On a Modified Nine-Tails Problem

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Base on the work [2] and [3] concerning the nine-tails problem, Fernando Castro G. made an interesting observation in his recent paper [1]. It claims that there exists a matrix over Z_2 such that its determinant is 0 but its row vectors are linearly independent. Let us first describe the matrix in question. For natural numbers $1 <= i, j <= 3$, C_{ij} denotes the 3x3 matrix over Z_2 with 1's on the i^{th} row and 1's on the j^{th} column, and 0's elsewhere. For example,

$$C_{11} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } C_{32} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

By joining the rows of C_{ij}, we obtain a row vector v_{ij} of $Z_2{}^9$. The example in [1] is the 9x9 matrix A consisting of the 9 vectors v_{ij}, $1 \le i, j \le 3$.

In spite of the fact that we primarily study matrices and vector spaces over real or complex numbers in an undergraduate linear algebra course, almost all the results there do not depend on the characteristic of a field. The invertible matrix over Z_2 can be characterized in exactly the same way as the invertible matrix over real numbers. It is impossible to have a matrix over any field to have the property as claimed. The row vectors of A are in fact linearly dependent. One checks readily that

$$C_{33} + C_{31} + C_{22} + C_{12} = 0$$

and that matrix A has rank only 5.

What went wrong with [1] was that Castro G. considered a different nine-tails problem. The rule for the original nine-tails problem is that when a coin is turned, only its adjacent neighbors on the same row or column are to be flipped, not all the coins on the same row or on the same column. If we let D_{ij} denote the 3x3 matrix over Z_2 representing the flip of ij^{th} coin of the original problem, then the nine matrices D_{ij}'s are actually linearly independent. That confirms mathematically that the original nine-tails problem is solvable in all cases.

For Castro's modified nine-tails problem, a matrix over Z_2 can be changed into a zero matrix if and only if it is a linear combination of the C_{ij}'s. Since the rank of A is 5, there are only 32 cases where the modified nine-tails problem has solutions. Standard argument shows that a matrix is a solvable case if and only if it is a linear combination of C_{11}, C_{13}, C_{33}, and C_{22}; in turn, if and only if the sum of the entries in any of its two rows or two columns is 0.

Reviewed Papers

References

1. Fernando Castro G., "More on the Nine-Tails Problem," *SIGCSE Bulletin*, Vol.31 No.4, December 1999, p47.
2. F.Delahan, W.F. Klostermeyer, G. Trapp, "Another Way to Solve Nine-Tails," *SIGCSE Bulletin*, Vol.27 No.4, December 1995, p27-28.
3. P. Heck, "Dynamic programming for Pennies a Day," *SIGCSE Bulletin*, Vol.26, March 1995, p213-217.