# Almost-Optimal Sublinear Additive Spanners 

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#### Abstract

Given an undirected unweighted graph $G=(V, E)$ on $n$ vertices and $m$ edges, a subgraph $H \subseteq G$ is a spanner of $G$ with stretch function $f: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$, iff for every pair $s, t$ of vertices in $V$, $\operatorname{dist}_{H}(s, t) \leq f\left(\operatorname{dist}_{G}(s, t)\right)$. When $f(d)=d+o(d), H$ is called a sublinear additive spanner; when $f(d)=d+o(n), H$ is called an additive spanner, and $f(d)-d$ is usually called the additive stretch of $H$.

As our primary result, we show that for any constant $\delta>0$ and constant integer $k \geq 2$, every graph on $n$ vertices has a sublinear additive spanner with stretch function $f(d)=d+\bar{O}\left(d^{1-1 / k}\right)$ and $O\left(n^{1+\frac{1+\delta}{2^{k+1}-1}}\right)$ edges. When $k=2$, this improves upon the previous spanner construction with stretch function $f(d)=d+O\left(d^{1 / 2}\right)$ and $\tilde{O}\left(n^{1+3 / 17}\right)$ edges [Chechik, 2013]; for any constant integer $k \geq 3$, this improves upon the previous spanner construction with stretch function $f(d)=d+O\left(d^{1-1 / k}\right)$ and $O\left(n^{1+\frac{\left(3 / 44^{k-2}\right.}{7-2 \cdot(3 / 4)^{k-2}}}\right)$ edges [Pettie, 2009]. Most importantly, the size of our spanners almost matches the lower bound of $\Omega\left(n^{1+\frac{1}{2^{k+1}-1}}\right)$ [Abboud, Bodwin, Pettie, 2017].

As our second result, we show a new construction of additive spanners with stretch $O\left(n^{0.403}\right)$ and $O(n)$ edges, which slightly improves upon the previous stretch bound of $O\left(n^{3 / 7+\varepsilon}\right)$ achieved by linear-size spanners [Bodwin and Vassilevska Williams, 2016]. An additional advantage of our spanner is that it admits a subquadratic construction runtime of $\tilde{O}\left(m+n^{13 / 7}\right)$, while the previous construction in [Bodwin and Vassilevska Williams, 2016] requires all-pairs shortest paths computation which takes $O\left(\min \left\{m n, n^{2.373}\right\}\right)$ time.


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## 1 Introduction

Graph spanners are sparse subgraphs that approximately preserve pairwise shortest-path distances. Let $G=(V, E)$ be an undirected unweighted graph on $n$ vertices and let $f: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$be a function. We say that a subgraph $H \subseteq G$ is a spanner with stretch function $f$ iff for every pair $s, t \in V$, $\operatorname{dist}_{H}(s, t) \leq f\left(\operatorname{dist}_{G}(s, t)\right)$. The research on spanners focuses on the optimal trade-offs between the stretch function $f$ and the sparsity (the number of edges) of the spanner $H$.

One extreme case is that we allow $f(d)$ to be significantly greater than $d$, and such spanners are known as multiplicative spanners. It was shown that for every integer $k \geq 1$, there always exists a subgraph $H$ with $O\left(n^{1+1 / k}\right)$ edges and stretch function $f(d)=(2 k-1) d\left[\mathrm{ADD}^{+93}\right.$. Furthermore, this sparsity bound is tight under the Girth Conjecture of Erdös [Erd63].

Another extreme case is that we restrict $f$ to be very close to $d$. In particular, $f(d)=d+O(1)$. There has been a line of previous work studying the sparsity of spanners with such stretch functions. When $f(d)=d+2$, it was shown that graph $G$ always has a spanner with $O\left(n^{3 / 2}\right)$ edges [ACIM99]; when $f(d)=d+4$, a construction of spanner with $\tilde{O}\left(n^{7 / 5}\right)$ edges was proposed in Che13; when $f(d)=d+6$, spanners with $O\left(n^{4 / 3}\right)$ edges were known to exist by [BKMP10]. For the lower bound side, in a recent breakthrough AB17, it was proved that, for any constant $\varepsilon>0$, there are graphs such that any spanner with $O\left(n^{4 / 3-\varepsilon}\right)$ edges has stretch $n^{\Omega(1)}$. Hence, we already have an almost complete understanding of the spanner sparsity when $f(d)=d+O(1)$.

Besides the two extreme cases mentioned above, much less is known when $f$ lies in intermediate regimes. Two notable regimes studied in previous works are the sublinear additive regime where $f(d)=d+o(d)$, and the additive regime where $f(d)=d+o(n)$.

As for sublinear additive spanners, Thorup and Zwick [T06] were the first to design a nontrivial construction of sublinear additive spanners when $f(d)=d+O\left(d^{1-1 / k}\right)$ where $k \geq 2$ is a constant integer, and the number of edges in the spanner is bounded by $O\left(n^{1+1 / k}\right)$. This sparsity bound was later improved to $O\left(n^{1+\frac{\left(3 / 4 \cdot{ }^{k-2}\right.}{7-2 \cdot(3 / 4)^{k-2}}}\right)$ in [Pet09. The sparsity bound for the special case where $k=2$ was subsequently improved to $\tilde{O}\left(n^{20 / 17}\right)$ by Che13. These algorithms also work for a nonconstant $k$, but here we only focus on the case where $k$ is a constant, as we are mainly interested in the stretch/size dependency on $d$ and $n$. On the lower bound side, Abboud, Bodwin, and Pettie ABP18] proved that there exists hard instances where any spanner of stretch $f(d)=d+O\left(d^{1-1 / k}\right)$ must contain at least $\Omega\left(n^{1+\frac{1}{2^{k+1}-1}-o(1)}\right)$ edges for each constant $k \geq 2$. Thus, there still exists a large gap between sparsity upper and lower bounds for sublinear additive spanners.

For the additive regime where $f(d)=d+o(n)$, the tail term $f(d)-d$ is usually called the additive stretch. A natural question is to study the best additive stretch that can be achieved by spanners with $\tilde{O}(n)$ edges. The first nontrivial construction was given bet09 with an additive stretch of $O\left(n^{9 / 16}\right)$, which was improved subsequently by BW15, BW16 to $O\left(n^{3 / 7+\varepsilon}\right)$ for any constant $\varepsilon>0$. On the negative side, the first stretch lower bound of $\Omega\left(n^{1 / 22}\right)$ was proved in [AB17], and later on raised to $\Omega\left(n^{1 / 7}\right)$ by a sequence of works [HP21, LWWX22, BH22.

### 1.1 Our results

As our primary result, we construct sublinear additive spanners that almost match the lower bound from ABP18.

Theorem 1.1. For any undirected unweighted graph $G$ on $n$ vertices, any constant $\delta>0$ and any integer $k \geq 2$, there is a sublinear spanner $H \subseteq G$ with stretch function $f(d)=d+O_{\delta, k}\left(d^{1-1 / k}\right)$
and $O\left(n^{1+\frac{1+\delta}{2^{k+1}-1}}\right)$ edge. 1 .
Our second result is a slightly improved bound on linear size additive spanners upon the bound $O\left(n^{3 / 7+\varepsilon}\right)$ of BW16. In addition, we show that such a spanner can be computed in subquadratic time, while previous constructions in BW16, BW15 need the computation of all-pairs shortest paths in $G$ which takes time $O\left(\min \left\{m n, n^{2.373}\right\}\right)$.

Theorem 1.2. For any undirected unweighted graph $G$ on $n$ vertices and $m$ egdges, there exists a spanner with $O\left(n^{0.403}\right)$ additive stretch and $O(n)$ edges. Moreover, such a spanner can be computed in time $\tilde{O}\left(m+n^{13 / 7}\right)$.

### 1.2 Technical overview

The basic tool of our algorithms is a clustering algorithm in BW16. Let $G=(V, E)$ be the input undirected unweighted graph. Roughly speaking, for any radius parameter $R$, we can decompose the graph into a set $\mathcal{B}$ of balls, with the following properties.

- (Radius) The radius of each ball in $\mathcal{B}$ is roughly $R$.
- (Coverage) The union of all balls in $\mathcal{B}$ covers the whole graph $G$.
- (Disjointness) The total size of the balls in $\mathcal{B}$ is linear in $n$.

Next, we will describe how to utilize the above clustering to construct new sublinear additive spanners and additive spanners, respectively.

### 1.2.1 Sublinear additive spanners

Our starting point is a spanner with stretch function $f(d)=d+O\left(d^{1 / 2}\right)$ and $\tilde{O}\left(n^{8 / 7}\right)$ edges, which improves upon the previous sparsity bound of $\tilde{O}\left(n^{20 / 17}\right)$ in Che13. Consider any pair of vertices $s, t \in V$, and let $\pi$ be a shortest path between them of length $D$. We apply the clustering algorithm from [BW16] with radius parameter $R=D^{1 / 2}$ to $G$, and obtain a set $\mathcal{B}$ of balls. Intuitively, by the coverage property, and recalling that every ball in $\mathcal{B}$ has radius roughly $R=D^{1 / 2}$, we can choose a subset of $O(D / R)=O\left(D^{1 / 2}\right)$ balls from $\mathcal{B}$ whose union contains the entire shortest path $\pi$. See Figure $\mathbb{1}$ for an illustration.


Figure 1: An illustration of a covering of a shortest path $\pi$ between $s, t$ with at most $l=O\left(D^{1 / 2}\right)$ balls from $\mathcal{B}$. For simplicity, we assume that, for each index $0 \leq i \leq l-1$, the balls $\mathrm{B}\left(c_{i}, r_{i}\right)$ and $\mathrm{B}\left(c_{i+1}, r_{i+1}\right)$ shares exactly one vertex of $\pi$, denoted by $w_{i+1}$. We denote $s=w_{0}$ and $t=w_{l+1}$.

[^1]Following the notations in Figure $\mathbb{1}$, assume the sequence of balls divides the path into subpaths $\left\{\pi\left[w_{i}, w_{i+1}\right]\right\}_{0 \leq i \leq l-1}$. In order to preserve the distance between $s, t$ in $G$, a simple approach is to plant, within each subgraph $G\left[\mathrm{~B}\left(c_{i}, r_{i}\right)\right]$, a 6 -additive spanner from BKMP10 with $O\left(\left|\mathrm{~B}\left(c_{i}, r_{i}\right)\right|^{4 / 3}\right)$ edges. Then for each $i, \operatorname{dist}_{H}\left(w_{i}, w_{i+1}\right) \leq \operatorname{dist}_{G}\left(w_{i}, w_{i+1}\right)+6$, so $\operatorname{dist}_{H}(s, t)-\operatorname{dist}_{G}(s, t) \leq 6(l+1)=$ $O\left(D^{1 / 2}\right)$. If we further assume that each ball in $\mathcal{B}$ contains at most $n^{3 / 7}$ vertices, then by the disjointness property, the union of all these 6 -additive spanners contains at most $\sum_{\mathrm{B}(c, r) \in \mathcal{B}}|\mathrm{B}(c, r)|^{4 / 3}=$ $O\left(n^{8 / 7}\right)$ edges. So we only need to deal with large balls that contain more than $n^{3 / 7}$ vertices.

A natural approach to handling the large balls is taking a random subset $S \subseteq V$ of $10 n^{4 / 7} \log n$ vertices that hits all large balls with high probability, and try preserving pairwise distances between vertices in $S$, as preserving distances among a subset is conceivably easier than the whole graph. In fact, if this can be done, then the distance between all pairs $s, t \in V$ is also well-preserved. To see why this is true, assume for simplicity that $\mathrm{B}\left(c_{1}, r_{1}\right)$ and $\mathrm{B}\left(c_{l-1}, r_{l-1}\right)$ are the first and the last large ball, respectively. Then, by construction of $S$, there exist $u \in \mathrm{~B}\left(c_{1}, r_{1}\right) \cap S$ and $v \in \mathrm{~B}\left(c_{l-1}, r_{l-1}\right) \cap S$, and it is easy to verify using triangle inequality that the following path connecting $s$ to $t$ in $H$ has length at most $\operatorname{dist}_{G}(s, t)+O(R)$; the path is the concatenation of (see Figure 2 for an illustration):

- a path from $s$ to $w_{1}$ in the 6 -additive spanner within subgraph $G\left[\mathrm{~B}\left(c_{0}, r_{0}\right)\right]$; and similarly a path from $t$ to $w_{l}$ in the 6 -additive spanner within subgraph $G\left[\mathrm{~B}\left(c_{l}, r_{l}\right)\right]$;
- a path from $w_{1}$ to $u$ of length at most $2 R$, as we can afford to add to $H$ a breath-first search tree of $G\left[\mathrm{~B}\left(c_{1}, r_{1}\right)\right]$ beforehand; similarly, a path from $w_{l}$ to $v$ of length at most $2 R$; and
- the shortest path connecting $u, v$ in $H$.


Figure 2: If the balls $\mathrm{B}\left(c_{1}, r_{1}\right)$ and $\mathrm{B}\left(c_{l-1}, r_{l-1}\right)$ are large, then we can find a short path from $s$ to $t$ drawn as the orange wavy lines.

Therefore, it suffices to construct a spanner that faithfully preserves pairwise distances between vertices in $S$. Consider now a pair $s, t$ of vertices in $S$. We proceed similarly by first finding a sequence of $O\left(D^{1 / 2}\right)$ balls in $\mathcal{B}$ that covers the shortest path between $s$ and $t$, and then computing a sparse subgraph $H_{c_{i}} \subseteq G\left[\mathrm{~B}\left(c_{i}, r_{i}\right)\right]$ in order to preserve the distance between $w_{i}, w_{i+1}$ to within a constant additive error. Note that we would like that the graph $H_{c_{i}}$ also approximately preserves, for other pairs $s^{\prime}, t^{\prime} \in S$, the distance between their corresponding vertices $w_{j}^{\prime}, w_{j+1}^{\prime}$. The reason that this is easier to achieve is because there are fewer pairs $\left(w_{i}, w_{i+1}\right)$ assigned to the ball $\mathrm{B}\left(c_{i}, r_{i}\right)$ while processing all pairs in $S$, as $|S|$ itself is small. More formally, each ball $\mathrm{B}\left(c_{i}, r_{i}\right)$ is associated with a set $\mathcal{P}_{c_{i}} \subseteq V \times V$ of demand pairs, and we want to find a spanner $H_{c_{i}} \subseteq G\left[\mathrm{~B}\left(c_{i}, r_{i}\right)\right]$ that only preserves distances between vertex pairs in $\mathcal{P}_{c_{i}}$. The way we construct $\mathcal{P}_{c_{i}}$ is to enumerate all pairs $s, t \in S$, find the $s$ - $t$ shortest path and a set of balls that cover it, and if we have used the ball $\mathrm{B}\left(c_{i}, r_{i}\right)$ in the covering, then add the corresponding pair $\left(w_{i}, w_{i+1}\right)$ to $\mathcal{P}_{c_{i}}$. In the end, we can show that the size of each set $\mathcal{P}_{c_{i}}$ is at most $|S|=\tilde{O}\left(n^{4 / 7}\right)$.

This special type of spanners restricted to demand pairs are called pairwise spanners, and has been studied in Kav17. Applying their result in a black-box way, we can construct, for each ball $G\left[\mathrm{~B}\left(c_{i}, r_{i}\right)\right]$, a subgraph $H_{c_{i}}$ of size $O\left(\left|\mathrm{~B}\left(c_{i}, r_{i}\right)\right| \cdot\left|\mathcal{P}_{c_{i}}\right|^{1 / 4}\right)$ preserving distances between pairs in $\mathcal{P}_{c_{i}}$ up to an additive error of 6 . Then, the total size of all $H_{c_{i}}$ is $\sum_{\mathrm{B}(c, r) \in \mathcal{B}}|\mathrm{B}(c, r)| \cdot\left|\mathcal{P}_{c}\right|^{1 / 4}=\tilde{O}\left(n^{8 / 7}\right)$.

### 1.2.2 Pairwise sublinear additive spanners

The only missing component towards a sublinear additive spanner with stretch $f(d)=d+O\left(d^{1-1 / k}\right)$ for general $k \geq 2$ turns out to be a pairwise spanner with stretch $d+O\left(d^{1-1 /(k-1)}\right)$. As the previous work on pairwise spanners Kav17 only considered stretch function $f(d)=d+O(1)$, we need to generalize their result for general $k$. Assume we are given an undirected unweighted graph $G=(V, E)$ on $n$ vertices and a set $\mathcal{P} \subseteq V \times V$ of pairs, and the goal is to find a spanner $H \subseteq G$ with at most $\tilde{O}\left(n|\mathcal{P}|^{1 / 2^{k+1}}\right)$ edges, such that $\operatorname{dist}_{H}(s, t) \leq f\left(\operatorname{dist}_{G}(s, t)\right)$ for all $(s, t) \in \mathcal{P}$. For the purpose of this overview, we assume for simplicity that $\operatorname{dist}_{G}(s, t)=D$ for all $(s, t) \in \mathcal{P}$.

The construction of $H$ is inductive on $k \geq 1$. Assume we know how to construct such spanners for $k-1$. First, apply the clustering algorithm from BW16 with radius $R=D^{1-1 / k}$ to graph $G$, and obtain a set $\mathcal{B}$ of balls.

Uniform size. We start by considering a special case, where all balls in $\mathcal{B}$ have the same size $|\mathrm{B}(c, r)|=L$. By the disjointness property, the number of balls in $\mathcal{B}$ is approximately $n / L$. For each pair $(s, t) \in \mathcal{P}$, we compute a sequence of $D^{1 / k}$ balls and partition the shortest path connecting $s, t$, in a similar way as illustrated in Figure 1 . We wish to inductively build inside each ball $G\left[\mathrm{~B}\left(c_{i}, r_{i}\right)\right]$ a pairwise spanner with a better stretch $d+O\left(d^{1-1 /(k-1)}\right)$; if this is done, then the cumulative error along the shortest path connecting $s$ and $t$ is bounded by $D^{1 / k} \cdot R^{1-1 /(k-1)}=D^{1-1 / k}$.

In order to apply the inductive hypothesis, we will construct, for each ball $\mathrm{B}(c, r) \in \mathcal{B}$, a set $\mathcal{P}_{c}$ of demand pairs. But if we add, for all $i$, the pair $\left(w_{i}, w_{i+1}\right)$ to $\mathcal{P}_{c_{i}}$, then the sets $\left|\mathcal{P}_{c_{i}}\right|$ may become too large (as large as $|\mathcal{P}|$ ) to construct a spanner for. To circumvent this, we use the approach in [Kav17]. Observe that, if we add each pair $\left(w_{i}, w_{i+1}\right)$ to $\mathcal{P}_{c_{i}}$, then not only the distance between $s, t$ is approximately preserved (in this case we call the pair ( $s, t$ ) settled), but the distances between all pairs of ball centers $c_{i}, c_{j}$ are also approximately preserved. Now there are two cases.

- Case 1. There are at least $\beta \cdot(l+1)$ center pairs that were unsettled in $H$; recall that $l+1$ refers to the number of clusters as in Figure 1 .
Then after adding each $\left(w_{i}, w_{i+1}\right)$ as a demand pair to $\mathcal{P}_{c_{i}}$, the total number of such center pairs would drop by $\beta \cdot(l+1)$. In other words, each new demand pair $\left(w_{i}, w_{i+1}\right)$ contributes $\beta$ to the decrease of the number of unsettled center pairs on average.
- Case 2. There are fewer than $\beta \cdot(l+1)$ center pairs that were unsettled in $H$.

In this case, we show that there is always a "bridge" that helps settle the pair $(s, t)$. Specifically, we can prove there are indices $x \in[1, \beta], z \in[\beta+1, l+1-\beta], y \in[l-\beta+2, l+1]$ such that center pairs $\left(c_{x}, c_{z}\right)$ and $\left(c_{z}, c_{y}\right)$ were already settled. In this case, adding demand pairs $\left(w_{i}, w_{i+1}\right)$ to $\mathcal{P}_{c_{i}}$ only for $i \in[1, \beta] \cup[l-\beta+2, l+1]$ would be enough. See Figure 3 for an illustration.
To summarize, for each pair $(s, t) \in \mathcal{P}$, we either add at most $2 \beta$ demand pairs in total, or add a number of demand pairs such that the number of unsettled center pairs decreases by $\beta$ per pair. Therefore, the total number of demand pairs is at most $\beta|\mathcal{P}|+\frac{n^{2}}{L^{2} \beta}$, which is $O\left(\frac{n|\mathcal{P}|^{1 / 2}}{L}\right)$ if we set $\beta=\frac{n}{|\mathcal{P}|^{1 / 2}}$, and thus the size of sets $\left|\mathcal{P}_{c}\right|$ is bounded by $|\mathcal{P}|^{1 / 2}$ on average. Now apply the inductive hypothesis within each ball, and the total size of all these pairwise sublinear spanners is bounded by $\tilde{O}\left(n|\mathcal{P}|^{1 / 2^{k+1}}\right)$.


Figure 3: In this case, we can find a "bridge" that allows us to travel from $c_{x}$ to $c_{z}$ and then to $c_{y}$ via a near-shortest path. The orange segments represent new demand pairs assigned to the balls.

Non-uniform size. In order to provide some intuition for the general case, we consider a slightly more general case (compared with the uniform size case) where balls in $\mathcal{B}$ have size either $L_{1}$ or $L_{2}$, which naturally classify balls in $\mathcal{B}$ into level- 1 balls and level- 2 balls. Similar to the uniform size case, we process all pairs $(s, t) \in \mathcal{P}$ and find the balls that partition the shortest $s-t$ path. Assume there are $l_{1}$ level- 1 balls and $l_{2}$ level- 2 balls. To extend the technique from Kav17, this time we only count the number of center pairs separately for each level. For each $b \in\{1,2\}$, let $\Phi_{b}$ be the number of unsettled level- $b$ ball center pairs, and let $\Delta \Phi_{b}$ be the number of unsettled level- $b$ center pairs along the $s-t$ path. Similarly, we consider the following two cases.

- Case 1. $\Delta \Phi_{b} \geq \beta_{b} \cdot l_{b}$ for both $b \in\{1,2\}$.

In this case, by adding, for all $i$, the pair $\left(w_{i}, w_{i+1}\right)$ to $\mathcal{P}_{c_{i}}$, on average, $\Phi_{1}$ is decreased by $\beta_{1}$ per pair, and $\Phi_{2}$ is decreased by $\beta_{2}$ per pair.

- Case 2. (Without loss of generality) $\Delta \Phi_{1}<\beta_{1} \cdot l_{1}$.

Similar to the uniform size case, we are able to find a bridge of level-1 ball centers. Specifically, we can prove there are indices $x, y, z$ such that both pairs $\left(c_{x}, c_{z}\right)$ and $\left(c_{z}, c_{y}\right)$ are settled level- 1 ball centers. In this case, we add demand pairs $\left(w_{i}, w_{i+1}\right)$ to $\mathcal{P}_{c_{i}}$ where $\mathrm{B}\left(c_{i}, r_{i}\right)$ is among the first and the last $\beta_{1}$ level- 1 balls. See Figure 4 for an illustration. Then we still need to zoom into the intervals $[1, x]$ and $[y, l+1]$, and recurse. Now, since all the level- 1 balls have already been assigned their demand pairs within these two intervals, we have now reduced to the uniform size case within those two intervals.

Intuitively, in the general case where all balls in $\mathcal{B}$ have different sizes, we will first partition them into $O(1 / \varepsilon)$ groups according to their sizes, and then generalize the above approach for two ball sizes to multiple ball sizes. This will eventually introduce an additional factor $2^{O(k / \varepsilon)} n^{O(k \varepsilon)}$ in the sparsity.

### 1.2.3 Additive spanners

We now provide an overview for our additive spanner construction. In what follows we will mainly highlight the difference between the construction by [BW16] and ours.

First, we apply the clustering algorithm from [BW16] with radius $R$ to $G$ and obtain a set $\mathcal{B}$ of balls. Similar to the construction of sublinear and pairwise additive spanners, while processing


Figure 4: Orange segments are covered by level-1 balls, and cyan segments are covered by level- 2 balls; we only add the first and the last $\beta_{1}$ level- 1 segments as new demand pairs.
unsettled pairs of vertices, we will add demand pairs to the corresponding balls that partition their shortest paths. It has been shown that $O(\sqrt{n})$ demand pairs can be preserved exactly with $O(n)$ edges [CE06], and the bound $O(\sqrt{n})$ is cannot be improved, so we need to ensure that each ball B collects at most $O\left(|\mathrm{~B}|^{1 / 2}\right)$ demand pairs. On the one hand, the small balls B with $|\mathrm{B}| \leq R^{4 / 3}$ can be handled using previous work on subset distance preservers [CE06. On the other hand, for the large balls B with $|\mathrm{B}| \geq R^{4 / 3}$, the algorithm in BW 16 ] placed an $O\left(|\mathrm{~B}|^{1 / 2}\right)$ additive spanner (which follows from [BW15]) inside them. This implies that, whenever a new path $\pi$ passes through a ball B , along with the demand pair added to B , at least $R \cdot\left(R^{4 / 3} / R^{2 / 3}\right)=R^{5 / 3}$ new vertice $3^{2}$ are settled with B , and so a ball B may collect $O\left(n / R^{5 / 3}\right)$ pairs. When $R=n^{3 / 7}$, a ball B can collect at most $O\left(n / R^{5 / 3}\right)=O\left(n^{2 / 7}\right) \leq\left(R^{4 / 3}\right)^{1 / 2} \leq O\left(|\mathrm{~B}|^{1 / 2}\right)$ pairs.

We proceed differently. We start by showing via the above framework that any graph $G$ and any subset $S \subseteq V(G)$ has a $O\left(|S|^{3 / 2}\right)$-additive subset spanner of linear size with respect to $S$, which is a subgraph $H \subseteq G$, such that for every pair $s, s^{\prime} \in S$, $\operatorname{dist}_{H}\left(s, s^{\prime}\right) \leq \operatorname{dist}_{G}\left(s, s^{\prime}\right)+O\left(|S|^{3 / 2}\right)$. The algorithm is simple. Set $R=|S|^{3 / 2}$ and compute a clustering (and we can ignore balls B with $|\mathrm{B}| \leq\left(|S|^{3 / 2}\right)^{4 / 3}=|S|^{2}$ ); iteratively go through all shortest paths connecting pairs in $S$ and buy them whenever their endpoints are unsettled. Since whenever a cluster collects a new demand pair, it is settled with a new vertex in $S$, a ball B gets at most $|S|=O\left(|\mathrm{~B}|^{1 / 2}\right)$ demand pairs. Now instead of planting a $+O\left(|\mathrm{~B}|^{1 / 2}\right)$ additive spanner inside each ball, we plant a $+O\left(|S|^{3 / 2}\right)$ subset spanner inside each ball with respect to its boundary vertices. The stretch between them is now $O\left((|\mathrm{~B}| / R)^{3 / 2}\right)$, which is less than $|\mathrm{B}|^{1 / 2}$ when $R^{4 / 3}<|\mathrm{B}|<R^{3 / 2}$, this small advantage, together with some further utilization of the subset spanner, allows us to eventually improve their bound $O\left(n^{3 / 7+\varepsilon}\right)$ to $O\left(n^{0.403}\right)$.

### 1.3 Organization

We start with preliminaries in Section 2. We provide a construction for pairwise additive spanners in Section 3, which will be a building block in the proof of Theorem 1.1 which appears in Section 4 , Next, we show the construction of subset spanners in Section 5, which will be a crucial subroutine

[^2]used in the algorithm of Theorem 1.2. Lastly, we provide the proof of Theorem 1.2 in Section 6 .

## 2 Preliminaries

By default, all logarithms are to the base of 2 , and all graphs are undirected and unweighted.
Let $G=(V, E)$ be a graph. For a subset $S \subseteq V$ of vertices, we define $\operatorname{vol}_{G}(S)=\sum_{v \in S} \operatorname{deg}_{G}(v)$, where $\operatorname{deg}_{G}(v)$ is the degree of $v$ in $G$. For any pair $u, v \in V(G)$, let $\operatorname{dist}_{G}(u, v)$ be the shortest-path distance between $u$ and $v$ in $G$. For any $c \in V$ and $r>0$, we define the ball $\mathrm{B}_{G}(c, r)$ as the set of vertices in $G$ that are at distance at most $R$ from $c$, namely $\mathrm{B}_{G}(c, r)=\left\{v \in V \mid \operatorname{dist}_{G}(c, v) \leq r\right\}$; $r$ is called the radius of the ball, and we define the boundary of the ball $\mathrm{B}_{G}(c, r)$ as $\mathrm{B}_{\bar{G}}(c, r)=$ $\left\{v \in V \mid \operatorname{dist}_{G}(c, v)=r\right\}$. We will sometimes omit the subscript $G$ in the above notations when it is clear from context. Let $\mathcal{B}$ be a collection of balls. We say that a vertex $v$ is covered by $\mathcal{B}$ iff $v$ belongs to some ball in $\mathcal{B}$. We will use the following lemma, whose proof is a simple implementation of the breath-first search (BFS) algorithm, and is omitted.
Lemma 2.1. Given a graph $G$, a vertex $c \in V(G)$, and two integers $0<r_{1}<r_{2}<r$, we can find an integer $r_{1}<d \leq r_{2}$, such that $\left|\mathrm{B}^{=}(c, d) \cup \mathrm{B}^{=}(c, d+1)\right| \leq \frac{2 \cdot|\mathrm{~B}(c, r)|}{r_{2}-r_{1}}$, in time $O(\operatorname{vol}(\mathrm{~B}(c, r)))$.

Clustering in almost-linear time. One of the building blocks in the construction of $+O\left(n^{3 / 7+\varepsilon}\right)$ additive spanner in BW21 is a clustering procedure, which takes the input graph and computes a collections of balls with certain coverage and disjointness properties. We will also use this clustering procedure. However, the running time of algorithm for computing such a clustering in BW21 is a large polynomial in $n$. In order to design a sub-quadratic algorithm for additive spanners, we provide an almost-linear time algorithm for computing a clustering with slightly different parameters.

Lemma 2.2 (Almost-linear time algorithm for Lemma 13 in [BW21). There is an algorithm, that, given any undirected unweighted graph $G=(V, E)$ on $n$ vertices and $m$ edges, any parameter $\varepsilon>0$, and any integer $R>0$, computes in time $O\left(m n^{\varepsilon} / \varepsilon\right)$ a collection $\mathcal{B}$ of balls in $G$, such that

- the radius of each ball $\mathrm{B}(c, r) \in \mathcal{B}$ satisfies that $R \leq r \leq 2^{10 / \varepsilon} \cdot R$;
- all vertices in $V$ are covered by $\mathcal{B}$;
- the following coverage properties hold:

$$
\begin{aligned}
& -\sum_{\mathrm{B}(c, r) \in \mathcal{B}}|\mathrm{B}(c, r / 2)|=O(n / \varepsilon) ; \\
& - \text { for each } \mathrm{B}(c, r) \in \mathcal{B},|\mathrm{B}(c, 4 r)| \leq n^{\varepsilon} \cdot|\mathrm{B}(c, r / 2)| \text {, so } \sum_{\mathrm{B}(c, r) \in \mathcal{B}}|\mathrm{B}(c, 4 r)|=O\left(n^{1+\varepsilon} / \varepsilon\right) ; \\
& -\sum_{\mathrm{B}(c, r) \in \mathcal{B}} \operatorname{vol}(\mathrm{B}(c, 4 r))=O\left(m \cdot n^{\varepsilon} / \varepsilon\right) \text {. }
\end{aligned}
$$

The proof of Lemma 2.2 is deferred to Appendix A.
Consistent paths and distance preservers. Let $\pi$ be a path and let $x, y$ be two vertices of $\pi$. We denote by $\pi[x, y]$ the subpath of $\pi$ between $x$ and $y$, and we denote by $|\pi|$ the number of edges in $\pi$; we can also define notations $\pi(x, y), \pi(x, y], \pi[x, y)$ in the natural way. Let $\Pi$ be a collection of paths. We say that $\Pi$ is consistent, iff for any pair $\pi_{1}, \pi_{2} \in \Pi$, the intersection between $\pi_{1}$ and $\pi_{2}$ is a (possibly empty) subpath of both $\pi_{1}$ and $\pi_{2}$.

Let $\mathcal{P} \subseteq V \times V$ be a set of pairs of vertices in $G$. A subgraph $H \subseteq G$ is a distance preserver of $G$ with respect to $\mathcal{P}$, iff $\operatorname{dist}_{H}(s, t)=\operatorname{dist}_{G}(s, t)$ holds for every $(s, t) \in \mathcal{P}$. We will use the following previous results on distance preservers and +6 pairwise additive spanners.

Lemma 2.3 ([CE06]). Let $G=(V, E)$ be a graph on $n$ vertices, let $\mathcal{P} \subseteq V \times V$ be a set of pairs of its vertices, and let $\Pi$ be any consistent collection of paths in $G$ that contains, for each pair $(s, t) \in \mathcal{P}$, a path $\pi_{s, t}$ connecting s to $t$. Then $|E(\Pi)|=\left|E\left(\bigcup_{(s, t) \in \mathcal{P}} \pi_{s, t}\right)\right|=O(n+\sqrt{n} \cdot|\mathcal{P}|)$.

We use the following corollary of Lemma 2.3, whose proof is a straightforward implementation of the BFS algorithm and the standard edge-weight perturbation technique, and is omitted.

Corollary 2.4. There is an algorithm, that given a graph $G$ on $n$ vertices and $m$ edges, and a subset $S$ of its vertices, in time $O(m \cdot|S|)$, computes a consistent collection $\Pi$ of paths that contains, for each pair $s, t \in S$, a shortest path $\pi_{s, t}$ connecting s to $t$ in $G$, such that $|E(\Pi)|=\left|E\left(\bigcup_{s, t \in S} \pi_{s, t}\right)\right|=$ $O\left(n+\sqrt{n} \cdot|S|^{2}\right)$.

Lemma 2.5 (【Kav17]). There is an efficient algorithm, that, given any graph $G=(V, E)$ and any set $\mathcal{P} \subseteq V \times V$ of pairs, computes a subgraph $H \subseteq G$ with $|E(H)|=O\left(n|\mathcal{P}|^{1 / 4}\right)$, such that for every pair $(s, t) \in \mathcal{P}, \operatorname{dist}_{H}(s, t) \leq \operatorname{dist}_{G}(s, t)+6$.

## 3 Pairwise Sublinear Additive Spanners

In this section, we prove the following Lemma 3.1, which will serve as a building block for Theorem 1.1. Throughout this section, we use the following stretch function $f$ : for any parameter $\varepsilon>0$ and any integer $k \geq 1, f_{k, \varepsilon}(d)=d+2^{30 k / \varepsilon} d^{1-1 / k}$.

Lemma 3.1. For any undirected unweighted graph $G=(V, E)$ on $n$ vertices, any collection $\mathcal{P} \subseteq V \times$ $V$ of pairs of its vertices, any integer $k \geq 1$ and parameter $0<\varepsilon<1$, there is a subgraph $H \subseteq G$ with $O\left(2^{2 k / \varepsilon} n^{1+10 k \varepsilon}|\mathcal{P}|^{1 / 2^{k+1}}\right)$ edges, such that for every pair $(s, t) \in \mathcal{P}$, $\operatorname{dist}_{H}(s, t) \leq f_{k, \varepsilon}\left(\operatorname{dist}_{G}(s, t)\right)$.

### 3.1 Preparation and a subrountine

We prove Lemma 3.1 by an induction on $k$. The base case (when $k=1$ ) immediately follows from Lemma [2.5. We assume now that Lemma 3.1 is correct for $1, \ldots, k-1$. We will design an algorithm that computes a pairwise additive spanner required in Lemma 3.1. At a high level, our algorithm can be viewed as a combination of the clustering algorithm (Lemma [2.2) from BW21 and the path-buying schemes from Kav17.

For each integer $D \in\left\{1,2,2^{2}, \ldots, 2^{\lfloor\log n\rfloor}\right\}$, we will construct a subgraph $H_{D} \subseteq G$, such that for all pairs $(s, t) \in \mathcal{P}$ with $D \leq \operatorname{dist}_{G}(s, t)<2 D, \operatorname{dist}_{H_{D}}(s, t) \leq \operatorname{dist}_{G}(s, t)+2^{30 k / \varepsilon} D^{1-1 / k}$ holds. We will then let $H=\bigcup_{0 \leq i \leq\lfloor\log n\rfloor} H_{2^{i}}$ to finish the construction.

We now describe the construction of the subgraph $H_{D}$. We first apply the algorithm from Lemma 2.2 to graph $G$ with parameters $R=D^{1-1 / k}$ and $\varepsilon$; for convenience, we will assume $D^{1-1 / k}$ is an integer. Let $\mathcal{B}$ be the collection of balls we get. We will also iteratively construct, for each ball $\mathrm{B}(c, r) \in \mathcal{B}$, a set $\mathcal{P}_{c} \subseteq \mathrm{~B}(c, 4 r) \times \mathrm{B}(c, 4 r)$ of pairs, which is initially empty. Over the course of the algorithm, we maintain the graph $H_{D}$ as the union of the following edges over all balls $\mathrm{B}(c, r) \in \mathcal{B}$ :
(i) a BFS tree $T_{c}$ that is rooted at $c$ and spans all vertices in $G[\mathrm{~B}(c, 4 r)]$;
(ii) a pairwise spanner of $G[\mathrm{~B}(c, 4 r)]$ with respect to the set $\mathcal{P}_{c}$ of pairs, with stretch function $f_{k-1, \varepsilon}$ and size $O\left(2^{2 k / \varepsilon}|\mathrm{B}(c, 4 r)|^{1+10 k \varepsilon}\left|\mathcal{P}_{c}\right|^{1 / 2^{k+1}}\right)$, whose existence is guaranteed by the inductive hypothesis. When the sets $\left\{\mathcal{P}_{c}\right\}$ change during our construction algorithm, the graph $H_{D}$ evolves with them.

In order to construct the sets $\left\{\mathcal{P}_{c}\right\}$, we will take a path-buying approach. Specifically, we will iteratively find a short path that contains relatively few new edges and "settles" many pairs of clusters (by connecting them in a near-optimal way). As the construction of $H_{D}$ is recursive, we will distribute the task of "buying this path" to the clusters in $\mathcal{B}$ as in [BW21], which requires us to first chop up a shortest path into short segments. For this, we need the following subroutine called PathPartition, which was initially proposed in BW21.

Subroutine PathPartition. The input to subrountine PathPartition is a shortest path $\pi$ connecting a pair $s, t$ of vertices in a graph $G$ and a collection $\mathcal{B}$ of balls that covers all vertices in $G$, and the output is a partitioning of path $\pi$ as the sequential concatenation of its subpaths $\pi=\alpha_{1} \circ \alpha_{2} \circ \ldots \circ \alpha_{l}$, such that, for each $1 \leq i \leq l$, if we denote by $s_{i}, t_{i}$ the endpoints of $\alpha_{i}$ (so $\alpha_{i}=\pi\left[s_{i}, t_{i}\right]$ ), then there exists a ball $\mathrm{B}\left(c_{i}, r_{i}\right) \in \mathcal{B}$, such that $s_{i} \in \mathrm{~B}\left(c_{i}, r_{i}\right)$ and $t_{i} \in \mathrm{~B}\left(c_{i}, 2 r_{i}\right)$, and we say that the ball $\mathrm{B}\left(c_{i}, r_{i}\right)$ hosts the subpath $\alpha_{i}$.

We start by directing $\pi$ from $s$ to $t$ and setting $s_{1}=s$. Since $s_{1}$ is covered by $\mathcal{B}$, there exists a ball in $\mathcal{B}$ that contains $s_{1}$, and we designate this ball as $\mathrm{B}\left(c_{1}, r_{1}\right)$. We then find the last vertex on $\pi$ that lies in $\mathrm{B}\left(c_{1}, 2 r_{1}\right)$, and designate it as $t_{1}$. The first subpath is then defined to be $\alpha_{1}=\pi\left[s_{1}, t_{1}\right]$. We then set $s_{2}=t_{1}$ and repeat the process to find subpaths $\alpha_{2}, \ldots, \alpha_{l}$ until all edges are included in some subpath; that is, for a general index $i \geq 2$, if $s_{i} \neq t$, then find a ball $\mathrm{B}\left(c_{i}, r_{i}\right) \in \mathcal{B}$ containing $s_{i}$, and define $t_{i} \in \pi\left(s_{i}, t\right] \cap \mathrm{B}\left(c_{i}, 2 r_{i}\right)$ to be the vertex on $\pi$ which is closest to $t$. After that, if $t_{i} \neq t$, assign $s_{i+1}=t_{i}$ and repeat. See Figure 5 for an illustration.


Figure 5: A partitioning of the shortest path $\pi$ from $s$ to $t$ into subpaths $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{l}$ by balls in $\mathcal{B}$. Note that in general $G\left[\mathrm{~B}\left(c_{i}, 2 r_{i}\right)\right]$ does not necessarily contain the entire sub-path $\alpha_{i}$.

We prove the following simple properties of the subroutine.
Observation 3.2. All balls in $\left\{\mathrm{B}\left(c_{i}, r_{i}\right)\right\}_{1 \leq i \leq l}$ are distinct.
Proof. Suppose otherwise that a ball is selected twice by PathPartition, namely $\left(c_{i}, r_{i}\right)=\left(c_{j}, r_{j}\right)$ for some $1 \leq i<j \leq l$. Then, by the algorithm description we have $t_{j} \in \mathrm{~B}\left(c_{j}, 2 r_{j}\right)=\mathrm{B}\left(c_{i}, 2 r_{i}\right)$, which contradicts the choice of $t_{i}$ which is the closest-to- $t$ vertex in $\mathrm{B}\left(c_{i}, 2 r_{i}\right)$ on $\pi$.

Observation 3.3. For each $1 \leq i \leq l$, the subpath $\alpha_{i}$ is entirely contained in the ball $\mathrm{B}\left(c_{i}, 4 r_{i}\right)$.
Proof. From the description of the subroutine, for each $1 \leq i \leq l, s_{i} \in \mathrm{~B}\left(c_{i}, r_{i}\right)$ and $t_{i} \in \mathrm{~B}\left(c_{i}, 2 r_{i}\right)$. Therefore, for every vertex $u \in \pi\left[s_{i}, t_{i}\right]$, by triangle inequality:

$$
\operatorname{dist}_{G}\left(c_{i}, u\right) \leq \operatorname{dist}_{G}\left(c_{i}, s_{i}\right)+\operatorname{dist}_{G}\left(s_{i}, t_{i}\right) \leq \operatorname{dist}_{G}\left(c_{i}, s_{i}\right)+\operatorname{dist}_{G}\left(c_{i}, s_{i}\right)+\operatorname{dist}_{G}\left(c_{i}, t_{i}\right) \leq 4 r_{i},
$$

which implies that $u \in \mathrm{~B}\left(c_{i}, 4 r_{i}\right)$.
Observation 3.4. Let $R$ be the minimum radius of the balls in $\mathcal{B}$. Then $l \leq|\pi| / R+1$.
Proof. It suffices to show that, for each $1 \leq i<l,\left|\alpha_{i}\right| \geq R$. From the algorithm, vertex $s_{i}$ belongs to the ball $\mathrm{B}\left(c_{i}, r_{i}\right)$, and vertex $v_{i}$ is the last vertex on $\pi$ that lies in the ball $\mathrm{B}\left(c_{i}, 2 r_{i}\right)$, so if $t_{i} \neq t$ then $\operatorname{dist}_{G}\left(t_{i}, c_{i}\right)=2 r_{i}$ must hold. Therefore, $\left|\pi\left[s_{i}, t_{i}\right]\right| \geq \operatorname{dist}_{G}\left(c_{i}, t_{i}\right)-\operatorname{dist}_{G}\left(c_{i}, s_{i}\right) \geq 2 r_{i}-r_{i}=r_{i} \geq$ $R$.

Level of balls. Before we describe the algorithm for constructing the sets $\left\{\mathcal{P}_{c}\right\}$, we first classify all clusters according to their sizes as follows. Recall that we are initially given a set $\mathcal{P} \subseteq V \times V$ of pairs in $G$. For each integer $1 \leq i \leq\lceil 1 / \varepsilon\rceil$, we say that a ball $\mathrm{B}(c, r)$ is at level $i$ iff

$$
n^{1-i \varepsilon} / \sqrt{|\mathcal{P}|}<|\mathrm{B}(c, r)| \leq n^{1-(i-1) \varepsilon} / \sqrt{|\mathcal{P}|} .
$$

Denote by $\mathcal{B}_{i}$ the set of all level- $i$ balls in $\mathcal{B}$. We define the set of level- 0 balls as $\mathcal{B}_{0}=\{\mathrm{B}(c, r) \mid$ $|\mathrm{B}(c, r)|>n / \sqrt{|\mathcal{P}|}\}$, so $\mathcal{B}=\mathcal{B}_{0} \cup \mathcal{B}_{1} \cup \cdots \cup \mathcal{B}_{[1 / \varepsilon\rceil}$. A ball from $\mathcal{B}$ is called large if it is in $\mathcal{B}_{0}$, and small otherwise. Here is a simple observation.

Observation 3.5. For each $0 \leq i \leq\lceil 1 / \varepsilon\rceil,\left|\mathcal{B}_{i}\right| \leq O\left(n^{(i+1) \varepsilon} \sqrt{|\mathcal{P}|} / \varepsilon\right)$.
Proof. By Lemma 2.2, $\sum_{\mathrm{B}(c, r) \in \mathcal{B}_{i}}|\mathrm{~B}(c, r)| \leq \sum_{\mathrm{B}(c, r) \in \mathcal{B}_{i}}|\mathrm{~B}(c, 4 r)| \leq O\left(n^{1+\varepsilon} / \varepsilon\right)$, and since each ball in $\mathcal{B}_{i}$ has size at least $n^{1-i \varepsilon} / \sqrt{|\mathcal{P}|},\left|\mathcal{B}_{i}\right| \leq O\left(n^{(i+1) \varepsilon} \sqrt{|\mathcal{P}|} / \varepsilon\right)$.

### 3.2 Step 1. Handling large balls

We first take a uniformly random subset $S \subseteq V$ of size $\lceil 10 \sqrt{|\mathcal{P}|} \log n\rceil$. Since each ball in $\mathcal{B}_{0}$ contains at least $n / \sqrt{|\mathcal{P}|}$ vertices, with high probability, $S$ intersects all balls in $\mathcal{B}_{0}$.

We now proceed to iteratively construct the sets $\left\{\mathcal{P}_{c}\right\}$ of pairs. Throughout, we maintain, for each ball $\mathrm{B}(c, r) \in \mathcal{B}$, a set $\mathcal{P}_{c}$ of pairs of vertices in $\mathrm{B}(c, 2 r)$, and another set $U_{c} \subseteq S$ of vertices in $V$, which are initially empty sets. Intuitively, $U_{c}$ collects all vertices in $S$ whose distances from the ball center $c$ are already preserved by the spanner $H_{D}$ during the algorithm.

For every pair $s, t$ of vertices in $S$, we denote by $\pi_{s, t}$ an arbitrary shortest path connecting them in $G$, and we compute the set

$$
\Pi=\left\{\pi_{s, t} \mid s, t \in S, \operatorname{dist}_{G}(s, t)<2 D+4 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k}\right\} .
$$

We then process all paths in $\Pi$ sequentially in an arbitrary order. For each path $\pi_{s, t} \in \Pi$, we first apply the subroutine PathPartition to it and the collection $\mathcal{B}$ of balls, and obtain a partition $\pi_{s, t}=\alpha_{1} \circ \alpha_{2} \circ \cdots \circ \alpha_{l}$. For each $1 \leq i \leq l$, we denote by $\mathrm{B}\left(c_{i}, r_{i}\right)$ the ball that hosts the subpath $\alpha_{i}$. If there exists a ball $\mathrm{B}\left(c_{i}, r_{i}\right)$ such that $s, t \in U_{c_{i}}$, then we do nothing and move on to the next path in $\Pi$. Otherwise, for each $1 \leq i \leq l$, we add both vertices $s, t$ to the set $U_{c_{i}}$, and add the pair $\left(s_{i}, t_{i}\right)$ to the set $\mathcal{P}_{c_{i}}$ of pairs.

Stretch analysis of Step 1. We make use of the following simple observation.
Observation 3.6. At the end of Step 1, for each ball $\mathrm{B}(c, r) \in \mathcal{B},\left|\mathcal{P}_{c}\right| \leq|S|=\left\lceil 10|\mathcal{P}|^{1 / 2} \log n\right\rceil$.
Proof. By the algorithm description along with Observation 3.2, every time a new pair is added to $\mathcal{P}_{c}$, a new vertex from $S$ is also added to $U_{c}$. Since no more pair will be added to $\mathcal{P}_{c}$ as long as $U_{c}$ contains all vertices of $S$, we get that $\left|\mathcal{P}_{c}\right| \leq\left|U_{c}\right| \leq|S|=\left\lceil 10|\mathcal{P}|^{1 / 2} \log n\right\rceil$.

We now show in the following claims that, after the first step, all pairwise distances (in $G$ ) between vertices in $S$ are well-preserved in $H_{D}$. And as the set $S$ hits all balls in $\mathcal{B}_{0}$ with high probability, the distance between any pair of vertices from $V\left(\mathcal{B}_{0}\right)$ is also well-preserved.

Claim 3.7. After a vertex $s \in S$ is added to the set $U_{c}$ for some $\mathrm{B}(c, r) \in \mathcal{B}$, the following holds:

$$
\operatorname{dist}_{H_{D}}(s, c) \leq \operatorname{dist}_{G}(s, c)+49 \cdot 2^{(30 k-10) / \varepsilon} \cdot D^{1-1 / k}
$$

Proof. Assume that the shortest path $\pi_{s, t}$ was being processed when $s$ was added to $U_{c}$, and that $\mathrm{B}(c, r)$ was $\mathrm{B}\left(c_{i}, r_{i}\right)$ under the notation of the subroutine PathPartition. From the algorithm, for each $1 \leq j \leq l$, the pair $\left(s_{j}, t_{j}\right)$ was added to the collection $\mathcal{P}_{c_{j}}$. From the construction of $H_{D}$, for each $1 \leq j \leq l$, there exists a path $\phi_{j}$ in $H_{D}$ from $s_{j}$ to $t_{j}$, such that:

$$
\begin{aligned}
\left|\phi_{j}\right| \leq\left|\alpha_{j}\right|+2^{30(k-1) / \varepsilon} \cdot\left|\alpha_{j}\right|^{1-1 /(k-1)} & \leq\left|\alpha_{j}\right|+2^{30(k-1) / \varepsilon} \cdot\left(4 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k}\right)^{1-1 /(k-1)} \\
& \leq\left|\alpha_{j}\right|+4 \cdot 2^{(30 k-20) / \varepsilon} \cdot D^{1-2 / k}
\end{aligned}
$$

(we have used the fact that $\left.\left|\alpha_{j}\right|<\operatorname{dist}_{G}\left(s_{j}, t_{j}\right) \leq \operatorname{dist}_{G}\left(s_{j}, c_{j}\right)+\operatorname{dist}_{G}\left(t_{j}, c_{j}\right) \leq 4 r_{j} \leq 4 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k}\right)$. Taking a summation over all indices $1 \leq j \leq i$ and using Observation 3.4, we have:

$$
\begin{aligned}
\operatorname{dist}_{H_{D}}\left(s, c_{i}\right) & \leq \operatorname{dist}_{H_{D}}\left(c_{i}, t_{i}\right)+\operatorname{dist}_{H_{D}}\left(s, t_{i}\right) \\
& \leq 2 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k}+\operatorname{dist}_{G}\left(s, t_{i}\right)+48 \cdot 2^{(30 k-10) / \varepsilon} \cdot D^{1-1 / k} \\
& \leq 2 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k}+\left(\operatorname{dist}_{G}\left(s, c_{i}\right)+\operatorname{dist}_{G}\left(c_{i}, t_{i}\right)\right)+48 \cdot 2^{(30 k-10) / \varepsilon} \cdot D^{1-1 / k} \\
& \leq \operatorname{dist}_{G}\left(s, c_{i}\right)+49 \cdot 2^{(30 k-10) / \varepsilon} \cdot D^{1-1 / k} .
\end{aligned}
$$

Claim 3.8. For any $s, t \in S$ such that $\operatorname{dist}_{G}(s, t)<2 D+4 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k}$, we have:

$$
\operatorname{dist}_{H_{D}}(s, t) \leq \operatorname{dist}_{G}(s, t)+100 \cdot 2^{(30 k-10) / \varepsilon} \cdot D^{1-1 / k} .
$$

Proof. For any such pair of vertices $s, t \in S$, consider the moment when the shortest path $\pi_{s, t} \in \Pi$ was processed and partitioned into subpaths $\alpha_{1}, \ldots, \alpha_{l}$. We distinguish between the following two cases.

- Case 1. There existed an index $1 \leq i \leq l$ such that $s, t \in U_{c_{i}}$ at the moment.

In this case, from Claim 3.7:

$$
\begin{aligned}
& \operatorname{dist}_{H_{D}}\left(s, c_{i}\right) \leq \operatorname{dist}_{G}\left(s, c_{i}\right)+49 \cdot 2^{(30 k-10) / \varepsilon} \cdot D^{1-1 / k} \leq \operatorname{dist}_{G}\left(s, s_{i}\right)+50 \cdot 2^{(30 k-10) / \varepsilon} \cdot D^{1-1 / k}, \\
& \operatorname{dist}_{H_{D}}\left(c_{i}, t\right) \leq \operatorname{dist}_{G}\left(c_{i}, t\right)+49 \cdot 2^{(30 k-10) / \varepsilon} \cdot D^{1-1 / k} \leq \operatorname{dist}_{G}\left(s_{i}, t\right)+50 \cdot 2^{(30 k-10) / \varepsilon} \cdot D^{1-1 / k}
\end{aligned}
$$

Summing up these two inequalities finishes the proof.

- Case 2. There did not exist any $i$ such that $s, t \in U_{c_{i}}$ at the moment.

In this case, for each $1 \leq j \leq l$, the pair $\left(s_{j}, t_{j}\right)$ was added to the set $\mathcal{P}_{c_{j}}$. Similar to the proof of Claim 3.7, for each $1 \leq j \leq l$, there exists a path $\phi_{j}$ in $H_{D}$ from $s_{j}$ to $t_{j}$, such that:

$$
\begin{aligned}
\left|\phi_{j}\right| \leq\left|\alpha_{j}\right|+2^{30(k-1) / \varepsilon} \cdot\left|\alpha_{j}\right|^{1-1 /(k-1)} & \leq\left|\alpha_{j}\right|+2^{30(k-1) / \varepsilon} \cdot\left(4 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k}\right)^{1-1 /(k-1)} \\
& \leq\left|\alpha_{j}\right|+4 \cdot 2^{(30 k-20) / \varepsilon} \cdot D^{1-2 / k}
\end{aligned}
$$

Taking a summation for all indices $1 \leq j \leq l$ and using Observation 3.4, we get that:

$$
\operatorname{dist}_{H_{D}}(s, t) \leq \operatorname{dist}_{G}(s, t)+48 \cdot 2^{(30 k-10) / \varepsilon} \cdot D^{1-1 / k}
$$

### 3.3 Step 2. Handling small balls

We now process all pairs $(s, t) \in \mathcal{P}$ with $D \leq \operatorname{dist}_{G}(s, t)<2 D$ in an arbitrary order. Take any such pair $(s, t)$, and compute a shortest path $\pi_{s, t}$ between them in $G$. We then apply the subroutine PathPartition to it and the collection $\mathcal{B}$ of balls, and obtain a partitioning $\pi_{s, t}=\alpha_{1} \circ \alpha_{2} \circ \cdots \circ \alpha_{l}$ together with the balls $\left\{\mathrm{B}\left(c_{i}, r_{i}\right)\right\}_{1 \leq i \leq l}$ that host them. However, as the number of pairs $(s, t) \in \mathcal{P}$ with $D \leq \operatorname{dist}_{G}(s, t)<2 D$ can be very large, we cannot afford to add the pair $\left(s_{i}, t_{i}\right)$ to set $\mathcal{P}_{c_{i}}$ for all $1 \leq i \leq l$ as in Step 1, and will instead carefully pick a subset of indices in $[1, l]$ to do so.

We will now describe how to choose these indices. This is again done via an iterative process. Throughout, we will gradually construct a binary tree $\mathcal{T}$, which initially contains a single node, which is the root of $\mathcal{T}$. Each node of the tree is labeled by a subinterval of $[1, l]$. Initially, the root of $\mathcal{T}$ is labeled with $[1, l]$. Each index $i \in[1, l]$ is either marked active or inactive. Initially, all indices are inactive. The algorithm continues to be executed unless for each leaf of the tree $\mathcal{T}$, all indices in its associated interval are active. Over the course of the algorithm, whenever we mark some index $i$ active, we will simultaneously add the pair $\left(s_{i}, t_{i}\right)$ to the set $\mathcal{P}_{c_{i}}$. We say that an index $i \in[1, l]$ is at level $j$ iff its corresponding ball $\mathrm{B}\left(c_{i}, r_{i}\right)$ is at level $j$ (that is, if $\left.n^{1-j \varepsilon} / \sqrt{|\mathcal{P}|}<\left|\mathrm{B}\left(c_{i}, r_{i}\right)\right| \leq n^{1-(j-1) \varepsilon} / \sqrt{|\mathcal{P}|}\right)$.

As a pre-processing step, we first focus on level-0 indices (if there are none of them, then we skip the pre-processing). Let $i_{1}, i_{2}$ be the smallest and the largest level- 0 indices, respectively. We add two new nodes to the tree $\mathcal{T}$, connecting each of them to the root by an edge. One of the new nodes is labeled by interval $\left[1, i_{1}-1\right]$, and the other node is labeled by interval $\left[i_{2}+1, l\right]$. So now the levels of indices on $\left[1, i_{1}-1\right]$ and $\left[i_{2}+1, l\right]$ are at least 1 . Here we do not modify the active/inactive status of the indices.

We now describe an iteration. We take an arbitrary leaf of the tree $\mathcal{T}$ that whose associated interval contains an inactive index, and denote by $I$ the interval associated with this leaf node. We can assume that all levels of $I$ are at least 1 . For each $1 \leq L \leq\lceil 1 / \varepsilon\rceil$, let $i_{1}^{L}<i_{2}^{L}<\cdots<i_{p(L)}^{L}$ be
all level- $L$ indices in the interval $I$ (so the number of level- $L$ indices in $I$ is $p(L)$ ); we will ignore the superscript $L$ if it is clear from context. Consider the corresponding balls $\left\{\mathrm{B}\left(c_{i_{a}^{L}}, r_{i_{a}^{L}}\right)\right\}$ that host them. We say that a pair $\left(\mathrm{B}\left(c_{i_{a}^{L}}, r_{i_{a}^{L}}\right), \mathrm{B}\left(c_{i_{b}^{L}}, r_{i_{b}^{L}}\right)\right)$ of balls is tight, iff

$$
\operatorname{dist}_{H_{D}}\left(c_{i_{a}^{L}}, c_{i_{b}^{L}}\right) \leq \operatorname{dist}_{G}\left(c_{i_{a}^{L}}, c_{i_{b}^{L}}\right)+\left(3 \cdot 2^{(30 k-20) / \varepsilon}+10 \cdot 2^{10 / \varepsilon}\right) \cdot D^{1-1 / k}
$$

and we denote by $q(L)$ the number of pairs of balls in $\left\{\mathrm{B}\left(c_{i_{a}^{L}}, r_{i_{a}^{L}}\right)\right\}$ which are not tight. In addition, we define $\beta(L)=\left\lceil n^{(L+1) \varepsilon}\right\rceil$.

We will distinguish between the following two cases.
Case 1. For all $1 \leq L \leq\lceil 1 / \varepsilon\rceil, p(L) \leq \max \{4 \beta(L), 8 q(L) / \beta(L)\}$. In this case, we activate all indices $i \in I$ (and along with it add the corresponding pair $\left(s_{i}, t_{i}\right)$ to the set $\mathcal{P}_{c_{i}}$ and then update $H_{D}$ accordingly), and continue to the next iteration. We do not modify the tree $\mathcal{T}$ in this iteration. Note that, after this iteration, the associated interval of the leaf that we processed in this iteration no longer contains any inactive indices, so it will remain a leaf in $\mathcal{T}$ forever.

Case 2. There exists some $1 \leq L \leq\lceil 1 / \varepsilon\rceil$, such that $p(L)>\max \{4 \beta(L), 8 q(L) / \beta(L)\}$. In this case, we activate the $\beta(L)$ smallest and $\beta(L)$ largest level- $L$ indices in $I$ (and along with it add the corresponding pair $\left(s_{i}, t_{i}\right)$ to the set $\mathcal{P}_{c_{i}}$ and then update $H_{D}$ accordingly).

In order to describe the modification of $\mathcal{T}$ in this iteration, we need the following lemma.
Lemma 3.9. There exists three level-L indices $i_{x}, i_{y}, i_{z} \in I$ such that:
(i) $1 \leq x \leq \beta(L)$;
(ii) $p(L)-\beta(L)<y \leq p(L)$;
(iii) $\beta(L)<z \leq p(L)-\beta(L)$;
(iv) the pair $\left(\mathrm{B}\left(c_{i_{x}}, r_{i_{x}}\right), \mathrm{B}\left(c_{i_{z}}, r_{i_{z}}\right)\right)$ and the pair $\left(\mathrm{B}\left(c_{i_{y}}, r_{i_{y}}\right), \mathrm{B}\left(c_{i_{z}}, r_{i_{z}}\right)\right)$ of balls are both tight.

Proof. Assume for contradiction that there do not exist such three indices. Consider now any pair $a, b$ of indices such that $1 \leq a \leq \beta(L)$ and $\beta(L)<b \leq p(L)-\beta(L)$. From our assumption, for at least one pair of indices $\left(i_{a}^{L}, i_{b}^{L}\right)$ and $\left(i_{b}^{L}, i_{p(L)+1-a}^{L}\right)$, their corresponding balls are not tight. This implies that $q(L) \geq(p(L)-2 \beta(L)) \cdot \beta(L)$. Then either $p(L)<4 \beta(L)$ holds, or $q(L) \geq p(L) \cdot \beta(L) / 8$ holds, a contradiction of the initial assumption of Case 2.

To complete this iteration, we pick indices $i_{x}, i_{y}, i_{z}$ as in the above lemma. We then add two new nodes to the tree $\mathcal{T}$ (together with an edge connecting to the leaf) as the child nodes of this leaf. Denote $I=\left[i^{\prime}, i^{\prime \prime}\right]$, and we label the child nodes with intervals $\left[i^{\prime}, i_{x}\right]$ and $\left[i_{y}, i^{\prime \prime}\right]$, respectively. This completes the description of an iteration. See Figure 6 for an illustration.

Stretch analysis of Step 2. Consider any pair $(s, t) \in \mathcal{P}$ such that $D \leq \operatorname{dist}_{G}(s, t)<2 D$. Let $\pi$ be the shortest path connecting $s$ to $t$ that was processed in Step 2. We start by showing that, in this iteration, the depth of the tree $\mathcal{T}$ that we construct is small.

Observation 3.10. At the end of the iteration of processing $\pi, \mathcal{T}$ has depth at most $\lceil 1 / \varepsilon\rceil+1$.

activate the first $\beta(L)$ level $-L$ indices
activate the last $\beta(L)$ level $-L$ indices

Figure 6: The distances between $c_{i_{x}}, c_{i_{z}}$ and $c_{i_{y}}, c_{i_{z}}$ are approximately preserved in $H$; the orange segments correspond to level- $L$ indices; a prefix and a suffix of level- $L$ indices are activated in this iteration.

Proof. The pre-processing step can increase the depth by at most one. From the algorithm description, it is easy to observe that, as we walk down in any root-to-leaf path in $\mathcal{T}$, in each step, the number of levels with inactive indices decreases by one. Hence, the observation follows.

We are now ready to analyze the stretch (in $H_{D}$ ) of pairs in $\mathcal{P}$.
Lemma 3.11. For every pair $(s, t) \in \mathcal{P}, \operatorname{dist}_{H_{D}}(s, t) \leq \operatorname{dist}_{G}(s, t)+2^{30 k / \varepsilon} \cdot D^{1-1 / k}$.
Proof. We will utilize the tree structure of $\mathcal{T}$. Focus on the time when the construction of $\mathcal{T}$ was completed. For any tree node $N$, its height is defined to be the maximum depth of the subtree of $\mathcal{T}$ rooted at $N$. We first prove the following claim.

Claim 3.12. For each non-root tree node $N$ with height $h$ and associated interval $\left[i^{\prime}, i^{\prime \prime}\right]$,

$$
\operatorname{dist}_{H_{D}}\left(c_{i^{\prime}}, c_{i^{\prime \prime}}\right) \leq \operatorname{dist}_{G}\left(c_{i^{\prime}}, c_{i^{\prime \prime}}\right)+30 \cdot\left(2^{h+1}-1\right) \cdot 2^{(30 k-20) / \varepsilon} \cdot D^{1-1 / k}
$$

Proof of Claim 3.12. We prove the claim by induction on $h$. The base case is when $h=0$ and $N$ is a leaf in $\mathcal{T}$. From the algorithm, for each $i \in\left[i^{\prime}, i^{\prime \prime}\right]$, the pair $\left(s_{i}, t_{i}\right)$ is added into the set $\mathcal{P}_{c_{i}}$. Similar to the proof of Claim [3.7, for each $i^{\prime} \leq i \leq i^{\prime \prime}$, there exists a path $\phi_{i}$ in $H_{D}$, such that

$$
\begin{aligned}
\left|\phi_{j}\right| \leq\left|\alpha_{j}\right|+2^{30(k-1) / \varepsilon} \cdot\left|\alpha_{j}\right|^{1-1 /(k-1)} & \leq\left|\alpha_{j}\right|+2^{30(k-1) / \varepsilon} \cdot\left(4 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k}\right)^{1-1 /(k-1)} \\
& \leq\left|\alpha_{j}\right|+4 \cdot 2^{(30 k-20) / \varepsilon} \cdot D^{1-2 / k}
\end{aligned}
$$

Taking a summation and using Observation 3.4 (which implies that $l<3 D^{1 / k}$ ),

$$
\operatorname{dist}_{H_{D}}\left(s_{i^{\prime}}, t_{i^{\prime \prime}}\right) \leq \operatorname{dist}_{G}\left(s_{i^{\prime}}, t_{i^{\prime \prime}}\right)+12 \cdot 2^{(30 k-20) / \varepsilon} \cdot D^{1-1 / k}
$$

By triangle inequality, we have:

$$
\begin{gathered}
\operatorname{dist}_{H_{D}}\left(s_{i^{\prime}}, t_{i^{\prime \prime}}\right) \geq \operatorname{dist}_{H_{D}}\left(c_{i^{\prime}}, c_{i^{\prime \prime}}\right)-\operatorname{dist}_{H_{D}}\left(c_{i^{\prime}}, s_{i^{\prime}}\right)-\operatorname{dist}_{H_{D}}\left(c_{i^{\prime \prime}}, t_{i^{\prime \prime}}\right) \geq \operatorname{dist}_{H_{D}}\left(c_{i^{\prime}}, c_{i^{\prime \prime}}\right)-3 \cdot 2^{10 / \varepsilon} D^{1-1 / k} \\
\operatorname{dist}_{G}\left(s_{i^{\prime}}, t_{i^{\prime \prime}}\right) \leq \operatorname{dist}_{G}\left(c_{i^{\prime}}, c_{i^{\prime \prime}}\right)+\operatorname{dist}_{G}\left(c_{i^{\prime}}, s_{i^{\prime}}\right)+\operatorname{dist}_{G}\left(c_{i^{\prime \prime}}, t_{i^{\prime \prime}}\right) \leq \operatorname{dist}_{G}\left(c_{i^{\prime}}, c_{i^{\prime \prime}}\right)+3 \cdot 2^{10 / \varepsilon} D^{1-1 / k}
\end{gathered}
$$

Combining all three inequalities, we have:

$$
\begin{aligned}
\operatorname{dist}_{H_{D}}\left(c_{i^{\prime}}, c_{i^{\prime \prime}}\right) & \leq \operatorname{dist}_{G}\left(c_{i^{\prime}}, c_{i^{\prime \prime}}\right)+\left(12 \cdot 2^{(30 k-20) / \varepsilon}+6 \cdot 2^{10 / \varepsilon}\right) \cdot D^{1-1 / k} \\
& <\operatorname{dist}_{G}\left(c_{i^{\prime}}, c_{i^{\prime \prime}}\right)+30 \cdot 2^{(30 k-20) / \varepsilon} \cdot D^{1-1 / k}
\end{aligned}
$$

Assume now that the claim is true for $1, \ldots, h-1$. Consider a node $N$ at height $h$ with two children $N_{1}, N_{2}$ associated with intervals $\left[i^{\prime}, i_{x}\right]$ and $\left[i_{y}, i^{\prime \prime}\right]$ respectively. By inductive hypothesis, we have:

$$
\begin{aligned}
& \operatorname{dist}_{H}\left(c_{i^{\prime}}, c_{i_{x}}\right) \leq \operatorname{dist}_{G}\left(c_{i^{\prime}}, c_{i_{x}}\right)+30 \cdot\left(2^{h}-1\right) \cdot 2^{(30 k-20) / \varepsilon} \cdot D^{1-1 / k}, \\
& \operatorname{dist}_{H}\left(c_{i^{\prime \prime}}, c_{i_{y}}\right) \leq \operatorname{dist}_{G}\left(c_{i^{\prime \prime}}, c_{i_{y}}\right)+30 \cdot\left(2^{h}-1\right) \cdot 2^{(30 k-20) / \varepsilon} \cdot D^{1-1 / k} .
\end{aligned}
$$

From the algorithm, the node $N$ may only grow two children $N_{1}$ and $N_{2}$ in Case 2, and in this case there exists an index $i_{z}$ such that the pairs $\left(\mathrm{B}\left(c_{i_{x}}, r_{i_{x}}\right), \mathrm{B}\left(c_{i_{z}}, r_{i_{z}}\right)\right)$ and $\left(\mathrm{B}\left(c_{i_{y}}, r_{i_{y}}\right), \mathrm{B}\left(c_{i_{z}}, r_{i_{z}}\right)\right)$ of balls are both tight. In other words,

$$
\begin{aligned}
& \operatorname{dist}_{H_{D}}\left(c_{i_{x}}, c_{i_{z}}\right) \leq \operatorname{dist}_{G}\left(c_{i_{x}}, c_{i_{z}}\right)+\left(3 \cdot 2^{(30 k-20) / \varepsilon}+10 \cdot 2^{10 / \varepsilon}\right) D^{1-1 / k} \\
& \operatorname{dist}_{H_{D}}\left(c_{i_{z}}, c_{i_{y}}\right) \leq \operatorname{dist}_{G}\left(c_{i_{z}}, c_{i_{y}}\right)+\left(3 \cdot 2^{(30 k-20) / \varepsilon}+10 \cdot 2^{10 / \varepsilon}\right) D^{1-1 / k}
\end{aligned}
$$

The sum of the left-hand sides of the above four inequalities is at least dist $H_{H_{D}}\left(c_{i^{\prime}}, c_{i^{\prime \prime}}\right)$, from triangle inequality. For their right-hand sides, notice that:

$$
\begin{aligned}
\operatorname{dist}_{G}\left(c_{i^{\prime}}, c_{i_{x}}\right)+\operatorname{dist}_{G}\left(c_{i_{x}}, c_{i_{z}}\right) & \leq\left(\operatorname{dist}_{G}\left(s_{i^{\prime}}, s_{i_{x}}\right)+\operatorname{dist}_{G}\left(c_{i^{\prime}}, s_{i^{\prime}}\right)+\operatorname{dist}_{G}\left(c_{i_{x}}, s_{i_{x}}\right)\right) \\
& \left.+\operatorname{dist}_{G}\left(s_{i_{x}}, s_{i_{z}}\right)+\operatorname{dist}_{G}\left(c_{i_{x}}, s_{i_{x}}\right)+\operatorname{dist}_{G}\left(c_{i_{z}}, s_{i_{z}}\right)\right) \\
& \leq \operatorname{dist}_{G}\left(s_{i^{\prime}}, s_{i_{z}}\right)+4 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k} \\
& \leq \operatorname{dist}_{G}\left(c_{i^{\prime}}, c_{i_{z}}\right)+6 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k} .
\end{aligned}
$$

Symmetrically, we also have $\operatorname{dist}_{G}\left(c_{i^{\prime \prime}}, c_{i_{y}}\right)+\operatorname{dist}_{G}\left(c_{i_{y}}, c_{i_{z}}\right) \leq \operatorname{dist}_{G}\left(c_{i^{\prime \prime}}, c_{i_{z}}\right)+6 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k}$.
Note that $\operatorname{dist}_{G}\left(c_{i^{\prime}}, c_{i_{z}}\right)+\operatorname{dist}_{G}\left(c_{i^{\prime \prime}}, c_{i_{z}}\right) \leq \operatorname{dist}_{G}\left(c_{i^{\prime}}, c_{i^{\prime \prime}}\right)+2 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k}$ (again by triangle inequality). Altogether, we get that:

$$
\operatorname{dist}_{H_{D}}\left(c_{i^{\prime}}, c_{i^{\prime \prime}}\right) \leq \operatorname{dist}_{G}\left(c_{i^{\prime}}, c_{i^{\prime \prime}}\right)+30 \cdot\left(2^{h+1}-1\right) \cdot 2^{(30 k-20) / \varepsilon} \cdot D^{1-1 / k}
$$

Lastly, we consider the root $N_{0}$ of $\mathcal{T}$. Recall that it is associated with the interval $[1, l]$. If there are no level-0 indices in [1,l], then Claim 3.12 already implies Lemma 3.11. We assume from now on that there are level- 0 indices in $[1, l]$. From the algorithm, in the pre-processing step, $N_{0}$ grows two children $N_{1}, N_{2}$ associated with intervals $\left[1, i_{1}-1\right]$ and $\left[i_{2}+1, l\right]$, respectively, such that $i_{1}, i_{2}$ are the smallest and the largest level-0 indices. By definition of $S$, there exists $v_{1}, v_{2}$ such that $v_{1} \in\left(\mathrm{~B}\left(c_{i_{1}}, r_{i_{1}}\right) \cap S\right), v_{2} \in\left(\mathrm{~B}\left(c_{i_{2}}, r_{i_{2}}\right) \cap S\right)$. Notice that $\operatorname{dist}_{G}\left(v_{1}, v_{2}\right) \leq \operatorname{dist}_{G}\left(s_{i_{1}}, s_{i_{2}}\right)+4 \cdot 2^{10 / \varepsilon}$. $D^{1-1 / k}<2 D+4 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k}$. Then, from Claim 3.8, we have:

$$
\operatorname{dist}_{H_{D}}\left(v_{1}, v_{2}\right) \leq \operatorname{dist}_{G}\left(v_{1}, v_{2}\right)+100 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k}
$$

Suppose $N_{1}, N_{2}$ have height $h_{1}, h_{2} \leq\lceil 1 / \varepsilon\rceil$ respectively. By Claim 3.12,

$$
\begin{aligned}
& \operatorname{dist}_{H_{D}}\left(c_{1}, c_{i_{1}-1}\right) \leq \operatorname{dist}_{G}\left(c_{1}, c_{i_{1}-1}\right)+30 \cdot\left(2^{h_{1}+1}-1\right) \cdot 2^{(30 k-20) / \varepsilon} \cdot D^{1-1 / k}, \\
& \operatorname{dist}_{H_{D}}\left(c_{l}, c_{i_{2}+1}\right) \leq \operatorname{dist}_{G}\left(c_{l}, c_{i_{2}+1}\right)+30 \cdot\left(2^{h_{2}+1}-1\right) \cdot 2^{(30 k-20) / \varepsilon} \cdot D^{1-1 / k}
\end{aligned}
$$

Taking a summation of the left-hand sides of the above three inequalities:

$$
\begin{aligned}
& \operatorname{dist}_{H_{D}}\left(c_{1}, c_{i_{1}-1}\right)+\operatorname{dist}_{H_{D}}\left(c_{l}, c_{i_{2}+1}\right)+\operatorname{dist}_{H_{D}}\left(v_{1}, v_{2}\right) \\
& \geq \operatorname{dist}_{H_{D}}\left(c_{1}, c_{i_{1}-1}\right)+\operatorname{dist}_{H_{D}}\left(c_{l}, c_{i_{2}+1}\right)+\operatorname{dist}_{H_{D}}\left(c_{i_{1}}, c_{i_{2}}\right)-2 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k} \\
& \geq \operatorname{dist}_{H_{D}}\left(c_{1}, c_{l}\right)-8 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k} \\
& \geq \operatorname{dist}_{H_{D}}(s, t)-11 \cdot 2^{10 / \varepsilon} D^{1-1 / k}
\end{aligned}
$$

Taking a summation of their right-hand sides (ignoring the tails for now):

$$
\begin{aligned}
\operatorname{dist}_{G}\left(c_{1}, c_{i_{1}-1}\right)+\operatorname{dist}_{G}\left(c_{l}, c_{i_{2}+1}\right)+\operatorname{dist}_{G}\left(v_{1}, v_{2}\right) & \leq \operatorname{dist}_{G}\left(s, s_{i_{1}-1}\right)+\left(\operatorname{dist}_{G}\left(s, c_{1}\right)+\operatorname{dist}_{G}\left(c_{i_{1}-1}, s_{i_{1}}\right)\right) \\
& +\operatorname{dist}_{G}\left(t, s_{i_{2}+1}\right)+\left(\operatorname{dist}_{G}\left(t, c_{l}\right)+\operatorname{dist}_{G}\left(s_{i_{2}+1}, c_{i_{2}}\right)\right) \\
& +\operatorname{dist}_{G}\left(s_{i_{1}}, s_{i_{2}}\right)+\left(\operatorname{dist}_{G}\left(v_{1}, s_{i_{1}}\right)+\operatorname{dist}_{G}\left(v_{2}, s_{i_{2}}\right)\right) \\
& \leq \operatorname{dist}_{G}(s, t)+9 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k} .
\end{aligned}
$$

Altogether, $\operatorname{dist}_{H_{D}}(s, t) \leq \operatorname{dist}_{G}(s, t)+200 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k}<\operatorname{dist}_{G}(s, t)+2^{30 k / \varepsilon} D^{1-1 / k}$.

### 3.4 Size analysis

In this subsection, we complete the proof of Lemma 3.1 by showing that the spanner $H$ constructed by the algorithm in this section satisfies the desired size bound. We start by proving the following claim.

Claim 3.13. In the end, for each level $1 \leq i \leq\lceil 1 / \varepsilon\rceil, \sum_{\mathrm{B}(c, r) \in \mathcal{B}_{i}}\left|\mathcal{P}_{c}\right| \leq O\left(\frac{1}{\varepsilon^{2}} 2^{[1 / \varepsilon\rceil} n^{(i+1) \varepsilon}|\mathcal{P}| \log n\right)$.
Proof. By Observation [3.6, all sets $\mathcal{P}_{c}$ have size $O\left(|\mathcal{P}|^{1 / 2} \log n\right)$ before Step 2 begins. Hence, at the time when Step 1 finishes, we have $\sum_{\mathrm{B}(c, r) \in \mathcal{B}_{i}}\left|\mathcal{P}_{c}\right| \leq O\left(\frac{1}{\varepsilon} n^{(i+1) \varepsilon}|\mathcal{P}| \log n\right)$. So, we only need to analyze how much $\left|\mathcal{P}_{c}\right|$ has increased during Step 2.

Define $\Phi_{i}$ to be the number of pairs of level- $i$ balls that are not tight. So, at the beginning of Step 2, we have $\Phi_{i} \leq\left|\mathcal{B}_{i}\right|^{2} \leq O\left(n^{2(i+1) \varepsilon}|\mathcal{P}| / \varepsilon^{2}\right)$.

Consider the iteration in Step 2 when some pair $(s, t) \in \mathcal{P}$ was being processed, and let $\mathcal{T}$ be the binary tree constructed in that iteration. Let $N$ be a node of $\mathcal{T}$. If the algorithm added more than $4 \beta(i)$ pairs to $\bigcup_{\mathrm{B}(c, r) \in \mathcal{B}_{i}} \mathcal{P}_{c}$ in the round that $N$ was processed, then it must be through Case 1 and $N$ would remain a leaf in $\mathcal{T}$ till the end. Put in other way, at most $4 \beta(i)$ pairs were added to the collections $\bigcup_{\mathrm{B}(c, r) \in \mathcal{B}_{i}} \mathcal{P}_{c}$ in processing every non-leaf node in $\mathcal{T}$. Consider now a leaf node $N$ of $\mathcal{T}$. Using similar analysis in the proof of Claim 3.12, we can show that after the round of processing $N$, every pair of level- $i$ balls that host some segment of the shortest path processed in that iteration became tight, and so $\Phi_{i}$ would decrease by $p(i) \cdot \beta(i) / 8$. Therefore, over the course of the iteration, the sum $\sum_{\mathrm{B}(c, r) \in \mathcal{B}_{i}}\left|\mathcal{P}_{c}\right|$ increased by at most $2^{[1 / \varepsilon\rceil+2} \cdot 4 \beta(i)+8 \Delta \Phi_{i} / \beta(i)$, where $\Delta \Phi_{i}$ was the total decrease of $\Phi_{i}$ in this iteration. Therefore, during Step 2, the amount $\sum_{\mathrm{B}(c, r) \in \mathcal{B}_{i}}\left|\mathcal{P}_{c}\right|$ has increased by at most: $|\mathcal{P}| \cdot 2^{[1 / \varepsilon\rceil+2} \cdot 4 \beta(i)+8\left|\mathcal{B}_{i}\right|^{2} / \beta(i)=O\left(2^{[1 / \varepsilon]} n^{(i+1) \varepsilon}|\mathcal{P}| / \varepsilon^{2}\right)$.

Lemma 3.14. In the end, the spanner $H_{D}$ contains at most $\tilde{O}\left(2^{2 k / \varepsilon} \cdot n^{1+(10 k-8) \varepsilon}|\mathcal{P}|^{1 / 2^{k+1}}\right)$ edges.
Proof. For level 0 , for each ball $\mathrm{B}(c, r) \in \mathcal{B}_{0}$, since $\mathcal{P}_{c}$ stays unchanged during Step 2, by Observation 3.6, we have $\left|\mathcal{P}_{c}\right| \leq O\left(|\mathcal{P}|^{1 / 2} \log n\right)$ at the end of the algorithm. Therefore, by the inductive hypothesis on sublinear additive spanners with stretch function $f_{k-1, \varepsilon}(\cdot)$, the number of edges added to $H$ by balls in $\mathcal{B}_{0}$ is asymptotically (up to an $O(\log n)$ factor) bounded by:

$$
\sum_{\mathrm{B}(c, r) \in \mathcal{B}_{0}} 2^{2(k-1) / \varepsilon} \cdot|\mathrm{B}(c, 4 r)|^{1+10(k-1) \varepsilon} \cdot\left|\mathcal{P}_{c}\right|^{1 / 2^{k}} \leq \tilde{O}\left(2^{2 k} n^{1+(10 k-8) \varepsilon}|\mathcal{P}|^{1 / 2^{k+1}}\right) .
$$

For each level $1 \leq i \leq\lceil 1 / \varepsilon\rceil$, by the inductive hypothesis on sublinear additive spanners with stretch function $f_{k-1, \varepsilon}(\cdot)$, the number of edges added to $H$ by balls in $\mathcal{B}_{i}$ is asymptotically (up to
an $O(\log n)$ factor) bounded by:

$$
\begin{aligned}
& \sum_{\mathrm{B}(c, r) \in \mathcal{B}_{i}} 2^{2(k-1) / \varepsilon} \cdot|\mathrm{B}(c, 4 r)|^{1+10(k-1) \varepsilon} \cdot\left|\mathcal{P}_{c}\right|^{1 / 2^{k}} \\
\leq & 2^{2(k-1) / \varepsilon} \cdot\left(\frac{n^{1-(i-1) \varepsilon}}{|\mathcal{P}|^{1 / 2}}\right)^{1+10(k-1) \varepsilon} \cdot \sum_{\mathrm{B}(c, r) \in \mathcal{B}_{i}}\left|\mathcal{P}_{c}\right|^{1 / 2^{k}} \\
\leq & 2^{2(k-1) / \varepsilon} \cdot \frac{n^{1-(i-1) \varepsilon+10(k-1) \varepsilon}}{|\mathcal{P}|^{1 / 2}} \cdot\left|\mathcal{B}_{i}\right| \cdot\left(\frac{\sum_{\mathrm{B}(c, r) \in \mathcal{B}_{i}}\left|\mathcal{P}_{c}\right|}{\left|\mathcal{B}_{i}\right|}\right)^{1 / 2^{k}} \\
\leq & 2^{2(k-1) / \varepsilon} \cdot \frac{n^{1-(i-1) \varepsilon+10(k-1) \varepsilon}}{|\mathcal{P}|^{1 / 2}} \cdot O\left(n^{(i+1) \varepsilon}|\mathcal{P}|^{1 / 2} / \varepsilon^{2}\right) \cdot \tilde{O}\left(2^{\lceil 1 / \varepsilon\rceil}|\mathcal{P}|^{1 / 2}\right)^{1 / 2^{k}} \\
\leq & \tilde{O}\left(2^{2 k / \varepsilon} \cdot n^{1+(10 k-8) \varepsilon}|\mathcal{P}|^{1 / 2^{k+1}}\right) .
\end{aligned}
$$

where the second inequality uses concavity of $x^{1 / 2^{k}}$ and Jensen's inequality, and the third inequality uses Claim 3.13. Summing over all different levels $0 \leq i \leq\lceil 1 / \varepsilon\rceil$, we get that $\left|E\left(H_{D}\right)\right|=\tilde{O}\left(2^{2 k / \varepsilon}\right.$. $\left.n^{1+(10 k-8) \varepsilon}|\mathcal{P}|^{1 / 2^{k+1}}\right)$.

From Lemma 3.14 and by summing over all $D \in\left\{1,2, \ldots, 2^{\lfloor\log n\rfloor}\right\}$, we get that $|E(H)| \leq$ $\tilde{O}\left(2^{2 k / \varepsilon} \cdot n^{1+(10 k-8) \varepsilon} \cdot|\mathcal{P}|^{1 / 2^{k+1}}\right)=O\left(2^{2 k / \varepsilon} n^{1+10 k \varepsilon}|\mathcal{P}|^{1 / 2^{k+1}}\right)$, which completes the size analysis by induction.

## 4 Almost Optimal Sublinear Additive Spanners

In this section, we provide the proof of Theorem 1.1, In fact, we will prove the following theorem.
Theorem 4.1. For any undirected unweighted graph $G$ on $n$ vertices, any parameter $\varepsilon>0$ and any integer $k \geq 1$, there is a subgraph $H \subseteq G$ with $|E(H)| \leq O\left(n^{1+10 k \varepsilon+\frac{1}{2^{k+1}-1}}\right)$, such that for every pair $u, v \in V(G), \operatorname{dist}_{H}(u, v) \leq \operatorname{dist}_{G}(u, v)+2^{30 k / \varepsilon} \cdot \operatorname{dist}_{G}(u, v)^{1-1 / k}$.

Note that, for any constant parameter $\delta>0$, if we let $\varepsilon=\frac{\delta}{10 k\left(2^{k+1}-1\right)}$, then Theorem4.1 gives a sublinear spanner with stretch function $f(d)=d+2^{O\left(k^{2} 2^{k} / \delta\right)} \cdot d^{1-1 / k}$ on $O\left(n^{1+\frac{1+\delta}{2^{k+1}-1}}\right)$ edges, which implies Theorem 1.1.

In the remainder of this section, we provide the proof of Theorem 4.1 by induction on $k$, with the help of Lemma 3.1. The base case is when $k=1$. We note that it was shown by BKMP10] that any undirected unweighted graph admits an +6 -additive spanner of size $O\left(n^{4 / 3}\right)$, and the base case follows from this result. Assume from now on that Theorem 4.1 is true for $1, \ldots,(k-1)$. Similar to Section 3, we denote $f_{k, \varepsilon}(d)=d+2^{30 k / \varepsilon} \cdot d^{1-1 / k}$ for brevity.

Similar to the algorithm in Section 3, for each $D \in\left\{1,2,2^{2}, \ldots, 2^{\lfloor\log n\rfloor}\right\}$, we will construct a subgraph $H_{D} \subseteq G$, such that for all pairs $s, t \in V(G)$ with $D \leq \operatorname{dist}_{G}(s, t)<2 D$, $\operatorname{dist}_{H}(s, t) \leq$ $\operatorname{dist}_{G}(s, t)+O\left(D^{1-1 / k}\right)$ holds. We will then let $H=\bigcup_{0 \leq i \leq\lfloor\log n\rfloor} H_{2^{i}}$ to finish the construction. For convenience, we assume that $D^{1-1 / k}$ is an integer.

### 4.1 Algorithm description

We now describe the construction of graph $H_{D}$. We first apply the algorithm of Lemma 2.2 to $G$ with parameters $R=D^{1-1 / k}$ and $\varepsilon$. Let $\mathcal{B}$ be the set of balls we obtain. Let $L_{k}$ be a threshold
value to be determined later. We say that a ball $\mathrm{B}(c, r) \in \mathcal{B}$ is small iff $|\mathrm{B}(c, r)| \leq L_{k}$, otherwise we say it is large.

Similar to the algorithm in Section 3, the spanner $H_{D}$ is the union of the following graphs:
(i) for each ball $\mathrm{B}(c, r) \in \mathcal{B}$, a BFS tree that is rooted at $c$ and spans all vertices in $G[\mathrm{~B}(c, 4 r)]$;
(ii) for each small ball $\mathrm{B}(c, r) \in \mathcal{B}$, a spanner of the induced subgraph $G[\mathrm{~B}(c, 4 r)]$, with stretch function $f_{k-1, \varepsilon}$ and size $O\left(|\mathrm{~B}(c, 4 r)|^{1+10(k-1) \varepsilon+\frac{1}{2^{k-1}}}\right)$, whose existence is guaranteed by the inductive hypothesis;
(iii) for each ball $\mathrm{B}(c, r) \in \mathcal{B}$, a pairwise spanner with respect to a collection $\mathcal{P}_{c}$ of pairs in $\mathrm{B}(c, 2 r)$, with stretch function $f_{k-1, \varepsilon}$ and size $O\left(2^{2(k-1) / \varepsilon}|\mathrm{B}(c, 4 r)|^{1+10(k-1) \varepsilon}\left|\mathcal{P}_{c}\right|^{1 / 2^{k}}\right)$, whose existence is guaranteed by the inductive hypothesis; we will guarantee that for each demand pair $(s, t) \in \mathcal{P}_{c}$, any $s$ - $t$ shortest path in $G$ lies in the induced subgraph $G[\mathrm{~B}(c, 4 r)]$. The construction of sets $\mathcal{P}_{c}$ is iterative and described next.
Let $S$ be a random subset of $V$ of size $\left\lceil\frac{10 n}{L_{k}} \log n\right\rceil$, so with high probability, $S$ intersects all large balls in $\mathcal{B}$. We then compute $\Pi=\left\{\pi_{s, t} \mid s, t \in S\right.$, $\left.\operatorname{dist}_{G}(s, t)<2 D+4 \cdot 2^{10 / \varepsilon} D^{1-1 / k}\right\}$, where $\pi_{s, t}$ is an $s$-t shortest path in $G$. We now proceed to iteratively construct the sets $\left\{\mathcal{P}_{c}\right\}$ of pairs. Throughout, we maintain, for each large ball $\mathrm{B}(c, r) \in \mathcal{B}$, a set $\mathcal{P}_{c}$ of pairs of vertices in $\mathrm{B}(c, 2 r)$, and another set $U_{c}$ of vertices in $V$ which intuitively contains all vertices that are "settled with the ball $\mathrm{B}(c, r)$ ".

We then process all paths in $\Pi$ sequentially in an arbitrary order. For each path $\pi_{s, t} \in \Pi$, we first apply the subroutine PathPartition to it and the collection $\mathcal{B}$ of balls, and obtain a partitioning $\pi_{s, t}=\alpha_{1} \circ \alpha_{2} \circ \cdots \circ \alpha_{l}$. For each $1 \leq i \leq l$, we denote by $\mathrm{B}\left(c_{i}, r_{i}\right)$ the ball that hosts the subpath $\alpha_{i}$.

If either all the balls $\mathrm{B}\left(c_{1}, r_{1}\right), \mathrm{B}\left(c_{2}, r_{2}\right), \ldots, \mathrm{B}\left(c_{l}, r_{l}\right)$ are small, or there exists a large ball $\mathrm{B}\left(c_{i}, r_{i}\right)$ such that $s, t \in U_{c_{i}}$, then we do nothing and move on to the next path in $\Pi$. Otherwise, for each large ball $\mathrm{B}\left(c_{i}, r_{i}\right)$, we add vertices $s, t$ to set $U_{c_{i}}$, and add the pair $\left(s_{i}, t_{i}\right)$ to set $\mathcal{P}_{c}$. This completes the description of the construction of sets $\left\{\mathcal{P}_{c}\right\}$, and also finishes the description of the algorithm.

Before we proceed to the size and stretch analysis, we prove the following simple observation.
Observation 4.2. In the end, for each ball $\mathrm{B}(c, r) \in \mathcal{B},\left|\mathcal{P}_{c}\right| \leq|S|=\left\lceil\frac{10 n}{L_{k}} \log n\right\rceil$.
Proof. By the algorithm, each time a new pair is added to $\mathcal{P}_{c},\left|U_{c}\right|$ also increases by at least one. Therefore $\left|\mathcal{P}_{c}\right| \leq\left|U_{c}\right| \leq|S|=\left\lceil\frac{10 n}{L_{k}} \log n\right\rceil$.

### 4.2 Stretch analysis

The stretch analysis of the algorithm in Section 4 is quite similar to the Step 1 stretch analysis in Section 3 (Claim 3.7 and Claim 3.8). We start with the following claims.
Claim 4.3. In the end, for each large ball $\mathrm{B}(c, r) \in \mathcal{B}$ and each vertex $s \in U_{c}$,

$$
\operatorname{dist}_{H_{D}}(s, c) \leq \operatorname{dist}_{G}(s, c)+50 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k}
$$

Proof. Assume that the shortest path $\pi_{s, t}$ was being processed when $s$ was added to $U_{c}$, and that $\mathrm{B}(c, r)$ was $\mathrm{B}\left(c_{i}, r_{i}\right)$ under the notation of the subroutine PathPartition. Next, we will construct a short path from $s$ to $c$ in $H_{D}$. For each $1 \leq j<i$, consider the shortest path $\alpha_{j}=\pi\left[s_{j}, t_{j}\right]$ and the ball $\mathrm{B}\left(c_{j}, r_{j}\right)$. By Observation [3.4, $\alpha_{j}$ lies entirely within $G\left[\mathrm{~B}\left(c_{j}, 4 r_{j}\right)\right]$. Let $\phi_{j}$ be a shortest path from $s_{j}$ to $t_{j}$ in $H_{D}$. We distinguish between the following cases.

- $\mathrm{B}\left(c_{j}, r_{j}\right)$ is small.

Recall that graph $H_{D}$ contains a sublinear spanner of the induced subgraph $G\left[\mathrm{~B}\left(c_{j}, 4 r_{j}\right)\right]$ with stretch function $f_{k-1, \varepsilon}$. Therefore,

$$
\begin{aligned}
\left|\phi_{j}\right| \leq\left|\alpha_{j}\right|+2^{30(k-1) / \varepsilon} \cdot\left|\alpha_{j}\right|^{1-1 /(k-1)} & \leq\left|\alpha_{j}\right|+2^{30(k-1) / \varepsilon} \cdot\left(4 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k}\right)^{1-1 /(k-1)} \\
& \leq\left|\alpha_{j}\right|+4 \cdot 2^{(30 k-20) / \varepsilon} D^{1-2 / k}
\end{aligned}
$$

- $\mathrm{B}\left(c_{j}, r_{j}\right)$ is large.

From the algorithm description, the pair $\left(s_{j}, t_{j}\right)$ was added to set $\mathcal{P}_{c}$ in this iteration, and $\alpha_{j}$ is contained entirely within $G\left[\mathrm{~B}\left(c_{j}, 4 r_{j}\right)\right]$ (from Observation 3.4). Therefore,

$$
\left|\phi_{j}\right| \leq f_{k-1, \varepsilon}\left(\left|\alpha_{j}\right|\right) \leq\left|\alpha_{j}\right|+4 \cdot 2^{(30 k-20) / \varepsilon} D^{1-2 / k}
$$

As $t_{i-1}=s_{i} \in \mathrm{~B}\left(c_{i}, r_{i}\right)$ holds for every $1 \leq j<i$, we have $\operatorname{dist}_{G}\left(c, t_{i-1}\right) \leq r \leq 2^{10 / \varepsilon} D^{1-1 / k}$, and therefore

$$
\begin{aligned}
\sum_{j=1}^{i-1}\left|\phi_{j}\right| & \leq \sum_{j=1}^{i-1}\left(\left|\alpha_{j}\right|+4 \cdot 2^{(30 k-20) / \varepsilon} D^{1-2 / k}\right) \leq \sum_{j=1}^{i-1}\left|\alpha_{j}\right|+O_{\varepsilon}\left(D^{1-1 / k}\right) \\
& \leq \operatorname{dist}_{G}(s, c)+\operatorname{dist}_{G}\left(c, t_{i-1}\right)+i \cdot 4 \cdot 2^{(30 k-20) / \varepsilon} D^{1-2 / k} \\
& <\operatorname{dist}_{G}(s, c)+49 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k}
\end{aligned}
$$

Finally, we let $\phi_{i}$ be an arbitrary shortest path connecting $t_{i-1}$ to $c$ in $H_{D}$. Since $H_{D}$ contains a breath-first search tree rooted at $c$ that spans all vertices in $\mathrm{B}(c, 4 r),\left|\phi_{i}\right| \leq 4 r \leq 4 \cdot 2^{10 / \varepsilon} D^{1-1 / k}$. Therefore, $\rho=\phi_{1} \circ \phi_{2} \circ \cdots \circ \phi_{i}$ is a path in $H_{D}$ that connects $s$ and $c$, and moreover,

$$
\operatorname{dist}_{H_{D}}(s, c) \leq \sum_{j=1}^{i}\left|\phi_{j}\right| \leq \operatorname{dist}_{G}(s, c)+50 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k}
$$

Claim 4.4. For any pair of vertices $s, t \in S$ such that $\operatorname{dist}_{G}(s, t)<2 D+4 \cdot 2^{10 / \varepsilon} D^{1-1 / k}$,

$$
\operatorname{dist}_{H_{D}}(s, t) \leq \operatorname{dist}_{G}(s, t)+101 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k}
$$

Proof. For any such pair of vertices $s, t \in S$, consider the moment when the shortest path $\pi_{s, t} \in \Pi$ was processed and partitioned into $\alpha_{1}, \ldots, \alpha_{l}$. We distinguish between the following two cases.

- There existed an index $1 \leq i \leq l$ such that $s, t \in U_{c_{i}}$ at the moment.

In this case, by Lemma 4.3, $\operatorname{dist}_{H_{D}}\left(s, c_{i}\right) \leq \operatorname{dist}_{G}\left(s, c_{i}\right)+50 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k}$, and dist $H_{H_{D}}\left(c_{i}, t\right) \leq$ $\operatorname{dist}_{G}\left(c_{i}, t\right)+50 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k}$. By triangle inequality,

$$
\begin{aligned}
\operatorname{dist}_{G}\left(s, c_{i}\right)+\operatorname{dist}_{G}\left(c_{i}, t\right) & \leq\left(\operatorname{dist}_{G}\left(s, s_{i}\right)+\operatorname{dist}_{G}\left(s_{i}, c_{i}\right)\right)+\left(\operatorname{dist}_{G}\left(s_{i}, t\right)+\operatorname{dist}_{G}\left(s_{i}, c_{i}\right)\right) \\
& \leq \operatorname{dist}_{G}(s, t)+2 \cdot 2^{10 / \varepsilon} D^{1-1 / k} .
\end{aligned}
$$

Therefore, $\operatorname{dist}_{H_{D}}(s, t) \leq \operatorname{dist}_{H_{D}}\left(s, c_{i}\right)+\operatorname{dist}_{H_{D}}\left(c_{i}, t\right) \leq \operatorname{dist}_{G}(s, t)+101 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k}$.

- There did not exist any $i$ such that $s, t \in U_{c_{i}}$ at the moment.

In this case, for each $1 \leq j \leq l$, the pair $\left(s_{j}, t_{j}\right)$ is added to the set $\mathcal{P}_{c_{j}}$ after this iteration. According to the algorithm, in the resulting graph $H_{D}$, for each $1 \leq i \leq l$, there is a path $\phi_{i}$ in $H_{D}$ between $s_{i}, t_{i}$ such that $\left|\phi_{i}\right| \leq\left|\alpha_{i}\right|+4 \cdot 2^{(30 k-20) / \varepsilon} D^{1-2 / k}$. By Lemma 3.4,

$$
\begin{aligned}
\operatorname{dist}_{H_{D}}(s, t) & \leq \sum_{i=1}^{l}\left|\phi_{i}\right| \leq \sum_{i=1}^{l}\left(\left|\alpha_{i}\right|+4 \cdot 2^{(30 k-20) / \varepsilon} D^{1-2 / k}\right) \\
& \leq \operatorname{dist}_{G}(s, t)+48 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k} .
\end{aligned}
$$

In the following claim, we complete the analysis of stretch of the graph $H_{D}$ on pairs of vertices in $G$ at distance at most $2 D$.
Lemma 4.5. For any pair of vertices $s, t \in V$ such that $\operatorname{dist}_{G}(s, t)<2 D$, we have:

$$
\operatorname{dist}_{H_{D}}(s, t) \leq \operatorname{dist}_{G}(s, t)+2^{30 k / \varepsilon} D^{1-1 / k} .
$$

Proof. Let $\pi$ be an $s$ - $t$ shortest path. We apply the subroutine PathPartition to path $\pi$ and the set $\mathcal{B}$ of balls and obtain a partition $\pi=\alpha_{1} \circ \alpha_{2} \circ \cdots \circ \alpha_{l}$ and balls $\mathrm{B}\left(c_{1}, r_{1}\right), \mathrm{B}\left(c_{2}, r_{2}\right), \ldots, \mathrm{B}\left(c_{l}, r_{l}\right)$. If all these balls are small, note that for each $1 \leq i \leq l$, the subpath $\alpha_{i}$ lies entirely in $G\left[\mathrm{~B}\left(r_{i}, r_{i}\right)\right]$, so there exists a path $\phi_{i}$ in $H_{D}$, such that

$$
\begin{aligned}
\left|\phi_{i}\right| \leq\left|\alpha_{i}\right|+2^{30(k-1) / \varepsilon} \cdot\left|\alpha_{i}\right|^{1-1 /(k-1)} & \leq\left|\alpha_{i}\right|+2^{30(k-1) / \varepsilon} \cdot\left(4 \cdot 2^{10 / \varepsilon} \cdot D^{1-1 / k}\right)^{1-1 /(k-1)} \\
& \leq\left|\alpha_{i}\right|+4 \cdot 2^{(30 k-20) / \varepsilon} D^{1-2 / k} .
\end{aligned}
$$

Since $l \leq\left\lceil\frac{|\pi|}{D^{1-1 / k}}\right\rceil<2 D^{1 / k}+1<3 D^{1 / k}$, we get that $\operatorname{dist}_{H_{D}}(s, t) \leq \operatorname{dist}_{G}(s, t)+12 \cdot 2^{(30 k-20) \varepsilon} D^{1-1 / k}$.
Assume now on that, some ball among $\mathrm{B}\left(c_{1}, r_{1}\right), \mathrm{B}\left(c_{2}, r_{2}\right), \ldots, \mathrm{B}\left(c_{l}, r_{l}\right)$ is large. Let $x \leq y \in[1, l]$ be the smallest and the largest indices of large balls. By the hitting set property, there exist $u, v \in S$ such that $u \in \mathrm{~B}\left(c_{x}, r_{x}\right), v \in \mathrm{~B}\left(c_{y}, r_{y}\right)$. By triangle inequality,

$$
\operatorname{dist}_{G}(u, v) \leq \operatorname{dist}_{G}\left(c_{x}, c_{y}\right)+\operatorname{dist}_{G}\left(c_{x}, u\right)+\operatorname{dist}_{G}\left(c_{y}, v\right)<2 D+4 \cdot 2^{10 / \varepsilon} D^{1-1 / k}
$$

From Claim 4.4, $\operatorname{dist}_{H_{D}}(u, v) \leq \operatorname{dist}_{G}(u, v)+101 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k}$.
For each $j \in[1, x) \cup(y, l]$, since $\mathrm{B}\left(c_{j}, r_{j}\right)$ is small, there is a path $\phi_{j}$ in $H_{D}$ connecting $s_{j}$ to $t_{j}$, such that

$$
\left|\phi_{j}\right| \leq\left|\alpha_{j}\right|+2^{30(k-1) / \varepsilon} \cdot\left|\alpha_{j}\right|^{1-1 /(k-1)} \leq\left|\alpha_{j}\right|+4 \cdot 2^{(30 k-20) / \varepsilon} D^{1-2 / k} .
$$

We define $\phi_{x}$ ( $\phi_{y}$, resp.) to be the shortest path in $H_{D}$ connecting $s_{x}$ to $u\left(s_{y}\right.$ and $v$, resp.). Then,

$$
\operatorname{dist}_{H_{D}}(s, u)+\operatorname{dist}_{H_{D}}(v, t) \leq \operatorname{dist}_{G}(s, u)+\operatorname{dist}_{G}(v, t)+12 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k} .
$$

Finally, by triangle inequality,

$$
\begin{aligned}
\operatorname{dist}_{H_{D}}(s, t) & \leq \operatorname{dist}_{G}(s, u)+\operatorname{dist}_{G}(u, v)+\operatorname{dist}_{G}(v, t)+113 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k} \\
& \left.\leq \operatorname{dist}_{G}\left(s, s_{x}\right)+\operatorname{dist}_{G}\left(s_{x}, u\right)\right) \\
& \left.+\operatorname{dist}_{G}\left(s_{x}, s_{y}\right)+\operatorname{dist}_{G}\left(s_{x}, u\right)+\operatorname{dist}_{G}\left(s_{y}, v\right)\right) \\
& +\left(\operatorname{dist}_{G}\left(s_{y}, t\right)+\operatorname{dist}_{G}\left(v, s_{y}\right)\right)+113 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k} \\
& \leq \operatorname{dist}_{G}(s, t)+8 \cdot 2^{10 / \varepsilon} D^{1-1 / k}+113 \cdot 2^{(30 k-10) / \varepsilon} D^{1-1 / k} \\
& \leq \operatorname{dist}_{G}(s, t)+2^{30 k / \varepsilon} D^{1-1 / k} .
\end{aligned}
$$

### 4.3 Size analysis

From the algorithm, the edge set of $H_{D}$ contains three types of edges that are calculated below.
(1) For each ball $\mathrm{B}(c, r) \in \mathcal{B}$, graph $H_{D}$ contains a BFS tree $T_{c}$ that is rooted at $c$ and spans all vertices in $\mathrm{B}(c, 4 r)$. From Lemma 2.2, $\sum\left|E\left(T_{c}\right)\right|=\sum|\mathrm{B}(c, 4 r)|=O\left(n^{1+\varepsilon} / \varepsilon\right)$.
(2) For each small ball $\mathrm{B}(c, r) \in \mathcal{B}$, graph $H_{D}$ contains a sublinear additive spanner of the subgraph $G[\mathrm{~B}(c, 4 r)]$ with stretch function $f_{k-1, \varepsilon}$, which contains $O\left(|\mathrm{~B}(c, 4 r)|^{1+10(k-1) \varepsilon+\frac{1}{2^{k}-1}}\right)$ edges by inductive hypothesis. Summing over all small balls, the number of edges in these spanners is at most (ignoring constant factors):

$$
\sum_{\mathrm{B}(c, r) \text { is small }}|\mathrm{B}(c, 4 r)|^{1+10(k-1) \varepsilon+\frac{1}{2^{k}-1}} \leq n^{1+(10 k-5) \varepsilon} \cdot L_{k}^{\frac{1}{2 k-1}} .
$$

(3) For each large ball $\mathrm{B}(c, r) \in \mathcal{B}$, graph $H_{D}$ contains a pairwise sublinear additive spanner of the induced subgraph $G[\mathrm{~B}(c, 4 r)]$ with respect to the set $\mathcal{P}_{c}$ of pairs, with stretch function $f_{k-1, \varepsilon}$. By Lemma 3.1 and Observation 4.2, the number of edges in such a spanner is at most

$$
\tilde{O}\left(2^{2 k / \varepsilon}|\mathrm{B}(c, 4 r)|^{1+10(k-1) \varepsilon} \cdot\left(\frac{n}{L_{k}}\right)^{\frac{1}{2^{k}}}\right) .
$$

Summing over all large balls, the number of edges in all these spanners is at most

$$
\tilde{O}\left(\sum_{\mathrm{B}(c, r) \text { is large }} 2^{2 k / \varepsilon}|\mathrm{B}(c, 4 r)|^{1+10(k-1) \varepsilon} \cdot\left(\frac{n}{L_{k}}\right)^{\frac{1}{2^{k}}}\right) \leq 2^{2 k / \varepsilon} n^{1+(10 k-5) \varepsilon} \cdot\left(\frac{n}{L_{k}}\right)^{\frac{1}{2^{k}}} .
$$

Setting $L_{k}=n^{\frac{2^{k}-1}{k+1}-1}$, the total number of edges over all the above types is at most $O\left(2^{2 k / \varepsilon}\right.$. $\left.n^{1+(10 k-5) \varepsilon+\frac{1}{2^{k+1}-1}}\right)$. Summing over all $D \in\left\{1,2, \ldots, 2^{\lceil\log n\rceil}\right\}$, we can conclude that $|E(H)| \leq$ $O\left(n^{1+10 k \varepsilon+\frac{1}{2^{k+1}-1}}\right)$.

## 5 Subset Additive Spanners

In this section, we prove the following lemma, which will serve as a building block for Theorem 1.2 , For a graph $G=(V, E)$, a subgraph $H \subseteq G$ and a subset $U \subseteq V$, we say that $H$ is a subset spanner of $G$ on $U$ with additive error $f(n)$, iff for every pair $u, u^{\prime} \in U$, $\operatorname{dist}_{H}\left(u, u^{\prime}\right) \leq \operatorname{dist}_{G}\left(u, u^{\prime}\right)+f(n)$.

Lemma 5.1. For any $\varepsilon>0$, there is an algorithm that, given an unweighted undirected graph $G$ on $n$ vertices and $m$ edges, and a subset $U \subseteq V(G)$, in time $O\left(m\left(|U|+2^{O(1 / \varepsilon)}\right)\right)$, computes a subset spanner $H$ of $G$ on $U$ with additive error $O\left(|U|^{3 / 2} \cdot n^{\varepsilon}\right)$, such that $|E(H)|=2^{O(1 / \varepsilon)} \cdot n$.

It would be interesting to study if the additive error can be improved even if we put aside runtime concerns.

### 5.1 Algorithm description

We now describe the algorithm for Lemma 5.1. We first slightly perturb the (unit) weight of each edge, such that in the resulting graph, there is a unique shortest path between every pair of vertices. When calculating their distances, we ignore this perturbation. It is easy to verify that any subset spanner of this resulting graph immediately gives a subset spanner of the original graph with same stretch function. We rename the resulting graph by $G$.

We now describe the algorithm for constructing $H$. We first apply the algorithm from Lemma 2.2 to $G$ with parameters $R=\left\lceil|U|^{3 / 2}\right\rceil$ and $\frac{10}{\varepsilon \log _{2} n}$, and obtain a set $\mathcal{B}$ of balls in time $O\left(2^{10 / \varepsilon} m\right)$. Classify balls in $\mathcal{B}$ as two categories: (Small) $|\mathrm{B}(c, r)| \leq|U|^{2}$, (Large) $|\mathrm{B}(c, r)|>|U|^{2}$.

For each ball $\mathrm{B}(c, r) \in \mathcal{B}$, we compute a BFS tree $T_{c}$ that is rooted at $c$ and spans all vertices in $\mathrm{B}(c, 4 r)$, in time $O(\operatorname{vol}(\mathrm{~B}(c, 4 r)))$. From Lemma 2.2, $\sum_{c}\left|E\left(T_{c}\right)\right|=\sum_{c}|\mathrm{~B}(c, 4 r)|=O\left(2^{10 / \varepsilon} n / \varepsilon\right)$.

Handling small balls. Consider a small ball $\mathrm{B}(c, r) \in \mathcal{B}$. From Lemma 2.1, we can compute in time $O(\operatorname{vol}(\mathrm{~B}(c, 4 r)))$ an integer $d \in[r, 2 r]$ such that $\left|\mathrm{B}^{=}(c, d) \cup \mathrm{B}^{=}(c, d+1)\right| \leq 2|\mathrm{~B}(c, 4 r)| / r \leq$ $2|\mathrm{~B}(c, 4 r)| / R$. Further, denote:

$$
\mathcal{B}_{1}=\{\mathrm{B}(c, d) \mid \mathrm{B}(c, r) \in \mathcal{B} \text { is small }\} .
$$

We then apply the algorithm from Corollary 2.4 to the induced subgraph $G[\mathrm{~B}(c, 4 r)]$ with $S=\mathrm{B}^{=}(c, d) \cup \mathrm{B}^{=}(c, d+1)$, and obtain a collection $\Pi_{c}$ of shortest paths. We define $L_{c}=\bigcup_{\pi \in \Pi_{c}} \pi$. From Corollary 2.4 and Lemma 2.2.
$\sum_{c}\left|E\left(L_{c}\right)\right|=\sum_{c} O\left(|\mathrm{~B}(c, 4 r)|+\sqrt{|\mathrm{B}(c, 4 r)|} \cdot\left(\frac{|\mathrm{B}(c, 4 r)|}{R}\right)^{2}\right)=\sum_{c} O\left(2^{15 / \varepsilon}|\mathrm{B}(c, 4 r)|\right)=O\left(2^{25 / \varepsilon} n / \varepsilon\right)$.
Handling large balls. We first apply the algorithm from Corollary 2.4 to $G$ with $S=U$, and obtain a collection $\Pi$ of shortest paths connecting pairs of $U$. We will then iteratively construct, for each large ball $\mathrm{B}(c, r)$, a set $U_{c} \subseteq U$ of vertices. Initially, all sets $U_{c}$ are empty.

We then process all paths in $\Pi$ sequentially in an arbitrary order. Consider any path $\pi_{s, t} \in \Pi$ from $s$ to $t$. Then, the algorithm consists of the following steps.
(1) Recall that a vertex is covered by $\mathcal{B}_{1}$ if it is contained in some ball in $\mathcal{B}_{1}$. We first compute the set $\Sigma$ of all maximal subpaths of $\pi_{s, t}$ consisting of only vertices covered by $\mathcal{B}_{1}$, and let $\Sigma^{\prime}$ be the collection of subpaths of $\pi_{s, t}$ obtained by removing all edges of paths in $\Sigma$.
(2) Go over each vertex $x$ on $\pi_{s, t}$. Find an arbitrary ball $\mathrm{B}\left(c_{x}, r_{x}\right) \ni x$. If $\mathrm{B}\left(c_{x}, r_{x}\right)$ is large and $s, t \in U_{c_{x}}$, then we abort and continue on to the next path in $\Pi$.
(3) Otherwise, add all paths in $\Sigma^{\prime}$ to the final spanner $H$, and add $s, t$ to $U_{c_{x}}$ for each $x$ on $\pi_{s, t}$ where $\mathrm{B}\left(c_{x}, r_{x}\right)$ is large.

### 5.2 Stretch analysis

We now show that the graph $H$ obtained by the above algorithm satisfies the properties required in Lemma 5.1. Recall that the balls in $\mathcal{B}$ were obtained by applying the algorithm from Lemma 2.2 to graph $G$ with parameters $R=\left\lceil|U|^{3 / 2}\right\rceil$ and $\frac{10}{\varepsilon \log _{2} n}$. Note that, from Lemma 2.2, the radius of each ball in $\mathcal{B}$ is at most $R \cdot 2^{10 /\left(\frac{10}{\varepsilon \log _{2} n}\right)} \leq R n^{\varepsilon}$, so the radius (diameter, resp.) of every ball $\mathrm{B}(c, 2 r)$ is at most $2 R n^{\varepsilon}$ (4Rn $n^{\varepsilon}$, resp.).

Definition 5.1. A vertex $v$ is covered at boundary by $\mathcal{B}_{1}$, if $v$ belongs to the boundary of every ball in $\mathcal{B}_{1}$ that contains $v$.

Claim 5.2. Let $s, t$ be a pair of vertices such that all vertices on the shortest path between them are covered by $\mathcal{B}_{1}$. Then $\operatorname{dist}_{H}(s, t) \leq \operatorname{dist}_{G}(s, t)+8 n^{\varepsilon} \cdot R$. Furthermore, if both $s$ and $t$ are covered at boundary by $\mathcal{B}_{1}$, then $\operatorname{dist}_{H}(s, t)=\operatorname{dist}_{G}(s, t)$.

Proof. Let $\pi$ be the shortest path $\pi$ from $s$ to $t$ such that all vertices in $\pi$ are covered by $\mathcal{B}_{1}$. We first compute two sequences $\left(u_{0}, u_{1}, u_{2}, \ldots, u_{l-1}\right),\left(v_{0}, v_{1}, v_{2}, \ldots, v_{l-1}\right)$ of vertices in $\pi$, such that vertices $v_{0}=u_{0}=s, v_{1}, u_{1}, \ldots, v_{l-1}, u_{l-1}$ appear on path $\pi$ from in the direction from $s$ to $t$ in this order, and a sequence $\left(\mathrm{B}\left(c_{1}, r_{1}\right), \mathrm{B}\left(c_{2}, r_{2}\right), \ldots, \mathrm{B}\left(c_{l}, r_{l}\right)\right)$ of small balls in $\mathcal{B}$, such that $v_{0} \in \mathrm{~B}\left(c_{1}, d_{1}\right)$, and for all $1 \leq i \leq l-1, v_{i} \in \mathrm{~B}^{=}\left(c_{i+1}, d_{i+1}\right), u_{i} \in \mathrm{~B}^{=}\left(c_{i}, d_{i}+1\right)$; recall that integers $d_{i}$ were computed by Lemma [2.1. See Figure 7 for an illustration.


Figure 7: The construction of sequences $u_{1}, u_{2}, \ldots, u_{l}$ and $v_{1}, v_{2}, \ldots, v_{l-1}$.
This is done via an iterative process described as the following steps.
(1) Starting with $i=0$, and set $u_{0}=v_{0}=s$. If $u_{i}=t$ then we terminate the process and set $l=i+1$.
Otherwise, we find the ball in $\mathcal{B}_{1}$ that, among all balls in $\mathcal{B}_{1}$ that intersects $\pi\left[s, u_{i}\right]$, the one that contains a vertex on $\pi$ that is closest to $t$. In other words, if we denote $\pi$ as $(s=$ $\left.w_{0}, w_{1}, \ldots, w_{k-1}, w_{k}=t\right)$, then we find the ball that intersects $\pi\left[s, u_{i}\right]$, and, subject to this, contains a vertex $w_{j}$ with the largest index.
Note that such a ball always exists; for example, we can take an arbitrary ball in $\mathcal{B}_{1}$ that contains $u_{i}$, and we can also notice that $u_{i+1}$ should belong to $\pi\left(u_{i}, t\right]$.
We denote this ball by $\mathrm{B}\left(c_{i+1}, d_{i+1}\right)$ and set $u_{i+1}$ as the last vertex of $\pi$ that belongs to $\mathrm{B}\left(c_{i+1}, d_{i+1}+1\right)$.
(2) If $i \geq 1$, we then let $v_{i}$ be any vertex that belongs to both $\pi\left[u_{i-1}, u_{i}\right]$ and $\mathrm{B}^{=}\left(c_{i+1}, d_{i+1}\right)$. We will prove shortly that such $v_{i}$ always exists.
Then, increase $i \leftarrow i+1$ and go to Step (1).
We now turn to argue the existence of $v_{i}$.
Claim 5.3. For each $i \geq 1, \pi\left[u_{i-1}, u_{i}\right]$ intersects with $\mathrm{B}=\left(c_{i+1}, d_{i+1}\right)$, and if $u_{i+1} \neq t$, then $u_{i+1} \in$ $\mathrm{B}^{=}\left(c_{i+1}, d_{i+1}+1\right)$.

Proof of claim. First, if $\pi\left[s, u_{i}\right]$ does not intersect $\mathrm{B}=\left(c_{i+1}, d_{i+1}\right)$, as $\pi\left[s, u_{i}\right]$ intersects $\mathrm{B}\left(c_{i+1}, d_{i+1}\right)$, it has to lie entirely within $\mathrm{B}\left(c_{i+1}, d_{i+1}\right)$, a contradiction to the choice of $\mathrm{B}\left(c_{1}, d_{1}\right)$ and $u_{1}$. Therefore, $\pi\left[s, u_{i}\right]$ intersect $\mathrm{B}=\left(c_{i+1}, d_{i+1}\right)$.

We now claim any such intersection $v_{i}$ must belong to $\pi\left[u_{i-1}, u_{i}\right]$. Otherwise, if $v_{i}$ belongs to $\pi\left[u_{j-1}, u_{j}\right]$ for some index $1 \leq j<i$, then earlier when we were determining $u_{j+1}$, it should have been at least as close to $t$ as $u_{i+1}$, a contradiction to the property that $u_{i} \in\left(u_{j}, t\right]$.

Lastly, if $u_{i+1} \in \mathrm{~B}\left(c_{i+1}, d_{i+1}\right)$, then the next vertex on path $\pi$ should also belong to $\mathrm{B}\left(c_{i+1}, d_{i+1}+\right.$ 1 ), a contradiction to the choice of $u_{i+1}$.

It is clear that when the iterative process terminates, $u_{l}=t$. We now show that for each $2 \leq i \leq$ $l-1$, the subpath $\pi\left[v_{i-1}, u_{i}\right]$ is contained in $H$. Note that $v_{i-1} \in \mathrm{~B}^{=}\left(c_{i}, d_{i}\right)$ and $u_{i} \in \mathrm{~B}^{=}\left(c_{i}, d_{i}+1\right)$. Since $\mathrm{B}\left(c_{i}, r_{i}\right)$ is small, from the algorithm, we have constructed a subgraph $L_{c}$ in $G\left[\mathrm{~B}\left(c_{i}, 4 r_{i}\right)\right]$ that preserves all-pairs distances between vertices in $\mathrm{B}=\left(c_{i}, d_{i}\right) \cup \mathrm{B}=\left(c_{i}, d_{i}+1\right)$. As the shortest path between every pair of vertices is unique, the subpath $\pi\left[v_{i-1}, u_{i}\right]$ should be entirely contained in $L_{c}$.

Lastly, we consider the distance in $H$ between the endpoints $s, t$ of $\pi$. From the construction, we know that vertices $s, u_{1} \in \mathrm{~B}\left(c_{1}, d_{1}+1\right)$, and $v_{l-1}, t \in \mathrm{~B}\left(c_{l}, d_{l}+1\right)$. Therefore, $\operatorname{dist}_{H}\left(s, u_{1}\right) \leq 4 R n^{\varepsilon}$, and $\operatorname{dist}_{H}\left(v_{l-1}, t\right) \leq 4 R n^{\varepsilon}$. It follows that $\operatorname{dist}_{H}(s, t) \leq \operatorname{dist}_{G}(s, t)+8 R n^{\varepsilon}$. Furthermore, if both $s, t$ are covered at boundary by $\mathcal{B}_{1}$, then $s \in \mathrm{~B}^{=}\left(c_{1}, d_{1}\right)$ and $t \in \mathrm{~B}^{=}\left(c_{l}, d_{l}\right)$. Therefore, the entire path $\pi$ belongs to $H$, and so $\operatorname{dist}_{H}(s, t)=\operatorname{dist}_{G}(s, t)$.

We now focus on the algorithm for handling large balls. Consider an iteration in which a shortest path $\pi_{s, t}$ is processed for some pair $s, t \in U$. We prove the following claims.

Claim 5.4. Let $\pi_{s, t}[u, v]$ be a maximal subpath of $\pi_{s, t}$ consisting of only vertices covered by $\mathcal{B}_{1}$, such that $u \neq s$ and $v \neq t$. Then both $u, v$ are covered at boundary by $\mathcal{B}_{1}$.

Proof. Assume for contradiction that $u$ belongs to a ball $\mathrm{B}(c, d-1)$ for some small ball $\mathrm{B}(c, r) \in \mathcal{B}$. Let ( $w, u$ ) be the last edge of subpath $\pi_{s, t}[s, u]$, then $w \in \mathrm{~B}(c, d)$, and so $w$ is also covered. Therefore, $\pi_{s, t}[w, v]$ should be a covered subpath, which contradicts the maximality of subpath $\pi_{s, t}[u, v]$.

Claim 5.5. For any large ball $\mathrm{B}(c, r)$ and every vertex $s \in U_{c}$, $\operatorname{dist}_{H}(s, c) \leq \operatorname{dist}_{G}(s, c)+10 R n^{\varepsilon}$.
Proof. Let $\pi_{s, t}$ be the shortest path that was processed in the iteration where $s$ was added to $U_{c}$. By the algorithm, $\mathrm{B}(c, r)$ must be referring to some ball $\mathrm{B}\left(c_{x}, r_{x}\right)$ for some $x \in \pi_{s, t}$. As $H$ contains a BFS tree that is rooted at $c$ and contains $x$, and the radius of each ball is at most $2 R n^{\varepsilon}$, it suffices to show that $\operatorname{dist}_{H}(s, x) \leq \operatorname{dist}_{G}(s, x)+8 R n^{\varepsilon}$.

Recall that path $\pi_{s, t}$ was partitioned into sub-paths in sets $\Sigma$ and $\Sigma^{\prime}$. From the algorithm description, all subpaths in $\Sigma^{\prime}$ are entirely contained in $H$. For every subpath $\pi_{s, t}[y, z]$ of $\pi_{s, t}$ in $\Sigma$, if $y \neq s$, then from Claim 5.2 and Claim 5.4, $\operatorname{dist}_{H}(y, z)=\operatorname{dist}_{G}(y, z)$; if $y=s$, then from Claim 5.2, we get that $\operatorname{dist}_{H}(y, z) \leq \operatorname{dist}_{G}(y, z)+8 R n^{\varepsilon}$. Since there is at most one subpath $\pi_{s, t}[y, z]$ in $\Sigma$ with $y=s$, the overall additive error caused by subpaths in $\Sigma$ is at most $8 R n^{\varepsilon}$.

Claim 5.6 (additive error). For every pair $s, t \in U$, $\operatorname{dist}_{H}(s, t) \leq \operatorname{dist}_{G}(s, t)+24 R n^{\varepsilon}$.
Proof. Consider the iteration when $\pi_{s, t}$ was processed. If $\pi_{s, t}$ is entirely covered by $\mathcal{B}_{1}$, then the claim follows from Claim 5.2. If not, then from the algorithm, after this iteration some large ball $\mathrm{B}(c, r) \in \mathcal{B}$ intersecting $\pi_{s, t}$ must have added $s, t$ into its set $U_{c}$. Then from Claim 5.5,

$$
\begin{aligned}
\operatorname{dist}_{H}(s, c) & \leq \operatorname{dist}_{G}(s, c)+10 R n^{\varepsilon} \\
\operatorname{dist}_{H}(c, t) & \leq \operatorname{dist}_{G}(c, t)+10 R n^{\varepsilon} .
\end{aligned}
$$

Let $v$ be an arbitrary vertex in $\mathrm{B}(c, r) \cap V\left(\pi_{s, t}\right)$. By triangle inequality,

$$
\operatorname{dist}_{G}(s, c)+\operatorname{dist}_{G}(c, t) \leq \operatorname{dist}_{G}(s, v)+\operatorname{dist}_{G}(v, t)+2 \cdot \operatorname{dist}(v, c) \leq \operatorname{dist}_{G}(s, t)+4 R n^{\varepsilon} .
$$

Altogether, $\operatorname{dist}_{H}(s, t) \leq \operatorname{dist}_{G}(s, t)+24 R n^{\varepsilon}$.

### 5.3 Size and runtime analysis

Claim 5.7 (spanner size). $|E(H)|=2^{O(1 / \varepsilon)} \cdot n$.
Proof. It suffices to bound the total number of edges added to $H$ when handling large balls. For each large ball $\mathrm{B}(c, r) \in \mathcal{B}$, let us conceptually construct a set $\Pi_{c}$ of paths within $G[\mathrm{~B}(c, 4 r)]$ during handling large balls.

Initially, all sets $\Pi_{c}$ are empty. When processing a path $\pi_{s, t} \in \Pi$, suppose Step (3) is executed. Consider the set of large balls $\left\{\mathrm{B}\left(c_{x}, r_{x}\right) \mid x \in \pi_{s, t}\right\}$. For any ball $\mathrm{B}(c, r) \in\left\{\mathrm{B}\left(c_{x}, r_{x}\right) \mid x \in \pi_{s, t}\right\}$, let $y, z \in \mathrm{~B}(c, r+1) \cap \pi_{s, t}$ be the first and the last vertex on $\pi_{s, t}$. Then, add the sub-path $\pi_{s, t}[y, z]$ to $\Pi_{c}$. Note that this path $\pi_{s, t}$ lies entirely in $G[\mathrm{~B}(c, 4 r)]$.

We first show that in the end, all edges added to $H$ for handling large balls is a subset of $E\left(\bigcup_{\mathrm{B}(c, r) \in \mathcal{B}}\right.$ is large $\left.\Pi_{c}\right)$. In fact, When processing a path $\pi_{s, t} \in \Pi$, consider any vertex $x$ on a sub-path $\rho \in \Sigma^{\prime}$ which is not an endpoint of $\rho$. Then, since $x$ is not covered by $\mathcal{B}_{1}$, any ball that covers $x$ must be large, and in particular $\mathrm{B}\left(c_{x}, r_{x}\right)$ should be large. Suppose $y, z \in \mathrm{~B}\left(c_{x}, r_{x}+1\right)$ are the first and the last vertex on $\pi_{s, t}$. Then, $\pi_{s, t}[y, z]$ must contain all edges incident on $x$ on $\rho$. Hence, the union of all $\pi_{s, t}[y, z]$ ranging over all $x$ 's should contain all paths in $\Sigma^{\prime}$.

Next, we show that $\left|\Pi_{c}\right| \leq|U|$ for any $c$. In fact, each time we added a new path to $\Pi_{c}$, we must have added $s, t$ to $U_{c}$ which increased $\left|U_{c}\right|$. Since $U_{c} \subseteq U$, we know that $\left|\Pi_{c}\right| \leq|U|$ in the end.

Finally, it suffices to bound the number of edges in $E\left(\bigcup_{B(c, r) \in \mathcal{B}}\right.$ is large $\left.\Pi_{c}\right)$. Using Lemma 2.3, we have:

$$
E\left(\Pi_{c}\right) \leq O\left(|\mathrm{~B}(c, 4 r)|+\sqrt{\mathrm{B}(c, 4 r)} \cdot\left|\Pi_{c}\right|\right) \leq O(|\mathrm{~B}(c, 4 r)|) .
$$

Hence,

$$
E\left(\bigcup_{\mathrm{B}(c, r) \in \mathcal{B} \text { is large }} \Pi_{c}\right) \leq O\left(\sum_{\mathrm{B}(c, r) \in \mathcal{B} \text { is large }}|\mathrm{B}(c, 4 r)|\right)=2^{O(1 / \varepsilon)} n .
$$

Claim 5.8 (runtime). The runtime of the algorithm is $O\left(m\left(|U|+2^{O(1 / \varepsilon)}\right)\right)$.
Proof. From Lemma 2.2, computing all the balls in $\mathcal{B}$ takes time at most $O\left(2^{10 / \varepsilon} m / \varepsilon\right)$. From Corollary 2.4, constructing graphs $L_{c}$ in all small balls takes time at most $O\left(|U|^{1 / 2} \sum_{c} \operatorname{vol}_{G}(\mathrm{~B}(c, 4 r))\right)=$ $O\left(|U|^{1 / 2} m \cdot 2^{10 / \varepsilon} / \varepsilon\right)$, and computing the paths in $\Pi$ takes time $O(m|U|)$. As for the part with large balls, the runtime is dominated by scanning all shortest paths $\pi_{s, t}$ in $\Pi$, which takes time at most $O(n|U|)$. Overall, the runtime is $O\left(m\left(|U|+2^{O(1 / \varepsilon)}\right)\right)$.

## 6 Linear-Size Additive Spanners in Sub-quadratic Time

In this section we provide the proof of Theorem 1.2, We will first show in Section 6.1 a simple subquadratic time algorithm for computing an $+O\left(n^{3 / 7+\varepsilon}\right)$-additive spanner. Then we will slightly modify it to achieve better additive error bound in Section 6.2.

### 6.1 A subquadratic time algorithm for $O\left(n^{3 / 7+\varepsilon}\right)$-additive spanner

In this subsection, we prove the following theorem, which shows that an additive spanner with the current best error (as in BW21) can be computed in subquadratic time.

Theorem 6.1. There is an algorithm, that, given any undirected unweighted graph on $n$ vertices and $m$ edges, and any parameter $\varepsilon>0$, computes an spanner on $2^{O(1 / \varepsilon)} \cdot n$ edges with $+\tilde{O}\left(n^{3 / 7+\varepsilon}\right)$ error, in time $\tilde{O}\left(m+2^{O(1 / \varepsilon)} \cdot n^{13 / 7}\right)$.

We start with the following lemma for a preliminary sparsification of the input graph.
Lemma 6.2. There is an algorithm, that given any graph $G$ and parameter $1<d<n$, computes in $\tilde{O}(m)$ time an $+\tilde{O}(n / d)$ additive spanner $G^{\prime}$ of $G$ with $\left|E\left(G^{\prime}\right)\right| \leq O(n d)$ edges.

Proof. Graph $G^{\prime}$ is simply the union of (i) for each vertex $v \in V(G)$ with $\operatorname{deg}_{G}(v) \leq\lceil d\rceil$, all incident edges of $v$; and (ii) an $O(\log n)$-multiplicative spanner of $G$ with $O(n)$ edges; such a multiplicative spanner can be computed in $\tilde{O}(m)$ time as shown in BS07. Clearly, graph $G^{\prime}$ can be computed in $\tilde{O}(m)$ time.

Consider now any pair $s, t \in V(G)$ and let $\pi$ be an $s-t$ shortest path in $G$. It is easy to observe that at most $O(n / d)$ vertices in $\pi$ have degree more than $d$, so the number of edges in $E(\pi)$ that is incident to any degree $\leq d$ vertex is at most $O(n / d)$. As we have included in $G^{\prime}$ an $O(\log n)$ multiplicative spanner of $G$, the distance in $G^{\prime}$ between the pair of endpoints of every such edge is at most $O(\log n)$. Therefore, $\operatorname{dist}_{G^{\prime}}(s, t) \leq \operatorname{dist}_{G}(s, t)+\tilde{O}(n / d)$.

We now proceed to describe our algorithm for Lemma 6.9. Similar to Section [5, we assume that the (unit) edge weights are slightly perturbed so that for every pair $s, t$ there is a unique shortest path connecting them in $G$ (or any subgraph that contains both $s$ and $t$ ).

As a pre-processing step, if $|E(G)| \geq 10 \cdot n^{2-3 / 7}$, then we first apply the algorithm from Lemma 6.2 to $G$ with $d=n^{1-3 / 7}$, and get graph $G^{\prime}$, so $\left|E\left(G^{\prime}\right)\right|=O\left(n^{2-3 / 7}\right)$. If $|E(G)| \leq 10 \cdot n^{2-3 / 7}$, then we simply set $G^{\prime}=G$.

We then apply the algorithm from Lemma 2.2 to $G^{\prime}$ with parameter $R$ (to be determined later) and $\frac{10}{\varepsilon \log _{2} n}$, and let $\mathcal{B}$ be the set of balls we obtain. We say that a ball $\mathrm{B}(c, r) \in \mathcal{B}$ is small iff $|\mathrm{B}(c, r)| \leq R^{5 / 3}$; otherwise we say it is large. For each ball $\mathrm{B}(c, r) \in \mathcal{B}$, we compute a BFS tree $T_{c}$ that is rooted at $c$ and spans all vertices in $\mathrm{B}(c, 4 r)$, in time $O(\operatorname{vol}(\mathrm{~B}(c, 4 r)))$. From Lemma 2.2, $\sum_{c}\left|E\left(T_{c}\right)\right|=\sum_{c}|\mathrm{~B}(c, 4 r)|=O\left(2^{10 / \varepsilon} n / \varepsilon\right)$.

Handling small balls. Consider a small ball $\mathrm{B}(c, r) \in \mathcal{B}$. From Lemma 2.1, we can compute in time $O(\operatorname{vol}(\mathrm{~B}(c, 4 r)))$ an integer $d \in[r, 2 r]$ such that $\left|\mathrm{B}^{=}(c, d) \cup \mathrm{B}^{=}(c, d+1)\right| \leq 2|\mathrm{~B}(c, 4 r)| / r \leq$ $2|\mathrm{~B}(c, 4 r)| / R$. We denote $\mathcal{B}_{1}=\{\mathrm{B}(c, d) \mid \mathrm{B}(c, r) \in \mathcal{B}$ is small $\}$. Then, for each small ball $\mathrm{B}(c, r)$, apply Lemma 5.1] to compute a subset spanner $L_{c}$ of $G^{\prime}[\mathrm{B}(c, 4 r)]$ on the set $\mathrm{B}^{=}(c, d) \cup \mathrm{B}^{=}(c, d+1)$.

Handling large balls. Choose a random subset $S \subseteq V(G)$ of size $\left\lceil 10 R^{2 / 3} \log n\right\rceil$. We then apply Lemma 5.1 to compute a subset spanner $\hat{H}$ of $G^{\prime}$ on $S$.

The spanner construction. The output graph $H$ is simply defined to be the union of

- for each ball $\mathrm{B}(c, r) \in \mathcal{B}$, the tree $T_{c}$;
- for each small ball $\mathrm{B}(c, r) \in \mathcal{B}$, graph $L_{c}$;
- graph $\hat{H}$.


### 6.1.1 Stretch analysis

According to Lemma 6.2, in order to show that $H$ is a $+\tilde{O}\left(n^{3 / 7+\varepsilon}\right)$ spanner of $G$, it suffices to show that $H$ is a $+\tilde{O}\left(n^{3 / 7+\varepsilon}\right)$ spanner of $G^{\prime}$. For notational convenience, in this subsection we rename $G^{\prime}$ by $G$. The following statement, which is a generalization of Claim 5.2, is the key to the stretch analysis.

Claim 6.3. Let $s, t$ be a pair of vertices in $G$ and let $\pi$ be a shortest path connecting them. Then there exists (i) a path $\phi$ in $H$ connecting sto $t$; (ii) a sequence of balls $\mathrm{B}\left(c_{1}, r_{1}\right), \ldots, \mathrm{B}\left(c_{l}, r_{l}\right)$; (iiii) two sequence of vertices $\left(u_{0}=s, u_{1}, \ldots, u_{l}=t\right)$ and $\left(v_{0}=s, v_{1}, \ldots, v_{l-1}\right)$, and two sequence of paths $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{l}, \beta_{1}, \beta_{2}, \ldots, \beta_{l}$, with the following properties.
(a) Vertices $u_{1}, u_{2}, \ldots, u_{l}$ appear on path $\pi$ in this order in the direction from sto to
(b) $s \in \mathrm{~B}\left(c_{1}, d_{1}\right), t \in \mathrm{~B}\left(c_{l}, d_{l}+1\right)$, and $v_{i} \in \mathrm{~B}^{=}\left(c_{i+1}, d_{i+1}\right), u_{i} \in \mathrm{~B}^{=}\left(c_{i}, d_{i}+1\right), \forall 1 \leq i \leq l-1$.
(c) For each $1 \leq i \leq l, \alpha_{i}$ is a shortest path in $H$ connecting $v_{i-1}$ to $u_{i}$ that is entirely contained in $G\left[\mathrm{~B}\left(c_{i}, 4 r_{i}\right)\right]$, and moreover, vertex $v_{i}$ lies on $\alpha_{i}$.
(d) For each $1 \leq i \leq l-1, \beta_{i}=\alpha_{i}\left[v_{i-1}, v_{i}\right]$; and $\beta_{l}=\alpha_{l}$.
(e) $\phi=\beta_{1} \circ \cdots \circ \beta_{l} ;$ and $|\phi| \leq \operatorname{dist}_{G}(s, t)+15 \cdot R n^{\varepsilon}+\tilde{O}\left(2^{15 / \varepsilon} \cdot n^{\varepsilon} \cdot \sum_{i=2}^{l-1} \frac{\left|\mathrm{~B}\left(c_{i}, r_{i}\right)\right|}{R^{2 / 3}}\right)$.

Proof. We start with $i=0$. Before iteration $i$, suppose we have already computed vertices $u_{1}, u_{2}, \ldots, u_{i}$, and $v_{1}, v_{2}, \ldots, v_{i-1}$, and paths $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{i}$, and paths $\beta_{1}, \beta_{2}, \ldots, \beta_{i-1}$. During the algorithm, keep $\phi=\beta_{1} \circ \beta_{2} \cdots \circ \beta_{i-1} \circ \alpha_{i}$. So $\phi$ is a path in $H$ from $s$ to $u_{i}$.
(1) If $u_{i}=t$ then we terminate the process and let $l=i$.

Otherwise, find the ball in $\mathcal{B}_{1}$ that, among all balls in $\mathcal{B}_{1}$ that intersects with $\phi$, the one that contains a vertex on $\pi$ that is closest to $t$. Note that this ball always exists; for example we can pick an arbitrary one that contains $u_{i}$.
We denote this ball by $\mathrm{B}\left(c_{i+1}, d_{i+1}\right)$ and set $u_{i+1}$ as the last vertex of $\pi$ that belongs to $\mathrm{B}\left(c_{i+1}, d_{i+1}+1\right)$.
(2) Next, if $i=0$, then let $\alpha_{1}$ be the shortest path in $H$ from $s=v_{0}$ to $u_{1}$.

If $i \geq 1$, we then let $v_{i}$ be any vertex that belongs to both $\alpha_{i}$ and $\mathrm{B}=\left(c_{i+1}, d_{i+1}\right)$, and let $\alpha_{i+1}$ be the shortest path from $v_{i}$ to $u_{i+1}$ in $H$. Then, define $\alpha_{i+1}$ to be the shortest path from $v_{i}$ to $u_{i+1}$, and $\beta_{i}=\alpha_{i}\left[v_{i-1}, v_{i}\right]$. We will prove shortly the existence of $v_{i}$.
Finally, increase $i \leftarrow i+1$ and go to Step (1).
We now argue the existence of $v_{i}$.


Figure 8: The construction of vertex sequences $u_{1}, u_{2}, \ldots, u_{l}$ and $v_{1}, v_{2}, \ldots, v_{l-1}$; the orange paths belong to the spanner $H$.

Claim 6.4. For each $i \geq 1$, $\phi$ intersects with $\mathrm{B}^{=}\left(c_{i+1}, d_{i+1}\right)$, and the intersection point $v_{i}$ must belong to $\alpha_{i}$. Also, if $u_{i+1} \neq t$, then $u_{i+1} \in \mathrm{~B}^{=}\left(c_{i+1}, d_{i+1}+1\right)$.

Proof of Claim 6.4. First, if $\phi$ does not intersect $\mathrm{B}=\left(c_{i+1}, d_{i+1}\right)$, then as $\phi$ already intersects with $\mathrm{B}\left(c_{i+1}, d_{i+1}\right)$, the path should lie entirely within $\mathrm{B}\left(c_{i+1}, d_{i+1}\right)$, a contradiction with the choice of $\mathrm{B}\left(c_{1}, d_{1}\right)$ and $u_{1}$. Therefore, there exists a vertex $v_{i}$ which is an intersection point of $\phi$ and $\mathrm{B}^{=}\left(c_{i+1}, d_{i+1}\right)$.

Second, if $v_{i}$ does not belong to $\alpha_{i}$ but instead belongs to sub-path $\beta_{j}$ for some index $j<i$, then in an earlier iteration when we were determining $u_{j+1}, u_{j+1}$ should be at least as close to $t$ as $u_{i+1}$, which contradicts the fact that $u_{i+1} \in \pi\left(u_{j+1}, t\right]$.

Third, if $u_{i+1} \neq t$, while $u_{i+1} \in \mathrm{~B}\left(c_{i+1}, d_{i+1}\right)$, then the next vertex of $\pi$ should also belong to $\mathrm{B}\left(c_{i+1}, d_{i+1}+1\right)$, a contradiction to the fact that $u_{i+1}$ is the closest vertex to $t$.

It is easy to verify that properties $(a)-(d)$ hold from the above iterative process. It remains to prove property ( $e$ ), which we do in the next claim.
Claim 6.5. For each $2 \leq i \leq l-1,\left|\beta_{i}\right| \leq 5 R n^{\varepsilon}$, and

$$
\left|\beta_{i}\right| \leq\left|\pi\left[u_{i-1}, u_{i}\right]\right|+\left|\alpha_{i-1}\left[v_{i-1}, u_{i-1}\right]\right|-\left|\alpha_{i}\left[v_{i}, u_{i}\right]\right|+\tilde{O}\left(2^{15 / \varepsilon} \cdot n^{\varepsilon} \cdot \frac{\left|\mathrm{B}\left(c_{i}, r_{i}\right)\right|}{R^{2 / 3}}\right) .
$$

Proof of Claim 6.5. As the radius of $G\left[\mathrm{~B}\left(c_{i}, r_{i}\right)\right]$ is at most $R n^{\varepsilon}$, $\operatorname{dist}_{H}\left(v_{i-1}, u_{i}\right) \leq 5 R n^{\varepsilon}$.
From the iterative process, $v_{i-1}, u_{i} \in \mathrm{~B}^{=}\left(c_{i}, d_{i}\right) \cup \mathrm{B}^{=}\left(c_{i}, d_{i}+1\right)$. If the ball $\mathrm{B}\left(c_{i}, r_{i}\right)$ is small,
then by the construction of subset spanners,

$$
\begin{aligned}
\operatorname{dist}_{H}\left(v_{i-1}, u_{i}\right) & \leq \operatorname{dist}_{G}\left(v_{i-1}, u_{i}\right)+\tilde{O}\left(\left(\frac{\left|\mathrm{~B}\left(c_{i}, 4 r_{i}\right)\right|}{R}\right)^{3 / 2} \cdot n^{\varepsilon}\right) \\
& \leq \operatorname{dist}_{G}\left(v_{i-1}, u_{i}\right)+\tilde{O}\left(\frac{\left|\mathrm{~B}\left(c_{i}, r_{i}\right)\right|}{R^{2 / 3}} \cdot 2^{15 / \varepsilon} \cdot n^{\varepsilon}\right)
\end{aligned}
$$

If $\mathrm{B}\left(c_{i}, r_{i}\right)$ is large, then by definition, $\left|\mathrm{B}\left(c_{i}, r_{i}\right)\right| \geq R^{5 / 3}$, and therefore,

$$
\operatorname{dist}_{H}\left(v_{i-1}, u_{i}\right) \leq 5 R n^{\varepsilon} \leq \tilde{O}\left(\frac{\left|\mathrm{~B}\left(c_{i}, r_{i}\right)\right|}{R^{2 / 3}} \cdot n^{\varepsilon}\right)
$$

Finally, as $\operatorname{dist}_{H}\left(v_{i-1}, u_{i}\right)=\left|\beta_{i}\right|+\left|\alpha_{i}\left[v_{i}, u_{i}\right]\right|$, and dist ${ }_{G}\left(v_{i-1}, u_{i}\right) \leq\left|\pi\left[u_{i-1}, u_{i}\right]\right|+\left|\alpha_{i-1}\left[v_{i-1}, u_{i-1}\right]\right|$. The claim now follows by rearranging the terms yields the inequality.

Summing all $2 \leq i \leq l-1$ for the above claim, and note that $\left|\beta_{0}\right|,\left|\beta_{l}\right| \leq 5 R n^{\varepsilon},\left|\alpha_{1}\right| \leq 5 \cdot 5 R n^{\varepsilon}$. Property ( $e$ ) now follows.

Eventually, we are ready to analyze the additive error of any pairs of vertices in $V$.
Claim 6.6 (additive error). For every $s, t \in V$, $\operatorname{dist}_{H}(s, t) \leq \operatorname{dist}_{G}(s, t)+\tilde{O}\left(R n^{\varepsilon}\right)+\tilde{O}\left(n^{1+\varepsilon} / R^{4 / 3}\right)$.
Proof. Let $\pi$ be a shortest path between $s, t$ in $G$. We apply the algorithm from Claim 6.3 to $\pi$, and obtain a path $\phi$ as well as all the other auxiliary sequences, such that:

$$
|\phi| \leq \operatorname{dist}_{G}(s, t)+15 \cdot R n^{\varepsilon}+\tilde{O}\left(2^{15 / \varepsilon} \cdot n^{\varepsilon} \cdot \sum_{i=2}^{l-1} \frac{\left|\mathrm{~B}\left(c_{i}, r_{i}\right)\right|}{R^{2 / 3}}\right) .
$$

If $\sum_{i=1}^{l}\left|\mathrm{~B}\left(c_{i}, r_{i}\right)\right| \leq n / R^{2 / 3}$, then we are done. Otherwise, let $a$ be the smallest index such that $\sum_{i=1}^{a}\left|\mathrm{~B}\left(c_{i}, r_{i}\right)\right|>n / R^{2 / 3}$, and let $b$ be the largest index such that $\sum_{i=b}^{l}\left|\mathrm{~B}\left(c_{i}, r_{i}\right)\right|>n / R^{2 / 3}$. Then, by construction of $S$, with high probability, there exists indices $1 \leq x \leq a, b \leq y \leq l$ such that $\mathrm{B}\left(c_{x}, r_{x}\right) \cap S \neq \emptyset, \mathrm{B}\left(c_{y}, r_{y}\right) \cap S \neq \emptyset$; we can assume $x \leq y$ by selecting the smallest choice of $x$ and the largest choice of $y$. Take two vertices $s_{1} \in \mathrm{~B}\left(c_{x}, r_{x}\right) \cap S$ and $s_{2} \in \mathrm{~B}\left(c_{y}, r_{y}\right) \cap S$. Since $H$ contains a subset spanner $\hat{H}$ on $S$,

$$
\begin{aligned}
\operatorname{dist}_{H}\left(s_{1}, s_{2}\right) & \leq \operatorname{dist}_{G}\left(s_{1}, s_{2}\right)+\tilde{O}\left(|S|^{3 / 2} n^{\varepsilon}\right) \leq \operatorname{dist}_{G}\left(s_{1}, s_{2}\right)+\tilde{O}\left(R n^{\varepsilon}\right) \\
& \leq \operatorname{dist}_{G}\left(u_{x}, u_{y}\right)+\operatorname{dist}_{G}\left(u_{x}, s_{1}\right)+\operatorname{dist}_{G}\left(u_{y}, s_{2}\right)+\tilde{O}\left(R n^{\varepsilon}\right) \\
& \leq \operatorname{dist}_{G}\left(u_{x}, u_{y}\right)+\tilde{O}\left(R n^{\varepsilon}\right)
\end{aligned}
$$

Using similar arguments in the proof of Claim 6.3, we can show that:

$$
\begin{aligned}
& \operatorname{dist}_{H}\left(s, u_{x}\right) \leq \operatorname{dist}_{G}\left(s, u_{x}\right)+15 \cdot R n^{\varepsilon}+\left(2^{15 / \varepsilon} \cdot n^{\varepsilon} \cdot \sum_{i=2}^{x-1} \frac{\left|\mathrm{~B}\left(c_{i}, r_{i}\right)\right|}{R^{2 / 3}}\right) \\
& \operatorname{dist}_{H}\left(u_{y}, t\right) \leq \operatorname{dist}_{G}\left(u_{y}, t\right)+15 \cdot R n^{\varepsilon}+\left(2^{15 / \varepsilon} \cdot n^{\varepsilon} \cdot \sum_{i=y+1}^{l-1} \frac{\left|\mathrm{~B}\left(c_{i}, r_{i}\right)\right|}{R^{2 / 3}}\right)
\end{aligned}
$$

Summing over all three inequalities completes the proof.
Setting $R=\left\lceil n^{3 / 7}\right\rceil$, the above claim implies that the additive error of $H$ is $\tilde{O}\left(n^{3 / 7+\varepsilon}\right)$.

### 6.1.2 Size and runtime analysis

Claim 6.7 (spanner size). $|E(H)|=2^{O(1 / \varepsilon)} \cdot n$.
Proof. From Lemma [5.1, each subset spanner $L_{c}$ within $G[\mathrm{~B}(c, 4 r)]$ contains $2^{O(1 / \varepsilon)}|\mathrm{B}(c, 4 r)|$ edges, and so $\sum_{c}\left|E\left(L_{c}\right)\right|=2^{O(1 / \varepsilon)} \cdot 2^{O(1 / \varepsilon)} n=2^{O(1 / \varepsilon)} n$. Similarly, the glocal subset spanner $\hat{H}$ satisfies that $|E(\hat{H})|=2^{O(1 / \varepsilon)} n$. Additionally, $\sum_{c}\left|E\left(T_{c}\right)\right|=\sum_{c}|\mathrm{~B}(c, 4 r)|=2^{O(1 / \varepsilon)} n$ edges. Altogether, we get that $|E(H)|=2^{O(1 / \varepsilon)} \cdot n$.
Claim 6.8 (runtime). The runtime of the algorithm is $\tilde{O}\left(m+\left|E\left(G^{\prime}\right)\right| \cdot 2^{O(1 / \varepsilon)} \cdot R^{2 / 3}\right)$.
Proof. The runtime for computing $G^{\prime}$ is $\tilde{O}(m)$. From Lemma 2.2, the set $\mathcal{B}$ of balls can be computed in time $2^{O(1 / \varepsilon)} \cdot\left|E\left(G^{\prime}\right)\right|$. The runtime for computing BFS trees within balls in $\mathcal{B}$ is $2^{O(1 / \varepsilon)} \cdot\left|E\left(G^{\prime}\right)\right|$. From Lemma 5.1 the runtime for computing subset spanners within small balls is at most

$$
\begin{aligned}
\sum_{c} O\left(\operatorname{vol}_{G^{\prime}}(\mathrm{B}(c, 4 r)) \cdot\left(2^{O(1 / \varepsilon)}+\frac{2|\mathrm{~B}(c, 4 r)|}{R}\right)\right) & \leq \sum_{c} O\left(\operatorname{vol}_{G^{\prime}}(\mathrm{B}(c, 4 r)) \cdot 2^{O(1 / \varepsilon)} \cdot R^{2 / 3}\right) \\
& \leq m \cdot 2^{O(1 / \varepsilon)} \cdot R^{2 / 3}
\end{aligned}
$$

The claim now follows.
Since we set $R=\left\lceil n^{3 / 7}\right\rceil$, the above claim implies that runtime of the algorithm for computing $H$ is $\tilde{O}\left(m+2^{O(1 / \varepsilon)} \cdot n^{13 / 7}\right)$, as $\left|E\left(G^{\prime}\right)\right|=O\left(n^{11 / 7}\right)$.

### 6.2 Completing the proof of Theorem 1.2

In this subsection, we completing the proof of Theorem 1.2 by slightly modifying the algorithm in Section 6.1 and apply them recursively. Specifically, we first prove the following lemma.
Lemma 6.9. Let $f(\rho)=\frac{2 / 3-\rho}{4-(19 / 6) \rho}$ be a function. If there is an algorithm Alg, that given any graph $G$ on $n$ vertices and $m$ edges, in time $\tilde{O}\left(m+n^{\gamma}\right)$ computes an $+O\left(n^{\rho}\right)$ additive spanner of $G$ with at most Cn edges, such that $\gamma \geq 1+\frac{(3 / 2) f(\rho)(1-\rho)}{3 / 2-\rho}$; then for any parameter $\varepsilon>0$, there is an algorithm $\mathrm{Alg}^{\prime}$, that given any graph $G^{\prime}$ on $n$ vertices and $m$ edges, in time $\tilde{O}\left(m+2^{O(\gamma / \varepsilon)} n^{\gamma}\right)$ computes an $+O\left(n^{\varepsilon+f(\rho)}\right)$ additive spanner of $G^{\prime}$ with $2^{O(1 / \varepsilon)} \cdot$ Cn edges.

We now use Lemma 6.9 to prove Theorem 1.2, We set $\varepsilon>0$ as a small enough constant. Note that Theorem 6.1 in fact gives an algorithm with parameter $(\gamma=13 / 7, \rho=3 / 7+0.1, C=O(1))$ and we denote it by $\mathrm{Alg}_{0}$. We then apply Lemma 6.9 with $\mathrm{Alg}=\mathrm{Alg}_{0}$, and denote by $\mathrm{Alg}_{1}$ the algorithm that it produces, so $\mathrm{Alg}_{1}$ has produce an $+O\left(n^{\varepsilon+f(\rho)}\right)$ additive spanner. Note that the invariant point $\rho^{*}$ of the mapping $f$ (i.e., the value of $0<\rho^{*}<1$ such that $\left.f\left(\rho^{*}\right)=\rho^{*}\right)$ is $\rho^{*}=\frac{15-\sqrt{54}}{19}=0.4027 \ldots$. We then iteratively apply Lemma 6.9 for $K$ times (where $K$ is a large enough constant such that $f(f(\cdots f(3 / 7+0.1+\varepsilon) \cdots)+\varepsilon)+\varepsilon<0.403)$, and get algorithms $\operatorname{Alg}_{2}, \ldots, \operatorname{Alg}_{K}$. It is not hard to verify that the property $\gamma \geq 1+\frac{(3 / 2) f(\rho)(1-\rho)}{3 / 2-\rho}$ always holds. Eventually, $\mathrm{Alg}_{K}$ is the algorithm that we return. Note that the additive error of the spanner it produces is $+O\left(n^{0.403}\right)$, the running time is $\tilde{O}\left(m+n^{13 / 7} \cdot 2^{O((13 / 7) \cdot(K / \varepsilon))}\right)$, which is $\tilde{O}\left(m+n^{13 / 7}\right)$ as $1 / \varepsilon$ and $N$ are both constants, and the size of the smaller it produces is $2^{O((13 / 7) \cdot(K / \varepsilon))} \cdot C n=O(n)$.

We now sketch the proof of Lemma 6.9, highlighting the difference between the algorithm here and the algorithm in Section 6.1.

Proof Sketch of Lemma 6.9. The algorithm for Lemma 6.9 is very similar to the algorithm for Theorem 6.1, except for (i) an extra Step 4 below, which is a recursive call of an additive spanner algorithm; and (ii) more fine-grained tuning of parameters. We define the function $g(\rho)=\frac{(3 / 2) \cdot f(\rho)}{(3 / 2-\rho)}$.

Step 1. Sparsify $G$ to get $G^{\prime} \subseteq G$ using Lemma 6.2 with $d=O\left(n^{1-f(\rho)}\right)$ and $\left|E\left(G^{\prime}\right)\right|=O\left(n^{2-f(\rho)}\right)$.
Step 2. Compute a set $\mathcal{B}$ of balls using Lemma 2.2 with parameters $R=\left\lceil n^{f(\rho)}\right\rceil$ and $\frac{\varepsilon}{10 \log n}$. We say that a ball is small iff $|\mathrm{B}(c, r)| \leq n^{g(\rho)}$, otherwise we say it is large.

Step 3. For each small ball $\mathrm{B}(c, r)$, we apply Lemma 2.1 to compute an integer $d \in[r, 2 r]$ such that $\left|\mathrm{B}^{=}(c, d) \cup \mathrm{B}^{=}(c, d+1)\right| \leq 2|\mathrm{~B}(c, 4 r)| / r \leq 2|\mathrm{~B}(c, 4 r)| / R$, and then apply Lemma 5.1 to compute a subset spanner $L_{c}$ of $G^{\prime}[\mathrm{B}(c, 4 r)]$ on the set $\mathrm{B}^{=}(c, d) \cup \mathrm{B}^{=}(c, d+1)$.

Step 4. For each large ball $\mathrm{B}(c, r)$, we apply the algorithm Alg to compute a spanner of $G^{\prime}[\mathrm{B}(c, 4 r)]$ with error $+O\left(|\mathrm{~B}(c, r)|^{\rho}\right)$, that we denote by $L_{c}$.

Step 5. Sample a random subset $S \subseteq V(G)$ of $\left\lceil 10 R^{2 / 3} \log n\right\rceil$ vertices, and apply Lemma 5.1 to compute a subset spanner $\hat{H}$ of $G^{\prime}$ on $S$ with additive error $O\left(|S|^{3 / 2} \cdot n^{\varepsilon}\right)$.

The output graph $H$ is simply defined to be the union of

- for each ball $\mathrm{B}(c, r) \in \mathcal{B}$, a BFS tree $T_{c}$ that is rooted at $c$ and spans all vertices in $\mathrm{B}(c, 4 r)$;
- for each small or large ball $\mathrm{B}(c, r) \in \mathcal{B}$, graph $L_{c}$;
- graph $\hat{H}$.

The analysis is almost identical to the analysis in Section 6.1, with the following changes.
Stretch analysis. In Claim 6.5, the analysis would be changed to

$$
\begin{aligned}
\operatorname{dist}_{H}\left(v_{i-1}, u_{i}\right) & \leq \operatorname{dist}_{G}\left(v_{i-1}, u_{i}\right)+\tilde{O}\left(\max \left\{\left(\frac{\left|\mathrm{~B}\left(c_{i}, 4 r_{i}\right)\right|}{n^{f(\rho)}}\right)^{3 / 2} \cdot n^{\varepsilon},\left|\mathrm{B}\left(c_{i}, 4 r_{i}\right)\right|^{\rho}\right\}\right) \\
& \leq \operatorname{dist}_{G}\left(v_{i-1}, u_{i}\right)+\tilde{O}\left(\frac{\left|\mathrm{~B}\left(c_{i}, r_{i}\right)\right|}{n^{(1-\rho) \cdot g(\rho)}} \cdot 2^{O(1 / \varepsilon)} \cdot n^{\varepsilon}\right)
\end{aligned}
$$

Consequently, the analysis in Claim 6.6 would be changed to

$$
\begin{aligned}
\operatorname{dist}_{H}\left(s, u_{x}\right) & \leq \operatorname{dist}_{G}\left(s, u_{x}\right)+O\left(n^{\varepsilon+f(\rho)}\right)+O\left(2^{O(1 / \varepsilon)} \cdot n^{\varepsilon} \cdot \sum_{i=2}^{x-1} \frac{\left|\mathrm{~B}\left(c_{i}, r_{i}\right)\right|}{n^{(1-\rho) \cdot g(\rho)}}\right) \\
& \leq \operatorname{dist}_{G}\left(s, u_{x}\right)+O\left(n^{\varepsilon+f(\rho)}\right)+O\left(2^{O(1 / \varepsilon)} \cdot n^{\varepsilon} \cdot \frac{n^{1-\frac{2}{3} f(\rho)}}{n^{(1-\rho) \cdot g(\rho)}}\right) \\
& \leq \operatorname{dist}_{G}\left(s, u_{x}\right)+2^{O(1 / \varepsilon)} \cdot n^{\varepsilon+f(\rho)}
\end{aligned}
$$

and similarly we get that $\operatorname{dist}_{H}\left(u_{y}, t\right) \leq \operatorname{dist}_{G}\left(u_{y}, t\right)+2^{O(1 / \varepsilon)} \cdot n^{\varepsilon+f(\rho)}$. Therefore, the additive error of the algorithm is $2^{O(1 / \varepsilon)} \cdot n^{\varepsilon+f(\rho)}$.

Size and runtime analysis. Via similar arguments as in the proof of Claim 6.7, we can show that $|E(H)|=2^{O(1 / \varepsilon)} \cdot n$. We now analyze the runtime of the algorithm. Ignoring the new Step 4, we can show via similar arguments as in the proof of Claim 6.8 that the runtime is $\tilde{O}\left(m+2^{O(1 / \varepsilon)}\right.$. $\left.n^{1+\frac{(3 / 2) f(\rho)(1-\rho)}{3 / 2-\rho}}\right)$. The runtime of Step 4 is

$$
\sum_{c: \mathrm{B}(c, r) \text { large }}\left(\tilde{O}\left(\operatorname{vol}_{G}(\mathrm{~B}(c, 4 r))\right)+|\mathrm{B}(c, 4 r)|^{\gamma}\right)=\tilde{O}\left(m+2^{O(\gamma / \varepsilon)} \cdot n^{\gamma}\right) .
$$

As $\gamma \geq 1+\frac{(3 / 2) f(\rho)(1-\rho)}{3 / 2-\rho}$, the runtime of the whole algorithm is $\tilde{O}\left(m+2^{O(\gamma / \varepsilon)} \cdot n^{\gamma}\right)$.

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## A Proof of Lemma 2.2

Throughout, we use the parameter $\beta=n^{\varepsilon}$.
The algorithm iteratively builds the collection $\mathcal{B}$. Initially, $\mathcal{B}=\emptyset$. The algorithm continues to be executed as long as there exists a vertex that is not covered by the collection $\mathcal{B}$. We now describe an iteration. First, we pick an arbitrary vertex $c$ that is not covered by the current collection $\mathcal{B}$, and add a new ball $\mathrm{B}(c, r)$ to $\mathcal{B}$ centered at $c$. Its radius $r$ is determined by the following process:
(1) Start with $r=R$.
(2) Perform breath-first search from $c$ in $G$ to compute the ball $\mathrm{B}(c, 4 r)$.
(3) If $|\mathrm{B}(c, 4 r)| \leq \beta \cdot|\mathrm{B}(c, r / 2)|$ and $\operatorname{vol}(\mathrm{B}(c, 4 r)) \leq \beta \cdot \operatorname{vol}(\mathrm{B}(c, r / 2))$, then add the ball $\mathrm{B}(c, r)$ to $\mathcal{B}$, and terminate the iteration. Otherwise, update $r \leftarrow 4 r$ and repeat Steps (2) and (3).

We now proceed to analyze the algorithm. First, it is easy to see that the radius of every ball in $\mathcal{B}$ at the end of the algorithm is at least $R$, as the process of determining the radius of each new ball start with $r=R$ and only increases $r$ afterwards. Second, from the algorithm, when a new ball is added to $\mathcal{B}$, its radius $r$ is determined by an iterative process, where in each round, $r$ is increased by a factor of 4 whenever $|\mathrm{B}(c, 4 r)|>\beta \cdot|\mathrm{B}(c, r / 2)|$ or $\operatorname{vol}(\mathrm{B}(c, 4 r))>\beta \cdot|\mathrm{B}(c, r / 2)|$. As $|\mathrm{B}(c, 4 r)|$ and $\operatorname{vol}(\mathrm{B}(c, 4 r))$ are bounded by $n$ and $m$ respectively, the number of times that the radius $r$ is increased is at $\operatorname{most}\left\lceil\log _{\beta} n\right\rceil+\left\lceil\log _{\beta} m\right\rceil<5 / \varepsilon$. Therefore, in the end, $r \leq R \cdot 4^{5 / \varepsilon}=R \cdot 2^{10 / \varepsilon}$.

We next prove the following observation.

Observation A.1. At the end of the algorithm, for every vertex $v \in V$, there are at most $5 / \varepsilon$ balls $\mathrm{B}(c, r)$ in $\mathcal{B}$, such that $\operatorname{dist}_{G}(c, v) \leq r / 2$.

Proof. We first prove the following observation.
Observation A.2. When a new ball $\mathrm{B}(c, r)$ is added to the collection $\mathcal{B}$, for any other ball $\mathrm{B}\left(c^{\prime}, r^{\prime}\right)$ in $\mathcal{B}$ with $\mathrm{B}\left(c^{\prime}, r^{\prime} / 2\right) \cap \mathrm{B}(c, r / 2) \neq \emptyset, r^{\prime} \leq r / 4$ must hold.

Proof. Suppose otherwise that $r^{\prime}>r / 4$. As all radius are integral powers of $4, r^{\prime} \geq r$. Therefore, $\operatorname{dist}_{G}\left(c, c^{\prime}\right) \leq r^{\prime} / 2+r / 2 \leq r^{\prime}$, and so $c \in \mathrm{~B}\left(c^{\prime}, r^{\prime}\right)$, which means that $c$ was covered by the collection $\mathcal{B}$ before the ball $\mathrm{B}(c, r)$ is added, a contradiction.

We say that a vertex $v$ is captured by a ball $\mathrm{B}(c, r)$ iff $\operatorname{dist}_{G}(c, v) \leq r / 2$. From Observation A.2, when $v$ is captured by a new ball $\mathrm{B}(c, r)$, its radius $r$ is at least 4 times the radius of any other ball in $\mathcal{B}$ that captures $v$. As we have shown that the radius of every ball in $\mathcal{B}$ is at least $R$ and at most $R \cdot 4^{5 / \varepsilon}$, the number of balls in $\mathcal{B}$ that captures $v$ is at most $5 / \varepsilon$.

From Observation A.1, at the end of the algorithm, each vertex in $G$ is occupied by at most $O(1 / \varepsilon)$ balls in $\mathcal{B}$. Therefore, $\sum|\mathrm{B}(c, r / 2)| \leq O(n / \varepsilon) ;$ and $\sum|\mathrm{B}(c, 4 r)| \leq \beta \cdot \sum|\mathrm{B}(c, r / 2)| \leq$ $O\left(n^{1+\varepsilon} / \varepsilon\right)$. Similarly, $\sum \operatorname{vol}(\mathrm{B}(c, 4 r)) \leq \beta \cdot \sum \operatorname{vol}(\mathrm{B}(c, r / 2)) \leq \beta \cdot O(m / \varepsilon)=O\left(m \cdot n^{\varepsilon} / \varepsilon\right)$.

Finally, note that when we add a ball $\mathrm{B}(c, r)$ to $\mathcal{B}$, the running time of the algorithm in that iteration is $O(\operatorname{vol}(\mathrm{~B}(c, 4 r)))$. Therefore, algorithm terminates in time $O\left(\sum \operatorname{vol}(\mathrm{~B}(c, 4 r))\right) \leq$ $O\left(m \cdot n^{\varepsilon} / \varepsilon\right)$.


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[^1]:    ${ }^{1} O_{\delta, k}(\cdot)$ hides factors only dependent on constants $\delta, k$.

[^2]:    ${ }^{2}$ Say $\pi$ connects vertex $s$ to vertex $t$. Note that the $+O(\sqrt{n})$ spanners planted inside the balls settled a total of at least $R \cdot\left(R^{4 / 3} / R^{2 / 3}\right)$ vertices with $s$ in the balls passed through by $\pi$, which we call the prefix of $\pi$; and similarly they also settled at least $R \cdot\left(R^{4 / 3} / R^{2 / 3}\right)$ vertices with $s$ in the balls passed through by $\pi$, which we call the prefix of $\pi$. Also note that either the prefix or the suffix must be previously not settled with B and now settled with B, as otherwise the pair $(s, t)$ was already settled and we should not take $\pi$.

