A FAMILY OF TEST MATRICES

A family of test matrices with the following properties is described here: (a) an explicit inverse is given; (b) the characteristic polynomial is easily obtained; (c) a large measure of control over the eigenvalues is possible; (d) in special cases the eigenvalues and eigenvectors can be given explicitly, and the P-condition number can be arbitrarily assigned.

Consider a matrix of the form

$$Q = \begin{bmatrix} S & R \\ C & D \end{bmatrix},$$

where S is a scalar, R is a row-matrix $\{r_2, r_3, \cdots, r_n\}$, C is a column-matrix $\{c_2, c_3, \cdots, c_n\}^T$ and D is a diagonal matrix with elements $d_2, d_3, \cdots, d_n$. By use of the bordering method [1] the inverse is found to be

$$Q^{-1} = \begin{bmatrix} S' & R' \\ C' & M' \end{bmatrix},$$

where each submatrix of Q^{-1} has the same form as the corresponding submatrix of Q, except that M' is generally not diagonal. Letting the subscripts of R', C', M' run from 2 to n, we find that

$$S' = 1 \Big/ \left[S - \sum_2^n r_i\, c_i/d_i \right], \qquad c_i' = -S'c_i/d_i\,, \qquad r_i' = -S'r_i/d_i\,,$$

$$M'_{ij} = [\delta_{ij} - c_i\, r_j']/d_i\,,$$

where δ_{ij} is the Kronecker delta. The inversion can be performed in $2(n-1)(n+2)+1$ long operations; it might be possible to improve this figure with some ingenuity.

The eigenvalue problem. Let λ be an eigenvalue of Q, and let $\bar{x} = \{1, x_2, \cdots, x_n\}^T$ be the associated eigenvector. This leads to the following set of n equations:

$$S + \sum_2^n r_i x_i = \lambda, \qquad c_i + d_i x_i = \lambda x_i\,. \qquad (i \geqq 2)$$

On eliminating the x_i we obtain

$$\text{(1)} \qquad S + \sum_2 r_i c_i/(\lambda - d_i) - \lambda = 0.$$

If we write $\Pi(\lambda) = \Pi_2^n\,(\lambda - d_i)$, $\Pi_i\,(\lambda) = \Pi(\lambda)/(\lambda - d_i)$, then on clearing the fractions in (1) we obtain

$$\text{(2)} \qquad (\lambda - S)\Pi(\lambda) - \sum_2^n r_i c_i \Pi_i\,(\lambda) = 0.$$

This is the characteristic equation. The following statements can be made concerning the eigenvalues.

(a) If all $r_i c_i > 0$ and all d_i are distinct, then all the eigenvalues are real and are separated by the d_i.

(b) If all d_i are equal to d, then there are $n-2$ eigenvalues equal to d; the remaining two are zeros of the quadratic function $\lambda^2 - (S+d)\lambda + Sd - \Sigma_2^n\, r_i c_i$. These zeros are real if, and only if, $(S-d)^2 + 4\Sigma r_i c_i \geqq 0$.

(c) If all d_i are equal to d, then the eigenvectors associated with the multiple eigenvalue d have zero as their first component, and they are orthogonal to the vector $\{0, r_2, \cdots, r_n\}$. Eigenvectors corresponding to the other two eigenvalues are $\{\lambda_p - d, c_2, \cdots, c_n\}$, where λ_p is a zero of the quadratic given in (b).

Proof of (a). Let $H(\lambda)$ denote the left side of (1), and let $\{d_i'\}$ denote a reordering of the $\{d_i\}$ so that $d_i' < d_{i+1}'$. We note that $H(\lambda)$ is continuous in any interval which does not enclose any of the d_i', and that for sufficiently small ϵ, $H(d_i' + \epsilon) > 0$ and $H(d_{i+1}' - \epsilon) < 0$. Hence there is a zero of $H(\lambda)$ between each consecutive pair of the $\{d_i'\}$; moreover since $H(-\infty) > 0$ and $H(\infty) < 0$, there are two more real zeros of $H(\lambda)$ outside the interval (d_2', d_n').

Proof of (b). If all the d_i are equal to d, then $\Pi(\lambda) = (\lambda - d)^{n-1}$ and $\Pi_i(\lambda) = (\lambda - d)^{n-2}$. The characteristic equation (2) then reduces to $(\lambda - d)^{n-2}[(\lambda - S)(\lambda - d) - \Sigma_2^n\, r_i c_i] = 0$. The discriminant of the quadratic factor is $(S-d)^2 + 4\Sigma r_i c_i$. Statement (c) may be directly verified.

The P-condition number, i.e. the largest absolute ratio of two eigenvalues [2], can most conveniently be assigned by letting $d_i = d$; then, using statement (b), we can choose S and $\Sigma r_i c_i$ in such a way as to assign any desired zeros to the quadratic; hence any desired maximum ratio of eigenvalue magnitudes may be procured.

Remarks. If the inverse matrices are included along with the original family, then we have freedom within the family to specify sparse, nonsparse, symmetric, nonsymmetric, well- or ill-conditioned matrices; furthermore we can require that the eigenvalues shall be all real or mixed real and complex. This should provide sufficient versatility for most test purposes.

References

1. Fadeeva, V. N. *Computational Methods of Linear Algebra.* Dover, New York, 1959.
2. Marcus, M. Basic Theorems of Matrix Theory. N.B.S. Appl. Math. Ser. 57, 1960.

A.C.R. Newbery
University of Alberta
Calgary, Alberta, Canada

Received December, 1963

A CLASS OF MATRICES TO TEST INVERSION PROCEDURES

The test matrices given by M. L. Pei [*Comm. ACM 5*, 10 (Oct. 1962), 508] and R. D. Rodman [*Comm. ACM 6*, 9 (Sept. 1963, 515] are special cases of a general class of matrices with complex elements for which an explicit form of the inverse can be exhibited. This class of matrices is such that eigenvalues and a set of associated eigenvectors can also be obtained. Then not only inverses, but also eigenvalues of the Pei matrix given by W. S. Lasor [*Comm. ACM 6*, 3 (Mar. 1963), 102] and eigenvectors given by A. R. C. Newberry [*Comm. ACM 6*, 9 (Sept. 1963), 515], and eigenvalues of the Rodman matrix follow as special cases.

For the general case we let B be any matrix with complex elements, and let k be any real number, $k \neq -1$. Then the inverse of the matrix $I + kB^+B$, where B^+ is the Moore-Penrose generalized inverse of B, can be written as

$$(I + kB^+B)^{-1} = I - \frac{k}{k+1} B^+B.$$

Now if B is restricted to matrices with orthonormal rows, $B^+ = B^*$, and we have

$$(I + kB^*B)^{-1} = I - \frac{k}{k+1} B^*B$$

which provides a class of matrices to use in testing inversion procedures. Moreover, since

$$(I + kB^*B)B^* = (1+k)B^*$$

and

$$(I + kB^*B)(I - B^*B) = I - B^*B,$$