ALGORITHM 560
JNF, An Algorithm for Numerical Computation of the Jordan Normal Form of a Complex Matrix [F2]

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Language· Fortran

DESCRIPTION

1. Introduction

The routines given here are the actual Fortran implementation of the algorithm presented and discussed in [2]. We describe in detail how to use the Fortran subroutines and how to reach the results from a call. We also give some comments on the code that might be of value when implementing the subroutines on a particular machine. The subroutines have been checked with the PFORT verifier [4] and the notation from [2] is used.

2. The User-Written Routine DECIDE

DECIDE is a user-written subroutine which makes it possible for the user to change the grouping and/or the values of the numerical multiple eigenvalues.

This routine is useful when we have some information on the eigenvalues and their multiplicities in advance, and want the eigenvectors and the principal vectors for the given matrix. For instance, if we know of physical reasons why zero should be the only possible multiple eigenvalue and all others simple, then the contents of the parameters should be changed as indicated in Table I.

If the user does not want to influence the grouping or the values of the eigenvalues, it is enough to define a dummy subroutine, i.e.,

```
SUBROUTINE DECIDE (NM, N, NDEL, CSHTR, CSHTI, NBLOCK, HR, HI)
RETURN
END
```

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Table I. Example of Use of the Subroutine DECIDE

On entry N = 10 NBLOCK = 6		On exit NBLOCK = 7	
NDEL	CSHT	NDEL	CSHT
0	−1.5	0	−1.5
1	0.002	1	0.0
5	2.4	5	$\overline{H(6, 6)}$
7	3.5	6	$\overline{H(7, 7)}$
8	8.6	7	3.5
9	9.5	8	8.6
10		9	9.5
		10	

3. The Rank Determination Process and the RDEFL Routine

The rank determination process is described in [2, Section 2.2 and Section 3, Step 6 of the algorithm]. According to these two alternatives we have two subroutines RDEFL—one for the singular value decomposition strategy and one for the Kublanovskaya (RQ) decomposition strategy. Only the routine based on singular value decomposition is presented here.

In the RDEFL routine we use the CSVD algorithm [1] with one change; namely, we sort the singular values in increasing instead of decreasing order as in the original program.

4. How to Reach Any of the Vectors

The eigenvectors and principal vectors are stored in the arrays ZR, ZI(NM, N). To find the vectors, we use the information stored in NBLOCK, NDEL, NDB, NDEFL, and NXT (see the parameter list of subroutine JNF).

The column indices K in ZR and ZI for eigenvectors, principal vectors, and principal chains of an eigenvalue are listed below in Algol for-statement notation.

(a) Find the eigenvectors of the multiple eigenvalue
```
I = 1, . . . , NBLOCK.
J := NDB[I];
for K := NDEL[I] + 1 step 1 until NDEFL[J + 1] − 1 do;
```
(b) Find the principal vectors of grade P to the eigenvalue I = 1, . . . , NBLOCK.
```
J := NDB[I] + P − 1;
for K := NDEFL[J] step 1 until NDEFL[J + 1] − 1 do;
```
(c) Find the principal chain which ends with the eigenvector in column L.
```
First:
(A − EV[L]*I)*Z[L] = 0   (eigenvector)
Then:
K1 := L;
for K := NXT[K1] while K > 0 do
begin
Here:
(A − EV[K]*I)*Z[K] = SUPD[K1]*Z[K1];   (principal vector)
K1 := K
end;
```

Table II. Example of Output of Index Vectors

I	NDEL	NDB	NDEFL	NXT	SUPD
1	0	1	1	3	1.9903
2	4	3	3	4	48.080
3	9	6	5	0	0
4	10	0	7	0	0
5	0	0	9	7	3.1314
6	0	0	10	8	5.5354
7	0	0	11	0	0
8	0	0	0	9	1.3994
9	0	0	0	0	0
10	0	0	0	0	0

Note. Output for the first test matrix in [2] is displayed.

(Here we use the fact that the diagonal elements may be different as in [2, eq. (4.2)]. To get chains of the nilpotent B, EV[K] should be replaced by EV[L].)

In Table II we list the values of NDEL, NDEFL, NXT, NDB, and SUPD for the first test matrix in [2, Section 5].

5. Comments on the Fortran Code

Since we use the EISPACK routines CBAL, COMHES, COMLR2, and CBABK2 [5], we have their representation of complex arrays, i.e., the real parts are represented in one array with the name ending in R (e.g., HR, ZR) and the imaginary parts are represented in one array with the name ending in I (e.g., HI, ZI). This representation makes it possible to execute most of the computations in real arithmetic.

We use real arithmetic in all steps of the algorithm except Steps 4 and 7, i.e., the elimination processes, where we use some complex arithmetic.

Because of this complex arithmetic we also have to use the complex standard functions REAL(X), AIMAG(X), and CMPLX(X, Y).

If the actual Fortran compiler does not accept complex arithmetic, we can implement the functions CMUL, CDIV, CADD, and CSUB from [6] and substitute the actual statements with new statements.

A new parameter (NOBACK) has been added to the EISPACK routine COMLR2. We use this parameter (NOBACK = 0) as a control so that we do not get the eigenvectors from COMLR2. We get the transformations which transform the matrix to upper triangular form.

In the RDEFL routine we use complex arrays and arithmetic since the routine CSVD1 [1] is made for complex arithmetic.

6. Numerical Experiments

We have performed several numerical tests, both on matrices with a well-defined Jordan normal form, and on cases specially constructed in order to provide the routine with difficulties. Results from full Jordan normal form reduction are reported and discussed in [2]. In [2, Section 4] we describe how to choose tolerance parameters in the grouping procedure and in the procedure for determining the structures inside the invariant subspaces corresponding to different

numerical multiple eigenvalues. We also describe how to analyze the results from the program, given a combination of the tolerance parameters.

A large number of results from block diagonal reductions (ISTEP = 5 and 6) are reported and discussed in [3].

REFERENCES

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ALGORITHM

[Summary information and a part of the listing are printed here. The complete listing is available from the ACM Algorithms Distribution Service (see inside back cover for order form) or may be found in "Collected Algorithms from ACM."]

NAME(n): indicates a Fortran module with n records
NAME^T(n): indicates "NAME" is included for testing purposes
Contents: MAIN^T(328), RESID^T(75), DECIDE^T(56), MATRIS^T(43), MATRS1^T(32), MATRS3^T(55), MATRS4^T(62), MATRS5^T(42), JNF(834), RDEFL(132), CBAL(209), COMHES(138), COMLR2(405), CBABK2(91), CVSD1(384), OUTPUT^T(33), MATBLK^T(37), MATRS2^T(148)

```
      SUBROUTINE DECIDE (NM,N,NDEL,CSHTR,CSHTI,NBLOCK,HR,HI)
C     ************************************************************
C     *
C     *  DECIDE IS A USER WRITTEN SUBROUTINE WHICH MAKES IT POSSIBLE
C     *  FOR THE USER TO CHANGE THE GROUPING OF THE NUMERICAL MUL-
C     *  TIPLE EIGENVALUES,AND/OR CHANGE THE VALUES OF THE MULTIPLE
C     *  EIGENVALUES.
C     *  THE FORMAL PARAMETER LIST
C     *
C     *  NM        THE ROW DIMENSION OF THE TWO-DIMENSIONAL ARRAY
C     *            PARAMETERS(HR,HI)AS DECLARED IN THE CALLING
C     *            PROGRAM DIMENSION STATEMENT
C     *  N         THE ORDER OF THE ORIGINAL MATRIX
C     *  NBLOCK    THE NUMBER OF BLOCKS I.E THE NUMBER OF NUMERICAL
C     *            MULTIPLE EIGENVALUES
C     *  CSHTR,    THE REAL AND IMAGINARY PARTS,RESPECTIVELY,OF THE
C     *  CSHTI     NUMERICAL MULTIPLE EIGENVALUES
C     *  NDEL      INDICATES THE MULTIPLICITY OF THE NUMERICAL
C     *            MULTIPLE EIGENVALUES.NDEL(I+1)-NDEL(I)=THE
C     *            MULTIPLICITY OF EIGENVALUE I FOR I=1,..,NBLOCK
```

```
C   *  HR,HI     THE REAL AND IMAGINARY PARTS,RESPECTIVELY, OF          ØØØØ424Ø
C   *            THE UPPER TRIANGULAR MATRIX RESULTING FROM             ØØØØ425Ø
C   *            STEPS 1-3 OF THE ALGORITHM                             ØØØØ426Ø
C   *                                                                   ØØØØ427Ø
C   *  NOTE -- THE USER MUST NOT CHANGE HR,HI                           ØØØØ428Ø
C   *                                                                   ØØØØ429Ø
C   ****************************************************************   ØØØØ43ØØ
      INTEGER NDEL(N)                                                   ØØØØ431Ø
      REAL HR(NM,N),HI(NM,N),CSHTR(1),CSHTI(1)                          ØØØØ432Ø
      WRITE(6,9)                                                        ØØØØ433Ø
    9 FORMAT(1HØ,1X,47HTHE FOLLOWING OUTPUT (A,B AND C) ARE PRINTED BY, ØØØØ434Ø
     *          31HTHE USER WRITTEN ROUTINE DECIDE/                     ØØØØ435Ø
     *        1HØ,1X,31HSEE SECTION 2 OF THE ALGORITHM.)                ØØØØ436Ø
      WRITE(6,1Ø)                                                       ØØØØ437Ø
      WRITE(6,11) (HR(K,K),K=1,N)                                       ØØØØ438Ø
      WRITE(6,11) (HI(K,K),K=1,N)                                       ØØØØ439Ø
      WRITE(6,12)                                                       ØØØØ44ØØ
      DO 1 K=1,NBLOCK                                                   ØØØØ441Ø
      MULT=NDEL(K+1)-NDEL(K)                                            ØØØØ442Ø
      WRITE(6,13) NDEL(K+1),MULT,CSHTR(K),CSHTI(K)                      ØØØØ443Ø
    1 CONTINUE                                                          ØØØØ444Ø
      WRITE(6,14)                                                       ØØØØ445Ø
14    FORMAT(1HØ,1X,34HC--IN STEP 6 OF THE ALGORITHM THE ,              ØØØØ446Ø
     *  5ØHSTRUCTURE OF EACH MULTIPLE EIGENVALUE IS COMPUTED.           ØØØØ447Ø
     * /1HØ,1X,42HFOR THAT REASON RDEFL SUCCESIVELY COMPUTES,            ØØØØ448Ø
     *  44H SINGULAR VALUE DECOMPOSITIONS. RDEFL PRINTS                 ØØØØ449Ø
     * /1HØ,1X,39HTHE RESULTS BELOW(SEE ALSO COMMENTS IN ,              ØØØØ45ØØ
     *    35HRDEFL AND STEP 6 OF THE ALGORITHM).)                       ØØØØ451Ø
   1Ø FORMAT(1HØ,1X,43HA--ENTER DECIDE WITH EIGENVALUES COMPUTED  ,     ØØØØ452Ø
     *          36HBY COMLR2 (STEP 1 OF THE ALGORITHM ))                ØØØØ453Ø
   11 FORMAT(1ØF13.9)                                                   ØØØØ454Ø
12      FORMAT(1X,4ØHB--GROUPINGS OF THE EIGENVALUES,COMPUTED,          ØØØØ455Ø
     *  27H BY STEP 3 OF THE ALGORITHM)                                 ØØØØ456Ø
   13 FORMAT(12H DIVISION AT,I4, 7H MULT.=,I4, 8H CENTER=,2E2Ø.1Ø)      ØØØØ457Ø
      RETURN                                                            ØØØØ458Ø
      END                                                               ØØØØ459Ø

      SUBROUTINE JNF (NM,N,HR,HI,ZR,ZI,EVR,EVI,SUPD,NXT,NDEL,NDEFL,NDB, ØØØØ694Ø
     CNBLOCK,EIN,TOL,DELE,SM,IERR,ISTEP)                                ØØØØ695Ø
      REAL HR(NM,N),HI(NM,N),ZR(NM,N),ZI(NM,N),EVR(N),EVI(N),SUPD(N),   ØØØØ696Ø
     CDELE(N),SM(N)                                                     ØØØØ697Ø
      INTEGER NXT(N),NDEL(N),NDEFL(N),NDB(N)                            ØØØØ698Ø
C                                                                       ØØØØ699Ø
C   ****************************************************************   ØØØØ7ØØØ
C   *    THE FORMAL PARAMETER LIST                                      ØØØØ7Ø1Ø
C   *                                                                   ØØØØ7Ø2Ø
C   *    ON INPUT -                                                     ØØØØ7Ø3Ø
C   *                                                                   ØØØØ7Ø4Ø
C   *         NM    - MUST BE SET TO THE ROW DIMENSION OF THE TWO-      ØØØØ7Ø5Ø
C   *                 DIMENSIONAL ARRAY PARAMETERS AS DECLARED IN       ØØØØ7Ø6Ø
C   *                 THE CALLING PROGRAM                               ØØØØ7Ø7Ø
C   *         N     - IS THE ORDER OF THE MATRICES                      ØØØØ7Ø8Ø
C   *                                                                   ØØØØ7Ø9Ø
C   *         HR,HI - CONTAIN THE REAL AND IMAGINARY PARTS,RESPECTIVELY ØØØØ71ØØ
C   *                 OF THE COMPLEX MATRIX                             ØØØØ711Ø
C   *         EIN   - IS A TOLERANCE PARAMETER,CORRESPONDING TO         ØØØØ712Ø
C   *                 PERTURBATIONS OF HR,HI, AND IS USED IN THE        ØØØØ713Ø
C   *                 GROUPING OF NUMERICAL MULTIPLE EIGENVALUES        ØØØØ714Ø
C   *                                                                   ØØØØ715Ø
C   *         TOL   - IS A TOLERANCE PARAMETER USED IN THE CONSTRUCTION ØØØØ716Ø
C   *                 OF THE NILPOTENT MATRICES (IS USED IN RDEFL)      ØØØØ717Ø
C   *                                                                   ØØØØ718Ø
```

```
C   *          ISTEP  - RETURN FROM JNF AFTER STEP ISTEP(=1,2,3,4,5,6,7)ØØØØ719Ø
C   *                   FULL REDUCTION TO JORDAN FORM IF ISTEP .GE. 7   ØØØØ72ØØ
C   *                   OTHER CHOICES GIVE REDUCTION TO                 ØØØØ721Ø
C   *                                                                   ØØØØ722Ø
C   *           1       UPPER TRIANGULAR FORM                           ØØØØ723Ø
C   *                                                                   ØØØØ724Ø
C   *           2       UPPER TRIANGULAR FORM WITH THE EIGEN-           ØØØØ725Ø
C   *                   VALUES SORTED SUCH THAT CLOSE EIGEN-            ØØØØ726Ø
C   *                   VALUES APPEAR IN ADJACENT POSITIONS             ØØØØ727Ø
C   *                                                                   ØØØØ728Ø
C   *           3       THE SAME AS 2,IN ADDITION GROUPING OF           ØØØØ729Ø
C   *                   CLOSE EIGENVALUES INTO BLOCKS CORRE-            ØØØØ73ØØ
C   *                   SPONDING TO NUMERICAL MULTIPLE EIGEN-           ØØØØ731Ø
C   *                   VALUES                                          ØØØØ732Ø
C   *                                                                   ØØØØ733Ø
C   *           4       BLOCK DIAGONAL UPPER TRIANGULAR FORM            ØØØØ734Ø
C   *                                                                   ØØØØ735Ø
C   *           5       BLOCK DIAGONAL UPPER TRIANGULAR FORM            ØØØØ736Ø
C   *                   AND THE INVARIANT SUBSPACES CORRESPONDING       ØØØØ737Ø
C   *                   TO THE DIAGONALS BLOCKS HAVE UNITARY BASES      ØØØØ738Ø
C   *                                                                   ØØØØ739Ø
C   *           6       THE STRUCTURE OF EACH DIAGONAL BLOCK IS         ØØØØ74ØØ
C   *                   DETERMINED                                      ØØØØ741Ø
C   *    ON OUTPUT -                                                    ØØØØ742Ø
C   *                                                                   ØØØØ743Ø
C   *  THE PARAMETERS CONTAIN USEFUL INFORMATION DEPENDING ON           ØØØØ744Ø
C   *  THE VALUE OF THE INPUT PARAMETER ISTEP.                          ØØØØ745Ø
C   *                                                                   ØØØØ746Ø
C   *  NOTATION - (IF ISTEP .GE. N )                                    ØØØØ747Ø
C   *  WHERE N IS 1,2,3,4,5,6 OR 7                                      ØØØØ748Ø
C   *                                                                   ØØØØ749Ø
C   *                                                                   ØØØØ75ØØ
C   *         HR,HI - HAVE BEEN DESTROYED                               ØØØØ751Ø
C   *                                                                   ØØØØ752Ø
C   *         IERR  - IS A CONVERGENCE PARAMETER SET BY COMLR2          ØØØØ753Ø
C   *                 Ø  FOR NORMAL RETURN                              ØØØØ754Ø
C   *                 J  IF THE J-TH EIGENVALUE HAS NOT BEEN DETERMINED ØØØØ755Ø
C   *                 AFTER 3Ø ITERATIONS                               ØØØØ756Ø
C   *                 IF AN ERROR EXIT IS MADE (IERR.NE.Ø),NONE         ØØØØ757Ø
C   *                 OF THE FOLLOWING PARAMETERS CONTAIN MEANINGFUL    ØØØØ758Ø
C   *                 RESULTS                                           ØØØØ759Ø
C   *                (IF ISTEP.GE.1)                                    ØØØØ76ØØ
C   *         EVR,EVI CONTAIN THE REAL AND IMAGINARY PARTS,RESPECTIVELY ØØØØ761Ø
C   *                 OF THE EIGENVALUES.                               ØØØØ762Ø
C   *                 IF ISTEP.GE.1 BUT .LT.5 THE EIGENVALUES ARE       ØØØØ763Ø
C   *                 COMPUTED BY COMLR2.                               ØØØØ764Ø
C   *                 IF ISTEP.GE.6, EVR,EVI CONTAIN THE MAIN DIAGONAL  ØØØØ765Ø
C   *                 OF THE JORDAN MATRIX.                             ØØØØ766Ø
C   *                                                                   ØØØØ767Ø
C   *         NBLOCK- IS THE NUMBER OF NUMERICAL MULTIPLE EIGENVALUES   ØØØØ768Ø
C   *                 (IF ISTEP.GE.3)                                   ØØØØ769Ø
C   *                                                                   ØØØØ77ØØ
C   *         NDEL  - INDICATES THE MULTIPLICITIES OF THE NUMERICAL     ØØØØ771Ø
C   *                 MULTIPLE EIGENVALUES                              ØØØØ772Ø
C   *                 (IF ISTEP.GE.3)                                   ØØØØ773Ø
C   *                 NDEL(I+1)-NDEL(I)= MULTIPLICITY OF EIGENVALUE     ØØØØ774Ø
C   *                                    I FOR I=1,...,NBLOCK           ØØØØ775Ø
C   *         NDEFL - INDICATES THE STRUCTURE OF HR,HI                  ØØØØ776Ø
C   *                 (IF ISTEP.GE.6)                                   ØØØØ777Ø
C   *                                                                   ØØØØ778Ø
C   *         SUPD  - CONTAINS THE REAL SUPERDIAGONAL OF THE JORDAN     ØØØØ779Ø
C   *                 MATRIX (THE COUPLING ELEMENTS)                    ØØØØ78ØØ
C   *                 (IF ISTEP.GE.7)                                   ØØØØ781Ø
```

```
C   *                                                                    ØØØØ782Ø
C   *        NXT    - CONTAINS THE COLUMN INDICES OF SUPD               ØØØØ783Ø
C   *                 NXT(I)=J IMPLIES THAT THE VECTOR WITH COUPLING     ØØØØ784Ø
C   *                          ELEMENT SUPD(I) IS PLACED IN COLUMN       ØØØØ785Ø
C   *                          J OF ZR,ZI                                ØØØØ786Ø
C   *                 (IF ISTEP.GE.7)                                    ØØØØ787Ø
C   *        NDB    - CONTAINS POINTER REFERENCES                       ØØØØ788Ø
C   *                 NDB(I)=J MEANS THAT INFORMATION ABOUT THE          ØØØØ789Ø
C   *                          STRUCTURE OF THE NUMERICAL MULTIPLE       ØØØØ79ØØ
C   *                          EIGENVALUE I STARTS IN POSITION J         ØØØØ791Ø
C   *                          OF NDEFL                                  ØØØØ792Ø
C   *                 (IF ISTEP.GE.6)                                    ØØØØ793Ø
C   *        ZR,ZI - CONTAIN THE REAL AND IMAGINARY PARTS,RESPECTIVELY   ØØØØ794Ø
C   *                 OF THE ACCUMULATED TRANSFORMATIONS FROM STEP 1     ØØØØ795Ø
C   *                 TO 7 OF THE ALGORITHM.                             ØØØØ796Ø
C   *                 IF ISTEP.GE.7, ZR,ZI ARE THE EIGENVECTORS AND      ØØØØ797Ø
C   *                 THE PRINCIPAL VECTORS.                             ØØØØ798Ø
C   *                                                                    ØØØØ799Ø
C   *        DELE   - CONTAINS INFORMATION ABOUT DELETED ELEMENTS        ØØØØ8ØØØ
C   *                 DURING THE PROCESSES OF FINDING NILPOTENT MATRICESØØØØ8Ø1Ø
C   *                 DELE(I)= EUCLIDEAN NORM OF DELETED PART FOR BLOCK ØØØØ8Ø2Ø
C   *                          I FOR I=1,...,NBLOCK                      ØØØØ8Ø3Ø
C   *                 (IF ISTEP.GE.6)                                    ØØØØ8Ø4Ø
C   *                                                                    ØØØØ8Ø5Ø
C   *        SM     - CONTAINS ESTIMATES OF THE SPECTRAL PROJECTORS     ØØØØ8Ø6Ø
C   *                 CORRESPONDING TO DIFFERENT NUMERICAL MULTIPLE      ØØØØ8Ø7Ø
C   *                 EIGENVALUES                                        ØØØØ8Ø8Ø
C   *                 (IF ISTEP.GE.4)                                    ØØØØ8Ø9Ø
C   *                                                                    ØØØØ81ØØ
C   *    FOR COMPLETE DESCRIPTION SEE COMPUTATIONAL DETAILS IN           ØØØØ811Ø
C   *    THE TEXT                                                        ØØØØ812Ø
C   *                                                                    ØØØØ813Ø
```