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A Highly Parallel Algorithm for Approximating All Zeros of a Polynomial with Only Real Zeros

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An algorithm is described based on Newton's method which simultaneously approximates all zeros of a polynomial with only real zeros. The algorithm, which is conceptually suitable for parallel computation, determines its own starting values so that convergence to the zeros is guaranteed. Multiple zeros and their multiplicity are readily determined. At no point in the method is polynomial deflation used.

Key Words and Phrases: parallel numerical algorithms, real polynomials, real zeros, Newton's method, starting values, guaranteed convergence

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1. Introduction

In this paper we describe an algorithm which produces simultaneous approximations to all zeros of a polynomial with only real zeros. The algorithm, which is based on Newton's method, determines its own starting values. The starting values are determined so that convergence to the zeros of the polynomial is guaranteed. Furthermore the choice of starting values is such that at no stage in the algorithm is polynomial deflation required, a process which, in general, increases errors due to roundoff. Also the multiplicities of the approximated zeros are readily determined. As will be seen in the following, situations exist in which the algorithm will be useful.

Recently, some investigators (see e.g. Miranker [5]; Shedler [8]; Feldstein and Firestone [3]) have considered the development of parallel methods for approximating a zero of a function of a single variable. The work of these authors and others is nicely surveyed by Miranker [6]. In general, these methods consist of multiplexing extrapolation techniques or simple standard methods in order to speed up the process of determining an approximation to a zero of a nonlinear function. In the development of most of these methods it is assumed that initial conditions of the computation are such that convergence to a zero is guaranteed.

The parallel content of our method and its usefulness as a parallel method are clear from its description. We point out, in contrast to the above mentioned methods, that if r processors are available these may be obtaining approximations to r different zeros of the polynomial. However, as suggested by Dorn [2], more than one processor could be used executing Horner's algorithm for evaluating the polynomial and its derivative, in which case more than one processor would be used in obtaining approximations to a single zero.

We emphasize, although our method is highly parallel by nature, that it is equally suitable as a sequential method. This fact will also be clear from the descripion of the method.

2. Description of the Method

The method described here determines approximations to zeros of polynomials with only real zeros. It does, in some cases, produce approximations to complex zeros, but not in general. Examples will be given in Section 4.

Our method has the following properties: (1) its basic iteration function is Newton's iteration function; (2) it determines its own starting values so that convergence to a zero is guaranteed; (3) the multiplicities of zeros are readily determined; and (4) it avoids the process of deflation, which generally results in increased error due to roundoff.

The method is based on the fact that the zeros of the second derivative of a polynomial with only real zeros can serve as starting values for Newton's method with assured convergence to a zero of the polynomial. For such polynomials this fact follows from results of Barna [1], as shown by Patrick [7]. Gorn [4] remarks that the same is also true for the much larger class of functions with real zeros discussed in his paper.

Generally the method can be described as follows. Let $P_n(x)$ be the given polynomial. The method determines approximations to the zeros of $P_n^{(n-k)}(x)$, $P_n^{(n-k-2)}(x)$, ..., $P_n''(x)$ and, finally, $P_n(x)$, where $k = 1$ or 2 depending on whether n is odd or even. The zeros of $P_n^{(n-k)}(x)$ are found directly since it is either linear or quadratic. These are then used as starting values for Newton's method which produces approximations for k zeros of $P^{(n-k-2)}(x)$. Approximations of the remaining two zeros of $P^{(n-k-2)}(x)$ are determined by solving a quadratic equation. The quadratic equation arises from the fact that the sum of the zeros of $P^{(n-k-2)}(x)$ is equal to one of its coefficients, while the product of the zeros is equal to another coefficient; both coefficients are known. The zeros of $P^{(n-k-2)}(x)$ are then used to determine those of $P^{(n-k-4)}(x)$, etc.

More specifically, let

$$P_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \tag{2.1}$$

be a monic polynomial with real zeros not all necessarily simple. Let

$$\begin{aligned}\bar{P}_n^{(k)}(x) \\ = x^{n-k} + a_{n-k-1}^{(k)} x^{n-k-1} + \cdots + a_1^{(k)}x + a_0^{(k)}\end{aligned} \tag{2.2}$$

denote the kth derivative of $P_n(x)$ normalized so that its leading coefficient is 1. Also let $N = [n/2]$, if n is even, or $[(n + 1)/2]$, if n is odd, and let $m_j = n - (2N - 2j)$ for $j = 1, 2, ..., N$.

Recall that Newton's iteration function w.r.t. $f(x)$ is

$$N(f; x) = x - f(x)/f'(x) \tag{2.3}$$

and

$$N'(f; x) = f(x)f''(x)/(f'(x))^2. \tag{2.4}$$

We now define an algorithm for determining a sequence of vectors

$$\mathbf{r}^{(j)} = (r_1^{(j)}, r_2^{(j)}, ..., r_{m_j}^{(j)}) \tag{2.5}$$

for $j = 1, 2, ..., N$, where $r_1^{(j)} \leq r_2^{(j)} \leq \cdots \leq r_{m_j}^{(j)}$. We show in Section 3 that the components of $\mathbf{r}^{(j)}$ are approximations of the zeros of the polynomial $\bar{P}_n^{(2N-2j)}(x)$ of degree m_j. In particular, the components of $\mathbf{r}^{(N)}$ are approximations of the zeros of $P_n(x)$.

The vectors $\mathbf{r}^{(j)}$, $j = 1, 2, ..., N$ are determined sequentially as follows. For $j = 1$, $m_1 = 1$ or 2 which means $P_n^{(2N-2)}(x)$ is linear or quadratic. Let the components of $\mathbf{r}^{(1)}$ be the zeros of $P_n^{(2N-2)}(x)$. For each j, $j = 2, 3, ..., N$, we obtain the components $r_i^{(j)}$ of $\mathbf{r}^{(j)}$ in the following way. With $x_0^{(i)} = r_i^{(j-1)}$, $i = 1, 2, ..., m_{j-1}$, form the sequences $x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, ...,$ using the iteration formula

$$x_{k+1}^{(i)} = N(\bar{P}_n^{(2N-2j)}; x_k^{(i)}). \tag{2.6}$$

Let $r_{i+1}^{(j)} = \lim_{k\to\infty} x_k^{(i)}$, $i = 1, 2, ..., m_{j-1}$. We show in Section 3 that $r_{i+1}^{(j)}$ does actually exist and is a zero of $\bar{P}_n^{(2N-2j)}(x)$. In this way, for each j we obtain $r_2^{(j)}$, $r_3^{(j)}, ..., r_{m_j-1}^{(j)}$, which are approximations of $m_j - 2$ of the zeros of $\bar{P}_n^{(2N-2j)}(x)$. The remaining two approximations $r_1^{(j)}$ and $r_{m_j}^{(j)}$ are obtained from the equations

$$\begin{aligned} r_1^{(j)} + r_2^{(j)} + \cdots + r_{m_j}^{(j)} &= a_{m_j-1}^{(2N-2j)} \quad \text{and} \\ r_1^{(j)}r_2^{(j)} \cdots r_{m_j}^{(j)} &= a_0^{(2N-2j)}, \end{aligned} \tag{2.7}$$

in which $r_1^{(j)}$ and $r_{m_j}^{(j)}$ are the only unknowns. $a_{m_j-1}^{(2N-2j)}$ and $a_0^{(2N-2j)}$ are defined by (2.2). Since this process can be repeated for each j, we have, when $j = N$, the desired approximations of the zeros of $P_n(x)$.

Suppose in the above process for some j and some $i \neq 1$ or m_j the zero $r_i^{(j)}$ of $\bar{P}_n^{(2N-2j)}(x)$ is: (1) also a zero of $\bar{P}_n^{(2N-2j-1)}$ but not a zero of $\bar{P}^{(2N-2j+1)}(x)$, or (2) also a zero of $P^{(2N-2j+1)}(x)$ but not of $P_n^{(2N-2j+2)}(x)$. Using elementary calculus, Patrick [7] showed that $r_i^{(j)}$ in case (1) is a zero of multiplicity $2N - 2j + 1$ of $P_n(x)$, and in case (2) is a zero of $P_n(x)$ of multiplicity $2N - 2j + 2$. In case (1), $r_i^{(j)}$ is a simple zero of $\bar{P}_n^{(2N-2j)}$, and it was obtained from a zero of $\bar{P}_n^{(2N-2j+2)}(x)$ by Newton's method with a quadratic rate of convergence. In case (2), $r_i^{(j)}$ is a double zero of $\bar{P}_n^{(2N-2j)}(x)$, and it was obtained from a zero of $\bar{P}_n^{(2N-2j+2)}(x)$ by Newton's method with a linear rate of convergence. However, in either case after step j, the fact remains that $r_i^{(j)}$ is a multiple root of $P_n(x)$ and its multiplicity is determined simply by evaluating $\bar{P}_n^{(2N-2j-1)}(x)$ or $\bar{P}_n^{(2N-2j+1)}(x)$ at $x = r_i^{(j)}$. No further computation is necessary to produce the multiple root. Only the computation needed to compute approximations to the remaining $n - (2N - 2j + 2)$ or $n - (2N - 2j + 1)$ root is required. It follows, in going from step j to step $j + 1$, that $r_i^{(j)}$ will be a zero of $\tilde{P}_n^{(2N-2j-2)}(x)$ of multiplicity two greater than its multiplicity as a zero of $\tilde{P}_n^{(2N-2j)}$. Therefore, in addition to $r_i^{(j)}$ there are two other zeros of $\tilde{P}_n^{(2N-2j)}(x)$ which would produce $r_i^{(j)}$ as a zero of $\tilde{P}^{(2N-2j-2)}(x)$ if used as starting values for Newton's method. The fact that these zeros of $\tilde{P}_n^{(2N-2j)}(x)$ are $r_{i-1}^{(j)}$ and $r_{i+1}^{(j)}$ will be clear from our discussion of convergence in Section 3. So, in going from step j to step $j + 1$, $r_{i-1}^{(j)}$, $r_i^{(j)}$, and $r_{i+1}^{(j)}$ are not used as starting values for Newton's method, since they will simply produce $r_i^{(j)}$ as a multiple root of $\tilde{P}_n^{(2N-2j-2)}(x)$.

It is clear from the above discussion that if $P_n(x)$ has zeros of high multiplicity, then the method is quite practical. For example, in the extreme case, if $P_n(x)$ has a zero of multiplicity n or $n - 1$, no iteration with Newton's method is required. Only the solution of a linear or quadratic equation and appropriate derivative evaluations are needed.

3. Convergence of the Method

Using results of Barna [1], Patrick [7] showed that Newton's method, when using a zero of the second derivative of $P_n(x)$ as a starting value, would converge to a zero of $P_n(x)$. In this section we show, in addition, that such a sequence of Newton iterates converge monotonically to a zero of $P_n(x)$.

Let a and b satisfy $P_n'(a) = P_n'(b) = 0$, with $P_n(a) \neq 0$, $P_n(b) \neq 0$, and let $\alpha \in (a, b)$ be a simple zero of $P_n(x)$ where $P_n(x)$ is of the form (2.1). By Rolle's theorem there exists $\beta \in (a, b)$ such that

$$P_n''(b) = 0$$

and suppose $\alpha < \beta$. We further suppose that $P_n(a) < 0 < P_n(b)$, for if $P_n(a) > 0 > P_n(b)$ it would only be necessary to consider $-P_n(x)$ instead of $P_n(x)$, in which case our proof doesn't change.

To obtain our results we need only consider the Newton iteration function (2.3) for $P_n(x)$ on the interval $[\alpha, \beta]$. We have

$$P_n(x)P_n''(x) > 0 \text{ for } x \in (\alpha, \beta), \tag{3.1}$$

$$P_n(\alpha)P_n''(\alpha) = P_n(\beta)P_n''(\beta) = 0, \tag{3.2}$$

and

$$P_n(x)P_n'(x) > 0 \text{ for } x \in (\alpha, \beta). \tag{3.3}$$

Using (3.1), (3.2), and (2.4), it follows that $N(P_n ; x)$ is a strictly increasing function in the interval $[\alpha, \beta]$. This means, since from (2.3) $\alpha = N(P_n ; \alpha)$, that for $x \in (\alpha, \beta]$ $\alpha < N(P_n ; x)$. Also by (3.3) and (2.3) $N(P_n ; x) < x$ for $x \in (\alpha, \beta]$ so by combining we have

$$\alpha < N(P_n ; x) < x \text{ for } x \in (\alpha, \beta]. \tag{3.4}$$

Therefore, from (3.4), if we let $x_0 = \beta$, the sequence of iterates $x_0, x_1, x_2, \ldots$ produced by the iteration formula $x_{k+1} = N(P_n ; x_k)$ will be a monotonically decreasing sequence which is bounded below by α. Hence the sequence has a limit $\tilde{\alpha}$. From the continuity of $N(P_n ; x)$ on the interval $[\alpha, \beta]$ it follows that $\tilde{\alpha}$ is a zero of $P_n(x)$. But by assumption α is the only zero of $P_n(x)$ in $[a, b]$; hence $\tilde{\alpha} = \alpha$.

If $\beta < \alpha$ instead of $\alpha < \beta$ then (3.3) becomes

$$P_n(x)P_n'(x) < 0 \text{ for } x \in [\beta, \alpha) \tag{3.5}$$

and (3.4) becomes

$$x < N(P_n ; x) < \alpha \text{ for } x \in [\beta, \alpha). \tag{3.6}$$

Then from (3.5) and (3.6) the sequence of iterates produced using $x_0 = \beta$ is monotonically increasing and converges to α.

Next suppose that α is a multiple root of $P_n(x)$instead of a simple root. This means that $P_n'(a) = P_n'(\alpha) = P_n'(b) = 0$, and by Rolle's theorem there exist $\beta \in (a, \alpha)$ and $\gamma \in (\alpha, b)$ for which

$$P_n''(\beta) = P_n''(\gamma) = 0.$$

If α has multiplicity three or greater than three, also $P_n''(\alpha) = 0$. We consider the function $N(P_n ; x)$ on the interval $[\beta, \gamma]$. We have

$$P_n(x)P_n''(x) \geq 0 \text{ for } x \in (\beta, \gamma), \tag{3.7}$$

$$P_n(\beta)P_n''(\beta) = P_n(\gamma)P_n''(\gamma) = 0, \tag{3.8}$$

$$P_n(x)P_n'(x) < 0 \text{ for } x \in [\beta, \alpha), \tag{3.9}$$

and

$$P_n(x)P_n'(x) > 0 \text{ for } x \in (\alpha, \gamma]. \tag{3.10}$$

From these it follows, similarly as above that

$$x < N(P_n ; x) < \alpha \text{ for } x \in [\beta, \alpha) \tag{3.11}$$

and

$$\alpha < N(P_n ; x) < x \text{ for } x \in (\alpha, \gamma]. \tag{3.12}$$

As above it follows from (3.11) and (3.12) that both β and α, when used as starting values for Newton's method, will produce sequences which converge monotonically to α.

Since the function defined by (2.2) is a polynomial, it is clear from the above discussion that the vector $\mathbf{r}^{(N)}$, produced by the method described in Section 2, will have as components approximations of zeros of the polynomial (2.1).

4. Summary

We have described a method which simultaneously approximates all zeros of a polynomial with real zeros. The method is based on Newton's iteration function and determines its own starting values so that convergence to the zeros is guaranteed. Multiple zeros and their multiplicity are readily determined. At no point in the method is polynomial deflation used.

The method is not, in general, applicable to real polynomials with complex roots as is easily seen by considering the polynomial $f(x) = x^4 + 1$. The second derivative of this polynomial has a double zero at $x = 0$ which will not, if used as a starting value for Newton's method, yield convergence to a zero of $f(x)$.

However, there are real polynomials with complex zeros for which the method is applicable. For example, consider the polynomial

$$f(x) = (x - r_1)(x - r_2)(x - r_3) = x^3 - x^2 + x - 1.$$

Now $f''(x) = 6x - 2$ is zero at $x = 1/3$. It is easy to see, geometrically, that if $x_0 = 1/3$ is used as a starting value for Newton's method, the sequence produced will converge to the real root $r_2 = 1$. The complex roots result from the solution of the quadratic equation that appears in our method. Namely, we have

$$r_1 + r_2 + r_3 = 1 \quad \text{and} \quad r_1 r_2 r_3 = 1$$

but $r_2 = 1$ so we have

$$r_1 + r_3 = 0 \quad \text{and} \quad r_1 r_3 = 1$$

which leads to a quadratic equation and the zeros $r_1 = i$ and $r_3 = -i$.

We have not yet determined conditions under which the method produces approximations to complex zeros of polynomials.

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