# Kneser Graphs Are Hamiltonian

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## ABSTRACT

For integers $k \geq 1$ and $n \geq 2k+1$, the Kneser graph $K(n,k)$ has as vertices all $k$-element subsets of an $n$-element ground set, and an edge between any two disjoint sets. It has been conjectured since the 1970s that all Kneser graphs admit a Hamilton cycle, with one notable exception, namely the Petersen graph $K(5,2)$. This problem received considerable attention in the literature, including a recent solution for the sparsest case $n = 2k+1$. The main contribution of this paper is to prove the conjecture in full generality. We also extend this Hamiltonicity result to all connected generalized Johnson graphs (except the Petersen graph). The generalized Johnson graph $J(n,k,s)$ has as vertices all $k$-element subsets of an $n$-element ground set, and an edge between any two sets whose intersection has size exactly $s$. Clearly, we have $K(n,k) = J(n,k,0)$, i.e., generalized Johnson graphs include Kneser graphs as a special case. Our results imply that all known families of vertex-transitive graphs defined by intersecting set systems have a Hamilton cycle, which settles an interesting special case of Lovász' conjecture on Hamilton cycles in vertex-transitive graphs from 1970. Our main technical innovation is to study cycles in Kneser graphs by a kinetic system of multiple gliders that move at different speeds and that interact over time, reminiscent of the gliders in Conway's Game of Life, and to analyze this system combinatorially and via linear algebra.

## CCS CONCEPTS

• **Mathematics of computing → Combinatoric problems**; **Matchings and factors**.

## KEYWORDS

Kneser graph, Hamilton cycle, vertex-transitive graph, Lovász' conjecture, Johnson graph

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## 1 INTRODUCTION

For integers $k \geq 1$ and $n \geq 2k+1$, the *Kneser graph* $K(n,k)$ has as vertices all $k$-element subsets of $[n] := \{1, 2, \ldots, n\}$, and an edge between any two sets $A$ and $B$ that are disjoint, i.e., $A \cap B = \emptyset$. Kneser graphs were introduced by Lovász [42] in his celebrated proof of Kneser's conjecture. Using the Borsuk-Ulam theorem, he proved that the chromatic number of $K(n,k)$ equals $n-2k+2$, and his proof gave rise to the field of topological combinatorics. We proceed to list a few other important properties of Kneser graphs. The maximum independent set in $K(n,k)$ has size $\binom{n-1}{k-1}$ by the famous Erdős-Ko-Rado [22] theorem. Furthermore, the graph $K(n,k)$ is vertex-transitive, i.e., it 'looks the same' from the point of view of any vertex, and all vertices have degree $\binom{n-k}{k}$. Lastly, note that when $n < ck$, the Kneser graph $K(n,k)$ does not contain cliques of size $c$, whereas it does contain such cliques when $n \geq ck$. Many other properties of Kneser graphs have been studied, for example their diameter [61], treewidth [30], boxicity [8], and removal lemmas [24].

## 2 HAMILTON CYCLES IN KNESER GRAPHS

In this work we investigate Hamilton cycles in Kneser graphs, i.e., cycles that visit every vertex exactly once. Kneser graphs have long been conjectured to have a Hamilton cycle, with one notable exception, the Petersen graph $K(5,2)$ (see Figure 2), which only admits a Hamilton path. This conjecture goes back to the 1970s, and in the following we give a detailed account of this long history. As Kneser graphs are vertex-transitive, this is a special case of Lovász' famous conjecture [41], which asserts that every connected vertex-transitive graph admits a Hamilton path. A stronger form of the conjecture asserts that every connected vertex-transitive graph admits a Hamilton cycle, apart from five exceptional graphs, one of them being the Petersen graph. So far, the conjecture for Hamilton cycles in Kneser graphs has been tackled from two angles, namely for sufficiently dense Kneser graphs, and for the sparsest Kneser graphs. From the aforementioned results about the degree and cliques in $K(n,k)$, we see that $K(n,k)$ is relatively dense when $n$ is large w.r.t. $k$, and relatively sparse otherwise. The sparsest case is when $n = 2k+1$, and the graphs $O_k := K(2k+1,k)$ are also known as *odd graphs*. Intuitively, proving Hamiltonicity should be easier for the dense cases, and harder for the sparse cases.

We first recap the known results for dense Kneser graphs. Heinrich and Wallis [31] showed that $K(n,k)$ has a Hamilton cycle if $n \geq 2k + k/(\sqrt[k]{2}-1) = (1+o(1))k^2/\ln 2$. This was improved by B. Chen and Lih [10], whose results imply that $K(n,k)$ has a Hamilton cycle if $n \geq (1+o(1))k^2/\log k$; see [16]. In another breakthrough, Y. Chen [11] showed that $K(n,k)$ is Hamiltonian when $n \geq 3k$. A particularly nice and clean proof for the cases where $n = ck$,

$c \in \{3, 4, \ldots\}$, was obtained by Y. Chen and Füredi [13]. Their proof uses Baranyai's well-known partition theorem for complete hypergraphs [4] to partition the vertices of $K(ck, k)$ into cliques of size $c$. This proof method was extended by Bellmann and Schülke to any $n \geq 4k$ [5]. The asymptotically best result known to date, again due to Y. Chen [12], is that $K(n, k)$ has a Hamilton cycle if $n \geq (3k + 1 + \sqrt{5k^2 - 2k + 1})/2 = (1 + o(1))2.618 \ldots \cdot k$. With the help of computers, Shields and Savage [59] found Hamilton cycles in $K(n, k)$ for all $n \leq 27$ (except for the Petersen graph).

We now briefly summarize the Hamiltonicity story of the sparsest Kneser graphs, namely the odd graphs. Note that $O_k = K(2k + 1, k)$ has degree $k + 1$, which is only logarithmic in the number of vertices. The conjecture that $O_k$ has a Hamilton cycle for all $k \geq 3$ originated in the 1970s, in papers by Meredith and Lloyd [44, 45] and by Biggs [6]. Already Balaban [2] exhibited a Hamilton cycle for the cases $k = 3$ and $k = 4$, and Meredith and Lloyd described one for $k = 5$ and $k = 6$. Later, Mather [43] solved the case $k = 7$. Mütze, Nummenpalo and Walczak [51] finally settled the problem for all odd graphs, proving that $O_k$ has a Hamilton cycle for every $k \geq 3$. In fact, they even proved that $O_k$ admits double-exponentially (in $k$) many distinct Hamilton cycles. Already much earlier, Johnson [36] provided an inductive argument that establishes Hamiltonicity of $K(n, k)$ provided that the existence of Hamilton cycles is known for several smaller Kneser graphs. Combining his result with the unconditional results from [51] yields that $K(2k + 2^a, k)$ has a Hamilton cycle for all $k \geq 3$ and $a \geq 0$. These results still leave infinitely many open cases, the sparsest one of which is the family $K(2k + 3, k)$ for $k \geq 1$.

Another line of attack towards proving Hamiltonicity is to find long cycles in $K(n, k)$. To this end, Johnson [35] showed that there exists a constant $c > 0$ such that the odd graph $O_k$ has a cycle that visits at least a $(1 - c/\sqrt{k})$-fraction of all vertices, which is almost all vertices as $k$ tends to infinity. This was generalized and improved in [52], where it was shown that $K(n, k)$ has a cycle visiting a $2k/n$-fraction of all vertices. For $n = 2k + 1$ this fraction is $(1 - 1/(2k + 1))$, and more generally for $n = 2k + o(k)$ it is $1 - o(1)$.

The main contribution of this paper is to settle the conjecture on Hamilton cycles in Kneser graphs affirmatively in full generality.

Theorem 1. *For all $k \geq 1$ and $n \geq 2k+1$, the Kneser graph $K(n, k)$ has a Hamilton cycle, unless it is the Petersen graph, i.e., $(n, k) = (5, 2)$.*

In the following we present generalizations of this result that we establish in this paper, and we discuss how they extend previously known Hamiltonicity results. The relations between these results for different families of vertex-transitive graphs are illustrated in Figure 1. In fact, our proof of Theorem 1 enables us to settle all known instances of Lovász' conjecture for vertex-transitive graphs defined by intersecting set systems. As we shall see, Kneser graphs are the hardest cases among them to prove. Indeed, the more general families of graphs can be settled easily once Hamiltonicity is established for Kneser graphs.

## 3 GENERALIZED JOHNSON GRAPHS

The *generalized Johnson graph* $J(n, k, s)$ has as vertices all $k$-element subsets of $[n]$, and an edge between any two sets $A$ and $B$ that satisfy $|A \cap B| = s$, i.e., the intersection of $A$ and $B$ has size exactly $s$. To ensure that the graph is connected, we assume that $s < k$ and $n \geq 2k - s + \mathbf{1}_{[s=0]}$, where $\mathbf{1}_{[s=0]}$ denotes the indicator function that equals 1 if $s = 0$ and 0 otherwise. Generalized Johnson graphs are sometimes called 'uniform subset graphs' in the literature, and they are also vertex-transitive. Furthermore, by taking complements, we see that $J(n, k, s)$ is isomorphic to $J(n, n - k, n - 2k + s)$. Clearly, Kneser graphs are special generalized Johnson graphs obtained for $s = 0$. On the other hand, the graphs obtained for $s = k - 1$ are known as (ordinary) *Johnson graphs* $J(n, k) := J(n, k, k - 1)$.

Chen and Lih [10] conjectured that all graphs $J(n, k, s)$ admit a Hamilton cycle except the Petersen graph $J(5, 2, 0) = J(5, 3, 1)$, and this problem was reiterated in Gould's survey [26]. In their original paper, Chen and Lih settled the cases $s \in \{k-1, k-2, k-3\}$. It is known that a Hamilton cycle in the Johnson graph $J(n, k) = J(n, k, k - 1)$ can be obtained by restricting the binary reflected Gray code for bitstrings of length $n$ to those strings with Hamming weight $k$ [60]. In fact, for Johnson graphs $J(n, k)$ much stronger Hamiltonicity properties are known [34, 37]. Other properties of generalized Johnson graphs were investigated in [1, 14, 39, 62].

We generalize Theorem 1 further, by showing that all connected generalized Johnson graphs admit a Hamilton cycle. This resolves Chen and Lih's conjecture affirmatively in full generality.

Theorem 2. *For all $k \geq 1$, $0 \leq s < k$, and $n \geq 2k - s + \mathbf{1}_{[s=0]}$ the generalized Johnson graph $J(n, k, s)$ has a Hamilton cycle, unless it is the Petersen graph, i.e., $(n, k, s) \in \{(5, 2, 0), (5, 3, 1)\}$.*

## 4 GENERALIZED KNESER GRAPHS

The *generalized Kneser graph* $K(n, k, s)$ has as vertices all $k$-element subsets of $[n]$, and an edge between any two sets $A$ and $B$ that satisfy $|A \cap B| \leq s$, i.e., the intersection of $A$ and $B$ has size at most $s$. The definition is very similar to generalized Johnson graphs, only the equality condition on the size of the set intersection is replaced by an inequality. As a consequence, we clearly have $K(n, k, s) = \bigcup_{t \leq s} J(n, k, t)$, i.e., $K(n, k, s)$ has the same vertex set as $J(n, k, s)$, but more edges. In other words, $J(n, k, s)$ is a spanning subgraph of $K(n, k, s)$. Generalized Kneser graphs are also vertex-transitive, and they have been studied heavily in the literature; see e.g. [3, 15, 19, 23, 25, 33, 40, 47].

As $J(n, k, s)$ is a spanning subgraph of $K(n, k, s)$, Theorem 2 yields the following immediate corollary.

Corollary 3. *For all $k \geq 1$, $0 \leq s < k$, and $n \geq 2k - s + \mathbf{1}_{[s=0]}$ the generalized Kneser graph $K(n, k, s)$ has a Hamilton cycle, unless it is the Petersen graph, i.e., $(n, k, s) \in \{(5, 2, 0), (5, 3, 1)\}$.*

## 5 BIPARTITE KNESER GRAPHS AND THE MIDDLE LEVELS PROBLEM

For integers $k \geq 1$ and $n \geq 2k+1$, the *bipartite Kneser graph* $H(n, k)$ has as vertices all $k$-element and $(n - k)$-element subsets of $[n]$, and an edge between any two sets $A$ and $B$ that satisfy $A \subseteq B$. It is easy to see that bipartite Kneser graphs are also vertex-transitive. The following simple lemma shows that Hamiltonicity of $K(n, k)$ is harder than the Hamiltonicity of $H(n, k)$.

Lemma 4. *If $K(n, k)$ admits a Hamilton cycle, then $H(n, k)$ admits a Hamilton cycle or path.*

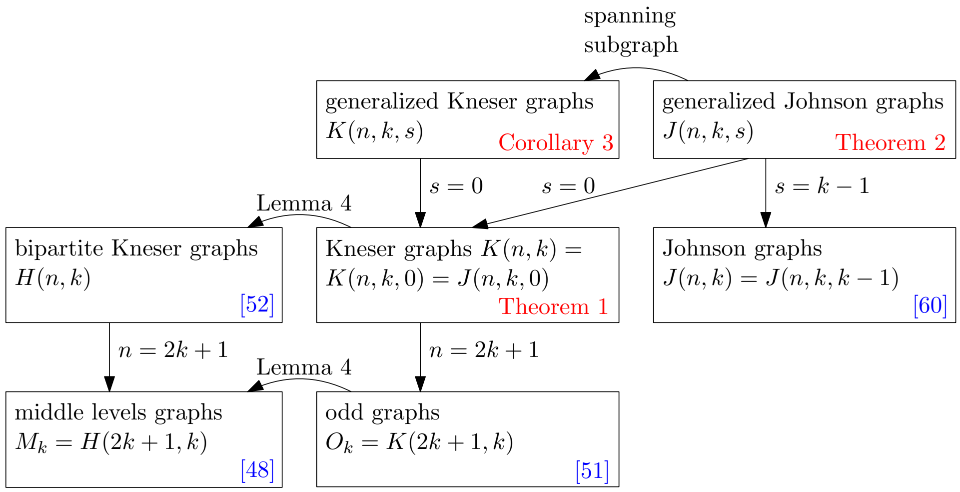


**Figure 1: Relation between Hamiltonicity results established in this paper and previous papers. Arrows indicate implications.**

Proof. Given a Hamilton cycle $C = (x_1, x_2, \ldots, x_N)$ in $K(n,k)$, the sequences $P := (x_1, \overline{x_2}, x_3, \overline{x_4}, \ldots)$ and $P' := (\overline{x_1}, x_2, \overline{x_3}, x_4, \ldots)$, where $\overline{x_i} := [n] \setminus x_i$, are two spanning paths in $H(n,k)$. Consequently, if $N = \binom{n}{k}$ is odd, then the concatenation $PP'$ is a Hamilton cycle in $H(n,k)$, and if $N$ is even, then $P$ and $P'$ are two disjoint cycles that together span the graph and that can be joined to a Hamilton path. □

The sparsest bipartite Kneser graphs $M_k := H(2k+1,k)$ are known as *middle levels graphs*, as they are isomorphic to the subgraph of the $(2k+1)$-dimensional hypercube induced by the middle two levels. The well-known *middle levels conjecture* asserts that $M_k$ has a Hamilton cycle for all $k \geq 1$. This conjecture was raised in the 1980s, settled affirmatively in [48], and a short proof was given in [29]. More generally, all bipartite Kneser graphs $H(n,k)$ were shown to have a Hamilton cycle in [52], via a short argument that uses the sparsest case $M_k$ as a basis for induction. These papers completed a long line of previous partial results on these problems; see the papers for more references and historical remarks. Via Lemma 4 and its proof shown before, our Theorem 1 thus also yields a new alternative proof for the Hamiltonicity of bipartite Kneser graphs. Consequently, our results in this paper settle Lovász' conjecture for all known families of vertex-transitive graphs that are defined by intersecting set systems.

## 6 ALGORITHMIC CONSIDERATIONS

A *combinatorial Gray code* [49, 57] is an algorithm that computes a listing of combinatorial objects such that any two consecutive objects in the list satisfy a certain adjacency condition. Many such algorithms are covered in depths in Knuth's book 'The Art of Computer Programming Vol. 4A' [38], and several of them correspond to computing a Hamilton cycle in a vertex-transitive graph, thus algorithmically solving one special case of Lovász' conjecture. For example, the classical binary reflected Gray code computes a Hamilton cycle in the $n$-dimensional hypercube, which can be seen as the Cayley graph of $\mathbb{Z}_2^n$ given by the standard generators. Another example is the well-known Steinhaus-Johnson-Trotter algorithm, which computes a Hamilton cycle in the Cayley graph of the symmetric group when the generators are adjacent transpositions. Similarly, the recent solution [58] of Nijenhuis and Wilf's sigma-tau problem [53, Ex. 6] computes a Hamilton cycle in the Cayley (di)graph of the symmetric group with the two generators being cyclic left-shift or transposition of the first two elements. Similar Gray code algorithms have been discovered for the symmetric group with other generators, such as prefix reversals [54, 63], prefix shifts [17, 18, 56], and for other groups such as the alternating group [27, 32].

Subsets of size $k$ of an $n$-element ground set are known as $(n,k)$*-combinations* in the Gray code literature. Many different algorithms have been devised for generating $(n,k)$-combinations by element exchanges, i.e., any two consecutive combinations differ in removing one element from the subset and adding another one [7, 9, 20, 21, 55, 60]. This is equivalent to saying that any two consecutive sets intersect in exactly $k-1$ elements, i.e., such a Gray code computes a Hamilton cycle in the Johnson graph $J(n,k)$.

Computing a Hamilton cycle in the Kneser graph $K(n,k)$ thus corresponds to computing a Gray code for $(n,k)$-combinations where the adjacency condition is disjointness. Our proof of the existence of a Hamilton cycle in $K(n,k)$ is constructive, and it translates straightforwardly into an algorithm for computing the cycle whose running time is polynomial in the size $N := \binom{n}{k}$ of the Kneser graph. It remains open whether there exists a more efficient algorithm, i.e., one with running time that is polynomial in $n$ and $k$ per generated combination (note that $N$ is exponential in $k$), similarly to the previously mentioned combination generation algorithms.

## 7 PROOF IDEAS

In Section 8 below we demonstrate how Theorem 1 can be used to establish Theorem 2 by a simple inductive construction. Consequently, the main work in this paper is to prove Theorem 1. In this extended abstract, we only sketch the main ideas for this proof. For details, see the full preprint version of this article [46].

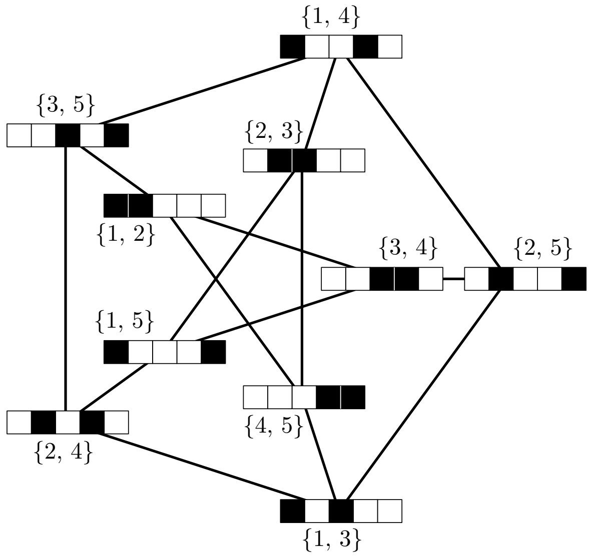


**Figure 2: The Petersen graph $K(5,2)$. The vertices are all 2-elements subsets of $[5] = \{1,2,3,4,5\}$, and in the corresponding bitstrings, 1s are represented by black squares and 0s by white squares.**

As mentioned before, Mütze, Nummenpalo and Walczak [51] proved that $K(n,k)$ has a Hamilton cycle for $n = 2k+1$ and all $k \geq 3$. Combining this result with Johnson's construction [36] shows that $K(n,k)$ has a Hamilton cycle for $n = 2k+2^a$ and all $k \geq 3$ and $a \geq 0$, in particular for $n = 2k+2$. The techniques developed in this paper work whenever $n \geq 2k+3$, and thus they settle all remaining cases of Theorem 1. It should be noted that our proof does not work in the cases $n = 2k+1$ and $n = 2k+2$, so the two earlier constructions do not become obsolete.

We follow a two-step approach to construct a Hamilton cycle in $K(n,k)$ for $n \geq 2k+3$. In the first step, we construct a *cycle factor* in the graph, i.e., a collection of disjoint cycles that together visit all vertices. In the second step, we join the cycles of the factor to a single cycle. In the following we discuss both of these steps in more detail, outlining the main obstacles and novel ingredients to overcome them.

## 7.1 Cycle Factor Construction

The starting point is to consider the characteristic vectors of the vertices of $K(n,k)$. For every $k$-element subset of $[n]$, this is a bitstring of length $n$ with exactly $k$ many 1s at the positions corresponding to the elements of the set. For example, the vertex $\{1,7,9\}$ of $K(9,3)$ is represented by the bitstring 100000101; see also Figure 2. In this figure and the following ones, 1s are often represented by black squares, and 0s by white squares. Clearly, two sets $A$ and $B$ that are vertices of $K(n,k)$ are disjoint if and only if the corresponding bitstrings have no 1s at the same positions.

Our construction of a cycle factor in the Kneser graph $K(n,k)$ uses the following simple rule based on parenthesis matching, which is a technique pioneered by Greene and Kleitman [28] (in a completely different context): Given a vertex represented by a bitstring $x$, we interpret the 1s in $x$ as opening brackets and the 0s as closing brackets, and we match closest pairs of opening and closing brackets in the natural way, which will leave some 0s unmatched. This matching is done *cyclically* across the boundary of $x$, i.e., $x$ is considered as a cyclic string. We write $f(x)$ for the vertex obtained from $x$ by complementing all matched bits, leaving the unmatched bits unchanged. For example, $x = 100000101$ is interpreted as $x = ()))))()( = ())$---$()($, where each - denotes an unmatched closing bracket, and then complementing matched bits (the first three and last three in this case) yields the vertex $f(x) = 011000010$. Repeatedly applying $f$ to every vertex partitions the vertices of the Kneser graph into cycles, and we write $C(x) := (x, f(x), f^2(x), \ldots)$ for the cycle containing $x$. For example, for $x$ from before we obtain $C(x) = (100000101,\ 011000010,\ 000110001,\ 100001100,\ 010000011,\ \ldots,\ 000011010)$. Figure 3 shows several more examples of cycles generated by this parenthesis matching rule. The reason that this rule indeed generates disjoint cycles is that $f$ is invertible and that $f(x) \neq x$ and $f^2(x) \neq x$. Indeed, $x$ is obtained from $f(x)$ by applying the same parenthesis matching procedure as before, but with interpreting the 1s as closing brackets and the 0s are opening brackets instead.

## 7.2 Analysis via Gliders

The next key step is to understand the structure of the cycles generated by $f$, as this is important for joining the cycles to a single Hamilton cycle. Unfortunately, the number of cycles and their lengths in our factor are governed by intricate number-theoretic phenomena, which we are unable to understand fully. Instead, we describe the evolution of a bitstring $x$ under repeated applications of $f$ combinatorially, which enables us to extract some important cycle properties and invariants (other than the number of cycles and the cycle lengths). Specifically, we describe this evolution by a kinetic system of multiple gliders that move at different speeds and that interact over time, reminiscent of the gliders in Conway's Game of Life. This physical interpretation and its analysis are one of the main innovations of this paper. Specifically, we view each application of $f$ as one unit of time moving forward. Furthermore, we partition the matched bits of $x$ into groups, and each of these groups is called a *glider*. A glider has a *speed* associated to it, which is given by the number of 1s in its group. As a consequence of this definition, the sum of speeds of all gliders equals $k$. For example, in the cycle shown in Figure 3 (a), there is a single matched 1 and the corresponding matched 0, and together these two bits form a glider of speed 1 that moves one step to the right in every time step. Applying $f$ means going down to the next row in the picture, so the time axis points downwards. Similarly, in Figure 3 (b), there are two matched 1s and the corresponding two matched 0s, and together these four bits form a glider of speed 2 that moves two steps to the right in every time step. As we see from these examples, a single glider of speed $v$ simply moves uniformly, following the basic physics law

$$s(t) = s(0) + v \cdot t,$$

where $t$ is the time (i.e., the number of applications of $f$) and $s(t)$ is the position of the glider in the bitstring as a function of time. The

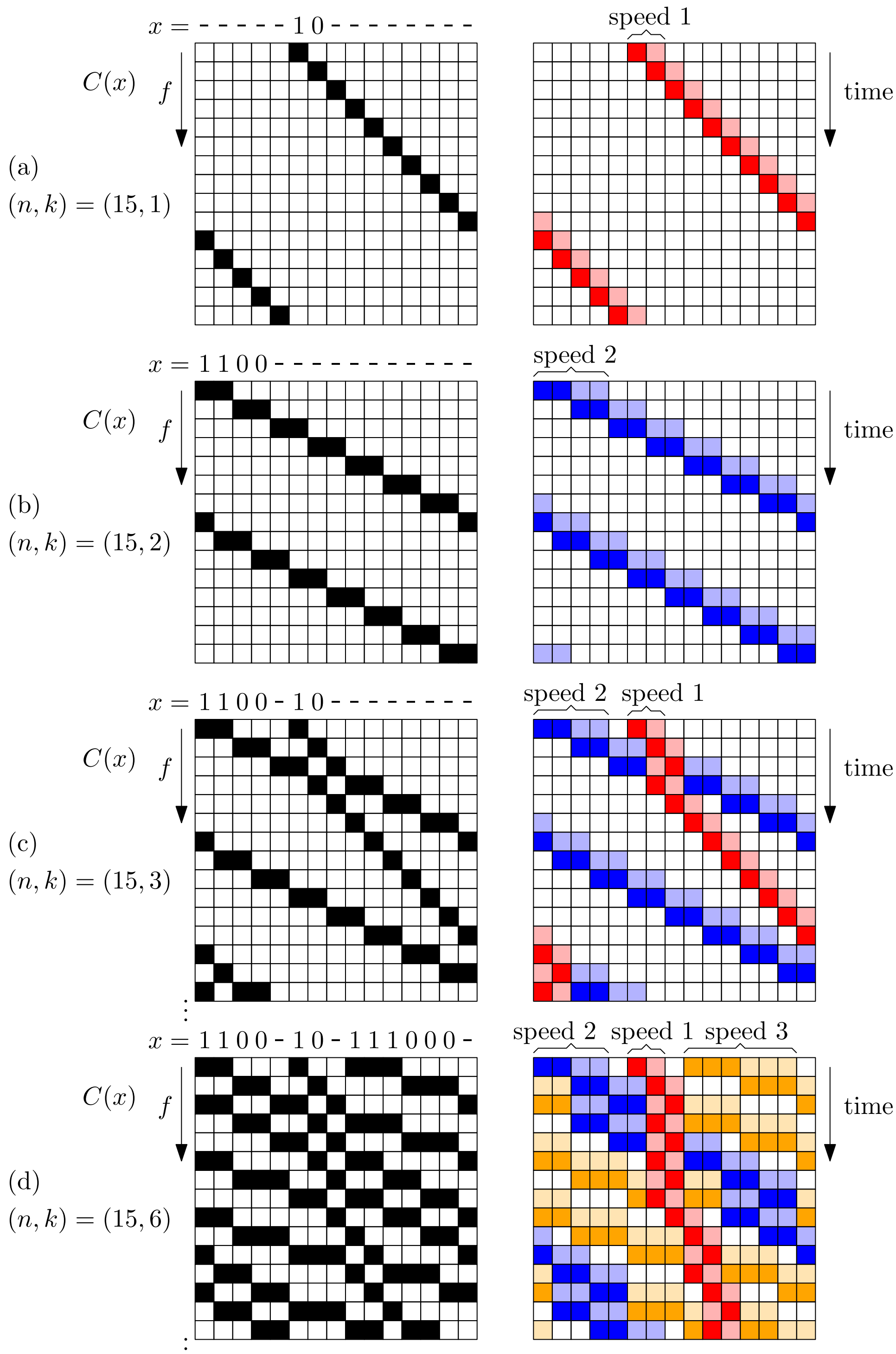


**Figure 3: Cycles of our factor in several different Kneser graphs $K(n,k)$. The cycles in (a) and (b) are shown completely, whereas in (c) and (d) only the first 15 vertices are shown. The right hand side shows the interpretation of certain groups of bits as gliders, and their movement over time. Matched bits belonging to the same glider are colored in the same color, with the opaque filling given to 1-bits, and the transparent filling given to 0-bits. (a) one glider of speed 1; (b) one glider of speed 2; (c) two gliders with speeds 1 and 2 that participate in an overtaking; (d) three gliders of speeds 1, 2 and 3 that participate in multiple overtakings. Animations of these examples are available at [50].**

position $s(t)$ has to be considered modulo $n$, as bitstrings are considered as cyclic strings and the gliders hence wrap around the boundary. The situation gets more interesting and complicated when gliders of different speeds interact with each other. For example, in Figure 3 (c), there is one glider of speed 2 and one glider of speed 1. As long as these groups of bits are separated, each glider moves uniformly as before. However, when the speed 2 glider catches up with the speed 1 glider, an overtaking occurs. During an overtaking, the faster glider receives a boost, whereas the slower glider is delayed. This can be captured by augmenting the corresponding equations of motion by introducing additional terms, making them non-uniform. In the simplest case of two gliders of different speeds, the equations become

$$s_1(t) = s_1(0) + v_1 \cdot t - 2v_1 c_{1,2},$$
$$s_2(t) = s_2(0) + v_2 \cdot t + 2v_1 c_{1,2},$$

where the subscript 1 stands for the slower glider and the subscript 2 stands for the faster glider, and the additional variable $c_{1,2}$ counts the number of overtakings. Note that the terms $2v_1c_{1,2}$ occur with opposite signs in both equations, capturing the fact that the faster glider is boosted by the same amount that the slower glider is delayed. This can be seen as 'energy conservation' in the system of gliders. Overall, the slower glider stands still for two time steps during an overtaking, as $v_1 \cdot 2 - 2v_1 \cdot 1 = 0$, and the faster glider's position changes by an additional amount of $2v_1$ (compared to its movement without overtaking). For more than two gliders, the equations of motion can be generalized accordingly, by introducing additional overtaking counters between any pair of gliders. Nevertheless, as the reader may appreciate from Figure 3 (d), in general it is highly nontrivial to recognize from an arbitrary bitstring $x$ which of its matched bits belong to which glider, and consequently which glider is currently overtaking which other glider. Note that in general the gliders will not be nicely separated, but will be involved in simultaneous interactions, so that the groups of bits forming the gliders will be interleaved in complicated ways. Our general rule that achieves the glider partition is based on a recursion that uses an interpretation of $x$ as a Motzkin path, where every matched 1 becomes an $\nearrow$-step in the Motzkin path, every matched 0 becomes a $\searrow$-step, and every unmatched 0 becomes a $\rightarrow$-step.

One important property that we extract from the aforementioned physics interpretation is that the number of gliders and their speeds are invariant along each cycle. For example, in Figure 3 (d), every bitstring along this cycle has three gliders of speeds 1, 2 and 3. Note in this example that the speeds do not necessarily correspond to the lengths of maximal sequences of consecutive 1s in the bitstrings, due to the interleaving of gliders. We also use the equations of motion to derive a seemingly innocent, but very crucial property, namely that no glider stands still forever, but will move eventually. Note that the speed 1 glider in Figure 3 (d) stands still between time steps 2–8, as during those steps it is overtaken once by the speed 2 glider, and twice by the speed 3 glider (wrapping around the boundary). We establish this fact by linear algebra, by showing that the determinant of the linear systems of equations that governs the gliders' movements is non-singular.

For the reader's entertainment, we programmed an interactive animation of gliders over time, and we encourage experimentation with this code, which can be found at [50]. In particular, this link contains animations of many examples used in figures from our paper, which greatly improves their educational value.

The cycle factor construction discussed before and our analysis via gliders actually work for all $n \geq 2k + 1$, not just for $n \geq 2k + 3$. The assumption $n \geq 2k + 3$ will become crucial in the next step, though.

### 7.3 Gluing the Cycles Together

To join the cycles of our factor to a single Hamilton cycle, we consider a 4-cycle $D$ that shares two opposite edges with two cycles $C, C'$ from our factor. Clearly, the symmetric difference of the edge sets $(C \cup C')\Delta D$ yields a single cycle on the same vertex set as $C \cup C'$. We may repeatedly apply such gluing operations, each time reducing the number of cycles in the factor by one, until the resulting factor has a single cycle, namely a Hamilton cycle. It turns out that the cycle factor defined by $f$ admits a lot of such gluing 4-cycles. Note that $K(n,k)$ does not have any 4-cycles for $n = 2k+1$, so the assumption $n \geq 2k + 2$ is needed here.

The two main technical obstacles we have to overcome are the following: (a) All of the 4-cycles used for the gluing must be edge-disjoint, so that none of the gluings interfere with each other. (b) We must use sufficiently many gluings to achieve connectivity, i.e., every cycle must be connected to every other cycle via a sequence of gluings. These two objectives are somewhat conflicting with each other, so satisfying both at the same time is nontrivial. The final gluings that we use and that satisfy both conditions are described by a set of nine intricate regular expressions.

The 4-cycles that we use for the gluings are based on local modifications of two bitstrings $x$ and $y$ that satisfy certain conditions and that lie on two different cycles $C(x)$ and $C(y)$ from our factor, by considering the gliders in $x$ and $y$. Specifically, this local modification changes the speed sets of the gliders in $x$ and $y$ in a controllable way. Recall that the speeds of gliders are invariant along each cycle, so these speeds will only change along the gluing 4-cycles. To control the gluing, we consider the speeds of gliders in a bitstring $x$ in non-increasing order. Recall that the sum of speeds equals $k$, so such a sorted sequence forms a number partition of $k$. To establish (b) we choose gluings that guarantee a lexicographic increase in those number partitions. This ensures that every cycle is joined, via a sequence of gluings, to a cycle that has the lexicographically largest number partition, namely the number $k$ itself. This corresponds to a single glider of maximum speed $k$, i.e., to a bitstring $x$ in which all 1s are consecutive.

For example, consider the two cycles $C(x)$ and $C(y)$ shown in Figure 4, which can be glued together using the 4-cycle $C_4(x,y) := (x, f(x), y, f(y))$. Note that in $C(x)$, there are two gliders of speed 1 and one glider of speed 3, whereas in $C(y)$ there is one glider of speed 2 and one of speed 3. Consequently, via the gluing we have moved from the number partition $(3,1,1)$ to the lexicographically larger partition $(3,2)$.

The general idea for choosing the gluings $C_4(x,y)$, which can already be seen in this example, is such that in $x$ we decrease the speed of a glider of minimum speed by 1, and instead we increase the speed of any other glider by 1, which ensures that the number partition associated with $y$ is lexicographically larger than that of $x$.

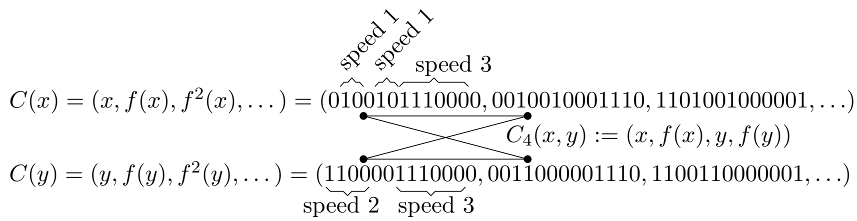


**Figure 4: Gluing of two cycles from the factor via a 4-cycle in $K(13, 5)$.**

Unfortunately, it is not always possible to use gluings that guarantee such immediate lexicographic improvement. In some cases we have to use gluings where a small lexicographic decrease occurs. It then has to be argued that subsequent gluings compensate for this defect such that the overall effect of the resulting sequence of gluings is again a lexicographic improvement. For example, from a vertex with associated number partition $(4, 4)$, the first gluing may lead to a vertex with number partition $(4, 3, 1)$, and the next gluing may lead to $(5, 3)$. While the step $(4, 4) \longrightarrow (4, 3, 1)$ is a lexicographic decrease instead of an increase, overall $(4, 4) \longrightarrow (4, 3, 1) \longrightarrow (5, 3)$ is a lexicographic increase. In this step of the proof the assumption $n \geq 2k+3$ finally enters the picture, as it gives us the necessary flexibility in choosing gluings that are guaranteed to achieve this improvement in all cases.

The arguments so far show that every cycle is connected, via a sequence of gluings, to a cycle in which all 1s are consecutive. Note however, that there may be several such cycles, depending on the values of $n$ and $k$. Specifically, there are exactly $\gcd(n, k)$ such cycles. To join those, we observe that the subgraph of $K(n, k)$ induced by those special cycles is isomorphic to a Cayley graph of $\mathbb{Z}/n\mathbb{Z}$, which admits many gluing 4-cycles to join them.

## 8 PROOF OF THEOREM 2

We show how Theorem 1 can be used to establish the more general Theorem 2 quite easily. Chen and Lih showed the following about generalized Johnson graphs.

Lemma 5 ([10, Thm. 1]). *If $J(n-1, k-1, s-1)$ and $J(n-1, k, s)$ have a Hamilton cycle, then $J(n, k, s)$ also has a Hamilton cycle.*

The proof of Lemma 5 given in [10] is based on a straightforward partitioning of the graph $J(n, k, s)$ into two subgraphs that are isomorphic to $J(n-1, k-1, s-1)$ and $J(n-1, k, s)$. Specifically, this partition is obtained by considering all vertices (=sets) that contain a fixed element, $n$ say, and those that do not contain it. One can then join the cycles in the two subgraphs to one, by taking the symmetric difference with a 4-cycle that has one edge in each of the two subgraphs, using the fact that Johnson graphs are edge-transitive, i.e., we can force each of the cycles in the two subgraphs to use this edge from the 4-cycle. All that is needed now for the proof of Theorem 2 is the following simple observation.

Lemma 6. *If $J(n, k, s)$ is a generalized Johnson graph, then either it is a Kneser graph or $J(n-1, k-1, s-1)$ and $J(n-1, k, s)$ are both generalized Johnson graphs.*

In the proof we will use the aforementioned observation that $J(n, k, s)$ is isomorphic to $J(n, n-k, n-2k+s)$.

Proof. Let $k \geq 1$, $0 \leq s < k$ and $n \geq 2k - s + \mathbf{1}_{[s=0]}$. If $s = 0$, then $J(n, k, s) = J(n, k, 0) = K(n, k)$ is a Kneser graph. This happens in particular if $k = 1$. If $s > 0$ and $n = 2k - s$, then $J(n, k, s) = J(n, n-k, n-2k+s) = J(n, k-s, 0) = K(n, k-s)$ is also a Kneser graph. Otherwise, we have $k > 1$, $s > 0$ and $n > 2k - s$, and we consider the graphs $H_1 := J(n-1, k-1, s-1)$ and $H_0 := J(n-1, k, s)$. From $k > 1$ we obtain $k - 1 \geq 1$, and from $s > 0$ and $s < k$ we obtain that $0 \leq s - 1 < k - 1$. Furthermore, the inequality $n > 2k - s$ is equivalent to $n - 1 > 2(k-1) - (s-1)$, which implies $n - 1 \geq 2(k-1) - (s-1) + 1$. Combining these observations shows that the graph $H_1$ is indeed a valid generalized Johnson graph. Similarly, the inequality $n > 2k - s$ implies that $n - 1 \geq 2k - s = 2k - s + \mathbf{1}_{[s=0]}$ (since $s > 0$), and consequently the graph $H_0$ is also a valid generalized Johnson graph. ◻

Proof of Theorem 2. Combine Lemmas 5 and 6, and use Theorem 1 and induction. Because of the exceptional cases $J(5, 2, 0) = J(5, 3, 1)$, in a few base cases the existence of a Hamilton cycle in $J(n, k, s)$ has to be checked directly, namely for $(n, k, s) \in \{(3, 1, 0), (4, 1, 0), (4, 2, 1), (5, 1, 0), (5, 2, 1), (6, 1, 0), (6, 2, 0), (6, 2, 1), (6, 3, 1), (6, 3, 2)\}$. Using that $J(n, k, s) = J(n, n-k, n-2k+s)$ this settles all cases with $n \leq 6$. ◻

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## REFERENCES

[1] L. A. Agong, C. Amarra, J. S. Caughman, A. J. Herman, and T. S. Terada. 2018. On the girth and diameter of generalized Johnson graphs. *Discrete Math.* 341, 1 (2018), 138–142. https://doi.org/10.1016/j.disc.2017.08.022

[2] A. T. Balaban. 1972. Chemical graphs. XIII. Combinatorial patterns. *Rev. Roumain Math. Pures Appl.* 17 (1972), 3–16. Issue 1.

[3] J. Balogh, D. Cherkashin, and S. Kiselev. 2019. Coloring general Kneser graphs and hypergraphs via high-discrepancy hypergraphs. *European J. Combin.* 79 (2019), 228–236. https://doi.org/10.1016/j.ejc.2019.03.004

[4] Zs. Baranyai. 1975. On the factorization of the complete uniform hypergraph. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I.* North-Holland, Amsterdam, 91–108. Colloq. Math. Soc. János Bolyai, Vol. 10.

[5] J. Bellmann and B. Schülke. 2021. Short proof that Kneser graphs are Hamiltonian for $n \geq 4k$. *Discrete Math.* 344, 7 (2021), Paper No. 112430, 2 pp. https://doi.org/10.1016/j.disc.2021.112430

[6] N. Biggs. 1979. Some odd graph theory. In *Second International Conference on Combinatorial Mathematics (New York, 1978).* Ann. New York Acad. Sci., Vol. 319. New York Acad. Sci., New York, 71–81.

[7] M. Buck and D. Wiedemann. 1984. Gray codes with restricted density. *Discrete Math.* 48, 2-3 (1984), 163–171. https://doi.org/10.1016/0012-365X(84)90179-1

[8] M. Caoduro and L. Lichev. 2021. On the boxicity of Kneser graphs and complements of line graphs. (2021). *arXiv:2105.02516*.

[9] P. Chase. 1989. Combination generation and graylex ordering. *Congr. Numer.* 69 (1989), 215–242. Eighteenth Manitoba Conference on Numerical Mathematics and Computing (Winnipeg, MB, 1988).

[10] B. Chen and K. Lih. 1987. Hamiltonian uniform subset graphs. *J. Combin. Theory Ser. B* 42, 3 (1987), 257–263. https://doi.org/10.1016/0095-8956(87)90044-X

[11] Y. Chen. 2000. Kneser graphs are Hamiltonian for $n \geq 3k$. *J. Combin. Theory Ser. B* 80, 1 (2000), 69–79. https://doi.org/10.1006/jctb.2000.1969

[12] Y. Chen. 2003. Triangle-free Hamiltonian Kneser graphs. *J. Combin. Theory Ser. B* 89, 1 (2003), 1–16. https://doi.org/10.1016/S0095-8956(03)00040-6

[13] Y. Chen and Z. Füredi. 2002. Hamiltonian Kneser graphs. *Combinatorica* 22, 1 (2002), 147–149. https://doi.org/10.1007/s004930200007

[14] Y. Chen and W. Wang. 2008. Diameters of uniform subset graphs. *Discrete Math.* 308, 24 (2008), 6645–6649. https://doi.org/10.1016/j.disc.2007.11.031

[15] Y. Chen and Y. Wang. 2008. On the diameter of generalized Kneser graphs. *Discrete Math.* 308, 18 (2008), 4276–4279. https://doi.org/10.1016/j.disc.2007.08.004

[16] W. E. Clark and M. E. H. Ismail. 1996. Binomial and $Q$-binomial coefficient inequalities related to the Hamiltonicity of the Kneser graphs and their $Q$-analogues. *J. Combin. Theory Ser. A* 76, 1 (1996), 83–98. https://doi.org/10.1006/jcta.1996.0089

[17] R. C. Compton and S. G. Williamson. 1993. Doubly adjacent Gray codes for the symmetric group. *Linear Multilinear Algebra* 35, 3-4 (1993), 237–293. https://doi.org/10.1080/03081089308818261

[18] P. F. Corbett. 1992. Rotator Graphs: An Efficient Topology for Point-to-Point Multiprocessor Networks. *IEEE Transactions on Parallel and Distributed Systems* 3 (1992), 622–626.

[19] T. Denley. 1997. The odd girth of the generalised Kneser graph. *European J. Combin.* 18, 6 (1997), 607–611. https://doi.org/10.1006/eujc.1996.0122

[20] P. Eades, M. Hickey, and R. C. Read. 1984. Some Hamilton paths and a minimal change algorithm. *J. Assoc. Comput. Mach.* 31, 1 (1984), 19–29. https://doi.org/10.1145/2422.322413

[21] P. Eades and B. McKay. 1984. An algorithm for generating subsets of fixed size with a strong minimal change property. *Inform. Process. Lett.* 19, 3 (1984), 131–133. https://doi.org/10.1016/0020-0190(84)90091-7

[22] P. Erdős, C. Ko, and R. Rado. 1961. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)* 12 (1961), 313–320. https://doi.org/10.1093/qmath/12.1.313

[23] P. Frankl. 1985. On the chromatic number of the general Kneser-graph. *J. Graph Theory* 9, 2 (1985), 217–220. https://doi.org/10.1002/jgt.3190090204

[24] E. Friedgut and O. Regev. 2018. Kneser graphs are like Swiss cheese. *Discrete Anal.* (2018), Paper No. 2, 18 pp. https://doi.org/10.19086/da

[25] I. García-Marco, K. Knauer, and L. P. Montejano. 2021. Chomp on generalized Kneser graphs and others. *Internat. J. Game Theory* 50, 3 (2021), 603–621. https://doi.org/10.1007/s00182-019-00697-x

[26] R. J. Gould. 1991. Updating the Hamiltonian problem—a survey. *J. Graph Theory* 15, 2 (1991), 121–157. https://doi.org/10.1002/jgt.3190150204

[27] R. J. Gould and R. Roth. 1987. Cayley digraphs and $(1, j, n)$-sequencings of the alternating groups $A_n$. *Discrete Math.* 66, 1–2 (1987), 91–102. https://doi.org/10.1016/0012-365X(87)90121-X

[28] C. Greene and D. J. Kleitman. 1976. Strong versions of Sperner's theorem. *J. Combin. Theory Ser. A* 20, 1 (1976), 80–88.

[29] P. Gregor, T. Mütze, and J. Nummenpalo. 2018. A short proof of the middle levels theorem. *Discrete Anal.* (2018), Paper No. 8, 12 pp. https://doi.org/10.19086/da.3659

[30] D. J. Harvey and D. R. Wood. 2014. Treewidth of the Kneser graph and the Erdős-Ko-Rado theorem. *Electron. J. Combin.* 21, 1 (2014), Paper 1.48, 11 pp.

[31] K. Heinrich and W. D. Wallis. 1978. Hamiltonian cycles in certain graphs. *J. Austral. Math. Soc. Ser. A* 26, 1 (1978), 89–98.

[32] A. E. Holroyd. 2017. Perfect snake-in-the-box codes for rank modulation. *IEEE Trans. Inform. Theory* 63, 1 (2017), 104–110. https://doi.org/10.1109/TIT.2016.2620160

[33] A. Jafari and M. J. Moghaddamzadeh. 2020. On the chromatic number of generalized Kneser graphs and Hadamard matrices. *Discrete Math.* 343, 2 (2020), 111682, 3 pp. https://doi.org/10.1016/j.disc.2019.111682

[34] M. Jiang and F. Ruskey. 1994. Determining the Hamilton-connectedness of certain vertex-transitive graphs. *Discrete Math.* 133, 1-3 (1994), 159–169. https://doi.org/10.1016/0012-365X(94)90023-X

[35] J. R. Johnson. 2004. Long cycles in the middle two layers of the discrete cube. *J. Combin. Theory Ser. A* 105, 2 (2004), 255–271. https://doi.org/10.1016/j.jcta.2003.11.004

[36] J. R. Johnson. 2011. An inductive construction for Hamilton cycles in Kneser graphs. *Electron. J. Combin.* 18, 1 (2011), Paper 189, 12 pp.

[37] M. Knor. 1994. Gray codes in graphs. *Math. Slovaca* 44, 4 (1994), 395–412.

[38] D. E. Knuth. 2011. *The Art of Computer Programming. Vol. 4A. Combinatorial Algorithms. Part 1.* Addison-Wesley, Upper Saddle River, NJ. xv+883 pages.

[39] V. Kozhevnikov and M. Zhukovskii. 2022. Large cycles in generalized Johnson graphs. (2022). *arXiv:2203.03006*.

[40] K. Liu, M. Cao, and M. Lu. 2022. Treewidth of the generalized Kneser graphs. *Electron. J. Combin.* 29, 1 (2022), Paper No. 1.57, 19 pp. https://doi.org/10.37236/10035

[41] L. Lovász. 1970. Problem 11. In *Combinatorial Structures and Their Applications (Proc. Calgary Internat. Conf., Calgary, AB, 1969)*. Gordon and Breach, New York.

[42] L. Lovász. 1978. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A* 25, 3 (1978), 319–324. https://doi.org/10.1016/0097-3165(78)90022-5

[43] M. Mather. 1976. The Rugby footballers of Croam. *J. Combin. Theory Ser. B* 20, 1 (1976), 62–63.

[44] G. H. J. Meredith and E. K. Lloyd. 1972. The Hamiltonian graphs $O_4$ to $O_7$. In *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*. Inst. Math. Appl., Southend-on-Sea, 229–236.

[45] G. H. J. Meredith and E. K. Lloyd. 1973. The footballers of Croam. *J. Combin. Theory Ser. B* 15 (1973), 161–166.

[46] A. Merino, T. Mütze, and Namrata. 2022. Kneser graphs are Hamiltonian. (2022). *arXiv:2212.03918*. Full preprint version of the present article.

[47] K. Metsch. 2022. On the treewidth of generalized Kneser graphs. (2022). *arXiv:2203.14036*.

[48] T. Mütze. 2016. Proof of the middle levels conjecture. *Proc. Lond. Math. Soc.* 112, 4 (2016), 677–713. https://doi.org/10.1112/plms/pdw004

[49] T. Mütze. 2022. Combinatorial Gray codes—an updated survey. (2022). *arXiv:2202.01280*.

[50] T. Mütze. 2023. Gliders in Kneser graphs. http://tmuetze.de/gliders.html.

[51] T. Mütze, J. Nummenpalo, and B. Walczak. 2021. Sparse Kneser graphs are Hamiltonian. *J. Lond. Math. Soc. (2)* 103, 4 (2021), 1253–1275. https://doi.org/10.1112/jlms.12406

[52] T. Mütze and P. Su. 2017. Bipartite Kneser graphs are Hamiltonian. *Combinatorica* 37, 6 (2017), 1207–1219. https://doi.org/10.1007/s00493-016-3434-6

[53] A. Nijenhuis and H. Wilf. 1975. *Combinatorial Algorithms.* Academic Press, New York-London. xiv+253 pages. Computer Science and Applied Mathematics.

[54] R. J. Ord-Smith. 1967. Algorithm 308: Generation of the permutations in pseudo-lexicographic order [G6]. *Commun. ACM* 10, 7 (1967), 452. https://doi.org/10.1145/363427.363478

[55] F. Ruskey. 1988. Adjacent interchange generation of combinations. *J. Algorithms* 9, 2 (1988), 162–180. https://doi.org/10.1016/0196-6774(88)90036-3

[56] F. Ruskey and A. Williams. 2010. An explicit universal cycle for the $(n-1)$-permutations of an $n$-set. *ACM Trans. Algorithms* 6, 3 (2010), Art. 45, 12 pp. https://doi.org/10.1145/1798596.1798598

[57] C. D. Savage. 1997. A survey of combinatorial Gray codes. *SIAM Rev.* 39, 4 (1997), 605–629. https://doi.org/10.1137/S0036144595295272

[58] J. Sawada and A. Williams. 2020. Solving the sigma-tau problem. *ACM Trans. Algorithms* 16, 1 (2020), Art. 11, 17 pp. https://doi.org/10.1145/3359589

[59] I. Shields and C. D. Savage. 2004. A note on Hamilton cycles in Kneser graphs. *Bull. Inst. Combin. Appl.* 40 (2004), 13–22.

[60] D. Tang and C. Liu. 1973. Distance-2 cyclic chaining of constant-weight codes. *IEEE Trans. Comput.* C-22 (1973), 176–180.

[61] M. Valencia-Pabon and J.-C. Vera. 2005. On the diameter of Kneser graphs. *Discrete Math.* 305, 1–3 (2005), 383–385. https://doi.org/10.1016/j.disc.2005.10.001

[62] D. Zakharov. 2020. Chromatic numbers of Kneser-type graphs. *J. Combin. Theory Ser. A* 172 (2020), 105188, 16 pp. https://doi.org/10.1016/j.jcta.2019.105188

[63] S. Zaks. 1984. A new algorithm for generation of permutations. *BIT* 24, 2 (1984), 196–204. https://doi.org/10.1007/BF01937486


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