subject but rather to sketch a background against which one can adequately assess Dijkstra's contribution to the subject.

When working on our book Apt et al. [2009], written jointly with Frank de Boer, we found that guarded commands form a natural 'glue' that allowed us to connect the chapters on parallel programs, distributed programs, and fairness into a coherent whole. This explains the regular references to this book in the second half of this chapter.

2 Avoiding nondeterminism

We begin with two early formalisms in which nondeterminism is present but the objective is to use them in such a way that it is not visible in the outcome.

Grammars

Mechanisms that are nondeterministic from the very start are grammars in the Chomsky hierarchy, Chomsky [1959]. For example, the set of arithmetic expressions with variables x, y, z, operators + and *, and brackets (and) as terminal symbols can be defined by the following context-free grammar using the start symbol S as its only nonterminal:

$$S ::= x | y | z | S + S | S * S | (S).$$
(1)

Here it is natural to postulate that in a derivation step \vdash , the nonterminal S can be replaced by any of the above right-hand sides. In particular, the arithmetic expression x + y * z has two different *leftmost derivations*, where always the leftmost occurrence of S is replaced:

$$S \vdash S + S \vdash x + S \vdash x + S * S \vdash x + y * S \vdash x + y * z \tag{2}$$

$$S \vdash S * S \vdash S + S * S \vdash x + S * S \vdash x + y * S \vdash x + y * z \tag{3}$$



Figure 1: Two different parsing trees for x + y * z using grammar (1).

In a compiler, derivation (2) corresponds to a parsing tree shown on the lefthand side of Figure 1, giving priority to the operator *, while derivation (3) corresponds to a parsing tree on the right-hand side, giving priority to +.

Thus by definition, the above grammar is *ambiguous*. When using contextfree grammars to define syntax of a programming language, one is interested in *unambiguous* grammars, meaning that each word (here: program) has only one parsing tree. So in grammars, nondeterminism (in the application of the production rules) is allowed, but the objective is that it does not lead to ambiguities in the above sense.

A context-free grammar that allows nondeterminism but is unambiguous uses three nonterminals, E (for 'expression'), T (for 'term'), and F (for 'factor'), where E is the start symbol, and the following production rules:

$$E ::= T | E + T$$

$$T ::= F | T * F$$

$$F ::= (E) | x | y | z$$
(4)

This grammar generates the same set of arithmetic expressions as the one above, but is unambiguous. In particular, the arithmetic expression x + y * z has now only one leftmost derivation corresponding to the unique parsing tree shown in Figure 2. The grammar encodes the fact that the operator * has a higher priority than + and that expressions with the same operator are evaluated from left to right. It can be generalized to a pattern dealing with any set of infix operators with arbitrary priority among them.

Abstract reduction systems

Another simple formalism that allows nondeterminism are abstract reduction systems. Formally, an *abstract reduction system* is a pair (A, \rightarrow) where A is a set and \rightarrow is a binary relation on A. If $a \rightarrow b$ holds, we say that a can be *replaced* by b. In this setting nondeterminism means that an element can be replaced in various ways.



Figure 2: Unique parsing tree for x + y * z using grammar (4).

There are several important examples of abstract reduction systems, in particular *term rewriting systems*, with *combinatory logic* and λ -calculus, and some functional languages as best known examples (see, e.g., Terese [2003]).

Let \rightarrow^* denote the reflexive transitive closure of \rightarrow . An element $a \in A$ is said to be in *normal form* if for no $b \in A$, $a \rightarrow b$ holds. If $a \rightarrow^* b$ and b is in normal form, then b can be viewed as a *value* of a obtained by means of an abstract computation consisting of a repeated application of the \rightarrow relation. In general one is interested in abstract reduction systems in which each element has at most one normal form, so that the notion of a value can be unambiguously defined. One says then that the system has the *unique normal form* (UN, in short).

To establish UN it suffices to establish the *Church-Rosser property* (CR, in short). It states that for all $a, b, c \in A$

$$a \\ * \swarrow \searrow * \\ b \qquad c$$

implies that for some $d \in A$

$$\begin{array}{c}
b & c \\
* \searrow \swarrow * \\
d.
\end{array}$$

Indeed, CR implies UN.

Several important term rewriting systems, including combinatory logic and λ -calculus, have CR, and hence UN, see, e.g., Terese [2003].

In this area the interest in UN means that one is interested in deterministic outcomes *in spite of* the nondeterminism that is present, that is, one aims at showing that the nondeterminism is *inessential*.

3 Angelic nondeterminism

We now proceed with a discussion of formalisms in which addition of nondeterminism allowed one to extend their expressiveness. These formalisms share a characteristic that one identifies successes and failures and only the former count. This kind of nondeterminism was later termed *angelic nondeterminism*.

Nondeterministic finite automata and Turing machines

The first definitions of finite-state automata required deterministic transition functions, as noted in Hopcroft and Ullman [1979]. It took the insight of Rabin and Scott to introduce *nondeterministic* finite automata in their seminal paper Rabin and Scott [1959]. Such an automaton has choices in its moves: at each transition it may select one of several possible next states. They motivated their definition as follows: "The main advantage of these machines is the small number of internal states that they require in many cases and the ease in which specific machines can be described." They proved by their famous *power set construction* (that uses as the set of states the powerset of the original set of states) that for each nondeterministic automaton a deterministic one can be constructed that accepts the same set of finite words, however the deterministic one may have exponentially more states. Crucial is here their definition of acceptance: a nondeterministic automaton *accepts* a word if there is *some* successful run of the automaton from an initial to a final state, processing the word symbol by symbol. Thus only a success counts, while failures do not matter.

This kind of nondeterminism has also been introduced for other types of machines, in particular pushdown automata and Turing machines, leading to the definition of various complexity classes discussed at the end of this section.

McCarthy's ambiguity operator

Probably the first proposal to add nondeterminism to a programming language is due to John McCarthy who introduced in McCarthy [1963] an 'ambiguity operator' amb(x, y) that, given two expressions x and y, nondeterministically returns the value of x or of y when both are defined, and otherwise whichever is defined. (McCarthy did not explain what happens when both x and y are undefined, but the most natural assumption is that amb(x, y) is then undefined, as well.) In particular, amb(1, 2) yields 1 or 2.

McCarthy was concerned with the development of a functional language, so in his formulation programs were expressions, possibly defined by recursion. As an example of a program that uses the ambiguity operator he introduced the function less(n) that assigns to each natural number n any nonnegative integer less than n. The function was defined recursively by:

$$less(n) = amb(n-1, less(n-1)).$$

McCarthy did not discuss this function in detail, but note that its definition involves a subtlety because of its use of undefined values. For the smallest value in its domain, namely 1, we get less(1) = amb(0, less(0)) = 0, since by definition less(0) is undefined. For the next value, so 2, we get less(2) = amb(1, less(1)) = amb(1, 0), which yields 0 or 1. Next, less(3) = amb(2, less(2)) = amb(2, amb(1, 0)), which yields 0 or 1 or 2, etc. This may be the first example of a program that uses nondeterminism.

McCarthy's used his ambiguity operator to extend computable functions by nondeterminism to what he called *computably ambiguous functions*. His work soon inspired first proposals of systems and programming languages that incorporated some form of nondeterminism, mainly to express concisely search problems, see, e.g., Smith and Enea [1973] for an account of some of them.

Sometime later, the authors of Zabih et al. [1987] extended a dialect of Lisp with McCarthy's nondeterministic operator denoted by AMB. The addition of nondeterminism was coupled with a dependency-directed backtracking, triggered by a special expression FAIL that has no value. The resulting language was called SCHEMER. This addition of nondeterminism to Lisp was later discussed in the book Abelson and Sussman [1996] that used a dialect of Lisp called Scheme.

Floyd's approach to nondeterministic programming

Another approach to programming that corresponds to the type of nondeterminism used in the nondeterministic Turing machines was proposed by Robert Floyd in Floyd [1967]. He began his article by a fitting quotation from a famous poem 'The Road Not Taken' by Robert Frost:

Two roads diverged in a yellow wood, And sorry I could not travel both And be one traveler, long I stood And looked down one as far as I could To where it bent in the undergrowth;

Floyd's idea was to add nondeterminism to the conventional flowcharts by

• using a nondeterministic assignment

$$x := choice(t),$$

that assigns to the variable x an arbitrary positive integer of at most the value of the integer expression t,

• labeling all termination points as *success* or *failure*.

Thus augmented flowcharts can generate several execution sequences, however, only those that terminate in a node labeled *success* are considered as the computations of the presented algorithm.

Using a couple of examples Floyd showed how these two additions can lead to simple flowchart programs that search for a solution through exhaustive enumeration and which otherwise would have to be programmed using backtracking combined with appropriate bookkeeping. Jacques Cohen illustrated in Cohen [1979] Floyd's approach using conventional programs in which one uses the above nondeterministic assignment statement and a **fail** statement that corresponds to a node labeled *failure*. Any computation that reaches the **fail** statement terminates improperly (it *aborts*). The label *success* is unneeded as it is implicitly modeled by a terminating computation that does not abort. We call such computations *successful*. In this approach a program is correct if *some* successful computation establishes the assumed postcondition.

We illustrate this approach by means of one of Floyd's examples, the problem of *eight queens* in which one is asked to place 8 queens on the chessboard so that they do not attack each other. The program looks as follows, where a, band c are integer arrays with appropriate bounds, and initialized for all indices to 0:

```
for col := 1 to 8 do

row := choice(8);

if a[row] = 1 \lor b[row + col] = 1 \lor c[row - col] = 1 then fail fi;

a[row] := 1;

b[row + col] := 1;

c[row - col] := 1

od
```

Subscripted variables have the following interpretation:

a[i] = 1 means that a queen was placed in the *i*th row,

b[i] = 1 means that a queen was placed in the *i*th \searrow diagonal,

c[i] = 1 means that a queen was placed in the *i*th \swarrow diagonal,

where the \searrow diagonals are the ones for which the sum of the coordinates is the same and the \swarrow diagonals are the ones for which the difference of the coordinates is the same. Upon successful termination a solution is produced in the form of a sequence of 8 values that are successively assigned to the variable *row*; these values correspond to the placements of the queens in the columns 1 to 8.

Thanks to the nondeterministic assignment and the **fail** statement this program can generate several computations, including ones that abort. The successful computations generate precisely all solutions to the eight queens problem.

Floyd showed that one can convert his augmented flowcharts to conventional flowcharts by means of a generic transformation that boils down to an implementation of backtracking. This way one obtains a deterministic algorithm that generates a solution to the considered problem but it is easy to modify his transformation so that all solutions are generated. Of course, such a transformation can also be defined for the programs considered here.

Logic programming and Prolog

In the early seventies angelic nondeterminism was embraced in a novel approach to programming that combined the idea of *automatic theorem proving* based on the use of relations, with built-in automatic backtracking. It was realized in the programming language Prolog, conceived and implemented by Alain Colmerauer and his team (see the historic account in Colmerauer and Roussel [1996]), while its theoretical underpinnings, called *logic programming*, were provided by Robert Kowalski in Kowalski [1974].

A detailed discussion of logic programming and Prolog is out of scope of this chapter, but to illustrate the nondeterminism present in this approach to programming consider a simple logic programming program that appends two lists. It is defined by means of two *clauses*, the first one unconditional and the second one conditional:

append([], Xs, Xs).

append([X | Xs], Ys, [X | Zs]) \leftarrow append(Xs, Ys, Zs).

Here append is a name of a relation, X, Xs, Ys and Zs are the variables, [] denotes the empty list, and [X | Xs] denotes the list with the head X and the tail Xs.

The first clause states that the result of appending [] and the list Xs is Xs. The second clause is a reverse implication stating that if the result of appending the lists Xs and Ys is Zs, then the result of appending the lists [X | Xs] and Ys is [X | Zs].

In general, a program is a set of clauses which are built from *atomic queries*. In the above program atomic queries are of the form append(s, t, u), where s, t, u are expressions built out of the variables and the constant [] using the list formation operation [.].]. A *query* is a conjunction of atomic queries. A program is activated by executing a query, which is a request to evaluate it w.r.t. the considered program. We do not discuss here the underlying computation model; it suffices to know that the computation searches for an instance of the query that logically follows from the program. If such an instance is found, one says that a query *succeeds* and otherwise that it *fails*.

In logic programming nondeterminism arises in two ways, by the fact that relations can be defined using several clauses, and by the choice of the atomic query to be evaluated first. At the abstract level this is the same form of nondeterminism as the one used in Floyd's approach: a computation succeeds if at all choice points the right selections are made.

Since a query can succeed in several ways, several solutions can be generated. Consider for example the query

append(_, Zs, [mon, tue, wed, thu, fri]), append(Xs, _, Zs),

in which we use Prolog's convention according to which each occurrence of '_' stands for a different (*anonymous*, i.e, irrelevant) variable and the comma is used (between the atomic queries) instead of the conjunction sign. Intuitively, this query stipulates that Xs is a prefix of a suffix of the list [mon, tue, wed, thu, fri]. Successful computations of this query w.r.t. the above program generate in Xs all possible sublists of this list.

In Prolog nondeterminism present in the computation model of logic programming is resolved by stipulating that the first clause and the first atomic query from the left are selected, and by providing a built-in automatic backtracking that allows the computation to recover from a failure.

Does angelic nondeterminism add expressive power?

After this discussion of nondeterministic programming let us look at the idea of angelic nondeterminism through the lens of computing and structural complexity.

We mentioned already in the introduction that nondeterminism was not considered in the original formalizations of the notion of a computation. But once one considers restricted models of computability, nondeterminism naturally arises. An early example is the characterization of formal languages in the Chomsky hierarchy of formal grammars. It distinguishes four types of grammars that correspond to four levels of formal languages of increasing complexity: regular, context-free, context-sensitive, and recursively enumerable languages, denoted by Type-3, Type-2, Type-1, and Type-0, respectively.

The need for nondeterminism arises in connection with the characterization of these classes by means of an automaton. Whereas Type-0 languages at the top of this hierarchy are the ones accepted by the deterministic Turing machines, and Type-3 languages at the bottom, i.e., regular languages, are the ones accepted by the deterministic finite automata, the characterization of the remaining two levels calls for the use of nondeterministic automata or machines.

In particular Type-2 languages, i.e., context-free languages, cannot be accepted by deterministic pushdown automata. A pushdown automaton extends a finite automaton by an unbounded stack or pushdown list that is manipulated in a 'first in – last out' fashion while scanning a given input word letter by letter. Such an automaton is called *deterministic* if for each control state, each symbol at the top of the stack, and each input symbol, it has at most one possible move; it is called *nondeterministic* if more than one move is allowed.

A standard example is the language of all *palindromes* over letters a and b, i.e., words that read the same forward and backward, like *abba*. This language is context-free, i.e., it can be generated by a context-free grammar, but it cannot be accepted by a deterministic pushdown automaton. The intuitive reason is that while checking an input word letter by letter, one has to 'guess' when the middle of the word has been reached, so that one can test that from now on the letters occur in the reverse order, by referring to the constructed pushdown list.

However, context-free languages can be characterized by means of nondeterministic pushdown automata. Such a characterization refers to angelic nondeterminism because it states that a word is generated by the language iff it can be accepted by *some* computation of the automaton.

Similarly, nondeterminism is used to characterize Type-1 languages, i.e., context-sensitive languages: they are the ones that can be recognized by linear bounded nondeterministic Turing machines.

While deterministic and nondeterministic Turing machines accept the same class of languages, the Type-0 languages, there is a difference when time complexity is considered. Probably the most known are the complexity classes P and NP of problems that can be solved in polynomial time by deterministic Turing machines and nondeterministic Turing machines, respectively. The class NP was introduced by Stephen Cook in Cook [1971] and Leonid Levin in Levin

[1973]. The intuition is that a problem is in NP if it can be solved by first (nondeterministically) guessing a candidate solution and then checking in polynomial time whether it is indeed a solution. Following the paradigm of angelic nondeterminism, wrong guesses do not count. The famous open problem posed by Cook is whether P = NP holds.

In contrast, when instead of time, space is considered as the complexity measure, it is known that there is no difference between the resulting classes PSPACE and NSPACE of problems that can be solved in polynomial space by deterministic Turing machines and nondeterministic Turing machines, respectively. This is a consequence of a result by Savitch [1970].

We conclude that the addition of angelic nondeterminism can, but does not have to, increase expressive power of the considered model of computability.

4 Guarded commands

One of us (KRA) met Edsger Dijkstra for the first time in Spring 1975, while looking for an academic job in computer science in the Netherlands. During a meeting at his office at the Technical University of Eindhoven, Dijkstra handed him a copy of his EWD472 titled *Guarded Commands, Nondeterminacy and Formal Derivation of Programs.* It appeared later that year as Dijkstra [1975]. This short, five-pages long, paper introduced two main ideas: a small programming language, now called *guarded commands*, and its semantics, now called the *weakest precondition semantics.* Both were new ideas of great significance.

In this chapter we focus on the guarded commands; Chapters Gries [2022] and Hähnle [2022] in the book Apt and Hoare [2022] discuss the weakest precondition semantics. The essence of guarded commands boils down to two new programming constructs:

• alternative command

$$S ::= \mathbf{if} \ B_1 \to S_1 \| \dots \| B_n \to S_n \ \mathbf{fi},$$

• repetitive command

$$S ::= \operatorname{do} B_1 \to S_1 \| \dots \| B_n \to S_n \operatorname{od}.$$

We sometimes abbreviate these commands to

if
$$[\![n_{i=1} \ B_i \to S_i \ \mathbf{fi} \ \mathbf{and} \ \mathbf{do} \]\![n_{i=1} \ B_i \to S_i \ \mathbf{od}.$$

A Boolean expression B_i within S is called a guard and the construct $B_i \to S_i$ is called a guarded command.

The symbol $[\!]$ represents a nondeterministic choice between the guarded commands $B_i \to S_i$. The alternative command

if
$$B_1 \to S_1 |\!| \dots |\!| B_n \to S_n$$
 fi

is executed by executing a statement S_i for which the corresponding guard B_i evaluates to true. There is no rule saying which statement among those whose guard evaluates to true should be selected. If all guards B_i evaluate to false, the alternative command *aborts*.

The selection of guarded commands in the context of a repetitive command

do
$$B_1 \to S_1 \llbracket \ldots \llbracket B_n \to S_n$$
 od

is performed in a similar way. The difference is that after termination of a selected statement S_i the whole command is repeated starting with a new evaluation of the guards B_i . Moreover, in contrast to the alternative command, the repetitive command properly terminates when all guards evaluate to false.

Dijkstra did not establish the notation of guarded commands directly. Two earlier EWDs reveal that he first considered other options, also about their intended semantics.

In EWD398 Dijkstra [1973a] he first used ',' to separate guarded commands, but changed it halfway to [], reporting criticism of Don Knuth and stating "this whole report <u>is</u> an experiment in notation!" Also, he wrote B : S instead of $B \to S$ adopted in Dijkstra [1975] and used by him thereafter. Further, for the repetitive command he wrote that

In the case of more than one executable command, it is again undefined which one will be selected, we postulate, however, that then they will be selected in "fair random order", i.e. we disallow the non-determinacy permanent neglect of a permanently executable guarded command from the list.

However, a day later, in EWD399 Dijkstra [1973b], he admitted that this decision

[...] was a mistake: for such constructs we prefer now not to exclude non-termination. It is just too tricky if the termination — and in particular: the proof of the termination— has to rely on the fair randomness of the selection and we had better restrict ourselves to constructs w[h]ere each guarded command, when executed, implies a further approaching of the terminal state.

We shall return to this problem of fairness shortly. But first let us focus on the main feature of guarded commands, the nondeterminism they introduce. However, this nondeterminism is of a different type than the one we discussed so far: by definition a guarded command program establishes the desired postcondition if *all* possible executions establish it. This kind of nondeterminism was later termed *demonic nondeterminism*.

This seems at the first sight like a flawed decision: why should one complicate the matters by adding to the program more possible executions paths, when one will suffice? But, as we shall soon see, there are good reasons for doing it. An often cited example in favour of the guarded commands language is the formalization of Euclid's algorithm that computes the greatest common divisor (gcd) of two positive integers x and y

do
$$x > y \rightarrow x := x - y \parallel x < y \rightarrow y := y - x$$
 od

that terminates with the gcd of the initial values of x and y equal to their final, common, value.

However, this program is not nondeterministic: for any initial value of the variables x and y there is only one possible program execution. This program actually illustrates something else: that guarded commands allow one to write more elegant algorithms. Here the variables x and y are treated symmetrically which is not the case when a deterministic program is used.

In another simple example from Dijkstra's paper one is asked to compute the maximum max of two numbers, x and y:

if
$$x \ge y \to max := x \parallel y \ge x \to max := y$$
 if

It illustrates the nondeterminism in a mildest possible form: when x = y two executions are possible, but the outcome is still the same.

A slightly more involved example of a nondeterministic program with a deterministic outcome is Dijkstra's solution to the following problem: assign to the variables x_1, x_2, x_3 , and x_4 an ordered permutation of the values X_1, X_2, X_3 , and X_4 , i.e., one such that $x_1 \leq x_2 \leq x_3 \leq x_4$ holds. The program uses a parallel assignment that forms part of the guarded commands language:

$$\begin{array}{l} x_1, x_2, x_3, x_4 := X_1, X_2, X_3, X_4; \\ \textbf{do} \ x_1 > x_2 \to x_1, x_2 := x_2, x_1 \\ \| \ x_2 > x_3 \to x_2, x_3 := x_3, x_2 \\ \| \ x_3 > x_4 \to x_3, x_4 := x_4, x_3 \\ \textbf{od} \end{array}$$

Upon exit all guards evaluate to false, i.e., $x_1 \leq x_2 \leq x_3 \leq x_4$ holds, as desired. The relevant invariant is that x_1, x_2, x_3, x_4 is a permutation of X_1, X_2, X_3, X_4 ,

Finally, the following example of Dijkstra results in a program with a nondeterministic outcome. The problem is to find for a fixed n > 0 and a fixed integer-valued function f defined on $\{0, ..., n-1\}$ a maximum point of f, i.e., a value k such that

$$k \in \{0, \dots, n-1\} \land \forall i \in \{0, \dots, n-1\} : f(k) \ge f(i).$$

A simple solution is the following program:

 \mathbf{od}

It scans the values of f starting with the argument 0, updates the value of k in case a new maximum is found (when f(j) < f(k)), and optionally updates the value of k in case another current maximum is found (when f(j) = f(k)). The relevant invariant is here

$$k \in \{0, \dots, j-1\} \land j \le n \land \forall i \in \{0, \dots, j-1\} : f(k) \ge f(i).$$

Indeed, it is established by the initial assignment, maintained by each loop iteration, and upon termination it implies the desired condition on k. Note that the program can compute in k any maximum point of f.

All these small examples (and there are no others in Dijkstra's paper) do not provide convincing reasons for embracing nondeterminism provided by the guarded commands language. A year after Dijkstra [1975] appeared, Dijkstra published his book Dijkstra [1976] in which he derived several elegant guarded command programs, including a more efficient version of the above program which avoids the recomputation of f(j) and f(k). But inspecting these programs we found only a few examples in which guards were not mutually exclusive and only two programs with a nondeterministic outcome.

So why then has demonic nondeterminism, as present in the guarded command language, turned out to be so influential? In what follows we discuss subsequent developments that provide some answers to this question. Many accounts of the guarded command language discuss it, as Dijkstra originally did, together with its weakest precondition semantics. But to appreciate the nondeterminism Dijkstra introduced in our view it is useful to separate the language from its weakest precondition semantics.

5 Some considerations on guarded commands

Dijkstra's famous article Go To Statement Considered Harmful Dijkstra [1968a] shows that he was aware of an ancestor of his alternative command in the form of a conditional expression $(B_1 \rightarrow e_1, \ldots, B_n \rightarrow e_n)$ introduced by John McCarthy in McCarthy [1963]. Here B_is are Boolean expressions and e_is are expressions. The Boolean expressions do not need to be mutually exclusive, but the conditional expressions are deterministic: when executed the $(B_1 \rightarrow e_1, \ldots, B_n \rightarrow e_n)$ yields the value of the first expression e_i for which B_i evaluates to true. When all B_is evaluate to false, the conditional expression is supposed to be undefined. So $(B_1 \rightarrow e_1, \ldots, B_n \rightarrow e_n)$ is a shorthand for a nested **if-thenelse** statement.

Dijkstra's explicit introduction of an abort, as opposed to McCarthy's reference to 'undefined', is useful because it provides a simple way of implementing an **assert** B statement that checks whether assertion B holds and causes an abort when this is not the case.

McCarthy worked within the framework of a functional language, so he was constrained to use recursion instead of a looping construct. As a result Euclid's algorithm is formalized in his notation as follows:

 $gcd(m,n) = (m > n \rightarrow gcd(m-n,n), n > m \rightarrow gcd(m,n-m), m = n \rightarrow m),$

which is less elegant than Dijkstra's solution, due to the need for the final component of the conditional expression.

In contrast, as already explained, Dijkstra did not prescribe any order in which the guards are selected and ensured that his repetitive command was not defined using the alternative command. In Dijkstra [1975] he motivated his introduction of nondeterminism (called by him nondeterminacy) as follows:

Having worked mainly with hardly self-checking hardware, with which nonreproducing behavior of user programs is a very strong indication of a machine malfunctioning, I had to overcome a considerable mental resistance before I found myself willing to consider nondeterministic programs seriously. [...] Whether nondeterminacy is eventually removed mechanically—in order not to mislead the maintenance engineer—or (perhaps only partly) by the programmer himself because, at second thought, he does care—e.g, for reasons of efficiency—which alternative is chosen is something I leave entirely to the circumstances. In any case we can appreciate the nondeterministic program as a helpful stepping stone.

But soon he overcame this resistance and one year later he wrote:

Eventually, I came to regard nondeterminacy as the normal situation, determinacy being reduced to a —not even very interesting special case. [Dijkstra, 1976, page xv]

McCarthy's semantics of conditional expression can be viewed as an example of such a 'mechanical removal' of nondeterminism. However, keeping nondeterminism intact often leads to simpler and more natural programs even if the outcome is deterministic. In some programs the considered alternatives do not need to be mutually exclusive as long as all cases are covered.

A beautiful example was provided by David Gries in his book Gries [1981]. Consider the following problem due to Wim Feijen. (We follow here the presentation of Gries.)

Given are three magnetic tapes, each containing a list of different names in alphabetical order. The first contains the names of people working at IBM Yorktown Heights, the second the names of students at Columbia University and the third the names of people on welfare in New York City. It is known that at least one person is on all three lists. The problem is to locate the alphabetically first such person.

In Gries [1981] the following elegant program solving this problem was systematically derived. We assume here that the lists of names are given in the form of ordered arrays a[0:M], b[0:M], and c[0:M]:

```
\begin{array}{l} i := 0; \ j := 0; \ k := 0; \\ \mathbf{do} \ a[i] < b[j] \rightarrow \ i := i+1 \\ \| \ b[j] < c[k] \rightarrow \ j := j+1 \\ \| \ c[k] < a[i] \rightarrow \ k := k+1 \end{array}
```

Note that upon termination of the loop a[i] = b[j] = c[k] holds. The appropriate invariant is

$$0 \le i \le i_0 \land 0 \le j \le j_0 \land 0 \le k \le k_0 \land r$$

where r states that the arrays a[0:M], b[0:M], and c[0:M] are ordered, $i_0, j_0, k_0 \leq M$ and (i_0, j_0, k_0) is the lexicographically smallest triple such that $a[i_0] = b[j_0] = c[k_0].$

This program uses nondeterministic guards, so various computations are possible. Still, it has a deterministic outcome.

In general, as soon as two or more guards are used in a loop, in the customary, deterministic, version of the program one is forced to use a, possibly nested, **if-then-else** statement, like in McCarthy's 'determinisation' approach. It imposes an evaluation order of the guards, destroys symmetry between them, and does not make the resulting programs easier to verify.

6 Modeling parallel programs

Concurrent programs, introduced in Dijkstra's *Cooperating sequential processes* paper Dijkstra [1968b], can share variables, which makes it difficult to reason about them. Therefore, starting with Ashcroft and Manna [1971] and Flon and Suzuki [1978, 1981], various authors proposed to analyze them at the level of nondeterministic programs, where the nondeterminism reflects existence of various component programs. Such a reduction is possible if one assumes that no concurrent reading and writing of variables takes place.

Using guarded commands it is possible to make the link between these two classes of programs explicit by a transformation. The precise transformation is a bit laborious, see Flon and Suzuki [1978], so we illustrate it by an example taken from Apt et al. [2009]. Consider the following concurrent program due to Owicki and Gries [1976] that searches for a positive value in an integer array ia[0:N]:

$$i := 1; \ j := 2; \ oddtop := N + 1; \ eventop := N + 1; [S_1 || S_2]; k := min(oddtop, eventop),$$

where S_1 and S_2 are deterministic components S_1 and S_2 scanning the odd and the even subscripts of ia, respectively:

$$S_1 \equiv a$$
: while $i < min(oddtop, eventop)$ do
b: if $ia[i] > 0$ then c: $oddtop := i$ else d: $i := i + 2$ for od

and

$$S_2 \equiv a$$
: while $j < min(oddtop, eventop)$ do
 b : if $ia[j] > 0$ then c : $eventop := j$ else d : $j := j + 2$ for
od

Upon termination of both components, the minimum of two shared integer variables *oddtop* and *eventop* is checked. The labels a, b, c, d, and e are added here to clarify the transformation. The parallel composition $S \equiv [S_1 || S_2]$ is transformed into the following guarded commands program T(S) with a single repetitive command that employs the control variables cv_1 and cv_2 for S_1 and S_2 that can assume the values of the labels:

$$T(S) \equiv cv_1 := a; cv_2 := a;$$

$$do cv_1 = a \land i < min(oddtop, eventop) \rightarrow cv_1 := b$$

$$\| cv_1 = a \land \neg(i < min(oddtop, eventop)) \rightarrow cv_1 := e$$

$$\| cv_1 = b \land ia[i] > 0 \rightarrow cv_1 := c$$

$$\| cv_1 = b \land \neg(ia[i] > 0) \rightarrow cv_1 := d$$

$$\| cv_1 = c \rightarrow oddtop := i; cv_1 := a$$

$$\| cv_2 = a \land j < min(oddtop, eventop) \rightarrow cv_2 := b$$

$$\| cv_2 = a \land \neg(j < min(oddtop, eventop)) \rightarrow cv_2 := e$$

$$\| cv_2 = b \land ia[j] > 0 \rightarrow cv_2 := c$$

$$\| cv_2 = b \land \neg(ia[j] > 0) \rightarrow cv_2 := d$$

$$\| cv_2 = d \rightarrow j := j + 2; cv_2 := a$$

$$\| cv_2 = d \rightarrow j := j + 2; cv_2 := a$$

$$d;$$

if $cv_1 = e \land cv_2 = e \rightarrow skip$ fi

Note that the repetitive command exhibits nondeterminism. For example, when $cv_1 = cv_2 = a$, two guarded commands can be chosen next. This corresponds to the *interleaving semantics* of concurrency that we assume here. When the repetitive command has terminated, the final alternative command checks whether this termination is the one intended by the original concurrent program S. This is the case when both cv_1 and cv_2 store the value e. In the current example, this check is trivially satisfied and thus the alternative command could be omitted.

However, for concurrent programs with synchronization primitives, a termination of the repetitive command may be due to a deadlock in the concurrent program. Then the final alternative command is used to transform the deadlock into a failure, indicating an undesirable state at the level of nondeterministic programs. For details of this transformation we refer to Chapter 10 of Apt et al. [2009].

This transformation allows us to clarify that the nondeterminism resulting from parallelism is the one used in the guarded commands language. However, this example also reveals a drawback of the transformation: the structure of the original parallel program gets lost. The resulting nondeterministic program represents a single loop at the level of an assembly language with atomic actions explicitly listed. The assignments to the control variables correspond to **go to** statements, which explains why reasoning about the resulting program is difficult. Interestingly, this problem does not arise for the transformation of the CSP programs that we give in the next section.

7 Communicating Sequential Processes and their relation to guarded commands

Dijkstra's quoted statement, "In any case we can appreciate the nondeterministic program as a helpful stepping stone" suggests that he envisaged some extensions of the guarded command language. But in his book Dijkstra [1976] he only augmented it with local variables by providing an extensive notation for various uses of local and global variables, and added arrays. In his subsequent research he only used the resulting language.

However, his discussion of the guarded commands program formalizing Euclid's algorithm suggests that he also envisaged some connection with concurrency. He suggested that the program could be viewed as a synchronization of two cyclic processes **do** x := x - y **od** and **do** y := y - x **od** in such a way that the relation $x > 0 \land y > 0$ is kept invariantly true. Still, he did not pursue this idea further.

Subsequent research showed that guarded command programs can be viewed as a natural layer lying between deterministic and concurrent programs. This was first made clear in 1978 by Tony Hoare who introduced in Hoare [1978] an elegant language proposal for distributed programming that he called Communicating Sequential Processes (abbreviated to CSP) in clear reference to Dijkstra's *Cooperating sequential processes* paper Dijkstra [1968b].

Hoare stated seven essential aspects of his proposal, mentioning as the first one Dijkstra's guarded commands "as the sole means of introducing and controlling nondeterminism". The second one also referred to Dijkstra, namely to his parallel command, according to which, "All the processes start simultaneously, and the parallel command ends only when they are all finished." It is useful to discuss CSP in some detail to see how each of these two aspects results in the same type of nondeterminism.

In Dijkstra's cooperating sequential processes model processes communicate with each other by updating global variables. By contrast, in CSP processes communicate solely by means of the *input* and *output* commands, which are atomic statements that are executed in a synchronized fashion. So CSP processes do not share variables.

For the purpose of communication CSP processes have names. The input command has the form P?x, which is a request to process (named) P to provide a value to the variable x, while the output command has the form Q!t, which is a granting of the value of the expression t to process (named) Q. When the types of x and t match and the processes refer to each other, we say that the considered input and output commands *correspond*. They are then executed simultaneously; the effect is that of executing the assignment $x := t.^1$

In CSP a single input command is also allowed to be part of a guard. The restriction to input commands was dictated by implementation considerations. But once it was clarified how to implement the use of output commands in

¹Note that not all assignments can be modelled this way. For instance, the assignment x := x + 1 cannot be reproduced since processes do not share variables.

guards, they were admitted as part of a guard, as well. So, in the sequel we admit both input and output commands in guards. Thus guards are of the form $B; \alpha$, where B is a Boolean expression and α is an input or output command, i/o command for short. We assume that such an extended guard fails when the Boolean part evaluates to false.²

To illustrate the language consider an example, taken from Apt et al. [2009], which is a modified version of an example given in Hoare [1978]. In what follows we refer to the repetitive commands of CSP as **do** loops.

We wish to transmit a sequence of characters from the process SENDER to the process RECEIVER with all blank characters (represented by ' ') deleted. To this end we employ an intermediary process FILTER and consider a distributed program

[SENDER || FILTER || RECEIVER]

The sequence of characters is initially stored by the process *SENDER* in the array a[0: M - 1] of characters, with '*' as the last character. The process *FILTER* uses an array b[0: M - 1] of characters as an intermediate store for processing the character sequence and the process *RECEIVER* has an array c[0: M - 1] of the same type to store the result of the filtering process. For coordinating its activities the process *FILTER* uses two integer variables *in* and *out* pointing to elements in the array a[0: M - 1]. The processes are defined as follows:

Note that the processes *SENDER* and *RECEIVER* are deterministic, in the sense that each extended guarded command either has just one guard or the Boolean parts of the used guards are mutually exclusive (this second case does not occur here), while *FILTER* is nondeterministic as it uses a **do** loop with two extended guards the Boolean parts of which are not mutually exclusive. They represent two possible actions for *FILTER*: to communicate with *SENDER* or with *RECEIVER*.

Hoare presented in his article several elegant examples of CSP programs. In some of them the processes are deterministic. But even then, if there are four or

 $^{^{2}}$ Hoare also allowed an extended guard to fail when its i/o command refers to a process that terminated. For simplicity do not adopt this assumption here.

more processes, the resulting program is nondeterministic, since it admits more than one computation.

By assumption a CSP program is correct if all of its computations properly terminate in a state that satisfies the assumed postcondition. So this is exactly demonic nondeterminism, like in the case of parallel programs.

This makes it possible to translate CSP programs in a simple way to guarded command programs. In Apt et al. [2009] we provided such a transformation for a fragment of CSP, in which the above example is written. The CSP programs in this fragment are of the form

$$S \equiv [S_1 \| \dots \| S_n],$$

where each process S_i is of the form

$$S_i \equiv S_{i,0}; \text{ do } [\!]_{i=1}^{m_i} B_{i,j}; \alpha_{i,j} \rightarrow S_{i,j} \text{ od},$$

each $S_{i,j}$ is a guarded command program, each $B_{i,j}$ is a Boolean expression, and each $\alpha_{i,j}$ is an i/o command. So each process S_i has a single **do** loop in which i/o commands appear only in the guards. No further i/o commands are allowed in the initialization part $S_{i,0}$ or in the bodies $S_{i,j}$ of the guarded commands.

As shown in Apt et al. [1987] and Zöbel [1988] each CSP program can be transformed into a program in this fragment by introducing some control variables.

As abbreviation we introduce

$$\Gamma = \{ (i, j, r, s) \mid \alpha_{i,j} \text{ and } \alpha_{r,s} \text{ correspond and } i < r \}.$$

According to the CSP semantics two generalized guards from different processes can be passed jointly when their i/o commands correspond and their Boolean parts evaluate to true. Then the communication between the i/o commands takes place. The effect of a communication between two corresponding i/o commands $\alpha_1 \equiv P?x$ and $\alpha_2 \equiv Q!t$ is the assignment x := t. Formally, for two such commands we define

$$Eff(\alpha_1, \alpha_2) \equiv Eff(\alpha_2, \alpha_1) \equiv x := t.$$

We transform S into the following guarded commands program T(S):

$$\begin{array}{lll} T(S) &\equiv& S_{1,0}; \ & \dots; \ S_{n,0}; \\ & \mathbf{do} \parallel_{(i,j,r,s)\in\Gamma} B_{i,j} \wedge B_{r,s} \rightarrow & Eff(\alpha_{i,j},\alpha_{r,s}); \\ & & S_{i,j}; \ S_{r,s} \end{array}$$

where we use of elements of Γ to list all guards in the **do** loop. In the degenerate case when $\Gamma = \emptyset$ we drop this loop from T(S).

For example, for $SFR \equiv [SENDER \parallel FILTER \parallel RECEIVER]$ we obtain the following guarded commands program:

$$\begin{split} T(SFR) \; \equiv \; i := 0; \; in := 0; \; out := 0; \; x := ` `; \; j := 0; \; y := ` `; \\ \mathbf{do} \; i \neq M \land x \neq `*` \rightarrow x := a[i]; \; i := i + 1; \\ & \mathbf{if} \; x = ` ` \rightarrow \; skip \\ & \parallel \; x \neq ` ` \rightarrow \; b[in] := x; \; in := in + 1 \\ \mathbf{fi} \\ & \parallel \; out \neq in \land y \neq `*` \rightarrow y := b[out]; \; out := out + 1; \; c[j] := y; \; j := j + 1 \\ & \mathbf{od} \end{split}$$

The semantics of S and T(S) are not identical because their termination behavior is different. However, the final states of properly terminating computations of S and the final states of properly terminating computations of T(S)that satisfy the condition

$$TERM \equiv \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m_i} \neg B_{i,j}$$

coincide. An interested reader can consult Chapter 11 of Apt et al. [2009].

The above transformation makes precise what we already mentioned when discussing an example CSP program: the CSP language introduces nondeterminism in two ways. The first one comes from allowing guarded commands; in the transformed program these are the programs $S_{a,b}$. The second one results from synchronous communication, modelled in the transformation by means of the outer repetitive command. So both ways are instances of demonic nondeterminism.

Thanks to the special form of CSP programs the transformed program does not introduce any new variables. As a result this transformation suggests a simple way to reason about correctness of CSP programs, by considering the translated guarded commands program, see Chapter 11 of Apt et al. [2009].

8 Fairness

One of the programs in Dijkstra [1976] with a nondeterministic outcome is the following one:

$$\begin{array}{l} goon := \mathbf{true}; \ x := 1; \\ \mathbf{do} \ goon \to x := x + 1 \\ \| \ goon \to goon := \mathbf{false} \\ \mathbf{od} \end{array}$$

The problem Dijkstra was addressing was that of writing a program that sets x to any natural number. He concluded using his weakest precondition semantics that no guarded commands program exists that sets x to any natural number *and* always terminates. Note that the above program can set x to an arbitrary natural number but also can diverge.

As noted in Plotkin [1976], Dijkstra's conclusion can be obtained in a more direct way by appealing to any operational semantics that formalizes the notion of a computation, by representing the computations in the form of a tree. Branching models the execution of a guarded command; each branch corresponds to a successful evaluation of a guard. Given an input state of a guarded command program we have then a finitely branching tree representing all possible computations. Let us call it a *computation tree*.

Denote a program that sets x to any natural number by x :=?. Suppose that it can be represented by a guarded commands program S. Then for any input state the computations of S form a finitely branching computation tree with infinitely many leaves, each of them corresponding to a different natural number assigned to x. We can now appeal to König's Lemma which states that any finitely branching infinite tree has an infinite path König [1927]. It implies that in every computation tree of S an infinite computation exists, i.e., that for every input state the program S can diverge.

Dijkstra's proof contained in Chapter 9 of Dijkstra [1976] proceeds differently. He first showed that his predicate transformer wp is continuous in the predicate argument and then that the specification of x :=? in terms of wp contradicts continuity. In his terminology, the program x :=? introduces unbounded nondeterminism, which means that no a priori upper bound for the final value of x can be given.

The program x :=? occupied his attention a number of times. In EWD673 published as Dijkstra [1982b] he noted that to prove termination of guarded command programs augmented with the program x :=? it does not suffice to use integer-valued bound functions and one has to resort to well-founded relations. In turn, in EWD675 published as Dijkstra [1982c] he noticed that the converse implication holds, as well: the existence of a program for which wp is not continuous implies the existence of a program with unbounded nondeterminism.

Soon more in-depth studies of the program x :=? and the consequences of its addition to deterministic programs followed. In particular, Chandra showed in Chandra [1978] that the halting problem for programs admitting x :=? is Π_1^1 complete, so of higher complexity than the customary halting problem for computable functions (see also Spaan et al. [1989] for a further discussion of this problem), while Back [1980] and Boom [1982] advocated use of x :=? as a convenient form of program abstraction that deliberately ignores the details of an implementation. One may note here that Hilbert's ϵ notation essentially serves a similar purpose: $\epsilon x \phi$ picks an x that satisfies the formula ϕ . We wish to remain within the realm of guarded commands, so limit ourselves to a clarification of the relation of the program x :=? with the notion of fairness.

If we assume that the guards are selected in 'fair random order', then the program given at the beginning of this section always terminates and can set x to any natural number. Here fairness refers to *weak fairness*, or *justice*, which requires that each guard that continuously evaluates to true is eventually chosen. So the assumption of weak fairness for guarded commands allows us to implement the program x := ?.

As shown in Boom [1982] and independently, though later, in Apt and Olderog [1983] the converse holds, as well. We follow here a presentation from the latter paper. We call a guarded commands program *deterministic* if in each alternative or repetitive command used in it the guards are mutually exclusive. We

call a guarded commands program

$$S \equiv S_0; \text{ do } [n_{i=1}^n B_i \to S_i \text{ od}]$$

one-level nondeterministic if S_0, S_1, \ldots, S_n are deterministic programs.

Given now such a one-level nondeterministic program we transform it into the following guarded commands program that uses the x :=? programs:

$$\begin{array}{rcl} T_{wf}(S) &\equiv& S_0; \ z_1 :=?; \ \ldots; \ z_n :=?; \\ & \mathbf{do} \ \|_{i=1}^n \ B_i \wedge z_i = min \ \{z_k \mid 1 \leq k \leq n\} \rightarrow \\ & z_i :=?; \\ & \mathbf{for \ all} \ j \in \{1, \ldots, n\} \setminus \{i\} \ \mathbf{do} \\ & \mathbf{if} \ B_j \ \mathbf{then} \ z_j := z_j - 1 \ \mathbf{else} \ z_j :=? \ \mathbf{fi} \\ & \mathbf{od}; \\ & S_i \\ & \mathbf{od}, \end{array}$$

where z_1, \ldots, z_n are integer variables that do not occur in S.

Intuitively, the variables z_1, \ldots, z_n represent priorities assigned to the *n* guarded commands in the repetitive command of *S*. A guarded command *i* has higher priority than a command *j* if $z_i < z_j$. Call a guarded command *enabled* if its guard evaluates to true.

Initially, the commands are assigned arbitrary priorities. During each iteration of the transformed repetitive command an enabled command with the maximal priority, i.e., with the minimum value of z_i , is selected. Subsequently, its priority gets reset arbitrarily, while the priorities of other commands are appropriately modified: if the command is enabled then its priority gets increased and otherwise it gets reset arbitrarily. The idea is that by repeatedly increasing the priority a continuously enabled guarded command we ensure that it will be eventually selected.

This way the transformation models weak fairness. More precisely, if we ignore the values of the variables z_1, \ldots, z_n , the computations of $T_{wf}(S)$ are exactly the weakly fair computations of S. A similar transformation can be shown to model a more demanding form of fairness (called *strong fairness* or *compassion*) according to which each guard that infinitely often evaluates to true is also infinitely often selected. An interested reader can consult Apt and Olderog [1983] or Chapter 12 of Apt et al. [2009].

But are there some interesting guarded command programs for which the assumption of fairness is of relevance? The answer is 'yes'. A nice example was provided to us some time ago by Patrick Cousot who pointed out that a crucial algorithm in their landmark paper Cousot and Cousot [1977] which introduced the idea of abstract interpretation relies on fairness. The authors were interested in computing a least fixed point of a monotonic operator in an asynchronous way by means of so-called chaotic iterations.

Recall that a *partial order* is a pair (A, \sqsubseteq) consisting of a set A and a reflexive, antisymmetric and transitive relation \sqsubseteq on A. Consider the *n*-fold

Cartesian product A^n of A for some $n \ge 2$ and extend the relation \sqsubseteq componentwise from A to A^n . Then (A^n, \sqsubseteq) is a partial order.

Next, consider a function

$$F: A^n \to A^n,$$

and the *i*th component functions $F_i : A^n \to A$, where $i \in \{1, ..., n\}$, each defined by

$$F_i(\bar{x}) = y_i$$
 iff $F(\bar{x}) = \bar{y}_i$

Suppose now that F is *monotonic*, that is, whenever $\bar{x} \sqsubseteq \bar{y}$ then $F(\bar{x}) \sqsubseteq F(\bar{y})$. Then the functions F_i are monotonic, as well.

Further, assume that A^n is finite and has the \sqsubseteq -least element that we denote by \emptyset . By the Knaster and Tarski Theorem Tarski [1955] F has a \sqsubseteq -least fixed point $\mu F \in A^n$. As in Cousot and Cousot [1977] we wish to compute μF asynchronously. This is achieved by means the following guarded commands program:

$$\bar{x} := \emptyset;$$

do $\llbracket_{i=1}^n \bar{x} \neq F(\bar{x}) \rightarrow x_i := F_i(\bar{x})$ od

This program can diverge, but it always terminates under the assumption of weak fairness. This is a consequence of a more general theorem proved in Cousot and Cousot [1977]. An assertional proof of correctness of this program under the fairness assumption is given in Apt et al. [2009].

Dijkstra had an ambiguous attitude to fairness. As noted in Chapter Emerson [2022] of the book Apt and Hoare [2022] Dijkstra stated in his EWD310, that appeared as Dijkstra [1971], that sequential processes forming a parallel program should "proceed with speed ratios, unknown but for the fact that the speed ratios would differ from zero" and referred to this property as 'fairness'.

In EWD391 dating from 1973 and published as Dijkstra [1982a], when introducing self-stabilization he wrote:

In the middle of the ring stands a demon, each time giving, in "fair random order" one of the machines the command "to adjust itself". (In "fair random order" means that in each infinite sequence of successive commands issued by the d[a]emon, each machine has received the command to adjust itself infinitely often.)

In the two-page journal publication Dijkstra [1974] that soon followed the qualification 'fair random order' disappeared, but 'daemon' that ensures it remained:

In order to model the undefined speed ratios of the various machines, we introduce a central daemon $[\ldots]$.

However, as we have seen when discussing the origin of guarded commands, he rejected in EWD399 Dijkstra [1973b] fairness at the level of guarded commands. In EWD798 Dijkstra [1981] one can find the following revealing comment: David Park (Warwick University) spoke as a last minute replacement for Dana Scott on "Fairness". The talk was well-prepared and carefully delivered, but I don't care very much for the topic.

Further, in his EWD1013 Dijkstra [1987], titled *Position paper on "fairness*", he plainly turned against fairness and ended his informal discussion by stating that "My conclusion [...] is that fairness, being an unworkable notion, can be ignored with impunity."

It is easy to check that the transformation given in Section 6 translates fairness for parallel programs assumed in Dijkstra [1971] to weak fairness for guarded commands. We are bound to conclude that Dijkstra's opinions on fairness were not consistent over the years.

At the time Dijkstra wrote his controversial note Dijkstra [1987] fairness was an accepted and a well-studied concept, see, e.g., Francez [1986]. The note did not change researchers' perception. It was soon criticized, in particular by Leslie Lamport and Fred Schneider who concluded in Lamport and Schneider [1988] that Dijkstra's arguments against fairness apply equally well against termination, or more generally, against any liveness notion.

9 Nondeterminism: further developments

Dijkstra's guarded commands language was not the last word on nondeterminism in computer science. Subsequent developments, to which he did not contribute, brought new insights, notably by clarifying the consequences of both angelic and demonic nondeterminism in the context of parallelism. In what follows we provide a short account of this subject.

Taxonomy of nondeterminism

Among several papers dealing with nondeterminism in the wake of Dijkstra's guarded commands we would like to single out two. In Harel and Pratt [1978] Harel and Pratt investigated nondeterministic programs in the context of Dynamic Logic and related their work to the weakest precondition approach of Dijkstra. In their approach the programs were built up from assignments to simple variables using a set of basic operators: \cup for nondeterministic choice and ; for sequential composition, B? for testing a Boolean condition B, and * for iteration. An alternative command if $[]_{i=1}^n B_i \to S_i$ fi can be viewed as an abbreviation for the program $\cup_{i=1}^n (B_i?; S_i)$ and a repetitive command do $[]_{i=1}^n B_i \to S_i$ od an abbreviation for the program $((\bigvee_{i=1}^n B_i?); (\cup_{i=1}^n (B_i?; S_i)))^*; \neg(\bigvee_{i=1}^n B_i?)$, see, e.g., de Bakker [1980].

Semantically, each program denotes a binary relation on states, augmented with the symbols \perp representing *divergence* (nonterminating computations) and f representing *failure*, i.e., a test evaluating to false without having any immediate alternative to pursue. Such a relation describes the input-output behavior of a given program. It is defined by induction on the structure of programs. For

example, the relation associated with the program $S_1 \cup S_2$ is the union of the relations associated with the programs S_1 and S_2 .

The input/output relation does not describe how it is computed in a stepby-step manner. When executed in a given state, the program $S_1 \cup S_2$ chooses either S_1 or S_2 to compute the successor state. For a given initial state, these nondeterministic choices can be systematically represented in a *computation* tree. Harel and Pratt distinguished four methods how such a computation tree can be traversed: (1) depth first, (2) depth first with backtracking when a failure state in encountered, (3) breadth first, (4) breadth first combined with ignoring failure states. For each method, they showed how to express the notion of total correctness in dynamic logic.

In the context of algebraic specifications of programming languages, Broy and Wirsing considered different kinds of nondeterminism in their paper Broy and Wirsing [1981]. They called them: (1) backtrack nondeterminism, (2) choice nondeterminism, (3) unbounded nondeterminism, and (4) loose nondeterminism. Option (1) computes the whole set of possible outcomes, where any possibility of nontermination must be taken. The choice of the output comes 'after' the computation of the set of all possible outputs. Option (2) corresponds to choices 'during' the execution of alternative statements. Option (3) applies 'prophetic' choices during the computation to avoid any nonterminating computations, thereby typically creating unboundedly many good outcomes. Finally, option (4) corresponds to choices 'before' the execution of the program.

Broy and Wirsing were also early users of the terminology of *angelic* nondeterminism, which they identified with (3), *demonic* nondeterminism, which they identified with (1), and *erratic* nondeterminism, which they identified with (2). We could not trace who first introduced this terminology, though it has been often attributed to Tony Hoare.

Nondeterminism in a context

In his books on CCS and the II-calculus Milner [1980, 1999], Robin Milner gave a simple example of two finite automata, one deterministic and one nondeterministic, that are equivalent when the accepted languages are compared. However, Milner argued that they are essentially different, when they are considered as processes interacting with a user or an environment. The essence of the example is shown in Figure 3, adapted from Milner [1999]. Milner took this observation as a motivation to develop a new notion of equivalence between processes, called *bisimilarity* and based on the following notion of bisimulation, which is sensitive to nondeterminism.

Processes are like nondeterministic automata, with states and transitions between states that are labeled by action symbols. We write $p \xrightarrow{a} q$ for a transition from a state p to a state q labeled by a. A process has an initial state and may have infinitely many states and thus transitions.

A bisimulation between processes P and Q is a binary relation \mathcal{R} between the states of P and Q such that whenever $p\mathcal{R}q$ holds, then every transition from p can be simulated by a transition from q with the same label, such that the successor states are again in the relation \mathcal{R} , and vice versa, every transition from q can be simulated by a transition from p with the same label such that the successor states are again in the relation \mathcal{R} . Processes are called *bisimilar* if there exists a bisimulation relating the initial states of the processes.

The processes shown in Figure 3 are *not* bisimilar. Indeed, suppose that \mathcal{R} is a bisimulation with $p_1 \mathcal{R}q_1$. Then the transition $p_1 \stackrel{i}{\longrightarrow} p_2$ can be simulated only by $q_1 \stackrel{i}{\longrightarrow} q_2$, which implies $p_2 \mathcal{R}q_2$. However, now the transition $p_2 \stackrel{c}{\longrightarrow} p_4$ cannot be simulated from q_2 because there is no transition with label c. Contradiction. This formalizes the intuition that only process P offers both tea and coffee, whereas Q offers either tea or coffee.



Figure 3: Two automata, P being deterministic and Q being nondeterministic on input of i, accept the same language consisting of the words it and ic. However, when viewed as processes interacting with a user, they are different. Suppose the process models a vending machine, i corresponds to the user's action of inserting a coin into the machine, and t and c to the user's choice of tea or coffee. Then, after insertion of the coin, P is in state p_2 and offers both tea and coffee to the user. However, Q makes a tacit choice by moving either to state q_2 or to state q'_2 after a coin is inserted. In state q_2 it offers only tea, and in state q'_2 only coffee, never both. Thus, from the user's perspective, the deterministic automaton is better because when using it, no decision is taken without consulting her or him.

This new notion of equivalence triggered a copious research activity resulting in various process equivalences that are sensitive to nondeterminism but differing in various other aspects, see for example van Glabbeek [2001]. Of particular interest is the idea of *testing* processes due to De Nicola and Hennessy [1984, 1988]. In these works, the interaction of a (nondeterministic) process and a user is explicitly formalized using a synchronous parallel composition. The user is formalized by a *test*, which is a process with some states marked as a *success*. For an example see Figure 4. The authors distinguish between two options: a process may or must pass a test. A process P may pass a test T if in *some* maximal parallel computation with P, synchronizing on transitions with the same label, the test T reaches a *success* state. A process P must pass a test T if in *all* such computations the test T reaches a *success* state.

This leads to *may* and *must* equivalences. Two processes are *may* equivalent if each test that one process may pass the other may pass as well, and analogously for the *must* equivalence. So *may* equivalence corresponds to angelic nondeterminism, and *must* equivalence to demonic nondeterminism.

As an example, consider the processes P and Q from Figure 3 and the test T from Figure 4. Then P both may and must pass the test T, whereas Q only may pass it because in a synchronous parallel computation it can get stuck in the state pair (q_2, t_2) , without being able to reach *success*. So in this simple example, both bisimilarity and *must* equivalence reveal the same difference between the deterministic process P and the nondeterministic process Q. In general, bisimilarity is finer than the testing equivalences, see again van Glabbeek [2001].

$$T: \qquad \underbrace{t_1} \xrightarrow{i} \underbrace{t_2} \xrightarrow{c} \underbrace{success}$$

Figure 4: This test T checks whether a process can engage in first i and then c.

Also CSP, originally built on Dijkstra's guarded commands as explained in Section 7, was developed further into a more algebraically oriented language that for clarity we call here 'new CSP'. It is described in Hoare's book Hoare [1985]. While guarded commands and the original CSP were notationally close to programming language constructs, where the nondeterminism appears only within the alternative command or the **do** loop, the new CSP introduced separate operators for each concept of the language. These can be freely combined to build up processes. We focus here on two nondeterministic operators introduced by Hoare.

Internal nondeterminism is denoted by the binary operator \sqcap , called nondeterministic or in Hoare [1985]. Informally, a process $P \sqcap Q$ "behaves like P or like Q, where the selection between them is made arbitrary, without knowledge or control of environment." In a formal operational semantics in the style of Plotkin [1980], this is modeled by using different labels of transitions. The special label τ appears at *internal* or *hidden* transitions, denoted by $p \xrightarrow{\tau} q$, which cannot be controlled or even seen by the environment. Labels $a \neq \tau$ appear at *external* or visible transitions, denoted by $p \xrightarrow{a} q$, and represent actions in which the environment can participate. The selection of $P \sqcap Q$ is modeled by the internal transitions $P \sqcap Q \xrightarrow{\tau} P$ and $P \sqcap Q \xrightarrow{\tau} Q$ Roscoe [1998, 2010]. Thus after this first hidden step, $P \sqcap Q$ behaves as P or as Q.

External nondeterminism or alternation is denoted by the binary operator [], called general choice in Hoare [1985]. The idea of a process P [] Q is that "the environment can control which of P and Q will be selected, provided that this control is exercised on the very first action." The formal operational semantics of the operator [] in the style of Plotkin 1980 is more subtle than the one for \Box , see Olderog and Hoare [1986], Roscoe [1998, 2010]. In applications, P [] Q is performed in the context of a synchronous parallel composition with another

process R modeling a user or an environment. Then the first visible transition with a label $a \neq \tau$ of R has to synchronize with a first visible transition P [] Qwith the same label a, thereby selecting P or Q of the alternative P [] Q. This formalizes Hoare's idea stated above.

These two nondeterministic operators have also been studied by De Nicola and Hennessy in the context of Milner's CCS De Nicola and Hennessy [1987]. The authors write \oplus for internal nondeterminism and keep [] for external nondeterminism. When defining the operational semantics of the two operators, they write \longrightarrow instead of $\xrightarrow{\tau}$ and speak of "CCS without τ 's" because τ is not present in this process algebra. Subsequently, they introduce a testing semantics and provide a complete algebraic characterization of the two operators.

To assess the effect of nondeterminism, the new CSP introduced a new equivalence between processes due to Brookes et al. [1984], called *failure equivalence*. A *failure* of a process is a pair consisting of a trace, i.e., a finite sequence of symbols that label transitions, and a set of symbols that after performing the trace the process can *refuse*. Processes with the same set of failures are called *failure equivalent*. Besides an equivalence, new CSP also provides a notion of *refinement* among processes. A process P *refines* a process Q if the set of failures of P is a subset of the set of failures of Q. Informally, this means that P is *more deterministic* than Q. Thus by definition, processes that refine each other are failure equivalent.

As an example consider again the processes P and Q in Figure 3. They are not failure equivalent, but P refines Q. This example shows that failure equivalence is sensitive to nondeterminism. It turns out that failure equivalence coincides with the *must* equivalence for 'strongly convergent' processes, i.e., those without any divergences De Nicola [1987]. So it represents demonic nondeterminism.

10 Conclusions

As explained in this chapter, nondeterminism is a natural feature of various formalisms used in computer science. The proposals put forward prior to Dijkstra's seminal paper Dijkstra [1975] are examples of what is now called angelic nondeterminism. Dijkstra's novel approach, now called demonic nondeterminism, was clearly motivated by his prior work on concurrent programs that are inherently nondeterministic in their nature. His guarded command language became a simplest possible setting allowing one to study demonic nondeterminism, unbounded nondeterminism, and fairness.

Its versatility was demonstrated by subsequent works on diverse topics. In Martin [1986] correct delay-insensitive VLSI circuits were derived by means of a series of semantics-preserving transformation starting with a distributed programming language. In some aspects the language is similar to CSP. In its sequential part it uses a subset of guarded commands with an appropriately customized semantics. To study randomized algorithms and their semantics an extension of the guarded commands language with a probabilistic choice operator was investigated in a number of papers, starting with He et al. [1997]. More recently, guarded commands emerged in the area of quantum programming, as a basis for quantum programming languages, see, e.g., Ying [2016].

As explained, the guarded commands language can also be viewed as a stepping stone towards a study of concurrent programs. In fact, it can be seen as a logical layer that lies between deterministic and concurrent programs.

The viability of Dijkstra's proposal can be best viewed by consulting statistics provided by Google Scholar. They reveal that the original paper, Dijkstra [1975], has been most often cited in the past decade.

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