Sum-of-Squares Lower Bounds for Densest *k*-Subgraph

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Abstract

Given a graph and an integer k, Densest k-Subgraph is the algorithmic task of finding the subgraph on k vertices with the maximum number of edges. This is a fundamental problem that has been subject to intense study for decades, with applications spanning a wide variety of fields. The state-of-the-art algorithm is an $O(n^{1/4+\varepsilon})$ -factor approximation (for any $\varepsilon > 0$) due to Bhaskara et al. [STOC '10]. Moreover, the so-called *log-density framework* predicts that this is optimal, i.e. it is impossible for an efficient algorithm to achieve an $O(n^{1/4-\varepsilon})$ -factor approximation. In the average case, Densest k-Subgraph is a prototypical noisy inference task which is conjectured to exhibit a *statistical-computational gap*.

In this work, we provide the strongest evidence yet of hardness for Densest *k*-Subgraph by showing matching lower bounds against the powerful Sum-of-Squares (SoS) algorithm, a meta-algorithm based on convex programming that achieves state-of-art algorithmic guarantees for many optimization and inference problems. For $k \le n^{\frac{1}{2}}$, we obtain a degree n^{δ} SoS lower bound for the hard regime as predicted by the log-density framework.

To show this, we utilize the modern framework for proving SoS lower bounds on average-case problems pioneered by Barak et al. [FOCS '16]. A key issue is that small denser-than-average subgraphs in the input will greatly affect the value of the candidate pseudoexpectation operator around the subgraph. To handle this challenge, we devise a novel matrix factorization scheme based on the *positive minimum vertex separator*. We then prove an intersection tradeoff lemma to show that the error terms when using this separator are indeed small.

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1 Introduction

In the Densest *k*-Subgraph problem, we are given an undirected graph *G* on *n* vertices and an integer *k* and we want to output the subgraph on *k* vertices with the most edges, or in other words, the subgraph on *k* vertices with the highest edge density. This is a natural generalization of the *k*-clique problem [Kar72] and has been subject to a long line of work for decades [FS⁺97, SW98, FPK01, FL01, AHI02, Fei02, Kho06, GL09, BCC⁺10, RS10, AAM⁺11, BCG⁺12, Bar15, HWX15, Ame15, HWX16, BKRW17, Man17, BA20, KL20]. This problem has been the subject of intense study partly because of its numerous connections to other problems and fields (e.g. [HJ06, HJL⁺06, KS07, Pis07, KMNT08, AC09, CHK11, HIM11, LNV14, CMVZ15, CL15, CLLR15, SFL16, CZ17, TV17, Lee17, CDK⁺18, MWZ23]) The best known approximation algorithm for this problem yields an approximation factor of $O(n^{1/4+\varepsilon})$ for any constant $\varepsilon > 0$, due to [BCC⁺10]. On the other hand, it is conjectured that no efficient algorithm can achieve an $O(n^{1/4-\varepsilon})$ approximation.

Densest *k*-Subgraph is a compelling problem because random instances (Erdős-Rényi graphs) are conjectured and widely believed to be the "hardest" instances for algorithms. In fact, the insight that "worst case is average case" was crucial to the aforementioned algorithm in [BCC⁺10]. Their idea of going from average-case instances to worst-case instances was generalized into the *log-density framework* (more in Section 1.2), which has been further applied to various other problems [CDK12, CDM17, CMMV17]. Since an algorithm for random instances seems to be the crucial conceptual step needed to solve the problem on all instances, understanding these random instances is a pressing topic.

As stated in [BCC⁺10, BCG⁺12, BKRW17, Man17], Densest *k*-Subgraph on a random graph is a landmark question in the field of average-case complexity. Moreover, the conjectured hardness of this problem on random instances (which is the focus of our work) has been used for applications in finance [ABB⁺10] and cryptography [ABW10]. However, evidence of hardness for Densest *k*-Subgraph stands to be improved, both in the average-case and worst-case settings. For example, even in the worst-case setting, no work has been able to show that Densest *k*-Subgraph is hard to n^{ε} -approximate for a fixed $\varepsilon > 0$ using any reasonable complexity-theoretic assumption (although some works come close, see Section 1.4). In the more interesting average-case setting of random graphs, relatively little progress has been made to justify hardness, let alone match the log-density framework.

In this work, we study the hardness of Densest *k*-Subgraph on random graphs through a generic, powerful algorithm for optimization known as the Sum-of-Squares (SoS) hierarchy [Sho87, Nes00, Par00, Gri01, Las01]. The SoS hierarchy is a family of semidefinite programming relaxations for polynomial optimization problems which implements a certain type of "sum-of-squares reasoning". Arguably at the center stage of average-case complexity in recent years, SoS has proven to be a highly effective tool for combinatorial and continuous optimization. Indeed, the SoS hierarchy is rich enough to capture the state-of-the-art convex relaxations for Sparsest Cut [ARV04], Max-Cut [GW95], all Max *k*-CSPs [Rag08], etc. Sum-of-Squares has also led to new algorithms for

approximating CSPs [AJT19, BBK⁺21, BHKL22] and breakthroughs in robust statistics [KS17, HL18, RSS18, KKM18, Hop20, BP21, BK20], a highlight being the resolution of longstanding open problems in Gaussian mixture learning (over a decade of work culminating in [BDJ⁺20, LM21]). Moreover, for a large class of problems, it has been shown that SoS algorithms are the most effective among all semidefinite programming relaxations [LRS15]. Therefore, understanding the limits of SoS algorithms is an important research endeavour and lower bounds against SoS serve as strong evidence for algorithmic hardness [HKP⁺17, Hop18, Kun21].

In this paper, we prove that for $k \le n^{\frac{1}{2}}$, SoS of degree n^{δ} does not offer any significant improvement in the conjectural hard regime of random instances for Densest *k*-Subgraph as predicted by the log-density framework. This settles the open questions raised in the works [BCG⁺12, Raj18, CM18]. Considering that the algorithm of Bhaskara et al. [BCC⁺10] matching the log-density framework is captured by SoS, our lower bound completes the picture of the performance of SoS for Densest *k*-Subgraph for $k \le n^{\frac{1}{2}}$. This gives solid evidence that the conjectured approximability thresholds for Densest *k*-Subgraph are correct.

1.1 Our contributions

We will now describe our results on SoS lower bounds for Densest *k*-Subgraph that match the predictions of the log-density framework (to be described in Section 1.2).

Consider the following hypothesis testing variant of the Densest *k*-Subgraph problem. For an integer *n* and a real $p \in [0, 1]$, let $\mathcal{G}_{n,p}$ denote the Erdős-Rényi random distribution where a graph on *n* vertices is sampled by choosing each edge to be present independently with probability *p*. For parameters $n, k \in \mathbb{N}$ and $p, q \in [0, 1]$, we are given a graph *G* sampled either from

- 1. The null distribution $\mathcal{G}_{n,p}$ or
- 2. The alternative distribution where we first sample $G \sim \mathcal{G}_{n,p}$, then a set $H \subseteq V(G)$ is chosen by including each vertex with probability $\frac{k}{n}$, and finally we replace H by a sample from $\mathcal{G}_{|H|,q}$.

and our goal is to correctly identify which distribution it came from, with non-negligible probability.

The hypothesis testing question is a "planted model" of Densest *k*-Subgraph which is conjectured to exhibit a *statistical-computational gap* [BB20, BBH⁺20]. With high probability, for *q* slightly larger than *p*, the subgraph *H* in the alternative distribution is truly the densest subgraph of *G* with size *k* (hence the null and alternative distributions are statistically distinguishable), but it is conjecturally computationally impossible to distinguish the two cases (in the parameter regime below).

Studying algorithms for this hypothesis testing variant was crucial to the log-density framework [BCC⁺10], which both generalizes an algorithm for the hypothesis testing variant into a worst-case algorithm, and predicts the relationships between n, k, p, q for which the hypothesis testing problem is hard. In particular, consider the setting

$$k = n^{\alpha}$$
, $p = n^{-\beta}$, $q = n^{-\gamma}$

for constants $\alpha \in (0, 1/2], \beta \in (0, 1), \gamma \in (0, 1)$, a notation that we will use throughout this paper. According to the framework, it's algorithmically hard to solve the problem if

$$\gamma > \alpha \beta$$

That is, in this regime, no polynomial-time algorithm can distinguish the two distributions with probability at least 2/3 of success.¹

To state our result, we recall that the SoS hierarchy is a family of convex semidefinite programming relaxations parameterized by an integer D_{SoS} called the *degree* or *level* of SoS. The relaxation gets tighter as D_{SoS} increases but the runtime also increases at the rate² of approximately $n^{O(D_{SoS})}$ for degree D_{SoS} SoS. Thus, conceptually degree O(1) corresponds to polynomial time, and degree n^{δ} to subexponential time algorithms. In this work, we study the performance of degree $D_{SoS} = n^{\delta}$ Sum-of-Squares on the Densest *k*-Subgraph problem for a constant $\delta > 0$ and obtain strong lower bounds.

Because of the well-known duality between SoS programs and pseudo-expectation operators, to show a lower bound, it suffices to show a feasible pseudo-expectation operator $\widetilde{\mathbb{E}}$ satisfying the constraints. For a formal definition of SoS, see Section 2.1. We are now ready to state our result.

Theorem 1.1. For all constants $\alpha \in (0, 1/2], \beta \in (0, 1), \gamma \in (0, 1)$ such that $\gamma > \alpha\beta$, there exists $\delta > 0$ such that with high probability over $G = (V, E) \sim \mathcal{G}_{n,p}$, there exists a degree n^{δ} pseudo-expectation operator $\widetilde{\mathbb{E}}$ on SoS program variables $\{\mathbf{X}_u\}_{u \in V}$ such that

- 1. (Normalization) $\widetilde{\mathbb{E}}[1] = 1 \pm o(1)$.
- 2. (Subgraph on k vertices) $\widetilde{\mathbb{E}}[\sum_{v \in V} \mathbf{X}_v] = k(1 \pm o(1)).$
- 3. (Large density) $\widetilde{\mathbb{E}}[\sum_{\{u,v\}\in E} \mathsf{X}_u \mathsf{X}_v] = \frac{k^2 q}{2}(1 \pm o(1))$
- 4. (Feasibility) The moment matrix **M** corresponding to $\widetilde{\mathbb{E}}$ is positive semidefinite.

¹When $\alpha > \frac{1}{2}$, i.e. $k = \omega(\sqrt{n})$, spectral algorithms beat the log-density threshold [BCC⁺10, KL20]. Spectral algorithms are captured by degree-2 SoS. Various works have also studied other special settings (e.g. when q = 1, or when p, q are constants). See Section 1.4.

²In pathological cases, there may be issues with bit complexity [O'D17, RW17]

This in particular implies that, in the predicted hard regime of the log-density framework, SoS cannot be used to solve the Densest *k*-Subgraph problem as stated above. As discussed earlier, these SoS lower bounds offer strong evidence that for $k \leq \sqrt{n}$, it is unlikely that efficient algorithms can beat the predictions of the log-density framework for Densest *k*-Subgraph.

By setting $\alpha = 1/2$, $\beta = 1/2$ and $\gamma = 1/4 + \varepsilon$, we obtain the following important corollary.

Corollary 1.2. For any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that degree- n^{δ} Sum-of-Squares exhibits an integrality gap of $O(n^{1/4-\varepsilon})$ for the Densest k-Subgraph problem.

This corollary essentially matches the best known algorithmic guarantees for the Densest *k*-subgraph problem [BCC⁺10], namely an efficient $O(n^{1/4+\varepsilon})$ -factor approximation algorithm, thereby completing the picture for Sum-of-Squares.

1.2 The log-density framework

For more context on our results, we give a brief description of the log-density framework [BCC⁺10]. See [KL20, Section 1.3] or [CM18] for a more detailed treatment.

The log-density framework is a relatively recent technique that devises worst-case algorithms for problems by studying algorithms for average-case instances. It was introduced in the context of the Densest *k*-Subgraph problem and has been since utilized for many other problems such as Lowest Degree 2-Spanner, Smallest p-Edge Subgraph (SpES) [CDK12], Small Set Bipartite Vertex Expansion (SSBVE) [CDM17], Label Cover, 2-CSPs [CMMV17], etc.

Formally, for a graph on *n* vertices with average degree *d*, we define its log-density to be $\frac{\log d}{\log n}$. Consider the hypothesis testing problem from Section 1.1. The log-density framework predicts that it is possible to algorithmically distinguish the distributions and solve the hypothesis testing problem if and only if the log-density of the planted subgraph is larger than the log-density of the original graph before planting. Since the average degree of a graph sampled from $G_{n,p}$ is $\approx np$, this framework predicts that the distributions are distinguishable if for some constant $\varepsilon > 0$,

$$\frac{\log(kq)}{\log k} \ge \frac{\log(np)}{\log n} + \varepsilon \qquad \Longleftrightarrow \qquad \gamma \le \alpha\beta - \varepsilon'$$

for some constant $\varepsilon' > 0$.

Moreover, and of extreme importance to us, the framework also predicts algorithmic hardness if the other direction of the inequality holds. That is, if

$$\gamma \ge \alpha\beta + \varepsilon$$

for some constant $\varepsilon > 0$, the log-density framework predicts that no efficient algorithm can distinguish the two distributions. For the sake of clarity, let's look at the special case $\alpha = 1/2$, $\beta = 1/2$ and $\gamma = 1/4 + \varepsilon$. Then, we expect it to be hard for efficient algorithms to distinguish the following distributions,

- 1. The null distribution $\mathcal{G}_{n,\frac{1}{\overline{c}}}$
- 2. The alternative distribution where we first sample $G \sim \mathcal{G}_{n,\frac{1}{\sqrt{n}}}$, then a set $H \subseteq V(G)$ is chosen by including each vertex with probability $\frac{1}{\sqrt{n}}$ (so $|H| \approx \sqrt{n}$), and finally we replace H by a sample from $\mathcal{G}_{|H|,\frac{1}{n!/4+\epsilon}}$.

As an aside, note that in this case, since the average degree of the densest *k*-subgraph for the null distribution is $\widetilde{O}(\sqrt{n})$ and that of the alternative distribution is $\widetilde{\Omega}(n^{3/4-\varepsilon})$, hardness of the distinguishing problem implies $n^{1/4-\varepsilon}$ -factor approximation hardness for Densest *k*-Subgraph.

1.3 Our approach

Since Sum-of-Squares is a convex program, in order to prove a lower bound, it suffices to construct a feasible point, i.e. a *pseudoexpectation operator* or *moment matrix*, which is a large nonlinear random matrix that depends on the input. At a high level, our proof leverages an existing strategy for proving lower bounds against the Sum-of-Squares algorithm on random inputs: use *pseudocalibration* [BHK⁺16] to construct a candidate moment matrix, then study the spectrum of the candidate matrix using *graph matrices* [AMP20]. This strategy has been successfully applied in several contexts [BHK⁺16, PR20, GJJ⁺20, JPR⁺22], although in each case, including ours, significant additional insights have been required.

Given a random input graph, the first step is to construct the candidate pseudoexpectation operator or moment matrix. Pseudocalibration suggests a candidate matrix, which we can use here without further thinking. Recall that a semidefinite program optimizes over the cone of positive semi-definite (PSD) matrices; the main challenge is showing that the candidate moment matrix is feasible (PSD) with high probability over the random input.

The main issue we face is that matrix factorization strategies in prior works do not obviously lead to dominant PSD terms in our setting. There are several steps in the existing framework:

- 1. Express the candidate moment matrix Λ in the graph matrix (i.e. Fourier) basis;
- 2. Identify a class of spectrally dominant graph matrices in Λ which are together approximately PSD;

- 3. Perform an approximate PSD decomposition to create PSD terms plus additional error terms;
- 4. Show that all non-dominant terms and error terms can be charged to the dominant PSD terms, i.e. they are "negligible".

For the purposes of the current discussion, it is enough to know that each graph matrix in step (1) measures how a fixed small subgraph, or *shape*, contributes to the candidate moment matrix, and furthermore that the spectral norm of a graph matrix can be read off of combinatorial properties of the small shape graph. It was shown in [JPR⁺22, RT23] that the norm of a graph matrix is determined up to lower-order factors by the *Sparse Minimumweight Vertex Separator (SMVS)* of the shape (Theorem 2.21). For intuition, shapes with smaller, denser separators have larger norms.

In order to identify the class of norm-dominant shapes in step (2), previous work decomposes shapes using their leftmost and rightmost *MVS* (in contrast to *SMVS*), yielding for each shape an approximately PSD term that spectrally dominates the original graph matrix. Using the norm bounds, combinatorial arguments about vertex separators are then employed to show that all deviation terms in step (4) are small.

Although prior work has avoided using the SMVS as the decomposition criterion and used the MVS instead, the SMVS is a necessity in our setting, because Densest *k*-Subgraph is sensitive to small, local structures in the input. To explain, for a fixed set of vertices *U*, if many vertices in *U* have a common exterior neighbor or are part of a denser-than-average subgraph, then this greatly increases the algorithm's belief that *U* is part of the dense subgraph. Using the SMVS can be thought of as pinpointing, for each shape, the small dense subgraph which has the strongest effect on the graph matrix's norm.

A decomposition based on SMVS poses new conceptual challenges. Surprisingly, the SMVS decomposition, without extra care, may rather lead to some supposedly "PSD" terms being negative instead. We address these technical challenges, alongside our solution using the *Positive Minimum-weight Vertex Separator* (see Section 3.1 for a technical overview) after providing the definitions needed for working with graph matrices.

Once we have properly identified the dominant PSD terms, what remains is to prove that the error terms in the decomposition are small using an *intersection tradeoff lemma*. This is also one of our novel contributions as it is significantly different from intersection lemmas in prior works. This combinatorial lemma is the most crucial part of the proof, as it ensures that the error terms in the approximate PSD decomposition have small enough norms.

It's worth highlighting that the log-density criterion $\gamma > \alpha\beta$ occurs multiple times throughout our proof, which is fascinating to the authors. A partial explanation is that if we look at the contribution of each Fourier character in Lemma 4.8, the quantity $\gamma - \alpha\beta$ measures the decay as the degree of the Fourier character increases, i.e. it's the edge decay in a shape. Therefore, this has a dampening effect on the higher Fourier levels in the decomposition. Such a Fourier decay is ubiquitous in the analysis of the low-degree likelihood ratio [HKP⁺17, Hop18, KWB22] and has been important in prior average-case SoS lower bounds [BHK⁺16, PR22, GJJ⁺20, JPR⁺22].

1.4 Related work

Algorithms Algorithms for the Densest *k*-Subgraph problem have been widely studied, e.g. [FS⁺97, SW98, FPK01, FL01, AHI02, ST08, GL09, BCC⁺10, MM15, Ame15, Bar15, BKRW17, BA20, KL20], and we do not attempt to give an overview of them (see e.g. [KL20] for a nice overview of some of them). For general graphs, the work [KP93] (which also introduced the problem) gave a polynomial time $\tilde{O}(n^{0.3885})$ -factor approximation algorithm. This was later improved to a $O(n^{1/3-\varepsilon})$ -factor approximation (for a constant $\varepsilon \approx 1/60$) in [FPK01] and to a $O(n^{0.3159})$ -factor approximation in [GL09] respectively. The seminal work of [BCC⁺10], which also proposed the log-density framework improved this to give an algorithm that achieves a $n^{1/4+\varepsilon}$ -factor approximation in time $n^{O(1/\varepsilon)}$, for all constants $\varepsilon > 0$. This is conjectured to be the best achievable by efficient algorithms.

Lower bounds for Densest *k***-Subgraph** Because of its conceptual significance and wide applicability, studying lower bounds against the Densest *k*-Subgraph problem is an important research endeavour. We give a non-exhaustive list of such prior works below.

Conditional hardness: It's well known that Densest *k*-Subgraph is NP-hard to solve exactly, but to the best of our knowledge, NP-hardness of even constant factor approximation is unknown. There are various other conditional hardness results assuming more than P ≠ NP, e.g. [Fei02, Kho06, RS10, AAM⁺11, BKRW17, Man17]. We highlight the influential work of Manurangsi [Man17], who assuming the Exponential Time Hypothesis showed almost-polynomial factor hardness for this problem. See the same paper for a more detailed list of other conditional hardness results. It's worth noting that none of these results achieve polynomial factor hardness.

These approaches argue that Densest *k*-Subgraph is hard by reduction. One source of difficulty is that reductions are not as successful for average-case problems, since a reduction tends to distort the input distribution and produce somewhat pathological outputs. Proving hardness of Densest *k*-Subgraph may be possible using a reduction to a novel non-random instance, but, if it is true that random (or sufficiently pseudo-random) graphs are the *only* hard instances of Densest *k*-Subgraph, then a stronger theory of average-case reductions may be a prerequisite. Some recent works make exciting progress on realizing average-case reductions [BBH18, BB20, BABB21, HS21].

The remaining lower bounds, including ours, are unconditional results that do not rely on any conjectures.

2. Sherali-Adams hardness: An integrality gap of $\Omega(n^{\alpha(1-\alpha)-o(1)})$ was shown for the degree- $\tilde{\Omega}(\log n)$ Sherali-Adams hierarchy (which is a family of linear programming

relaxations) in [BCG⁺12, CM18]. Our result is stronger than these Sherali-Adams lower bounds in three important ways. First, we consider SoS rather then Sherali-Adams. The SoS hierarchy captures the Sherali-Adams hierarchy and is known to be much stronger in many cases (e.g., see [KV15, DKSV06, CMM09, CLRS16, KMR17] in conjunction with [GW95, ARV09]) therefore we imply their results. Second, we obtain an n^{δ} degree lower bound as opposed to an $\tilde{\Omega}(\log n)$ degree lower bound. Finally, while these Sherali-Adams lower bounds are for the particular setting where $\beta = \alpha$ (the setting that maximizes the integrality gap for a fixed α), our lower bounds work for the entire range of parameters α , β , γ .

3. SoS hardness: Worst-case SoS lower bounds have been exhibited in [BCG⁺12, MM15, CMMV17] obtained by reducing from Max *k*-CSP hardness results, within the SoS framework as pioneered by [Tul09]. However, these SoS lower bounds were not optimal even for worst-case instances, since they didn't match known algorithmic guarantees (to be more precise, they showed an $n^{1/14-O(\varepsilon)}$ -factor lower bound for degree n^{ε} SoS, whereas $n^{1/4-O(\varepsilon)}$ -factor hardness is conjectured). Our work on the other hand studies average-case instances (as opposed to worst-case) and matches the guarantees of known algorithms. Therefore, we significantly improve these prior hardness results and close the gap. Moreover, our results can be reduced à la [Tul09] to show SoS hardness for other problems such as Densest *k*-Subhypergraph [Raj18, Theorem 3.17] and also potentially Minimum *p*-Union [Raj18].

Average-case Sum-of-Squares lower bounds Sum-of-Squares lower bounds for averagecase problems have proliferated in the last decade, for example, Planted Clique [HKP15, MPW15, BHK⁺16], Sherrington-Kirkpatrick Hamiltonian [MRX20, GJJ⁺20, Kun20], Sparse and Tensor PCA [HKP⁺17, PR20, PR22] and Max *k*-CSPs [KMOW17]. Most of these works have been in the colloquial "dense" regime where the random inputs are sampled from $\mathcal{G}_{n,1/2}$ or the standard normal distribution $\mathcal{N}(0,1)$. Recently, average-case SoS lower bounds have been shown for the sparse setting, i.e. inputs sampled from $\mathcal{G}_{n,p}$ where p = o(1), for the problem of Maximum Independent Set [JPR⁺22, RT23]. The common thread underlying recent SoS lower bounds, including ours, is spectral analysis of large random matrices. See the works [PR20, Raj22, Jon22] for additional background and intuition on the matrix analysis framework used in these lower bounds.

The low-degree likelihood ratio hypothesis We add that similar predictions as the log-density framework for the threshold of algorithmic distinguishability may possibly be obtained by analyzing the *low-degree likelihood ratio* [HKP⁺17, Hop18, KWB22]. The low-degree likelihood ratio is used in the context of noisy statistical inference problems to predict, among other things, the existence of statistical-computational gaps, i.e. when the signal (the planted dense subgraph) is information-theoretically detectable (and hence recoverable by a brute-force search), but is not detectable by efficient algorithms. In the same context, the low-degree likelihood ratio is used to predict the distinguishing power

of low-degree polynomial algorithms. In [SW22], they analyze the low-degree likelihood ratio for certain parameter regimes of Densest *k*-Subraph, but their results do not seem to recover the predictions of the log-density framework precisely. Our Proposition 2.52 can be interpreted as showing that the low-degree likelihood ratio is 1 + o(1) in the entire hard regime for the log-density framework.

Planted Dense Subgraph and Planted Clique conjectures In our work, we have focused on the regime $\alpha \in (0, 1/2], \beta, \gamma \in (0, 1)$. Other instantiations of these parameters have also been subject to intense study in recent years and various conjectures predicting the limits of efficient algorithms have been proposed, broadly referred to as the Planted Dense Subgraph conjecture or in the case $\gamma = 0$, the Planted Clique conjecture. Furthermore, assuming these conjectures, inapproximability results have been derived for various problems such as Sparse PCA, Stochastic Block model, Biclustering, etc. See e.g. [HWX15, CX14, BBH18, BBH19, BB19, MRS20, PR20, PR22] and references therein. Densest *k*-Subgraph lies at the heart of many of these reductions, therefore it's plausible that our hardness result can be exploited to derive better inapproximability results for various other problems, which we leave for future work.

1.5 Organization of the paper

The rest of the paper is organized as follows. We start with a brief overview in Section 2 of graph matrices, which are at the heart of our spectral analysis, using it to construct our candidate moment matrix following the pseudo-calibration framework in Section 2.9. With the matrix in hand, we then delve into the extensive PSDness analysis that forms the bulk of work. We motivate and discuss our conceptually novel PMVS decomposition in Section 3, and show the combinatorial analysis for the key "charging" arguments of the PSDness proof in Section 4. We defer the formal details and other technical verifications to appendices.

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2 Preliminaries

2.1 The Sum-of-Squares algorithm

We now formally describe the Sum-of-Squares hierarchy. For a detailed treatment and survey of SoS, see e.g. [RSS18, FKP⁺19, Sch17, Hop18, Jon22].

SoS is used to check feasibility of a system of polynomials. Given a graph G = (V, E), the simplest polynomial formulation for the existence of a subgraph with *k* vertices and *m* edges encodes the 0/1 indicator of the subgraph:

Variables:
$$\mathbf{X}_{v}, \forall v \in V$$

Constraints:

$$\sum_{v \in V} \mathbf{X}_{v} = k$$
 (Vertex count)

$$\sum_{\{u,v\} \in E} \mathbf{X}_{u} \mathbf{X}_{v} = m$$
 (Edge count)
 $\mathbf{X}_{v}^{2} = \mathbf{X}_{v}$ $\forall v \in V$ (Boolean)

The sum-of-squares algorithm is parameterized by the *degree* $D_{SoS} \in \mathbb{N}$. We assume D_{SoS} is even. For formal variables X_1, \ldots, X_n , let $\mathbb{R}^{\leq D_{SoS}}[X_1, \ldots, X_n]$ denote the set of polynomials with degree at most D_{SoS} .

Definition 2.1 (Pseudoexpectation). *Given a set of variables* X_1, \ldots, X_n , *a* degree- D_{SoS} pseudoexpectation operator *is a linear functional* $\widetilde{\mathbb{E}} : \mathbb{R}^{\leq D_{SoS}}[X_1, \ldots, X_n] \to \mathbb{R}$ such that $\widetilde{\mathbb{E}}[1] = 1$.

Definition 2.2 (Satisfying an equality constraint). A degree- D_{SoS} pseudoexpectation operator $\widetilde{\mathbb{E}}$ satisfies a polynomial constraint " $f(\mathbf{X}) = 0$ " if $\widetilde{\mathbb{E}}[f(\mathbf{X})p(\mathbf{X})] = 0$ for all polynomials $p(\mathbf{X})$ such that $\deg(p) + \deg(f) \le D_{SoS}$.

Definition 2.3 (SoS-feasible). A degree- D_{SoS} pseudoexpectation operator \mathbb{E} is SoS-feasible if for every polynomial $p \in \mathbb{R}^{\leq D_{SoS}/2}[X_1, \dots, X_n], \mathbb{E}[p(\mathbf{X})^2] \geq 0.$

Definition 2.4 (Sum-of-squares algorithm). Given a system of polynomial constraints { $f_i(\mathbf{X}) = 0$ } in *n* variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$, the degree- D_{SoS} Sum-of-Squares algorithm checks for the existence of an SoS-feasible degree- D_{SoS} pseudoexpectation operator \mathbb{E} that satisfies the constraints. If \mathbb{E} exists, the algorithm outputs "may be feasible", otherwise it outputs "infeasible". This can be done algorithmically by solving a semidefinite program of size $n^{O(D_{SoS})}$ that searches for a feasible moment matrix (Definition 2.7).

If no pseudoexpectation operator exists, then SoS successfully refutes the polynomial system (i.e., it proves that there is no dense subgraph in the input). On the other hand, if a pseudoexpectation operator exists, SoS cannot rule out that the polynomial system is feasible (the pseudoexpectation operator fools SoS, but it may or may not correspond to a true distribution on feasible points). A lower bound against SoS consists of a feasible pseudoexpectation operator in the case when the system is actually infeasible.

2.2 Moment matrices

Analysis of the SoS algorithm on an *n*-variable polynomial system is typically accomplished by formulating it in terms of large matrices indexed by subsets of [n], known as

moment matrices.

Definition 2.5 (Matrix index). Let *I* be the set of ordered subsets of [n] of size at most $D_{SoS}/2$.

Remark 2.6. Another reasonable definition of I uses subsets of [n] and not ordered subsets. For technical simplifications, we include an ordering.

The degree- D_{SoS} sum-of-squares algorithm can be equivalently formulated in terms of $\mathbb{R}^{I \times I}$ matrices, which are called *moment matrices*.

Definition 2.7 (Moment matrix). The moment matrix $\Lambda = \Lambda(\widetilde{\mathbb{E}})$ associated to a degree- D_{SoS} pseudoexpectation $\widetilde{\mathbb{E}}$ is an *I*-by-*I* matrix defined as

$$\mathbf{\Lambda}[I,J] := \widetilde{\mathbb{E}}\left[\mathbf{X}^{I} \cdot \mathbf{X}^{J}\right].$$

Fact 2.8. $\widetilde{\mathbb{E}}$ is SoS-feasible if and only if $\Lambda(\widetilde{\mathbb{E}}) \geq 0$.

Definition 2.9 (SoS-symmetric). A matrix $\Lambda \in \mathbb{R}^{I \times I}$ is SoS-symmetric if $\Lambda[I, J]$ depends only on the disjoint union $I \sqcup J$ as an unordered multiset. Along with the additional constraint $\Lambda[\emptyset, \emptyset] = 1$, this characterizes $\Lambda \in \mathbb{R}^{I \times I}$ which are moment matrices of degree- D_{SoS} pseudoexpectation operators.

In the presence of Boolean constraints " $X_i^2 = X_i$ ", a moment matrix satisfies these constraints if and only if $\Lambda[I, J]$ depends only on the union $I \cup J$ as an unordered set (ignore duplicates).

2.3 *p*-biased Fourier analysis and graph matrices

We are interested in matrices which depend on a random graph $G \sim \mathcal{G}_{n,p}$. To analyze these as functions of *G*, we encode *G* via its edge indicator vector in $\{0, 1\}^{\binom{n}{2}}$ and perform *p*-biased Fourier analysis.

Definition 2.10 (Fourier character). χ denotes the *p*-biased Fourier character,

$$\chi(0) = -\sqrt{\frac{p}{1-p}}, \qquad \chi(1) = \sqrt{\frac{1-p}{p}}.$$
(1)

For *H* a subset or multi-subset of $\binom{[n]}{2}$, let $\chi_H(G) := \prod_{e \in H} \chi(G_e)$.

Definition 2.11 (Ribbon). A ribbon is a tuple $R = (A_R, B_R, E(R))$ where $A_R, B_R \in I$ and $E(R) \subseteq {\binom{[n]}{2}}$. The corresponding matrix $\mathbf{M}_R \in \mathbb{R}^{I \times I}$ is:

$$\mathbf{M}_{R}[I,J] = \begin{cases} \chi_{E(R)}(G) & I = A_{R}, J = B_{R} \\ 0 & otherwise . \end{cases}$$

The ribbon matrices \mathbf{M}_R are mean-zero, orthonormal under the expectation of the Frobenius inner product on matrices, and form a basis for all $\mathbb{R}^{I \times I}$ -valued functions of *G*. They are the natural Fourier basis for random matrices that depend on *G*.

In the matrices that we study, the coefficient on a ribbon will not depend on the particular labels of the ribbon's vertices, but only on the graphical structure of the ribbon. This graphical structure is called the *shape*.

Definition 2.12 (Shape). A shape α is an equivalence class of ribbons under relabeling of the vertices (equivalently, permutation by S_n). Each shape is associated with a representative graph $(U_{\alpha}, V_{\alpha}, E(\alpha))$. We let $V(\alpha) := U_{\alpha} \cup V_{\alpha} \cup V(E(\alpha))$.

For an example of a shape, see Fig. 1. We use the convention of Greek letters such as α , γ , τ for shapes and Latin letters *R*, *L*, *T* for ribbons.

Definition 2.13 (Embedding). Given a shape α and an injective function $\varphi : V(\alpha) \rightarrow [n]$, we let $\varphi(\alpha)$ be the ribbon obtained by labeling α in the natural way (preserving the order on U_{α} and V_{α}).

A ribbon *R* has shape α if and only if *R* can be obtained by an embedding of *V*(α) into [*n*]. Note that different embeddings may produce the same ribbon.

Definition 2.14 (Graph matrix). *Given a shape* α *, the graph matrix* \mathbf{M}_{α} *is*

$$\mathbf{M}_{lpha} = \sum_{injective \; \varphi: V(lpha)
ightarrow [n]} \mathbf{M}_{arphi(lpha)}$$

The entries of a graph matrix are degree- $|E(\alpha)|$ monomials in the variables G_e , therefore we think of graph matrices as low-degree polynomial random matrices in *G*. We call them "nonlinear" to distinguish them from the degree-1 case, which is well-studied (being essentially the adjacency matrix of *G*).

Definition 2.15 (Trivial). *A ribbon or shape* α *is* trivial *if* $E(\alpha) = \emptyset$.

Definition 2.16 (Diagonal). A ribbon or shape α is diagonal if $V(\alpha) = U_{\alpha} = V_{\alpha}$.

A diagonal shape is only nonzero on the diagonal entries of the matrix in the block corresponding to U_{α} . Note that there are additional shapes which have the same support, namely shapes which potentially have additional edges and vertices outside of $U_{\alpha} = V_{\alpha}$. The diagonal shapes as we have defined them are the most important contributors to the diagonal entries of the matrix.

Definition 2.17 (Transpose). The transpose of a ribbon or shape swaps A_R , B_R or U_α , V_α respectively. This has the effect of transposing the matrix for the ribbon/shape.

2.4 Norm bounds

Definition 2.18 (Weight of a set). For a graph S, let $w(S) = |V(S)| - \log_n(1/p)|E(S)|$.

Definition 2.19 (Vertex separator). A vertex separator of two sets A, B in a graph G is a set $S \subseteq V(G)$ such that all paths from A to B pass through S.

Definition 2.20 (Sparse minimum vertex separator (SMVS)). *Given a ribbon or shape* α , *a* sparse minimum vertex separator (SMVS) *is a minimizer of* w(S) *over* $S \subseteq V(\alpha)$ *which separate* U_{α} *and* V_{α} .

Observe that up to lower-order factors, $w(S) = \log_n (\mathbb{E}[\# \text{ of copies of graph } S \text{ in } \mathcal{G}_{n,p}])$. The SMVS is thus the rarest separator of α .

Theorem 2.21 (Norm bound, informal [JPR⁺22, RT23]). With high probability, for all proper shapes α :

$$\|\mathbf{M}_{\alpha}\| \leq \widetilde{O}\left(n^{\frac{|V(\alpha)| - |w(S_{\min})|}{2}}\right)$$

where S_{\min} is the SMVS of α .

2.5 Graph matrix calculus: factoring

In light of the relevance of vertex separators to the spectrum of a graph matrix, a key ingredient underlying our machinery is that each shape admits an (approximate) factorization into three pieces based on its vertex separators. For the following discussion, fix a shape α . The separators of a shape α have a natural partial order as follows.

Definition 2.22 (Left and right). A vertex separator S is left (respectively right) of a vertex separator S' if S separates U_{α} and S' (resp. S' and V_{α}).

We will define the *leftmost SMVS* to be the SMVS which is left of all other SMVS for α , and similarly for the *rightmost SMVS*. For an example, see Fig. 1. We will decompose each shape into three pieces: the "left shape" between U_{α} and the leftmost SMVS, the "middle shape" between the leftmost and rightmost SMVS, and the "right shape" between the rightmost SMVS and V_{α} . We now work towards making this formal.

Unfortunately, it is not always true that there is a *unique* SMVS that is left of every SMVS. Nonetheless, we can define the leftmost SMVS in a natural and canonical way using the following proposition, whose proof is in Appendix A.1.

Proposition 2.23. Every shape has an SMVS which is left of every SMVS. Furthermore, there is a unique SMVS left of every SMVS with minimum vertex size.

Definition 2.24 (Leftmost and rightmost SMVS). *The* leftmost SMVS *is the SMVS which is left of every SMVS and has minimum vertex size. The* rightmost SMVS *is defined analogously.*

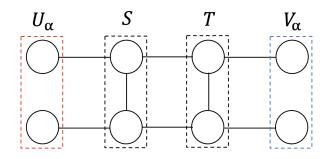


Figure 1: Example of a shape α with its leftmost SMVS *S* and rightmost SMVS *T*. This shape has norm $\tilde{O}(n^{\frac{8-2}{2}}\sqrt{\frac{1-p}{p}}) = \tilde{O}(\frac{n^3}{\sqrt{p}})$

Definition 2.25 (Left shape). A shape σ is a left shape if the unique SMVS is V_{σ} (hence, it is both leftmost and rightmost).

Definition 2.26 (Middle shape). A shape τ is a middle shape if U_{τ} is the leftmost SMVS, and V_{τ} is the rightmost SMVS.

Definition 2.27 (Right shape). A right shape σ is the transpose of a left shape.

We also extend these definitions to ribbons. By splitting a ribbon across its leftmost and rightmost SMVS, we have the following canonical decomposition theorem for ribbons, to be presented formally in the next section.

Proposition 2.28 (informal version of Proposition 2.33). Every ribbon *R* can be expressed uniquely as the composition of a left, middle, and right ribbon.

2.6 Graph matrix calculus: composition

Multiplying graph matrices can be carried out "diagramatically" by *composing* the ribbons or shapes.

Definition 2.29 (Composing ribbons). Two ribbons R, S are composable if $B_R = A_S$. The composition $R \circ S$ is the ribbon $T = (A_R, B_S, E(R) \cup E(S))$. Although it never occurs in the current work, see the footnote³ for the case $E(R) \cap E(S) \neq \emptyset$.

Fact 2.30. If R, S are composable ribbons, then $\mathbf{M}_{R\circ S} = \mathbf{M}_R \mathbf{M}_S$. Otherwise, $\mathbf{M}_R \mathbf{M}_S = 0$.

Therefore, when two matrices expressed as a linear combination of ribbons are multiplied, the effect is to compose every pair of ribbons.

³In this case, define $R \circ S$ as an improper ribbon (Definition 2.39) whose edge multiset is the disjoint union of E(R) and E(S).

The composition of two ribbons R, S can be easily visualized by drawing the two ribbons next to each other, then identifying the sets B_R and A_S . However, if the vertex sets of two composable ribbons R, S overlap nontrivially (i.e., beyond the "necessary" overlap $B_R = A_S$), then the resulting ribbon is smaller than this picture suggests. We will call these types of ribbons *intersection terms* and classify them based on their *intersection pattern*. The intersection terms are error terms in our analysis, but carefully bounding them is the most important and difficult conceptual step of the proof.

Definition 2.31 (Properly composable). *Composable ribbons* $R_1, ..., R_k$ are properly composable *if there are no intersections beyond the necessary ones* $B_{R_i} = A_{R_{i+1}}$.

With the above definitions and Proposition 2.23, we can deduce the main proposition about shape and ribbon factoring. For a ribbon *R* and a set of edges $F \subseteq {[n] \choose 2}$, we use the notation $R \setminus F$ to denote the ribbon $(A_R, B_R, E(R) \setminus F)$.

Definition 2.32 (Floating component). *The connected components of a ribbon* R *which are not connected to* $A_R \cup B_R$ *are called* floating components.

Proposition 2.33 (Ribbon decomposition). *Every ribbon R can be expressed as*

$$R = (L \setminus E(B_L)) \circ M \circ (R' \setminus E(A_{R'}))$$

where L, M, R' are properly composable left, middle, and right ribbons respectively, such that $E(B_L) = E(A_M)$ and $E(B_M) = E(A_{R'})$. Up to the orderings of $B_L = A_M$ and $B_M = A_{R'}$ and the floating components, the decomposition is unique.

Proof. The existence of the decomposition follows by splitting *R* across the leftmost and rightmost SMVS. Edges inside the SMVS should be put into the middle ribbon. Any floating components can be put into the middle ribbon.

To argue uniqueness, suppose $R = (L \setminus E(B_L)) \circ M \circ (R' \setminus E(A_{R'}))$. Then $B_L = A_M$ is an SMVS of *R* which is left of all other SMVS of *R*. By the structural result Proposition A.1, in order for *L* to have a unique SMVS, it must be that B_L is the leftmost SMVS of *R*. The same holds for *R'* with respect to the rightmost SMVS.

Remark 2.34. *When we decompose a ribbon, we will always put the floating components into the middle part.*

2.7 Graph matrix calculus: intersections

Based on the decomposition theorem for ribbons, it would be ideal if for any shape $\alpha = \sigma \circ \tau \circ \sigma'$, we also had an exact matrix equality

$$\mathbf{M}_{\alpha} = \mathbf{M}_{\sigma} \cdot \mathbf{M}_{\tau} \cdot \mathbf{M}_{\sigma'}$$
.

Unfortunately this fails as we have may "surprise" intersections beyond the necessary ones along the boundary. Let us describe in further detail these "intersection terms".

Definition 2.35 (Composing shapes). Given shapes α , β , we call them composable if $V_{\alpha} = U_{\beta}$ as subsets of I. The composition $\alpha \circ \beta$ is the shape whose multigraph is the result of gluing together α and β along V_{α} and U_{β} (following their respective orders)⁴, whose left side is U_{α} , and whose right side is V_{β} . See the footnote⁵ for an additional technicality.

Definition 2.36 (Intersection pattern). For composable shapes $\alpha_1, \alpha_2, ..., \alpha_k$, let $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$. An intersection pattern *P* is a partition of $V(\alpha)$ such that for all *i* and $v, w \in V(\alpha_i)$, *v* and *w* are not in the same block of the partition. We say that a vertex "intersects" if its block has size at least 2 and let $V_{intersected}(\alpha_i)$ denote the set of intersecting vertices in α_i .

Let $\mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_k}$ be the set of intersection patterns between $\alpha_1,\alpha_2,\ldots,\alpha_k$.

Definition 2.37 (Intersection shape). For composable shapes $\alpha_1, \alpha_2, ..., \alpha_k$ and an intersection pattern $P \in \mathcal{P}_{\alpha_1,\alpha_2,...,\alpha_k}$, let $\alpha_P = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$ then identify all vertices in blocks of P, i.e. contract them into a single vertex. Keep all edges.⁶

Proposition 2.38. For composable shapes $\alpha_1, \alpha_2, \ldots, \alpha_k$,

$$\mathbf{M}_{lpha_1}\cdots\mathbf{M}_{lpha_k} = \sum_{P\in\mathcal{P}_{lpha_1,\dots,lpha_k}} \mathbf{M}_{lpha_P}$$

Proof. The claimed statement expands to

L

$$\prod_{i=1}^{\kappa} \sum_{\text{injective } \varphi_i: V(\alpha_i) \to [n]} \mathbf{M}_{\varphi_i(\alpha_i)} = \sum_{P \in \mathcal{P}_{\alpha_1, \dots, \alpha_k}} \sum_{\text{injective } \varphi: V(\alpha_P) \to [n]} \mathbf{M}_{\varphi(\alpha_P)}.$$

Each of the ribbons on the left-hand side is an injective embedding of α_i into [n]; however, the joint embedding need not be injective. The intersection pattern P cases on which vertices overlap.

2.8 Graph matrix calculus: improper shapes and linearization

We will need to manipulate matrices expressed in the ribbon or shape basis, for example by casing on whether certain edges exist or multiplying two matrices together. To simplify the intermediate manipulations, we will allow ribbons to be *improper*. We lose uniqueness of representation in the ribbon basis (there are multiple ways to express a given matrix as a linear combination of improper ribbons), but the augmentation lets us easily track combinatorial features such as the presence of specific edges or subgraphs. At the end of the proof, we convert improper ribbons back into proper ones by *linearizing* them.

Our ribbons may be improper in three ways: *edge indicators, products of Fourier characters,* and *isolated vertices*. These are all contained in the following general definition.

⁴Although this never occurs in the current work, if an edge occurs inside both V_{α} and U_{β} , the composition $\alpha \circ \beta$ is an improper shape (Definition 2.39).

⁵If a vertex in V_{α} and U_{β} has degree 0, it becomes an isolated vertex in $\alpha \circ \beta$ (Definition 2.41).

⁶Keep duplicated edges with multiplicity; α_P may be improper (Definition 2.39).

Definition 2.39 (Improper ribbon). An improper ribbon is a tuple R given by $R = (A_R, B_R, V(R), E(R), \text{Yes}(R))$ where $A_R, B_R \in I$ as before, and additionally $A_R \cup B_R \subseteq V(R) \subseteq [n]$, also E(R) is a multigraph on V(R) without self-loops, and $\text{Yes}(R) \subseteq {\binom{V(R)}{2}}$. We extend the definition of $\mathbf{M}_R \in \mathbb{R}^{I \times I}$ to:

$$\mathbf{M}_{R}[I,J] = \begin{cases} \prod_{e \in \mathsf{Yes}(R)} \mathbf{1}_{e \in E(G)} \prod_{e \in \binom{V(R)}{2}} \chi_{e}(G)^{multiplicity of e in E(R)} & I = A_{R}, J = B_{R} \\ 0 & otherwise \,. \end{cases}$$

Definition 2.40 (Improper shape). An improper shape α is defined, as before, as an equivalence class of improper ribbons under relabeling of the vertices. Each improper shape is associated with a representative tuple $(U_{\alpha}, V_{\alpha}, V(\alpha), E(\alpha), \text{Yes}(\alpha))$. Equivalently, a ribbon *R* has shape α if *R* can be obtained by labeling $V(\alpha)$ by an injective mapping $\varphi : V(\alpha) \rightarrow [n]$.

Definition 2.41 (Isolated vertices). Let $Iso(\alpha)$ be the set of vertices in $V(\alpha) \setminus (U_{\alpha} \cup V_{\alpha})$ which are not incident to any edges in $E(\alpha)$ or $Yes(\alpha)$.

We linearize an improper ribbon by using identities

$$\chi_e^k = c_0 + c_1 \cdot \chi_e$$
$$\mathbf{1}_{e \in E(G)} = c'_0 + c'_1 \cdot \chi_e$$

for some coefficients c_0, c_1, c'_0, c'_1 . The equalities hold for inputs from $\{0, 1\}$.

Definition 2.42 (Linearization). *Given an improper ribbon R, we* linearize *R by using the equality*

$$\mathbf{M}_R = \sum_{proper \ ribbons \ S} c_S \mathbf{M}_S$$

where c_S are the Fourier coefficients of \mathbf{M}_R . The ribbons *S* which appear with nonzero coefficient c_S are called the linearizations of *R*.

When we linearize just the multiedges χ_e^k into either 1 or χ_e , we use the following proposition to bound the new coefficient.

Proposition 2.43. *Given an improper ribbon R, if we linearize the multiedges, the coefficient on a resulting ribbon S satisfies*

$$|c_{S}| \leq \left(\sqrt{\frac{1-p}{p}}\right)^{\sum_{e \in mul(R)} mult(e) - 1 - \mathbf{1}_{e \text{ vanishes}}}$$

where mul(R) is the set of multi-edges in R.

Proof. The linearization coefficients are $\chi_e^k = \mathbb{E}[\chi_e^k] + \mathbb{E}[\chi_e^{k+1}]\chi_e$. By induction, $|\mathbb{E}[\chi_e^k]| \le \left(\sqrt{\frac{1-p}{p}}\right)^{k-2}$ for all $k \ge 2$, which yields the claim.

We will estimate the norm of an improper shape by linearizing it and taking the maximum norm among all of its linearizations.

2.9 Pseudocalibration

Pseudocalibration is a heuristic used to construct candidate pseudoexpectation operators $\widetilde{\mathbb{E}}$ for SoS lower bounds, introduced in the context of SoS lower bounds for Planted Clique [BHK⁺16]. See e.g. [BHK⁺16, RSS18, GJJ⁺20] for a formal description.

The pseudocalibrated operator $\widetilde{\mathbb{E}}[\mathbf{X}^{I}]$ is defined using the Fourier coefficients of the corresponding function $\mathbf{X}^{I}(H)$ evaluated on the planted distribution. First we need to compute these Fourier coefficients. A similar computation was performed by [CM18] to exhibit integrality gaps for the Sherali-Adams hierarchy.

Lemma 2.44. Let $X^{I}(H)$ be the 0/1 indicator function for I being in the planted solution i.e. $I \subseteq H$. Then, for all $I \subseteq [n]$ and $\alpha \subseteq {[n] \choose 2}$,

$$\mathbb{E}_{(G,H)\sim\mathcal{D}_{pl}}[\mathbf{X}^{I}(H)\cdot\chi_{\alpha}(\widetilde{G})] = \left(\frac{k}{n}\right)^{|V(\alpha)\cup I|} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(\alpha)|}$$

Proof. First observe that if any vertex of $V(\alpha) \cup I$ is outside H, then the expectation is 0. This is because either I is outside H, in which case $\mathbf{X}^{I}(H) = 0$, or an edge of α is outside H, in which case the expectation of this Fourier character is 0. Now, each vertex of $V(\alpha) \cup I$ is in H independently with probability $\frac{k}{n}$. Conditioned on this event happening, each edge independently evaluates to

$$\mathbb{E}_{e \sim \text{Bernoulli}(q)} \chi(e) = q \cdot \chi(1) + (1-q) \cdot \chi(0) = \frac{q-p}{\sqrt{p(1-p)}}$$

Putting these together gives the result.

Pseudocalibration suggests transferring the low-degree Fourier coefficients from the planted distribution, in order for the pseudoexpectation operator to emulate a true expectation operator from a planted distribution. As long as the size of the Fourier coefficients is larger than the SoS degree, then SoS should not notice that we are using a truncation.

We will truncate up to shapes of size D_V for a parameter $D_V = O(D_{SoS})$ which is formally specified in Bound C.1.

Definition 2.45 (S). Let S be the set of (proper) ribbons R such that:

- (*i*) (*Degree bound*) $|A_R|$, $|B_R| \le D_{SoS}/2$
- (*ii*) (*Size bound*) $|V(R)| \le D_V$

We will sometimes use $\alpha \in S$ *as the set of shapes with the same properties, following the convention of using Latin letters for ribbons and Greek letters for shapes.*

Definition 2.46 (M). Define the pseudocalibrated candidate moment matrix

$$\mathbf{M} = \sum_{R \in \mathcal{S}} \left(\frac{k}{n}\right)^{|V(R)|} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|} \mathbf{M}_R$$

For technical convenience, we adjust the parameters β and γ slightly so that

$$n^{-\beta} = \frac{p}{1-p}, \qquad \qquad n^{-\gamma} = \frac{q-p}{1-p}$$

This change does not formally affect the statement of Theorem 1.1.

For the purposes of analyzing the spectrum of **M** in later sections, it is more convenient to rescale the entries so that $\widetilde{\mathbb{E}}[\mathbf{X}^{I}]$ has order 1 for all $I \subseteq [n]$. This will be the matrix $\mathbf{\Lambda}$.

Definition 2.47 (λ_{α}). *Given a shape or ribbon* α *, let*

$$\lambda_{\alpha} = \left(\frac{k}{n}\right)^{|V(\alpha)| - \frac{|U_{\alpha}| + |V_{\alpha}|}{2}} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(\alpha)|}$$
$$= n^{(\alpha-1)\left(|V(\alpha)| - \frac{|U_{\alpha}| + |V_{\alpha}|}{2}\right) + (\frac{\beta}{2} - \gamma)|E(\alpha)|}.$$

Lemma 2.48. *If* R, S *are properly composable ribbons, then* $\lambda_{R \circ S} = \lambda_R \lambda_S$.

Definition 2.49 (A). *Define* $\Lambda = \sum_{R \in S} \lambda_R \mathbf{M}_R$.

Lemma 2.50. $\mathbf{M} \geq 0$ *if and only if* $\mathbf{\Lambda} \geq 0$ *.*

Proof. We have $\mathbf{M} = \mathbf{D}\mathbf{\Lambda}\mathbf{D}$ where \mathbf{D} is a diagonal matrix with positive entries $\mathbf{D}[I, I] = \left(\frac{k}{n}\right)^{\frac{|I|}{2}}$. Hence $x^{\mathsf{T}}\mathbf{M}x \ge 0$ for all $x \in \mathbb{R}^{I}$ if and only if $x^{\mathsf{T}}\mathbf{\Lambda}x \ge 0$ for all $x \in \mathbb{R}^{I}$.

Lemma 2.51. M *is* SoS-symmetric and satisfies the constraints " $X_i^2 = X_i$ ".

Proposition 2.52. With high probability, we have $\widetilde{\mathbb{E}}[1] = 1 \pm o(1)$.

Proof (formal version in Appendix F).

$$\widetilde{\mathbb{E}}[1] - 1 = \Big| \sum_{\substack{\alpha \in \mathcal{S}: \\ U_{\alpha} = V_{\alpha} = \emptyset, \\ E(\alpha) \neq \emptyset}} \lambda_{\alpha} \mathbf{M}_{\alpha} \Big| \le \sum_{\substack{\alpha \in \mathcal{S}: \\ U_{\alpha} = V_{\alpha} = \emptyset, \\ E(\alpha) \neq \emptyset}} \lambda_{\alpha} || \mathbf{M}_{\alpha} ||$$

Up to a subpolynomial factor that is offset by edge decay, the norm bounds are (to be computed later in Lemma 4.8),

$$\lambda_{\alpha} \|M_{\alpha}\| \leq n^{-(\frac{1}{2}-\alpha)w(\alpha) - \frac{w(S)}{2} - (\gamma - \alpha\beta)|E(\alpha)|}.$$

With high probability over the random graph $G \sim \mathcal{G}_{n,p}$, all small subgraphs H have $w(H) \ge 0$ (up to a small error term). Therefore, a conditioning argument will get rid of any small shape such with $w(\alpha) < 0$ or w(S) < 0. The remaining shapes have both $w(\alpha) \ge 0$ and $w(S) \ge 0$. Since the parameters satisfy $\frac{1}{2} - \alpha \ge 0$ and $\gamma - \alpha\beta > 0$, the exponent of n is negative and the entire sum is o(1).

Remark 2.53. $\mathbb{E}[1] = 1 \pm o(1)$ fails with inverse polynomial probability. This is equivalent to the observation that there is a low-degree distinguisher that succeeds with inverse polynomial probability. Specifically, if we consider a constant-size subgraph which is unlikely to appear in $\mathcal{G}_{n,p}$ (e.g. K_{10} when p is sufficiently small), the probability that the dense subgraph $\mathcal{G}_{k,q}$ contains a copy of the subgraph is larger by a poly(n) factor. That said, even when $\mathbb{E}[1] \gg 1$, we can still show that $\Lambda \geq 0$ with high probability. For details, see the appendix.

Proposition 2.54. With high probability,

$$\left|\sum_{i=1}^{n} \widetilde{\mathbb{E}}[\mathbf{X}_i] - k\right| = o(k),$$

and

$$\sum_{\{i,j\}\in E(G)}\widetilde{\mathbb{E}}[\mathbf{X}_i\mathbf{X}_j] - \frac{k^2q}{2} = o(k^2q)$$

Proof (formal version in Appendix F). For each i, $\widetilde{\mathbb{E}}[\mathbf{X}_i] = \mathbf{M}[(i), \emptyset]$. For most vertices i, the dominant term in $\mathbf{M}[(i), \emptyset]$ is $\frac{k}{n}\mathbf{M}_R[(i), \emptyset]$ where R is the ribbon such that $V(R) = \{i\}$, $A_R = (i), B_R = \emptyset$, and $E(R) = \emptyset$. $\mathbf{M}_R[(i), \emptyset] = 1$ so this gives a contribution of $\frac{k}{n}$. Summing this over all $i \in [n]$ gives k. In Appendix F, we verify that the contribution from the other terms is o(k) with high probability.

For each $i, j \in V(G)$ such that i < j, $\mathbf{1}_{\{i,j\}\in E(G)} \widetilde{\mathbb{E}}[\mathbf{X}_i\mathbf{X}_j] = \mathbf{1}_{\{i,j\}\in E(G)}\mathbf{M}[(i), (j)]$. For most i, j, the dominant term in $\mathbf{1}_{\{i,j\}\in E(G)}\mathbf{M}[(i), (j)]$ is $\frac{k^2(q-p)}{n^2\sqrt{p(1-p)}}\mathbf{1}_{\{i,j\}\in E(G)}\mathbf{M}_R[(i), (j)]$ where R is the ribbon such that $V(R) = \{i, j\}, A_R = (i), B_R = (j), \text{ and } E(R) = \{\{i, j\}\}.$

 $\mathbf{1}_{\{i,j\}\in E(G)}\mathbf{M}_R[(i),(j)] = \sqrt{\frac{1-p}{p}}$ if $\{i,j\} \in E(G)$ and 0 if $\{i,j\} \notin E(G)$. Summing over all *i* < *j* gives a total contribution of $\frac{k^2(q-p)}{n^2p}|E(G)| \approx \frac{k^2q}{2}$. In Appendix F, we verify that the contribution from the other terms is $o(k^2q)$ with high probability. ■

3 Positive Minimum Vertex Separator Decomposition

3.1 Motivation for the positive minimum vertex separator

After pseudocalibration, to complete the proof of Theorem 1.1, we need to show that the rescaled candidate moment matrix is PSD with high probability,

$$\mathbf{\Lambda} = \sum_{\alpha \in \mathcal{S}} \lambda_{\alpha} \cdot \mathbf{M}_{\alpha} \geq 0 \,.$$

For each graph matrix $\lambda_{\alpha} \mathbf{M}_{\alpha}$ in $\mathbf{\Lambda}$, we want to find an approximately-PSD term which spectrally dominates it. Previous work led to the following idea: for each shape α , we can split it across the leftmost and rightmost minimum vertex separators so that α is decomposed into three parts,

 $\alpha = \sigma \circ \tau \circ \sigma'^{\intercal} .$

Then the target spectral upper bound is given by

$$\lambda_{\sigma}^2 \mathbf{M}_{\sigma \circ \sigma^{\intercal}} + \lambda_{\sigma'}^2 \mathbf{M}_{\sigma' \circ \sigma'^{\intercal}}$$

This is approximately PSD since $\mathbf{M}_{\sigma\circ\sigma^{\intercal}} \approx \mathbf{M}_{\sigma}\mathbf{M}_{\sigma}^{\intercal} \geq 0$. To make this strategy work, we need to prove that the middle shape \mathbf{M}_{τ} is spectrally dominated by the corresponding identity via combinatorial charging. In previous work, it has been essentially possible to charge all middle shapes to the identity matrix, but this breaks down in the setting of Densest *k*-Subgraph. In the baby case, this is evident in our calculation for $\sum_{(u,v)\in E(G)} \widetilde{\mathbb{E}}[\mathbf{X}_{u}\mathbf{X}_{v}]$ in Proposition 2.54, where the dominant term is no longer the trivial shape but instead the shape with an edge in between *i* and *j*.

A second, related issue is the presence of edges inside the separator. Concretely, say that $(U_{\tau}, E(U_{\tau}))$ and $(V_{\tau}, E(V_{\tau}))$ are the leftmost/rightmost SMVS of a middle shape τ , and we hope to charge τ to the diagonal matrix corresponding to the leftmost/rightmost SMVS. Concretely, letting U_{τ} also denote the diagonal shape with edges $E(U_{\tau})$, we want to charge

$$\lambda_{\tau}(\mathbf{M}_{\tau} + \mathbf{M}_{\tau}^{\dagger}) \leq \lambda_{U_{\tau}}\mathbf{M}_{U_{\tau}} + \lambda_{V_{\tau}}\mathbf{M}_{V_{\tau}}.$$

However, this strategy crucially requires that $\lambda_{U_{\tau}} \cdot \mathbf{M}_{U_{\tau}}$ and $\lambda_{V_{\tau}} \cdot \mathbf{M}_{V_{\tau}}$ are PSD by themselves in order to conclude that the result is PSD. Since λ_{α} is non-negative, this boils down to the PSD-ness of the diagonal shape $(U_{\tau}, E(U_{\tau}))$ for the SMVS. This latter

matrix is easy to verify as the non-zero diagonal entries are given by, for a ribbon *R* of the corresponding shape U_{τ} ,

$$\chi_{E(R)}(G) = \prod_{e \in E(R)} \chi_e(G)$$

and recall that we are working on the *p*-biased Fourier basis,

$$\chi_e(1) = \sqrt{\frac{1-p}{p}}, \quad \chi_e(0) = -\sqrt{\frac{p}{1-p}}$$

At this point, we observe that the instantiation of the SMVS edges E(R) plays a crucial role as they determine whether our candidate "PSD" mass is truly positive. If all edges of E(R) are present in G, then the diagonal entry is positive,

$$\prod_{e\in E(R)}\chi_e(G)=\sqrt{\frac{1-p}{p}}|E(R)|\geq 0.$$

On the other hand, if an edge is missing, then positivity is not guaranteed. Ignoring this bad case for now, we have the following sufficient criterion for finding a PSD dominant term. If *T* is a ribbon of shape τ , and *R* is the restricted ribbon to U_{τ} , then if $E(R) \subseteq E(G)$, we must charge $\lambda_{\tau} \mathbf{M}_T$ to $\lambda_{U_{\tau}} \mathbf{M}_R$.

When an edge is missing inside the SMVS, then we need to look harder. Despite the candidate PSD term not being truly positive, it is not yet time to panic. In this case, (1) a missing edge scales down the matrix, in line with the intuition that subgraphs with edges present are the highest-norm terms, therefore (2) we look in the remainder of the shape for the new SMVS, to determine the new matrix norm. This creates a recursive process, and when all edges inside the candidate SMVS are actually present in the graph, we terminate, calling this the *Positive Minimum-weight Vertex Separator (PMVS)*.

Let us give an example. The graph matrix at the top of Fig. 2 appears on the diagonal of our moment matrix. In this example shape, the only vertex separator is the entire shape, and so the SMVS contains the edge G(a, b). We check whether or not the edge appears in the graph. In the "yes" outcome on the left, we have a PSD matrix whose (a, b)-th diagonal entry is $\mathbf{1}_{(a,b)\in E(G)}\sqrt{\frac{1-p}{p}}$. In the "no" outcome on the right, the (a, b)-th diagonal entry is $-\mathbf{1}_{(a,b)\notin E(G)}\sqrt{\frac{p}{1-p}}$, which is negative and therefore the matrix is not PSD. This matrix comes with a small coefficient of approximately \sqrt{p} and hence it can be charged to the corresponding identity matrix, whose (a, b)-th diagonal entry is just 1. In this example, the recursion terminates after just one step, but in larger shapes, we would need to find the new SMVS for the case on the right.

It would have been cleaner if one can define the PMVS in "one shot", rather than through a recursion. As described above, the recursion outputs the minimizer of the

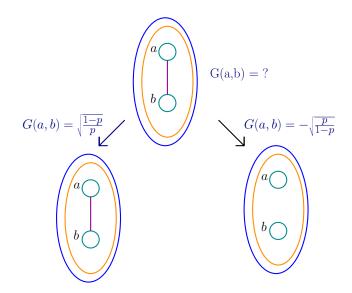


Figure 2: PMVS Search

weight function defined in Appendix B. However, we need to slightly modify the recursive process described above so that it always "moves left", in order for the crucial Remark 3.7 to hold.

3.2 PMVS subroutine

We make the following extended definition of the *Positive Minimum Vertex Separator* (*PMVS*) of a ribbon *R*.

Definition 3.1 (Left and right indicators). We say that a ribbon R has left indicators if R has edge indicators for every edge $e \in E(A_R)$. Similarly, we say that a ribbon R has right indicators if R has edge indicators for every edge $e \in E(B_R)$.

Our goal is to have composable triples of ribbons R_1, R_2, R_3 with the following properties:

Definition 3.2 (Ribbons with PMVS identified). *A composable triple of ribbons* R_1 , R_2 , R_3 *has PMVS identified if:*

- (*i*) R_1 is a left ribbon and R_3 is a right ribbon.
- (ii) R_1, R_2, R_3 are properly composable.
- (iii) R_1 has right indicators, R_2 has both left and right indicators, and R_3 has left indicators.
- (iv) The edges and edge indicators agree on $B_{R_1} = A_{R_2}$ and $B_{R_2} = A_{R_3}$.
- (v) R_1, R_2, R_3 have no other edge indicators.

When these properties hold, we say that the left PMVS is A_{R_2} and the right PMVS is B_{R_2} .

Remark 3.3. The left and right PMVS may not have the same size or weight. In fact, they may not even be an SMVS of R_2 . We will bound the difference between the PMVS and the SMVS in Section 4.

At the beginning, we take each ribbon R and decompose it into ribbons R_1 , R_2 , R_3 based on the leftmost and rightmost SMVS using Proposition 2.33. This gives us a composable triple of ribbons R_1 , R_2 , R_3 such that

- (i) R_1 is a left ribbon, R_2 is a middle ribbon, and R_3 is a right ribbon.
- (ii) R_1, R_2, R_3 are properly composable.
- (iv) The edges agree on $B_{R_1} = A_{R_2}$ and $B_{R_2} = A_{R_3}$.
- (v) R_1 , R_2 , and R_3 have no edge indicators.

Remark 3.4. The ribbon encoded by the triple R_1, R_2, R_3 is $(R_1 \setminus E(B_{R_1})) \circ R_2 \circ (R_3 \setminus E(A_{R_3}))$ rather than $R_1 \circ R_2 \circ R_3$ because edges inside $B_{R_1} = A_{R_2}$ should not be duplicated.

In order to satisfy the condition that R_1 has right indicators, R_2 has both left and right indicators, and R_3 has left indicators, we repeat the following sequence of operations as many times as needed.

- 1. Adding left and right indicators operation: To add indicators to $B_{R_1} = A_{R_2}$ and $B_{R_2} = A_{R_3}$, we replace each edge $e \in E(A_{R_2}) \cup E(B_{R_2})$ that does not yet have an indicator using⁷ the equation $\chi_e = \frac{1}{1-p} \mathbf{1}_{e \in E(G)} \chi_e \sqrt{\frac{p}{1-p}}$. This leads to two possible new ribbons which have different edge structure, one with *e* still present and the other with *e* removed.
- 2. **PMVS operation:** After adding the edge indicators to $B_{R_1} = A_{R_2}$ and $B_{R_2} = A_{R_3}$, we check if R_1 is still a left ribbon and R_3 is still a right ribbon. If so, we stop and exit the loop. If not, we let A' be the leftmost SMVS separating A_{R_1} from B_{R_1} and we let B' be the rightmost SMVS separating A_{R_3} from B_{R_3} . We then replace R_1 , R_2 , and R_3 with the ribbons R'_1 , R'_2 , and R'_3 where
 - (a) R'_1 is the part of R_1 between A_{R_1} and A'.
 - (b) R'_2 is the composition of the part of $R_1 \setminus E(B_{R_1})$ between A' and B_{R_1} , R_2 , and the part of $R_3 \setminus E(A_{R_3})$ between A_{R_3} and B'.
 - (c) R'_3 is the part of R_3 between B' and B_{R_3} .

⁷The high-level overview of the PMVS alluded to the slightly different formula $\chi_e = \mathbf{1}_{e \in E(G)} \chi_e + \mathbf{1}_{e \notin E(G)} \chi_e$. These are morally equivalent, but the formula here is simpler to analyze.

3. **Removing middle edge indicators operation:** If R₂ has one or more edge indicators which are now outside of A_{R_2} and B_{R_2} , we re-convert them back into Fourier characters using the equation $\frac{1}{1-p}\mathbf{1}_{e\in E(G)}\chi_e = \sqrt{\frac{p}{1-p}} + \chi_e$.

We call this repeated sequence of operations the **Finding PMVS subroutine**, which takes a triple of composable ribbons R_1, R_2, R_3 which have all the needed properties except having left and right indicators (some but not all indicators may be present) and gives us a triple of composable ribbons with all of the needed properties.

Remark 3.5. Note that each triple R_1 , R_2 , R_3 leads to many triples R'_1 , R'_2 , R'_3 depending on which summand is taken in each equation. The recursion proceeds on every term except for the one in which every χ_e is replaced by $\frac{1}{1-p} \mathbf{1}_{e \in E(G)} \chi_e$.

Remark 3.6. At first glance, checking whether or not edges inside A_{R_2} and B_{R_2} are present leads to a complicated dependence on the input graph G. In order to mathematically express the recursion in a G-independent way, we formally use the edge indicator function to express the two cases.

Intersection term operation 3.3

Once we have these triples of ribbons R_1 , R_2 , R_3 , we can apply an approximate factorization across the PMVS. When we do this, we will obtain error terms which can be described by triples of ribbons R_1 , R_2 , and R_3 which have at least one non-trivial intersection (they are not properly composable) but satisfy the other four properties in Definition 3.2. We handle this as follows.

- 1. Intersection term decomposition operation: Let A' be the leftmost SMVS between A_{R_1} and $B_{R_1} \cup V_{intersected}(R_1)$ and let B' be the rightmost SMVS between $A_{R_3} \cup V_{intersected}(R_3)$ and B_{R_3} . We now replace R_1 , R_2 , and R_3 with the ribbons R'_1, R'_2 , and R'_3 where
 - (a) R'_1 is the part of R_1 between A_{R_1} and A'.
 - (b) To obtain R'_2 , we improperly compose the part of $R_1 \setminus E(B_{R_1})$ between A' and B_{R_1} , R_2 , and the part of $R_3 \setminus E(A_{R_3})$ between A_{R_3} and B'. We then linearize the multi-edges, replacing $\chi_e^k = c_0 + c_1 \chi_e$ using the appropriate coefficients c_0, c_1 .

In the edge case that a multi-edge also has an edge indicator (for example, because an each inside A_{R_2} intersects with an edge from R_3), we instead use the equation $\mathbf{1}_e \chi_e^k = \left(\sqrt{\frac{1-p}{p}}\right)^{k-1} \mathbf{1}_e \chi_e$.

- (c) R'_3 is the part of R_3 between B' and B_{R_3} .
- 2. We apply the **Removing middle edge indicators operation** to *R*₂.

The ribbon R'_2 is defined to "grow" R_2 so that it includes the intersections. After these steps, we are in essentially the same situation as we started. More precisely, we have a triple of ribbons R_1 , R_2 , R_3 such that

- (i) R_1 is a left ribbon and R_3 is a right ribbon.
- (ii) R_1, R_2, R_3 are properly composable.
- (iv) The edges and edge indicators agree on $B_{R_1} = A_{R_2}$ and $B_{R_2} = A_{R_1}$.
- (v) R_1, R_2, R_3 have no edge indicators outside of $B_{R_1} = A_{R_2}$ and $B_{R_2} = A_{R_3}$.

At this point, we can repeat the operations, applying the **Finding PMVS subroutine** to identify a new PMVS, approximately factoring, then decomposing intersection terms, as many times as needed.

3.4 Summary of the operations and overall decomposition

We now summarize our procedure.

Finding PMVS subroutine: repeat the following until convergence,

- 1. Apply the **Adding left and right indicators operation** to add indicators to $B_{R_1} = A_{R_2}$ and $B_{R_2} = A_{R_3}$.
- 2. Apply the **PMVS operation** to ensure that R_1 is a left ribbon and R_3 is a right ribbon. If no change is made to R_1 or R_3 , then we have identified the PMVS.
- 3. Apply the **Removing middle edge indicators operation** to R_2 to ensure that R_2 has no middle indicators.

Overall decomposition procedure:

- 1. We start with triples of composable ribbons R_1, R_2, R_3 which have all the needed properties except having left and right indicators.
- 2. We apply the Finding PMVS subroutine.
- 3. Recursive factorization: We apply the following procedure repeatedly until there are no more error terms.
 - 1. We approximate the sum over the composable triples of ribbons R_1, R_2, R_3 by enlarging the sum to include all left ribbons R_1 and right ribbons R_3 (not necessarily properly composable with R_2 or with each other). This yields a matrix LQ_iL^{T} where L sums over left ribbons and Q_i sums over the ribbons R_2 on the

*i*th iteration of the loop. We then move to the triples of ribbons R_1 , R_2 , R_3 for the intersection error terms, if any.⁸

- 2. We apply the **Intersection term decomposition operation** to obtain a triple of ribbons *R*₁, *R*₂, *R*₃ which are properly composable.
- 3. We apply the **Removing middle edge indicators operation** to R_2 to ensure that R_2 has no middle indicators.
- 4. We apply the **Finding PMVS subroutine**.

Remark 3.7. As with previous SoS lower bounds using graph matrices, a key observation is that the PMVS operation and the intersection term decomposition operation are unaffected by replacing R'_1 with a different left ribbon R''_1 or replacing R'_3 with a different right ribbon R''_3 as long as $B_{R''_1} = B_{R'_1} = A_{R'_2}$ and $A_{R''_3} = A_{R'_3} = B_{R'_2}$. This ensures that all left ribbons R'_1 and right ribbons R'_3 appear in the matrices **L** and **L**[†].

Carrying out this process, the overall decomposition of the moment matrix is then

$$\mathbf{\Lambda} = \mathbf{L} \left(\sum_{i=0}^{D_V} \mathbf{Q}_i \right) \mathbf{L}^{\mathsf{T}} \pm \text{truncation error} \,.$$

Therefore, the main requirement for $\Lambda \geq 0$ is to show that $\sum_{i=0}^{D_V} \mathbf{Q}_i \geq 0$. We will show that the norm-dominant terms are the diagonal shapes (Definition 2.16). By virtue of the PMVS factorization, these shapes are PSD, as we can easily check.

Lemma 3.8. If R_1, R_2, R_3 are ribbons with PMVS identified, such that R_2 is diagonal, then $\lambda_{R_2} \mathbf{M}_{R_2} \geq 0$.

Proof. $\lambda_{R_2} \ge 0$ and R_2 is diagonal with one nonzero entry, so we need that the entry is nonnegative. Since R_2 has edge indicators, the entry is

$$\prod_{e \in E(R_2)} \mathbf{1}_{e \in E(G)} \chi_e(G) \, .$$

Any time the entry is nonzero, its value is $\chi(1)^{|E(R_2)|} = \left(\sqrt{\frac{1-p}{p}}\right)^{|E(R_2)|} \ge 0.$

In the next section, we will prove that the norm of any individual term in the Q_i is small relative to these PSD terms, which is the key remaining component of the PSDness proof. Summing over all the terms and bounding the truncation error is done in Appendix E.

⁸There are also additional error terms for the truncation error, as the maximum size of the left ribbons R_1 , R_3 will be slightly smaller for intersection terms. This must be handled separately.

⁹We enlarge the sum to include only ribbons R_1 such that $B_{R_1} = A_{R_2}$ and R_3 such that $A_{R_3} = B_{R_2}$. For this reason, the matrix **L** is slightly more restricted than including all left ribbons.

So far, we have described how the ribbons are manipulated. Each ribbon also comes with a coefficient that we need to track. Initially, the coefficient of every ribbon *R* is λ_R . Since the coefficients satisfy $\lambda_{R\circ S} = \lambda_R \lambda_S$ (Lemma 2.48), we may factor λ_R whenever we factor the ribbon. Doing so, the left and right ribbons R_1 and R_3 always come with the factors λ_{R_1} and λ_{R_3} .

The coefficient on R_2 is initially λ_{R_2} , but it accrues extra factors in some steps of the process.

Definition 3.9 (c_R). Given ribbons R_1, R_2, R_3 which produce ribbons R'_1, R'_2, R'_3 let $c_{R'_2}$ be such that the final coefficient is $c_{R'_2}\lambda_{R'_1}\lambda_{R'_2}\lambda_{R'_3}$. Note that $c_{R'_2}$ does not depend on R'_1 or R'_3 , but it does depend on the cases in the decomposition process.

For example, during the **Adding left and right indicators operation**, $c_{R'_2}$ accrues a factor of $\frac{1}{1-p}$ for each edge indicator or a factor $-\sqrt{\frac{p}{1-p}}$ if an edge is removed. It changes in the same way during the **Removing middle edge indicators operation**. It will also be multiplied by the excess edge and vertex factors during the **Intersection term decomposition operation**, as well as the linearization coefficient either c_0 or c_1 .

4 Combinatorial Norm Charging Arguments

Now that we have identified the dominant PSD terms, we analyze the norms of the nondominant terms that appear during the decomposition process in Section 3.4 and show that they are small.

Each graph matrix making up Λ has norm poly(*n*) times additional log factors. It is most important to perform the proof at the coarse level of poly(*n*), ignoring the log factors and other relatively small¹⁰ combinatorial factors such as poly(D_{SoS}). In this section, we will work at the coarse level by defining away all of the subpolynomial factors in order to focus on the key combinatorial arguments. Only the first subsection will involve *n*, and the remaining subsections will be pure combinatorics on shapes. The lower-order factors will be formally incorporated in Appendix E.

4.1 Setup

In this section we will use the parameterization α , β , γ instead of k, p, q.

Remark 4.1. Take note that there is a notation clash between the size of the dense subgraph $\alpha \in (0, 1/2)$ and a generic shape α , and also between the edge density of the planted subgraph $\gamma \in (0, 1)$ and a left shape γ which participates in an intersection. It should be clear from context whether the symbol refers to a shape or a real number.

¹⁰In order to study the regime where the random graph has subpolynomial average degree, or improve the SoS degree above n^{δ} , we would need to carefully track log factors and D_{SoS} respectively.

Let τ be the shape of a ribbon R_2 that appears in a matrix \mathbf{Q}_i in Section 3.4. The contribution of this term to \mathbf{Q}_i is $c_{\tau}\lambda_{\tau}\mathbf{M}_{\tau}$. We wish to show that when τ is not a diagonal shape, this expression has small norm.

Recall that the pseudocalibrated coefficients are

$$\lambda_{\tau} = n^{(\alpha-1)\left(|V(\tau)| - \frac{|U_{\tau}| + |V_{\tau}|}{2}\right) + (\frac{\beta}{2} - \gamma)|E(\tau)|}$$

Recall the weight function $w(S) = |S| - \beta \cdot |E(S)|$.

Definition 4.2 (Approximate norm bound). *Given a shape* α (*possibly improper*), *let:*

$$\left\|\mathbf{M}_{\alpha}^{\approx}\right\| = n^{\frac{|V(\alpha)| - w(S_{\min}) + |\operatorname{Iso}(\alpha)|}{2}}$$

It would be more proper to write $\|\mathbf{M}_{\alpha}\|^{\approx}$ although we use this version for more compact notation.

Definition 4.3 (Approximate coefficient change, informal). *Given a shape* τ , *let* $c_{\tau}^{\approx} = |c_{\tau}|$ *when ignoring subpolynomial factors.*

For the shape τ , we view U_{τ} and V_{τ} also as diagonal shapes which include only edges with both endpoints inside U_{τ} or V_{τ} . We want to bound $c_{\tau}\lambda_{\tau}\mathbf{M}_{\tau}$ using the diagonal shapes $\lambda_{U_{\tau}}\mathbf{M}_{U_{\tau}}$ and $\lambda_{V_{\tau}}\mathbf{M}_{V_{\tau}}$. In order to do this, we need to have that

$$c_{\tau}^{\approx}\lambda_{\tau}\left\|\mathbf{M}_{\tau}^{\approx}\right\| \leq \sqrt{\lambda_{U_{\tau}}\lambda_{V_{\tau}}\left\|\mathbf{M}_{U_{\tau}}^{\approx}\right\|\left\|\mathbf{M}_{V_{\tau}}^{\approx}\right\|}.$$

It turns out that this inequality will hold with a poly(n) factor of slack, which furthermore increases for larger shapes τ . We will use this extra slack to control the subpolynomial factors in the formal analysis. To keep track of this extra slack, we define the following slack parameter.

Definition 4.4 (Slack). Given a shape τ with a coefficient c_{τ} , we define $\operatorname{slack}(\tau)$ so that $c_{\tau}^{\approx}\lambda_{\tau} \left\|\mathbf{M}_{\tau}^{\approx}\right\| = n^{-\operatorname{slack}(\tau)} \sqrt{\lambda_{U_{\tau}}\lambda_{V_{\tau}}} \left\|\mathbf{M}_{U_{\tau}}^{\approx}\right\| \left\|\mathbf{M}_{V_{\tau}}^{\approx}\right\|.$

By construction, $slack(\tau)$ is a combinatorial quantity that does not depend on *n*. It is crucial to prove that $slack(\tau) \ge 0$, and in the remaining subsections, we will prove the following positive lower bound on $slack(\tau)$, by proving combinatorially that the slack increases during each operation of the recursion.

Definition 4.5 ($V_{tot}(\tau)$ and $E_{tot}(\tau)$). During the recursion, some vertices and edges are lost during intersections, or when adding or removing indicators. Given a shape τ of a ribbon during the recursion, let $V_{tot}(\tau)$ be the vertex set without performing the intersections. Let $E_{tot}(\tau)$ be the enlargement of $E(\tau)$ to include all of the removed edges.

Remark 4.6. Identify $(U_{\tau}, V_{\tau}, V_{tot}(\tau), E_{tot}(\tau))$ with the shape $\gamma_j \circ \cdots \circ \gamma_1 \circ \tau_0 \circ \gamma_1'^{\tau} \circ \cdots \circ \gamma_j'^{\tau}$ where τ_0 is the "initial" middle shape, *j* is the level of the recursion, and $\gamma_i, \gamma_i'^{\tau}$ will be described in the following sections.

Theorem 4.7. (Slack lower bound). At all times in the decomposition procedure described in Section 3.4, letting τ be the shape of R_2 ,

slack
$$(\tau) \ge \varepsilon \left(|E_{tot}(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} + |V_{tot}(\tau)| - \frac{|U_{\tau}| + |V_{\tau}|}{2} \right)$$

where $\varepsilon = \min\left\{1 - \alpha, \frac{\gamma - \alpha\beta}{8}\right\}$.

We develop a combinatorial formula for the slack in the next few lemmas. **Lemma 4.8.** *For any shape* τ *, if S is an SMVS of* τ *then*

$$\lambda_{\tau} \left\| \mathbf{M}_{\tau}^{\approx} \right\| = n^{(1-\alpha)\left(\frac{|U_{\tau}|+|V_{\tau}|}{2}\right) - (\frac{1}{2}-\alpha)w(\tau) - \frac{w(s)}{2} + \frac{|\operatorname{Iso}(\tau)|}{2} - (\gamma - \alpha\beta)|E(\tau)|}$$

Proof.

$$\lambda_{\tau} \left\| \mathbf{M}_{\tau}^{\approx} \right\| = n^{(\alpha-1)\left(|V(\tau)| - \frac{|U_{\tau}| + |V_{\tau}|}{2}\right) + (\frac{\beta}{2} - \gamma)|E(\tau)|} n^{\frac{|V(\tau)| - w(S) + |\operatorname{Iso}(\tau)|}{2}}$$

= $n^{(1-\alpha)\left(\frac{|U_{\tau}| + |V_{\tau}|}{2}\right) - (\frac{1}{2} - \alpha)|V(\tau)| + (\frac{1}{2} - \alpha)\beta|E(\tau)| - (\gamma - \alpha\beta)|E(\tau)| - \frac{w(S)}{2} + \frac{|\operatorname{Iso}(\tau)|}{2}}$
= $n^{(1-\alpha)\left(\frac{|U_{\tau}| + |V_{\tau}|}{2}\right) - (\frac{1}{2} - \alpha)w(\tau) - (\gamma - \alpha\beta)|E(\tau)| - \frac{w(S)}{2} + \frac{|\operatorname{Iso}(\tau)|}{2}}$

Lemma 4.9. For any shape τ , if S is an SMVS of τ then

$$\frac{\lambda_{\tau} \left\| \mathbf{M}_{\tau}^{\approx} \right\|}{\sqrt{\lambda_{U_{\tau}} \lambda_{V_{\tau}} \left\| \mathbf{M}_{U_{\tau}}^{\approx} \right\| \left\| \mathbf{M}_{V_{\tau}}^{\approx} \right\|}} = n^{-(\frac{1}{2} - \alpha) \left(w(\tau) - \frac{w(U_{\tau}) + w(V_{\tau})}{2} \right) - (\gamma - \alpha\beta) \left(|E(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) + \frac{1}{2} \left(\frac{w(U_{\tau}) + w(V_{\tau})}{2} - w(S) \right) + \frac{|\operatorname{Iso}(\tau)|}{2}}.$$

Proof. For the diagonal shapes U_{τ} and V_{τ} , we have

$$\lambda_{U_{\tau}} \left\| \mathbf{M}_{U_{\tau}}^{\approx} \right\| = n^{(\frac{\beta}{2} - \gamma)|E(U_{\tau})|} n^{\frac{\beta}{2}|E(U_{\tau})|} = n^{(\beta - \gamma)|E(U_{\tau})|} \lambda_{V_{\tau}} \left\| \mathbf{M}_{V_{\tau}}^{\approx} \right\| = n^{(\frac{\beta}{2} - \gamma)|E(V_{\tau})|} n^{\frac{\beta}{2}|E(V_{\tau})|} = n^{(\beta - \gamma)|E(V_{\tau})|} .$$

Therefore,

$$\sqrt{\lambda_{U_{\tau}}\lambda_{V_{\tau}}\left\|\mathbf{M}_{U_{\tau}}^{\approx}\right\|\left\|\mathbf{M}_{V_{\tau}}^{\approx}\right\|} = n^{(\beta-\gamma)\left(\frac{|E(U_{\tau})|+|E(V_{\tau})|}{2}\right)}.$$

Multiplying with Lemma 4.8,

$$\begin{split} & \frac{\lambda_{\tau} \left\| \mathbf{M}_{\tau}^{\approx} \right\|}{\sqrt{\lambda_{U_{\tau}} \lambda_{V_{\tau}} \left\| \mathbf{M}_{U_{\tau}}^{\approx} \right\| \left\| \mathbf{M}_{V_{\tau}}^{\approx} \right\|}} \\ &= n^{(1-\alpha) \left(\frac{|U_{\tau}| + |V_{\tau}|}{2} \right) - \left(\frac{1}{2} - \alpha \right) w(\tau) - \frac{w(S)}{2} + \frac{|\mathrm{Iso}(\tau)|}{2} - (\gamma - \alpha\beta) |E(\tau)| - (\beta - \gamma) \left(\frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right)}{2} \\ &= n^{(1-\alpha) \left(\frac{|U_{\tau}| + |V_{\tau}|}{2} \right) - \left(\frac{1}{2} - \alpha \right) w(\tau) - \frac{w(S)}{2} + \frac{|\mathrm{Iso}(\tau)|}{2} - (\gamma - \alpha\beta) |E(\tau)| - (\alpha\beta - \gamma) \left(\frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) + (\alpha\beta - \beta) \left(\frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right)}{2} \\ &= n^{(1-\alpha) \left(\frac{w(U_{\tau}) + w(V_{\tau})}{2} \right) - \left(\frac{1}{2} - \alpha \right) w(\tau) - \frac{w(S)}{2} + \frac{|\mathrm{Iso}(\tau)|}{2} - (\gamma - \alpha\beta) \left(|E(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right)}{2} \\ &= n^{-\left(\frac{1}{2} - \alpha \right) \left(w(\tau) - \frac{w(U_{\tau}) + w(V_{\tau})}{2} \right) - (\gamma - \alpha\beta) \left(|E(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) + \frac{1}{2} \left(\frac{w(U_{\tau}) + w(V_{\tau})}{2} - w(S) \right) + \frac{|\mathrm{Iso}(\tau)|}{2} }{2} . \end{split}$$

As a corollary, we have the following combinatorial formula for the slack.

Lemma 4.10.

$$\begin{aligned} \operatorname{slack}(\tau) &= \\ & \left(\frac{1}{2} - \alpha\right) \left(w(\tau) - \frac{w(U_{\tau}) + w(V_{\tau})}{2} \right) + (\gamma - \alpha\beta) \left(|E(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) \\ & - \frac{1}{2} \left(\frac{w(U_{\tau}) + w(V_{\tau})}{2} - w(S) \right) - \frac{|\operatorname{Iso}(\tau)|}{2} - \log_n(c_{\tau}^{\approx}) \end{aligned}$$

4.2 Slack for middle shapes

We start by computing the slack for middle shapes. This is the slack at the start of the process.

Theorem 4.11. Let τ be a proper middle shape. Then:

$$\operatorname{slack}(\tau) \ge (\gamma - \alpha \beta) \left(|E(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right)$$

Proof. Since τ is a proper middle shape, $|Iso(\tau)| = 0$ and $c_{\tau} = 1$. By Lemma 4.10 we have

slack
$$(\tau) = (\frac{1}{2} - \alpha) \left(w(\tau) - \frac{w(U_{\tau}) + w(V_{\tau})}{2} \right) + (\gamma - \alpha\beta) \left(|E(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) - \frac{1}{2} \left(\frac{w(U_{\tau}) + w(V_{\tau})}{2} - w(S) \right).$$

Furthermore, since $w(\tau) \ge w(S) = w(U_{\tau}) = w(V_{\tau})$, the last term is 0, and the first term is non-negative. Thus,

$$\operatorname{slack}(\tau) \ge (\gamma - \alpha \beta) \left(|E(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right).$$

4.3 Slack for the PMVS subroutine

For this subsection, suppose that one iteration of the **Finding PMVS subroutine** replaces ribbon R_2 of shape τ by ribbon R'_2 of shape τ' .

We will need to consider the "removed edges" $E(R_2) \setminus E(R'_2)$, which intuitively are the edges of the ribbon that were queried and not present, and formally are the edges that disappear during either the **Adding left and right indicators operation** or the **Removing middle edge indicators operation**. Observe that all removed edges are in $U_{\tau} \cup V_{\tau}$.

Recall that in the **PMVS operation**, we take a triple of ribbons R_1 , R_2 , R_3 after indicators have been added, and split R_1 across the leftmost SMVS A' between A_{R_1} and B_{R_1} , and likewise split R_3 across the rightmost SMVS B' between A_{R_3} and B_{R_3} .

Definition 4.12 (γ and γ'). Let γ be the shape of the part of R_1 between A' and B_{R_1} . Let γ'^{\intercal} be the shape of the part of R_3 between A_{R_3} and B'.

Remark 4.13. Note that γ , τ , and γ'^{\intercal} should include all removed edges, whereas τ' has the edges removed.

Remark 4.14. While this γ is technically different than the γ for an intersection term in Section 4.4, it plays a similar role.

Lemma 4.15. γ , γ' are left shapes.

Proof. Suppose that *S* is a separator of γ . We claim that *S* is also a separator of R_1 . Let *P* be any path from A_{R_1} to B_{R_1} . Since $A' = U_{\gamma}$ is a separator for R_1 , *P* must pass through *A'*. Starting from the final vertex of the path in *A'* gives a path from $A' = U_{\gamma}$ to $B_{R_1} = V_{\gamma}$ which is entirely contained in γ . Finally, since *S* is a separator of γ , it must contain a vertex of the path *P*.

Since R_1 is a left ribbon, we conclude that $w(S) \ge w(V_{\gamma})$ and the unique SMVS of γ is V_{γ} .

Theorem 4.16. Let $R_2 \rightarrow R'_2$ be a ribbon that undergoes one iteration of the Finding PMVS subroutine. Let τ and τ' be their respective shapes. Then

 $slack(\tau') - slack(\tau)$

$$\geq \alpha \left(\frac{w(U_{\tau'}) + w(V_{\tau'}) - w(U_{\tau}) - w(V_{\tau})}{2} \right) \\ + (\gamma - \alpha \beta) \left(|E(\tau')| - \frac{|E(U_{\tau'})| + |E(V_{\tau'})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) \\ + (\gamma - \alpha \beta) x$$

where *x* is the total number of removed edges.

Proof. By Lemma 4.10,

$$\begin{aligned} \operatorname{slack}(\tau') &- \operatorname{slack}(\tau) \\ &= \left(\frac{1}{2} - \alpha\right) \left(w(\tau') - \frac{w(U_{\tau'}) + w(V_{\tau'})}{2} - w(\tau) + \frac{w(U_{\tau}) + w(V_{\tau})}{2} \right) \\ &+ \left(\gamma - \alpha\beta\right) \left(|E(\tau')| - \frac{|E(U_{\tau'})| + |E(V_{\tau'})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) \\ &- \frac{1}{2} \left(\frac{w(U_{\tau'}) + w(V_{\tau'})}{2} - w(S') - \frac{w(U_{\tau}) + w(V_{\tau})}{2} + w(S) \right) \\ &- \frac{|\operatorname{Iso}(\tau')| - |\operatorname{Iso}(\tau)|}{2} - \log_n(c_{\tau'}^{\approx}) + \log_n(c_{\tau}^{\approx}) \\ &= \left(\frac{1}{2} - \alpha\right) \left(w(\tau') - w(\tau) \right) + \alpha \left(\frac{w(U_{\tau'}) + w(V_{\tau'}) - w(U_{\tau}) - w(V_{\tau})}{2} \right) \\ &+ \frac{w(U_{\tau}) + w(V_{\tau}) - w(U_{\tau'}) - w(V_{\tau'}) + w(S') - w(S) + |\operatorname{Iso}(\tau)| - |\operatorname{Iso}(\tau')|}{2} \\ &+ \left(\gamma - \alpha\beta\right) \left(|E(\tau')| - \frac{|E(U_{\tau'})| + |E(V_{\tau'})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) - \log_n \left(\frac{c_{\tau'}^{\approx}}{c_{\tau}^{\approx}} \right) \end{aligned}$$

The large term multiplied by $\frac{1}{2}$ is the most dangerous term as it does not come with any small coefficient. We analyze the terms as follows.

Claim 4.17.
$$\frac{c_{\tau'}^{\approx}}{c_{\tau}^{\approx}} = n^{-\gamma x}$$

Proof of Claim 4.17. As noted after Definition 3.9, for each edge which is removed, we get a factor of magnitude $n^{-\frac{\beta}{2}}$. Furthermore, we get a factor of $n^{\frac{\beta}{2}-\gamma}$ shifted from λ_{τ} to $c_{\tau'}$. Multiplying these factors together gives a factor of magnitude $n^{-\gamma}$ per removed edge.

Claim 4.18. $w(\tau') \ge w(\tau) + \beta x$

Proof of Claim 4.18. Observe that

$$w(\tau') = w(\gamma) + w(\tau) + w(\gamma'\tau) - w(U_{\tau}) - w(V_{\tau}) + \beta x.$$

 $U_{\tau} = V_{\gamma}$ is an SMVS for γ so $w(\gamma) \ge w(U_{\tau})$. Similarly, $w(\gamma') \ge w(V_{\tau})$. Putting these equations together, we have that $w(\tau') \ge w(\tau) + \beta x$, as needed.

The next lemma is a tradeoff lemma for the PMVS, which will be proven in the next sub-subsection.

Lemma 4.19 (PMVS tradeoff lemma).

$$w(U_{\tau'}) + w(V_{\tau'}) - w(S') + |\operatorname{Iso}(\tau')| \le w(U_{\tau}) + w(V_{\tau}) - w(S) + |\operatorname{Iso}(\tau)| + \beta x$$

Multiplying Claim 4.18 by $\frac{1}{2} - \alpha$ and multiplying Lemma 4.19 by $\frac{1}{2}$, we have that

1.
$$(\frac{1}{2} - \alpha)(w(\tau') - w(\tau)) \ge \frac{\beta}{2}x - \alpha\beta x$$

2. $w(U_{\tau}) + w(V_{\tau}) - w(U_{\tau'}) - w(V_{\tau})$

$$\frac{w(U_{\tau}) + w(V_{\tau}) - w(U_{\tau'}) - w(V_{\tau'}) + w(S') - w(S) + |\operatorname{Iso}(\tau)| - |\operatorname{Iso}(\tau')|}{2} \ge -\frac{\beta}{2}x$$

Using these equations in the formula above, we have that

$$slack(\tau') - slack(\tau) \geq \frac{\beta}{2}x - \alpha\beta x + \alpha \left(\frac{w(U_{\tau'}) + w(V_{\tau'}) - w(U_{\tau}) - w(V_{\tau})}{2}\right) - \frac{\beta}{2}x + (\gamma - \alpha\beta) \left(|E(\tau')| - \frac{|E(U_{\tau'})| + |E(V_{\tau'})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2}\right) + \gamma x = (\gamma - \alpha\beta)x + \alpha \left(\frac{w(U_{\tau'}) + w(V_{\tau'}) - w(U_{\tau}) - w(V_{\tau})}{2}\right) + (\gamma - \alpha\beta) \left(|E(\tau')| - \frac{|E(U_{\tau'})| + |E(V_{\tau'})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2}\right)$$

as needed.

Corollary 4.20.

$$\operatorname{slack}(\tau') - \operatorname{slack}(\tau) \\ \geq (\gamma - \alpha\beta) \cdot \left(|E_{tot}(\tau')| - \frac{|E(U_{\tau'})| + |E(V_{\tau'})|}{2} - |E_{tot}(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right)$$

Proof. Using our slack calculation above, observe that we can remove the vertex factor as

$$\alpha\left(\frac{w(U_{\tau'}) + w(V_{\tau'}) - w(U_{\tau}) - w(V_{\tau})}{2}\right) \ge 0$$

since $U_{\tau'}$ (including the removed edges) is a larger separator than U_{τ} , as U_{τ} is the SMVS of R_1 at this point in the iteration (and likewise for $V_{\tau'}$ and V_{τ}). Removing edges only increases $w(U_{\tau'})$.

4.3.1 Proof of the PMVS tradeoff lemma

Lemma 4.19 (PMVS tradeoff lemma).

 $w(U_{\tau'}) + w(V_{\tau'}) - w(S') + |\operatorname{Iso}(\tau')| \le w(U_{\tau}) + w(V_{\tau}) - w(S) + |\operatorname{Iso}(\tau)| + \beta x$

Proof. To prove this, we construct and analyze the following sets of vertices.

- 1. We take X_0 to be the set of vertices in $V_{\gamma} \setminus S'$ which can be reached in γ from U_{γ} without passing through a vertex in S'. We then take $X = X_0 \cup (S' \cap V(\gamma))$.
- 2. We take Y_l to be the set of non-isolated vertices in $V_{\gamma} \setminus S'$ which are not reachable in γ from U_{γ} without passing through S'. Similarly, we take Y_r to be the set of non-isolated vertices in $U_{\gamma'^{\intercal}} \setminus S'$ which are not reachable in γ'^{\intercal} from $V_{\gamma'^{\intercal}}$ without passing through a vertex in S'. We then take $Y = Y_l \cup Y_r \cup (S' \cap V(\tau)) \cup (V_{\gamma} \cap U_{\gamma'^{\intercal}})$.
- 3. We take Z_0 to be the set of vertices in $U_{\gamma'^{\intercal}} \setminus S'$ which can be reached in γ'^{\intercal} from $V_{\gamma'^{\intercal}}$ without passing through a vertex in S'. We take Z_{extra} to be the set of non-isolated vertices in $(U_{\tau} \cap V_{\tau}) \setminus S'$ which are not reachable from $V_{\gamma'^{\intercal}}$ in γ'^{\intercal} . We then take $Z = Z_0 \cup Z_{extra} \cup (S' \cap V(\gamma'^{\intercal}))$.

Let x_{\cap} be the number of edges removed from $U_{\tau} \cap V_{\tau}$. We now observe that it is sufficient to show the following statements.

1.
$$w(X) \ge w(U_{\gamma}), w(Y) \ge w(S) + \beta x_{\cap}, \text{ and } w(Z) \ge w(V_{\gamma'^{\intercal}}).$$

2.

$$\begin{split} w(X) + w(Y) + w(Z) &\leq w(S') + w(V_{\gamma}) + w(U_{\gamma'^{\tau}}) - (|\operatorname{Iso}(\tau')| - |\operatorname{Iso}(\tau)|) \\ &= w(S') \\ &+ w(U_{\tau}) + \beta(\text{\# of edges removed from } U_{\tau}) \\ &+ w(V_{\tau}) + \beta(\text{\# of edges removed from } V_{\tau}) \\ &- (|\operatorname{Iso}(\tau')| - |\operatorname{Iso}(\tau)|) \end{split}$$

Using the three initial inequalities on the left-hand side of the second statement,

$$w(U_{\gamma}) + w(S) + \beta x_{\cap} + w(V_{\gamma'^{\tau}}) \le w(S') + w(U_{\tau}) + w(V_{\tau}) + \beta x + \beta x_{\cap} - |\operatorname{Iso}(\tau')| + |\operatorname{Iso}(\tau)|.$$

Rearranging this, we have

$$w(U_{\tau'}) + w(V_{\tau'}) - w(S') + |\operatorname{Iso}(\tau')| \le w(U_{\tau}) + w(V_{\tau}) - w(S) + |\operatorname{Iso}(\tau)| + \beta x$$

as needed.

Moving to the statements, to show that $w(X) \ge w(U_{\gamma})$ and $w(Z) \ge w(V_{\gamma'^{\intercal}})$, we observe that because of how we chose *X* and *Z*, *X* is a vertex separator of γ and *Z* is a vertex separator of γ'^{\intercal} .

To show that $w(Y) \ge w(S) + \beta x_{\cap}$, we first observe that Y is a vertex separator of τ (with or without the missing edges). To see this, assume that Y is not a vertex separator of τ . If so, there is a path P_m from a vertex $u \in U_{\tau}$ to a vertex $v \in V_{\tau}$ which does not intersect Y and thus does not intersect S'. Since $u \notin Y$, there is a path P_l from U_{γ} to u in γ which does not intersect S'. Similarly, since $v \notin Y$, there is a path P_r from v to $V_{\gamma'^{\tau}}$ in γ'^{τ} which does not intersect S'. Composing P_l , P_m , and P_r gives a path from $U_{\tau'}$ to $V_{\tau'}$ which does not intersect S' which is a contradiction as S' is an SMVS for τ' .

We now observe that when the missing edges are removed, all vertex separators of τ have their weight increased by at least βx_{\cap} . Thus, after the missing edges are deleted, all vertex separators of τ have weight at least $w(S) + \beta x_{\cap}$ so $w(Y) \ge w(S) + \beta x_{\cap}$.

To show that

$$w(X) + w(Y) + w(Z) \le w(S') + w(V_{\gamma}) + w(U_{\gamma'^{\intercal}}) - (|\operatorname{Iso}(\tau')| - |\operatorname{Iso}(\tau)|)$$

we show the following two statements:

- 1. For each vertex v, $\mathbf{1}_{v \in X} + \mathbf{1}_{v \in Y} + \mathbf{1}_{v \in Z} \leq \mathbf{1}_{v \in S'} + \mathbf{1}_{v \in V_{\gamma}} + \mathbf{1}_{v \in U_{\gamma'}} \mathbf{1}_{v \in \operatorname{Iso}(\tau') \setminus \operatorname{Iso}(\tau)}$.
- 2. For each edge *e*, $\mathbf{1}_{e \in E(X)} + \mathbf{1}_{e \in E(Y)} + \mathbf{1}_{e \in E(Z)} \ge \mathbf{1}_{e \in E(S')} + \mathbf{1}_{e \in E(V_{\gamma})} + \mathbf{1}_{e \in E(U_{\gamma'} \top)}$.

For the first statement, we make the following observations:

- 1. If $v \in S'$ then v is not isolated. If $v \notin V_{\gamma} \cup U_{\gamma'^{\mathsf{T}}}$ then v is in exactly one of X, Y, and Z depending on whether v is in $V(\gamma)$, $V(\tau)$, or $V(\gamma'^{\mathsf{T}})$. If $v \in V_{\gamma} \cup U_{\gamma'^{\mathsf{T}}}$ then $v \in Y$, $v \in X$ if and only if $v \in V_{\gamma}$ and $v \in Z$ if and only if $v \in U_{\gamma'^{\mathsf{T}}}$.
- 2. If $v \in \text{Iso}(\tau') \setminus \text{Iso}(\tau)$ then $v \notin X$ and $v \notin Z$. Moreover, v must be in V_{γ} or $U_{\gamma'^{\intercal}}$ and $v \in Y$ if and only if v is in both V_{γ} and $U_{\gamma'^{\intercal}}$.
- 3. If $v \in (V_{\gamma} \setminus S') \setminus U_{\gamma'^{\intercal}}$ and v is not isolated then $\mathbf{1}_{v \in S'} + \mathbf{1}_{v \in V_{\gamma}} + \mathbf{1}_{v \in U_{\gamma'^{\intercal}}} \mathbf{1}_{v \text{ is isolated}} = 1$ and $\mathbf{1}_{v \in X} + \mathbf{1}_{v \in Y} + \mathbf{1}_{v \in Z} = 1$ as $v \notin Z$ and either $v \in X$ or $v \in Y$ but not both.
- 4. If $v \in (U_{\gamma'^{\intercal}} \setminus S') \setminus V_{\gamma}$ and v is not isolated then $\mathbf{1}_{v \in S'} + \mathbf{1}_{v \in V_{\gamma}} + \mathbf{1}_{v \in U_{\gamma'^{\intercal}}} \mathbf{1}_{v \text{ is isolated}} = 1$ and $\mathbf{1}_{v \in X} + \mathbf{1}_{v \in Y} + \mathbf{1}_{v \in Z} = 1$ as $v \notin X$ and either $v \in Y$ or $v \in Z$ but not both.
- 5. If $v \in (V_{\gamma} \cap U_{\gamma'^{\intercal}}) \setminus S'$ and v is not isolated then $\mathbf{1}_{v \in S'} + \mathbf{1}_{v \in V_{\gamma}} + \mathbf{1}_{v \in U_{\gamma'^{\intercal}}} \mathbf{1}_{v \text{ is isolated}} = 2$ and $\mathbf{1}_{v \in X} + \mathbf{1}_{v \in Y} + \mathbf{1}_{v \in Z} = 2$ as $v \in Y$ and either $v \in X$ or $v \in Z$ but not both.

For the second statement, we make the following observations:

- 1. If $e \in E(S')$ and $e \notin E(V_{\gamma}) \cup E(U_{\gamma'^{\intercal}})$ then *e* is in exactly one of E(X), E(Y), and E(Z) depending on whether *e* is in $E(\gamma)$, $E(\tau)$, or $E(\gamma'^{\intercal})$. If $e \in E(V_{\gamma}) \cup E(U_{\gamma'^{\intercal}})$ then $e \in E(Y)$, $e \in E(X)$ if and only if $e \in E(V_{\gamma})$, and $e \in E(Z)$ if and only if $e \in E(U_{\gamma'^{\intercal}})$. In all of these cases, $\mathbf{1}_{e \in E(X)} + \mathbf{1}_{e \in E(Y)} + \mathbf{1}_{e \in E(Z)} \ge \mathbf{1}_{e \in E(S')} + \mathbf{1}_{e \in E(V_{\gamma})} + \mathbf{1}_{e \in E(U_{\gamma'^{\intercal}})}$.
- 2. If $e \in (E(V_{\gamma}) \setminus E(S')) \setminus E(U_{\gamma'^{\intercal}})$ then $\mathbf{1}_{e \in E(S')} + \mathbf{1}_{e \in E(V_{\gamma})} + \mathbf{1}_{e \in E(U_{\gamma'^{\intercal}})} = 1$ and $\mathbf{1}_{e \in E(X)} + \mathbf{1}_{e \in E(Y)} + \mathbf{1}_{e \in E(Z)} = 1$ as $e \notin E(Z)$ and either $e \in E(X)$ or $e \in E(Y)$ but not both.
- 3. If $e \in (E(U_{\gamma'^{\intercal}}) \setminus E(S')) \setminus E(V_{\gamma})$ then $\mathbf{1}_{e \in E(S')} + \mathbf{1}_{e \in E(V_{\gamma})} + \mathbf{1}_{e \in E(U_{\gamma'^{\intercal}})} = 1$ and $\mathbf{1}_{e \in E(X)} + \mathbf{1}_{e \in E(Y)} + \mathbf{1}_{e \in E(Z)} = 1$ as $e \notin E(X)$ and either $e \in E(Y)$ or $e \in E(Z)$ but not both.
- 4. If $e \in (E(V_{\gamma}) \cap E(U_{\gamma'^{\intercal}})) \setminus E(S')$ then $\mathbf{1}_{e \in E(S')} + \mathbf{1}_{e \in E(V_{\gamma})} + \mathbf{1}_{e \in E(U_{\gamma'^{\intercal}})} = 2$ and $\mathbf{1}_{e \in E(X)} + \mathbf{1}_{e \in E(Y)} + \mathbf{1}_{e \in E(Z)} = 2$ as $e \in E(Z)$ and either $e \in E(X)$ or $e \in E(Z)$ but not both.

4.4 Slack for intersection terms

We now analyze the slack for the **Intersection term decomposition operation**. The analysis is similar to that for the PMVS subroutine, albeit with additional considerations.

Suppose that the operation replaces a ribbon R_2 of shape τ by a ribbon R'_2 of shape τ_P . Recall that in the **Intersection term decomposition operation**, we take a triple of ribbons R_1, R_2, R_3 with intersections, and split R_1 across the leftmost SMVS A' between A_{R_1} and $B_{R_1} \cup V_{intersected}(R_1)$, and likewise split R_3 across the rightmost SMVS B' between $A_{R_3} \cup V_{intersected}(R_3)$ and B_{R_3} .

Definition 4.21 (γ and γ'). Let γ be the shape of the part of R_1 between A' and B_{R_1} . Let γ'^{\intercal} be the shape of the part of R_3 between A_{R_3} and B'.

The notation τ_P is used because τ_P is an intersection shape (Definition 2.37) for some intersection pattern $P \in \mathcal{P}_{\gamma,\tau,\gamma'^{\intercal}}$ (after linearization).

The edges that are linearized away into a constant term during the **Intersection term decomposition operation** are referred to as "vanishing edges".

Theorem 4.22. Let $R_2 \rightarrow R'_2$ be a ribbon that undergoes the **Intersection term decomposition** *operation*. Let τ and τ_P be their respective shapes. Then for γ, γ' as defined in Definition 4.21,

$$\begin{aligned} \operatorname{slack}(\tau_P) - \operatorname{slack}(\tau) &\geq \\ (1 - \alpha) \left(\frac{w(U_{\gamma}) + w(V_{\gamma'\tau}) - w(U_{\tau}) - w(V_{\tau})}{2} \right) \\ &+ (\gamma - \alpha\beta) \left(|E(\tau_P)| - \frac{|E(U_{\tau_P})| + |E(V_{\tau_P})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) \\ &+ (\gamma - \alpha\beta) \cdot edge reduction \end{aligned}$$

where edgereduction is the total number of vanishing edges.

Proof. By Lemma 4.10,

$$\begin{aligned} \operatorname{slack}(\tau_{P}) - \operatorname{slack}(\tau) \\ &= \left(\frac{1}{2} - \alpha\right) \left(w(\tau_{P}) - \frac{w(U_{\tau_{P}}) + w(V_{\tau_{P}})}{2} - w(\tau) + \frac{w(U_{\tau}) + w(V_{\tau})}{2} \right) \\ &+ \left(\gamma - \alpha\beta\right) \left(|E(\tau_{P})| - \frac{|E(U_{\tau_{P}})| + |E(V_{\tau_{P}})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) \\ &- \frac{1}{2} \left(\frac{w(U_{\tau_{P}}) + w(V_{\tau_{P}})}{2} - w(S') - \frac{w(U_{\tau}) + w(V_{\tau})}{2} + w(S) \right) \\ &- \frac{|\operatorname{Iso}(\tau_{P})| - |\operatorname{Iso}(\tau)|}{2} - \log_{n}(c_{\tau_{P}}^{\approx}) + \log_{n}(c_{\tau}^{\approx}) \\ &= \left(\frac{1}{2} - \alpha\right) \left(w(\tau_{P}) - w(\tau) \right) - \frac{1 - \alpha}{2} \left(w(U_{\tau_{P}}) + w(V_{\tau_{P}}) - w(U_{\gamma}) - w(V_{\gamma'\tau}) \right) \\ &+ \alpha \left(\frac{w(U_{\gamma}) + w(V_{\gamma'\tau}) - w(U_{\tau}) - w(V_{\tau})}{2} \right) \\ &+ \frac{w(U_{\tau}) + w(V_{\tau}) - w(U_{\gamma}) - w(V_{\gamma'\tau}) + w(S') - w(S) + |\operatorname{Iso}(\tau)| - |\operatorname{Iso}(\tau_{P})|}{2} \\ &+ \left(\gamma - \alpha\beta\right) \left(|E(\tau_{P})| - \frac{|E(U_{\tau_{P}})| + |E(V_{\tau_{P}})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) - \log_{n} \left(\frac{c_{\tau_{P}}^{\approx}}{c_{\tau}^{\approx}} \right) \end{aligned}$$

We now analyze the different terms that appear.

Claim 4.23.

$$\frac{c_{\tau_P}^{\approx}}{c_{\tau}^{\approx}} = n^{(\alpha-1)\cdot(\# \text{ of intersections})} \left(\prod_{e \in E_{tot}(\tau_P)} \left(n^{\frac{\beta}{2}-\gamma} \right)^{\operatorname{mult}(e)-1+\mathbf{1}_{e \text{ vanishes}}} \left(n^{\frac{\beta}{2}} \right)^{\operatorname{mult}(e)-1-\mathbf{1}_{e \text{ vanishes}}} \right)$$

Note that we consider the iteration $e \in E_{tot}(\tau_P)$ to yield each multiedge only once.

Proof of Claim 4.23. Observe that for each intersection, we have one fewer vertex factor of $n^{(\alpha-1)}$ in λ_{τ_P} so we need to add this factor to c_{τ_P} . For each multiedge in $E_{tot}(\tau_P)$, we have mult $(e) - 1 + \mathbf{1}_{e \text{ vanishes}}$ fewer factors of $n^{\frac{\beta}{2}-\gamma}$ in λ_{τ_P} so these factors need to be added to c_{τ_P} . Finally, for each multiedge in $e \in E_{tot}(\tau_P)$, when we express it as a linear combination of 1 and $\chi_{\{e\}}$, we gain mult $(e) - 1 - \mathbf{1}_{e \text{ vanishes}}$ factors of $n^{\frac{\beta}{2}}$ (Proposition 2.43) which also need to be added to c_{τ_P} . When a multiedge with an indicator is linearized, it never vanishes, and the coefficient is the same as linearizing a multiedge without an indicator.

For notational convenience, we define the following expressions:

linearization =
$$\sum_{e \in E_{tot}(\tau_P)} \operatorname{mult}(e) - 1 - \mathbf{1}_{e \text{ vanishes}}$$
.

edgereduction =
$$\sum_{e \in E_{tot}(\tau_P)} \operatorname{mult}(e) - 1 + \mathbf{1}_{e \text{ vanishes}}$$

With these expressions, we can express $\log_n\left(\frac{c_{\tau_p}^{\approx}}{c_{\tau}^{\approx}}\right)$ as follows.

Corollary 4.24.

$$\log_{n}\left(\frac{c_{\tau_{p}}^{\approx}}{c_{\tau}^{\approx}}\right) = (\alpha - 1) \cdot (\# of intersections) + (\frac{\beta}{2} - \gamma)(edge reduction) + \frac{\beta}{2}(linearization)$$

Claim 4.25.

$$w(\tau_P) \ge w(U_{\gamma}) + w(V_{\gamma'^{\tau}}) - w(U_{\tau}) - w(V_{\tau}) + w(\tau)$$

- (# of intersections) + β (edgereduction)

Proof of Claim 4.25. We first observe that

$$w(\tau_P) = w(\gamma) + w(\tau) + w(\gamma'^{\mathsf{T}}) - w(U_{\tau}) - w(V_{\tau}) - (\# \text{ of intersections}) + \beta(\text{edgereduction}).$$

To see this, note that if there were no intersections then we would have that $w(\tau_P) = w(\gamma) + w(\tau) + w(\gamma'^{\intercal}) - w(U_{\tau}) - w(V_{\tau})$. Each intersection reduces the number of vertices and thus decreases the weight of τ_P by 1. The change in the number of edges increases the weight of τ_P by β (edgereduction).

We now observe that $w(\gamma) \ge w(U_{\gamma})$ as otherwise γ would be a separator in γ between U_{γ} and $V_{\gamma} \cup V_{intersected}(\gamma)$ with smaller weight than U_{γ} . Following similar logic, $w(\gamma') \ge w(V_{\gamma'^{\intercal}})$. Thus, we have that

$$w(\tau_P) \ge w(U_{\gamma}) + w(V_{\gamma'^{\tau}}) - w(U_{\tau}) - w(V_{\tau}) + w(\tau)$$

- (# of intersections) + β (edgereduction)

as needed.

The next lemma is an intersection tradeoff lemma to be proven in the next sub-section.

Lemma 4.26 (Intersection tradeoff lemma). *Given is a shape* τ *, left shapes* γ *,* γ' *, an intersection pattern* $P \in \mathcal{P}_{\gamma^-,\tau,(\gamma'\tau)^-}$ *such that the following structural property holds:*

 U_{γ} is the leftmost SMVS of U_{γ} and $V_{\gamma} \cup V_{intersected}(\gamma)$, and $V_{\gamma'^{\intercal}}$ is the rightmost SMVS of $U_{\gamma'^{\intercal}} \cup V_{intersected}(\gamma'^{\intercal})$ and $V_{\gamma'^{\intercal}}$.

Let S be an SMVS of τ , let τ_P be the shape resulting from P followed by linearization, and let S' be an SMVS of τ_P . Then,

$$\begin{split} w(U_{\gamma}) + w(V_{\gamma'^{\intercal}}) - w(S') + |\operatorname{Iso}(\tau_P)| &\leq w(U_{\tau}) + w(V_{\tau}) - w(S) + |\operatorname{Iso}(\tau)| \\ &+ (\# \ of \ intersections) - \beta \ (linearization) \\ &- \beta (\# \ of \ vanishing \ edges \ in \ U_{\gamma}) \\ &- \beta (\# \ of \ vanishing \ edges \ in \ V_{\gamma'^{\intercal}}) \end{split}$$

Multiplying Claim 4.25 by $\frac{1}{2} - \alpha$ and multiplying Lemma 4.26 by $\frac{1}{2}$, we have that

1.

$$\begin{aligned} &\left(\frac{1}{2} - \alpha\right) \left(w(\tau_P) - w(\tau)\right) \ge \\ &\left(\frac{1}{2} - \alpha\right) \left(w(U_{\gamma}) + w(V_{\gamma'^{\tau}}) - w(U_{\tau}) - w(V_{\tau}) - (\text{\# of intersections})\right) \\ &+ \left(\frac{1}{2} - \alpha\right) \beta(\text{edgereduction}) \end{aligned}$$

2.

$$\frac{w(U_{\tau}) + w(V_{\tau}) - w(U_{\tau_{P}}) - w(V_{\tau_{P}}) + w(S') - w(S) + |\operatorname{Iso}(\tau)| - |\operatorname{Iso}(\tau_{P})|}{2}$$

$$\geq \frac{-1}{2} (\text{\# of intersections}) + \frac{\beta}{2} (\text{linearization})$$

$$+ \frac{\beta}{2} ((\text{\# of vanishing edges in } U_{\gamma}) + (\text{\# of vanishing edges in } V_{\gamma'^{\tau}}))$$

Using these equations in the formula above, we have that

$$\begin{aligned} \operatorname{slack}(\tau_{P}) - \operatorname{slack}(\tau) \\ &\geq \left(\frac{1}{2} - \alpha\right) \left(w(U_{\gamma}) + w(V_{\gamma'^{\intercal}}) - w(U_{\tau}) - w(V_{\tau}) - (\text{\# of intersections}) \right) \\ &+ \left(\frac{1}{2} - \alpha\right) \beta \left(\operatorname{edgereduction} \right) - \frac{1 - \alpha}{2} \left(w(U_{\tau_{P}}) + w(V_{\tau_{P}}) - w(U_{\gamma}) - w(V_{\gamma'^{\intercal}}) \right) \\ &+ \alpha \left(\frac{w(U_{\gamma}) + w(V_{\gamma'^{\intercal}}) - w(U_{\tau}) - w(V_{\tau})}{2} \right) \\ &+ \frac{\beta}{2} \left(\operatorname{linearization} \right) - \frac{1}{2} (\text{\# of intersections}) \\ &+ \frac{\beta}{2} ((\text{\# of vanishing edges in } U_{\gamma}) + (\text{\# of vanishing edges in } V_{\gamma'^{\intercal}})) \end{aligned}$$

$$\begin{split} &+ (\gamma - \alpha \beta) \left(|E(\tau_P)| - \frac{|E(U_{\tau_P})| + |E(V_{\tau_P})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) \\ &- (\alpha - 1) (\text{\# of intersections}) - \frac{\beta}{2} \left(\text{linearization} \right) \\ &- (\frac{\beta}{2} - \gamma) (\text{edgereduction}) \\ &= \frac{1 - \alpha}{2} \left(w(U_{\gamma}) + w(V_{\gamma'^{\intercal}}) - w(U_{\tau}) - w(V_{\tau}) \right) \\ &- \frac{1 - \alpha}{2} \left(w(U_{\tau_P}) + w(V_{\tau_P}) - w(U_{\gamma}) - w(V_{\gamma'^{\intercal}}) \right) + (\gamma - \alpha \beta) \left(\text{edgereduction} \right) \\ &+ \frac{\beta}{2} ((\text{\# of vanishing edges in } U_{\gamma}) + (\text{\# of vanishing edges in } V_{\gamma'^{\intercal}})) \\ &+ (\gamma - \alpha \beta) \left(|E(\tau_P)| - \frac{|E(U_{\tau_P})| + |E(V_{\tau_P})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) \end{split}$$

We now use that

$$\begin{split} & w(U_{\tau_P}) + w(V_{\tau_P}) - w(U_{\gamma}) - w(V_{\gamma'^{\intercal}}) \\ &= \beta(\text{\# of vanishing edges in } U_{\gamma}) + \beta(\text{\# of vanishing edges in } V_{\gamma'^{\intercal}}) \\ &- \beta(\text{\# of edges added to } U_{\gamma}) - \beta(\text{\# of edges added to } V_{\gamma'^{\intercal}}) \\ &\leq \beta(\text{\# of vanishing edges in } U_{\gamma}) + \beta(\text{\# of vanishing edges in } V_{\gamma'^{\intercal}}) \end{split}$$

Plugging this in, we have that

$$\begin{aligned} \operatorname{slack}(\tau_{P}) - \operatorname{slack}(\tau) \\ &\geq \frac{1-\alpha}{2} \left(w(U_{\gamma}) + w(V_{\gamma'^{\intercal}}) - w(U_{\tau}) - w(V_{\tau}) \right) \\ &+ (\gamma - \alpha\beta) \left(|E(\tau_{P})| - \frac{|E(U_{\tau_{P}})| + |E(V_{\tau_{P}})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) \\ &- \frac{\beta - \alpha\beta}{2} \left((\text{\# of vanishing edges in } U_{\gamma}) + (\text{\# of vanishing edges in } V_{\gamma'^{\intercal}}) \right) \\ &+ (\gamma - \alpha\beta) \left(\operatorname{edgereduction} \right) \\ &+ \frac{\beta}{2} \left((\text{\# of vanishing edges in } U_{\gamma}) + (\text{\# of vanishing edges in } V_{\gamma'^{\intercal}}) \right) \\ &= \frac{1-\alpha}{2} \left(w(U_{\gamma}) + w(V_{\gamma'^{\intercal}}) - w(U_{\tau}) - w(V_{\tau}) \right) \\ &+ (\gamma - \alpha\beta) \left(|E(\tau_{P})| - \frac{|E(U_{\tau_{P}})| + |E(V_{\tau_{P}})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) \\ &+ (\gamma - \alpha\beta) \cdot \operatorname{edgereduction} \\ &+ \frac{\alpha\beta}{2} \left((\text{\# of vanishing edges in } U_{\gamma'}) + (\text{\# of vanishing edges in } V_{\gamma'^{\intercal}}) \right) \end{aligned}$$

as needed.

Corollary 4.27.

$$\begin{aligned} \operatorname{slack}(\tau_P) - \operatorname{slack}(\tau) \\ &\geq (\gamma - \alpha\beta) \cdot \left(|E_{tot}(\tau_P)| - \frac{|E(U_{\tau_P})| + |E(V_{\tau_P})|}{2} - |E_{tot}(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) \\ &+ \frac{1 - \alpha}{2} \left(\# \text{ of vertices in } (U_{\tau_P} \cup V_{\tau_P}) \setminus (U_{\tau_P} \cap V_{\tau_P}) \text{ not incident to } E_{tot}(\tau_P) \right) . \end{aligned}$$

Proof. Following our slack calculation above, observe that

$$(1-\alpha)\left(\frac{w(U_{\gamma})+w(V_{\gamma'^{\intercal}})-w(U_{\tau})-w(V_{\tau})}{2}\right) \ge 0$$

since U_{γ} is a larger separator than $U_{\tau} = V_{\gamma}$, as γ is a left shape (and likewise for $V_{\gamma'\tau}$ and $V_{\tau} = U_{\gamma'\tau}$). Furthermore, this holds if we remove the degree-0 vertices from U_{γ} or $V_{\gamma'\tau}$, since U_{γ} remains a separator without these vertices. Therefore we may replace the right-hand side by

$$\frac{1-\alpha}{2} \left(\# \text{ of degree-0 vertices in } (U_{\gamma} \cup V_{\gamma'^{\intercal}}) \setminus (U_{\gamma} \cap V_{\gamma'^{\intercal}}) \right)$$

The edge factors follow immediately from the slack formula. Thus the claim holds.

4.4.1 Proof of the intersection tradeoff lemma

Lemma 4.26 (Intersection tradeoff lemma). *Given is a shape* τ *, left shapes* γ *,* γ' *, an intersection pattern* $P \in \mathcal{P}_{\gamma^-,\tau,(\gamma'\tau)^-}$ *such that the following structural property holds:*

 U_{γ} is the leftmost SMVS of U_{γ} and $V_{\gamma} \cup V_{intersected}(\gamma)$, and $V_{\gamma'^{\intercal}}$ is the rightmost SMVS of $U_{\gamma'^{\intercal}} \cup V_{intersected}(\gamma'^{\intercal})$ and $V_{\gamma'^{\intercal}}$.

Let S be an SMVS of τ *, let* τ_P *be the shape resulting from P followed by linearization, and let S' be an SMVS of* τ_P *. Then,*

$$\begin{split} w(U_{\gamma}) + w(V_{\gamma'^{\intercal}}) - w(S') + |\operatorname{Iso}(\tau_P)| &\leq w(U_{\tau}) + w(V_{\tau}) - w(S) + |\operatorname{Iso}(\tau)| \\ &+ (\# \ of \ intersections) - \beta \ (linearization) \\ &- \beta(\# \ of \ vanishing \ edges \ in \ U_{\gamma}) \\ &- \beta(\# \ of \ vanishing \ edges \ in \ V_{\gamma'^{\intercal}}) \end{split}$$

Proof. Let S'_{pre} be the preimage of S' before the intersections, as a subset of $V(\gamma \circ \tau \circ \gamma'^{\intercal})$. We construct sets $X \subseteq V(\gamma)$, $Y \subseteq V(\tau)$, and $Z \subseteq V(\gamma'^{\intercal})$ as follows.

1. We take X_0 to be the set of vertices in $V(\gamma) \setminus S'_{pre}$ which can be reached from U_{γ} by a path of non-vanishing edges in $\gamma \setminus S'_{pre}$, are either intersected or are in V_{γ} , and are not isolated in τ_P . We then take $X = X_0 \cup (S'_{pre} \cap V(\gamma))$.

- 2. We take Y_l to be the set of non-isolated vertices in $V(\tau)$ which are also in $V(\gamma)$ (either because they are in $V_{\gamma} = U_{\tau}$ or because of an intersection) and which are not reachable from U_{γ} by a path of non-vanishing edges in $\gamma \setminus S'_{pre}$. Similarly, we take Y_r to be the set of non-isolated vertices in $V(\tau)$ which are also in $V(\gamma'^{\tau})$ (either because they are in $V_{\tau} = U_{\gamma'^{\tau}}$ or because of an intersection) and which are not reachable from $V_{\gamma'^{\tau}}$ by a path of non-vanishing edges in $\gamma'^{\tau} \setminus S'_{pre}$. We then take $Y = Y_l \cup Y_r \cup V_{common} \cup (S'_{pre} \cap V_{\tau})$ where V_{common} is the set of vertices which appear in γ , τ , and γ'^{τ} .
- 3. We take Z_0 to be the set of vertices in $V(\gamma'^{\intercal}) \setminus S'_{pre}$ which can be reached from $V_{\gamma'^{\intercal}}$ by a path of non-vanishing edges in $\gamma'^{\intercal} \setminus S'_{pre}$, are either intersected or are in $U_{\gamma'^{\intercal}}$, and are not isolated in τ_P . We take Z_{extra} to be the set of vertices in $V(\gamma'^{\intercal}) \setminus S'_{pre}$ which are also in $V(\gamma)$ and which are not reachable from U_{γ} by a path of non-vanishing edges in $\gamma \setminus S'_{pre}$. We then take $Z = Z_0 \cup Z_{extra} \cup (S'_{pre} \cap V(\gamma'^{\intercal}))$.

Claim 4.28.

- 1. X separates U_{γ} from $V_{\gamma} \cup \{$ intersected vertices $\}$.
- 2. *Y* separates U_{τ} from V_{τ}
- 3. *Z* separates $U_{\gamma'^{\intercal}} \cup \{\text{intersected vertices}\}$ from $V_{\gamma'^{\intercal}}$

Proof of Claim 4.28. Statements 1 and 3 follow from the definitions of *X* and *Z*. For statement 2, assume there is a path *P* from U_{τ} to V_{τ} which does not intersect *Y*. Let *u* be the last vertex in *P* which is in γ (either because *u* is in $V_{\gamma} = U_{\tau}$ or because of an intersection) and let *v* be the next vertex in *P* which is in γ'^{τ} (either because *v* is in $V_{\tau} = U_{\gamma'^{\tau}}$ or because of an intersection). Since $u \notin Y$, there is a path of non-vanishing edges in $\gamma \setminus S'_{pre}$ from U_{γ} to *u*. Since $v \notin Y$, there is a path of non-vanishing edges in $\gamma'^{\tau} \setminus S'_{pre}$ from *v* to $V_{\gamma'^{\tau}}$.

Now consider the part of *P* between *u* and *v*. For the vertices in this part, only *u* is in $V(\gamma)$ and only *v* is in $V(\gamma'^{\dagger})$, so no edges in this part vanish. Since no vertex of *P* is in *Y*, none of these vertices are in S'_{pre} . Thus, we can compose these path segments to obtain a path of non-vanishing edges from U_{γ} to V_{γ} which does not intersect *S'*. This contradicts the fact that *S'* is the SMVS of τ_P .

We conclude that $w(X) \ge w(U_{\gamma}), w(Y) \ge w(S)$, and $w(Z) \ge w(V_{\gamma'^{\intercal}})$. To complete the proof of the intersection tradeoff lemma, we now need to show that

$$\begin{split} w(X) + w(Y) + w(Z) &\leq (\text{\# of intersections}) - (|\operatorname{Iso}(\tau_P)| - |\operatorname{Iso}(\tau)|) \\ &- \beta \Biggl(\sum_{e \in E_{tot}(\tau_P)} (\operatorname{mult}(e) - 1 - \mathbf{1}_{e \text{ vanishes}}) \Biggr) + w(U_{\tau}) + w(V_{\tau}) + w(S') \\ &- \beta ((\text{\# of vanishing edges in } U_{\gamma}) + (\text{\# of vanishing edges in } V_{\gamma'^{\tau}})) \end{split}$$

To show this, we consider the number of times vertices in $V(\tau_P)$ and edges in $E_{tot}(\tau_P)$ appear on both sides.

- 1. Vertices u in S' appear $\mathbf{1}_{u \in V(\gamma)} + \mathbf{1}_{u \in V(\tau)} + \mathbf{1}_{u \in V(\gamma'^{\intercal})}$ times on the left hand side and $1 + \mathbf{1}_{u \in U_{\tau}} + \mathbf{1}_{v \in V_{\tau}} + (\# \text{ of intersections for } u)$ times on the right hand side. It is not hard to check that these two expressions are equal.
- 2. Vertices *u* which are not in *S'* and are not isolated appear (# of intersections for *u*) + $\mathbf{1}_{u \in U_{\tau}} + \mathbf{1}_{v \in V_{\tau}}$ times on the right hand side and at most (# of intersections for *u*) + $\mathbf{1}_{u \in U_{\tau}} + \mathbf{1}_{v \in V_{\tau}} = \mathbf{1}_{u \in V(\gamma)} + \mathbf{1}_{u \in V(\gamma)} + \mathbf{1}_{u \in V(\gamma'^{\tau})} 1$ times on the left hand side. To see this, observe that *u* cannot be in both *X* and *Y*_{*l*}, cannot be in both *Y*_{*r*} and *Z*₀, and cannot be in both *X* and *Z*.
- 3. Vertices *u* which are in $Iso(\tau_P) \setminus Iso(\tau)$ (and thus not in *S'*) cannot appear in *X* or *Z* and appear in *Y* if and only if they appear in γ , τ , and γ'^{τ} . Thus, *u* appears on the left hand side $\mathbf{1}_{u \in V(\gamma)} + \mathbf{1}_{u \in V(\tau)} + \mathbf{1}_{u \in V(\gamma'^{\tau})} 2$ times and appears on the right hand side (# of intersections for *u*) + $\mathbf{1}_{u \in U_{\tau}} + \mathbf{1}_{v \in V_{\tau}} 1 = \mathbf{1}_{u \in V(\gamma)} + \mathbf{1}_{u \in V(\tau)} + \mathbf{1}_{u \in V(\gamma'^{\tau})} 2$ times.
- 4. Edges *e* between two vertices in *S'* appear $\mathbf{1}_{e \in E(\gamma)} + \mathbf{1}_{e \in E(\gamma'^{\intercal})} + \mathbf{1}_{e \in E(\gamma'^{\intercal})}$ times on the left hand side and
 - $\begin{aligned} \mathbf{1}_{e \text{ does not vanish}} + \mathbf{1}_{e \in E(U_{\tau})} + \mathbf{1}_{e \in E(V_{\tau})} + (\text{mult}(e) 1 \mathbf{1}_{e \text{ vanishes}}) \\ + \mathbf{1}_{e \text{ vanishes from } U_{\gamma}} + \mathbf{1}_{e \text{ vanishes from } V_{\gamma'^{\intercal}}} \\ = \mathbf{1}_{e \in E(\gamma)} + \mathbf{1}_{e \in E(\tau)} + \mathbf{1}_{e \in E(\gamma'^{\intercal})} 2 \cdot \mathbf{1}_{e \text{ vanishes }} + \mathbf{1}_{e \text{ vanishes from } U_{\gamma}} + \mathbf{1}_{e \text{ vanishes from } V_{\gamma'^{\intercal}}} \\ \leq \mathbf{1}_{e \in E(\gamma)} + \mathbf{1}_{e \in E(\tau)} + \mathbf{1}_{e \in E(\gamma'^{\intercal})} \end{aligned}$

times on the right hand side.

- 5. Non-vanishing edges *e* which are not between two vertices in *S'* appear $\mathbf{1}_{e \in E(U_{\tau})} + \mathbf{1}_{e \in E(V_{\tau})} + \mathbf{1}_{e \in E(V_{\tau})} + \mathbf{1}_{e \in E(V_{\tau})} + \mathbf{1}_{e \in E(Y)} + \mathbf{1}_{e$
- 6. Vanishing edges *e* which are not between two vertices in S' appear

$$\mathbf{1}_{e \in E(U_{\tau})} + \mathbf{1}_{e \in E(V_{\tau})} + \operatorname{mult}(e) - 2 + \mathbf{1}_{e \text{ vanishes from } U_{\gamma}} + \mathbf{1}_{e \text{ vanishes from } V_{\gamma'^{\mathsf{T}}}}$$
$$= \mathbf{1}_{e \in E(\gamma)} + \mathbf{1}_{e \in E(\gamma)} + \mathbf{1}_{e \in E(\gamma'^{\mathsf{T}})} - 2 + \mathbf{1}_{e \text{ vanishes from } U_{\gamma}} + \mathbf{1}_{e \text{ vanishes from } V_{\gamma'^{\mathsf{T}}}}$$

times on the right hand side and at least

$$\mathbf{1}_{e \in E(\gamma)} + \mathbf{1}_{e \in E(\tau)} + \mathbf{1}_{e \in E(\gamma'^{\tau})} - 2 + \mathbf{1}_{e \text{ vanishes from } U_{\gamma}} + \mathbf{1}_{e \text{ vanishes from } V_{\gamma'^{\tau}}}$$

times on the left hand side. To see this, we make the following observations:

- (a) $\mathbf{1}_{e \in E(Y)} \ge \mathbf{1}_{e \in E(\gamma)} + \mathbf{1}_{e \in E(\tau)} + \mathbf{1}_{e \in E(\gamma'^{\intercal})} 2$ as if *e* appears in $E(\gamma)$, $E(\tau)$, and $E(\gamma'^{\intercal})$ then $e \in E(Y)$.
- (b) $\mathbf{1}_{e \in E(X)} \ge \mathbf{1}_{e \text{ vanishes from } U_{\gamma}}$ as if *e* vanishes from U_{γ} then $e \in E(X)$.
- (c) $\mathbf{1}_{e \in E(Z)} \ge \mathbf{1}_{e \text{ vanishes from } V_{\gamma'^{\mathsf{T}}}}$ as if *e* vanishes from $V_{\gamma'^{\mathsf{T}}}$ then $e \in E(Z)$.

4.5 Slack for Removing Middle Edge Indicators

We analyze the slack for the **Removing middle edge indicators operation** after the **Intersection term decomposition operation**. To do this, we may imagine that an edge removed during the **Removing middle edge indicators operation** was removed immediately before modifying the sets U_{τ} and V_{τ} ; applying the **Intersection term decomposition operation** will result in the same shape whether the edge is removed before or after the operation.¹¹ ¹²

Theorem 4.29. Let $R_2 \rightarrow R'_2$ be a ribbon that undergoes the Adding left and right indicators operation. Let τ and τ' be their respective shapes. Then

$$\operatorname{slack}(\tau') - \operatorname{slack}(\tau) \ge \frac{\gamma}{2} (x + x_{\cap})$$

where x is the total number of removed edges, and x_{\cap} is the number of edges removed from $U_{\tau} \cap V_{\tau}$.

Proof. By Lemma 4.10,

$$\begin{aligned} \operatorname{slack}(\tau') &- \operatorname{slack}(\tau) \\ &= \left(\frac{1}{2} - \alpha\right) \left(w(\tau') - w(\tau)\right) - \left(1 - \alpha\right) \left(\frac{w(U_{\tau'}) + w(V_{\tau'}) - w(U_{\tau}) - w(V_{\tau})}{2}\right) \\ &+ \left(\gamma - \alpha\beta\right) \left(|E(\tau')| - \frac{|E(U_{\tau'})| + |E(V_{\tau'})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2}\right) \\ &+ \frac{w(S') - w(S)}{2} - \frac{|\operatorname{Iso}(\tau')| - |\operatorname{Iso}(\tau)|}{2} - \log_n \left(\frac{c_{\tau'}}{c_{\tau}^{\infty}}\right). \end{aligned}$$

We now make the following observations:

¹¹Formally, this requires the following check. When an edge indicator moves out of $U_{\tau} \cup V_{\tau}$ and into the middle, this means that the SMVS no longer includes the edge. Therefore, whether or not the edge is present is not affecting the calculation of the SMVS, and therefore the presence of the edge is not affecting the recursion.

¹²Edges removed by the **Removing middle edge indicators operation** in the **Finding PMVS subroutine** can be analyzed using either Theorem 4.29 or the analysis in Section 4.3.

1. For all vertex sets $X \subseteq V(\tau)$ equal to $X' \subseteq V(\tau')$, we have

 $w(X') - w(X) = \beta$ (# of edges removed from X).

In particular, $w(\tau') - w(\tau) = \beta x$, $w(U_{\tau'}) - w(U_{\tau}) = \beta$ (# of edges removed from U_{τ}), and $w(V_{\tau'}) - w(V_{\tau}) = \beta$ (# of edges removed from V_{τ}).

2. $\log_n(c_{\tau'}^{\approx}) = \log_n(c_{\tau}^{\approx}) - \gamma x$. Similarly to Claim 4.17, when the edge indicator is replaced by a constant term, we get a factor of magnitude $n^{-\frac{\beta}{2}}$ from the update equation. Furthermore, we get a factor of $n^{\frac{\beta}{2}-\gamma}$ shifted from λ_{τ} to $c_{\tau'}$. Multiplying these factors together gives a factor of magnitude $n^{-\gamma}$ per removed edge.

3.

$$E(\tau')| - \frac{|E(U_{\tau'})| + |E(V_{\tau'})|}{2} - |E(\tau)| + \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2}$$

= $-\frac{1}{2}((\text{# of edges removed from } U_{\tau} \setminus V_{\tau}) + (\text{# of edges removed from } V_{\tau} \setminus U_{\tau}))$
= $-\frac{1}{2}(x - x_{\cap})$

4. $w(S') \ge w(S) + \beta x_{\cap}$. To see this, let S'_{pre} be S' before the edges are removed. Since S'_{pre} is also a separator for τ , $w(S'_{pre}) \ge w(S)$. Since all separators for τ contain $U_{\tau} \cap V_{\tau}$, removing the edges increases the weight of all separators for τ by at least βx_{\cap} which implies that $w(S') \ge w(S'_{pre}) + \beta x_{\cap}$. Putting these pieces together, $w(S') \ge w(S) + \beta x_{\cap}$, as needed.

5.
$$\operatorname{Iso}(\tau') = \operatorname{Iso}(\tau)$$
.

Putting these pieces together, we have that

$$\operatorname{slack}(\tau') - \operatorname{slack}(\tau) \ge \left(\frac{1}{2} - \alpha\right)\beta x - \frac{(1 - \alpha)\beta}{2}(x + x_{\cap}) - \frac{\gamma - \alpha\beta}{2}(x - x_{\cap}) + \frac{\beta}{2}x_{\cap} + \gamma x$$
$$= \frac{\gamma}{2}(x + x_{\cap}).$$

as needed.

4.6 Final slack lower bound

Theorem 4.7. (Slack lower bound). At all times in the decomposition procedure described in Section 3.4, letting τ be the shape of R_2 ,

$$\operatorname{slack}(\tau) \geq \varepsilon \left(|E_{tot}(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} + |V_{tot}(\tau)| - \frac{|U_{\tau}| + |V_{\tau}|}{2} \right)$$

where $\varepsilon = \min \left\{ 1 - \alpha, \frac{\gamma - \alpha\beta}{8} \right\}.$

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Proof. Examining Corollary 4.20, Corollary 4.27, Theorem 4.29,

$$\begin{aligned} \operatorname{slack}(\tau) &\geq \frac{\gamma - \alpha \beta}{2} \left(|E_{tot}(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) \\ &+ \frac{1 - \alpha}{2} \left(\# \text{ of vertices in } (U_{\tau} \cup V_{\tau}) \setminus (U_{\tau} \cap V_{\tau}) \text{ not incident to } E_{tot}(\tau) \right) \,. \end{aligned}$$

Note that the latter term is initially zero and after the **Finding PMVS subroutine**, since otherwise a degree-0 vertex in $(U_{\tau'} \cup V_{\tau'}) \setminus (U_{\tau'} \cap V_{\tau'})$ could be removed to reduce the size of the separator.

Every vertex in $V_{tot}(\tau) \setminus (U_{\tau} \cup V_{\tau})$ is not isolated and so is incident to an edge in $E_{tot}(\tau) \setminus (E(U_{\tau}) \cup E(V_{\tau}))$. On the other hand, the vertices of $U_{\tau} \cup V_{\tau}$ which are not incident to an edge of $E_{tot}(\tau)$ are accounted for by the second term. In summary,

$$\begin{split} &\frac{\gamma - \alpha\beta}{2} \left(|E_{tot}(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) \\ &+ \frac{1 - \alpha}{2} \left(\text{\# of vertices in } (U_{\tau} \cup V_{\tau}) \setminus (U_{\tau} \cap V_{\tau}) \text{ not incident to } E_{tot}(\tau) \right) \\ &\geq \frac{\gamma - \alpha\beta}{4} \left(|E_{tot}(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} \right) \\ &+ \min\left\{ 1 - \alpha, \frac{\gamma - \alpha\beta}{8} \right\} \left(|V_{tot}(\tau)| - \frac{|U_{\tau}| + |V_{\tau}|}{2} \right) \end{split}$$

as needed.

5 Conclusion

In this work, we showed Sum-of-Squares lower bounds for Densest *k*-Subgraph. Our results lend strength to the conjecture that Densest *k*-Subgraph is truly a hard problem in the predicted "hard" parameter regime. Our results are in line with the log-density framework for Densest-*k*-Subgraph, complementing the extraordinary work of [BCC⁺10] from over a decade ago.

Our work provides a formal lower bound against a concrete class of algorithms for Densest *k*-Subgraph. For the optimistic algorithm designer that wishes to solve Densest *k*-Subgraph, what kind of algorithms could circumvent our lower bound? First, one could try to modify the constraints or objective of the semi-definite program. For example, "mismatching" the size of the hidden subgraph may be helpful for the related Planted Clique problem [AFdF21]. Our proof does not formally rule out non-standard SDP-based algorithms, although we believe it is likely that our proof could be modified into a lower bound against other SDPs. Second, algebraic approaches based on finite fields, Gaussian elimination, or lattice-based methods are not captured by Sum-of-Squares reasoning

[ZSWB22]. However, these techniques typically require a rigid "noise-free" structure in the problem which isn't present in Densest *k*-Subgraph, so such an algorithm would be unexpected.

There are some technical limitations to our work, which are also present in almost all existing SoS lower bounds. Technical improvements such as improving the SoS degree from n^{ε} to $\tilde{\Omega}(k)$, or tightening the slack $\gamma - \alpha\beta$ seem out of reach for our current techniques. We could also consider the closely related planted model where the size of the planted subgraph is not approximately but exactly *k*. Our analysis doesn't go through immediately in this setting for technical reasons, which is also the case in most existing SoS lower bounds. With additional work, this might be overcome, as Pang [Pan21] did for Planted Clique. That said, we believe that the behavior of SoS is qualitatively the same.

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A Additional Content on Graph Matrices

A.1 Proof of Proposition 2.23

Proposition 2.23. Every shape has an SMVS which is left of every SMVS. Furthermore, there is a unique SMVS left of every SMVS with minimum vertex size.

To prove this, we prove a stronger structural characterization of the leftmost SMVSs of a shape α . The leftmost SMVS of α with minimum vertex size is then *S* in the statement of Proposition A.1. The intuition for this structural characterization is that the sets *S_i* are "subgraphs of weight 0" that may be freely added to or removed from the "necessary core" SMVS *S*.

Proposition A.1. The collection of SMVS of α which are left of every SMVS has the following structure: there are disjoint vertex sets S and S_1, \ldots, S_k such that the collection is exactly S unioned with $\bigcup_{i \in I} S_i$ for all subsets $I \subseteq [k]$.

Proof. First, we prove the existence of a leftmost SMVS. Let S_1 , S_2 be two minimum vertex separators. Then we can construct a minimum vertex separator to the left of both of them as follows (see Fig. 3 for a picture). Since this process cannot continue indefinitely, it must terminate in a vertex separator which is left of all other vertex separators.

Let $L_1 \subseteq S_1$ be vertices of S_1 reachable from U_α without passing through S_2 , and likewise for $L_2 \subseteq S_2$. Then we take $S_L := L_1 \cup L_2 \cup (S_1 \cap S_2)$.

To show that S_L is a vertex separator, take a path P from U_α . Without loss of generality, P passes through S_1 before S_2 (or at the same time). Then L_1 (or $S_1 \cap S_2$) blocks P.

To show that S_L is minimum, observe that if we perform the analogous construction of S_R then $w(S_L) + w(S_R) \le w(S_1) + W(S_2)$. To see this note that $|S_L| + |S_R| = |S_1| + |S_2|$ and each edge *e* appears at least as many times in S_L and S_R as it does in S_1 and S_2 . In particular, if $e \in E(S_1)$ and $e \in E(S_2)$ then *e* is also in both $E(S_L)$ and $E(S_R)$. If $e \in E(S_1) \setminus E(S_2)$ then *e* must be in either $E(S_L)$ or $E(S_R)$ (note that *e* cannot go between L_1 and R_1 as otherwise S_2 would not be a separator, see Figure 2). Following similar logic, if $e \in E(S_2) \setminus E(S_1)$ then *e* must be in either $E(S_L)$ or $E(S_R)$.

Since S_1 and S_2 are minimum weight vertex separators and $w(S_L) + w(S_R) \le w(S_1) + w(S_2)$, we must have that $w(S_L) = w(S_R) = w(S_1) = w(S_2)$ so both S_L and S_R must be minimum weight vertex separators as well. This finishes the proof of existence of a leftmost SMVS.

Next, we prove the structural characterization. Suppose S_1 , S_2 are both SMVS which are left of every SMVS. Since S_1 is left of S_2 , we have $L_2 = \emptyset$, and likewise $L_2 = \emptyset$. The previous construction now shows that $S_L = S_1 \cap S_2$ is also an SMVS.

Furthermore, we claim that $S_1 \cap S_2$ is left of every SMVS. Suppose *T* is another SMVS, and *P* is a path from U_α to *T*. Since S_1 is left of *T*, *P* passes through $(S_1 \cap S_2) \cup R_1$, and

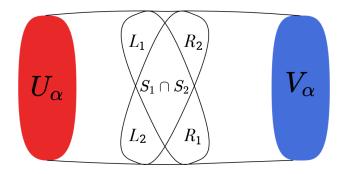


Figure 3: Leftmost and rightmost SMVS

likewise for $(S_1 \cap S_2) \cup R_2$. Since S_1 is left of S_2 and vice versa, path P must pass through both sets at the same time, therefore it passes through $S_1 \cap S_2$.

Therefore, the collection of SMVS which are left of every SMVS is closed under intersection. We may now produce the set *S* as the intersection of the family. The sets S_i are the refinement of the family under intersection. It must hold that $w(S \cup S_i) = w(S)$ for all *i*, since they are both SMVS. Therefore, there cannot be any edges between S_i and S_j , otherwise $S \cup S_i \cup S_j$ would have smaller weight than *S*. Therefore, unioning *S* with any number of the S_i always produces an SMVS with the same weight.

A.2 Additional definitions

Definition A.2 (Automorphism group). For a shape α , define Aut (α) to be the group of graph isomorphisms of $(V(\alpha), E(\alpha))$. Equivalently, Aut (α) is the stabilizer subgroup of S_n of any ribbon R with shape α .

Proposition A.3.
$$|\operatorname{Aut}(\alpha)| \sum_{\text{ribbons } R \text{ of shape } \alpha} \mathbf{M}_R = \sum_{\text{injective } \varphi: V(\alpha) \to [n]} \mathbf{M}_{\varphi(\alpha)}$$

We would like to enforce that all SMVS of a given shape are isomorphic graphs. This can be achieved by adding an infinitesimally small quantity to w(S) that breaks equality depending on the isomorphism class of *S*. Equivalently, we redefine the SMVS to minimize (w(S), S) lexicographically using an arbitrary and fixed total order of all graphs *S*. Either way, we will use the following proposition.

Proposition A.4. If τ is a middle shape, we may assume that U_{τ} and V_{τ} are isomorphic as graphs.

B Densest subgraph weight function

As pointed out in Section 3, whether or not edges are present inside a shape's separator affects the norm bound. We may find the norm bound using the following weight function for *S* being a graph on [n],

$$w_{densest}(S) = |S| - \log_n(1/p) \left| E(S) \cap E(G) \right| + \log_n(1/p) \left| E(S) \cap \overline{E(G)} \right| \,.$$

Letting S_{min} be the minimizer of $w_{densest}$ over separators of A_R and B_R in a shape α , then we would like:

$$\|\mathbf{M}_{\alpha}\| \leq \widetilde{O}\left(n^{\frac{|V(\alpha)|-w_{densest}(S_{min})}{2}}\right) = \widetilde{O}\left(n^{\frac{|V(\alpha)|-|S_{min}|}{2}}p^{\frac{|E(S_{min})\cap\overline{E(G)}|-|E(S_{min})\cap E(G)|}{2}}\right).$$

There is a problem with the above "formula". Observe that the definition of $w_{densest}(S)$ depends on the instantiation of *G* restricted to *S*. Hence two ribbons of the same shape may not have the same MVS, and it isn't possible to define the MVS with respect to $w_{densest}$ on a shape level instead of a ribbon level. This is also true of the PMVS as described in Section 3, although there we handle it by using edge and missing edge indicators (Remark 3.6). We might like to use edge indicators in a similar way to define $w_{densest}$. However, it is crucial in our analysis that the edge indicators are restricted to be in only the middle part, where they can be controlled, and are not allowed in the left and right parts.

The weight function $w_{densest}$ has the property that if it identifies an MVS with missing edge evaluations, then removing those Fourier characters still has the same MVS (and hence this is a positive ribbon that could potentially be used to norm-dominate the original ribbon). This is stated as follows.

Lemma B.1. If A_R is a minimizer of $w_{densest}$ in ribbon R, and A_R contains a missing edge χ_e , then A_R is a minimizer of $w_{densest}$ in $R' = (A_R, B_R, E(R) \setminus \{e\})$. The same holds for B_R .

Proof. The connectivity of A_R , B_R and $A_{R'}$, $B_{R'}$ is exactly the same, therefore the collection of left/right separators is the same. By removing the edge, $w_{densest}(A_R)$ decreases, and therefore it continues to be a minimizer.

C Requirements for Combinatorial Adjustment Terms

In order for the PSDness proof to go through, we need that for some $\varepsilon > 0$, the following bounds hold:

Bound C.1.

- 1. $\varepsilon \le \alpha \le \frac{1}{2}$
- 2. $\gamma \alpha \beta \ge \varepsilon$
- 3. $\beta \gamma \geq \varepsilon$
- 4. $\log_n(D_V) \leq \frac{\varepsilon}{20}$

- 5. $D_V \ge \frac{100}{\varepsilon} D_{SoS}$
- 6. For all proper middle shapes τ , $\operatorname{slack}(\tau) \geq \varepsilon \left(|E(\tau)| \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} + |V(\tau)| \frac{|U_{\tau}| + |V_{\tau}|}{2} \right)$. More generally, for any middle shape τ_P resulting from a sequence of interaction patterns,

slack
$$(\tau_P) \ge \varepsilon \left(|E_{tot}(\tau_P)| - \frac{|E(U_{\tau_P})| + |E(V_{\tau_P})|}{2} + |V_{tot}(\tau_P)| - \frac{|U_{\tau_P}| + |V_{\tau_P}|}{2} \right)$$

- 7. $B_{adjust}(\alpha)$ and $c(\alpha)$ are at most $n^{\frac{\varepsilon}{100}(|E(\alpha)| \frac{|E(U_{\alpha})| + |E(V_{\alpha})|}{2} + |V(\alpha)| \frac{|U_{\alpha}| + |V_{\alpha}|}{2})}$
- 8. For an interaction pattern P on γ , τ , γ'^{\intercal} , c(P) and N(P) are at most

$$m^{\frac{\varepsilon}{100}\left(|E(\gamma)|-\frac{|E(U_{\gamma})|+|E(V_{\gamma})|}{2}+|V(\gamma)|-\frac{|U_{\gamma}|+|V_{\gamma}|}{2}+|E(\gamma'^{\intercal})|-\frac{|E(U_{\gamma'^{\intercal}})|+|E(V_{\gamma'^{\intercal}})|}{2}+|V(\gamma'^{\intercal})|-\frac{|U_{\gamma'^{\intercal}}|+|V_{\gamma'^{\intercal}}|}{2}\right)}$$

The first five bounds are satisfied by the choice of parameters in the main Theorem 1.1. The lower bound on the slack was proven in Theorem 4.7. We will verify the bounds on the *c*-functions in Appendix E.5.

Remark C.2. While we generally think of $\varepsilon > 0$ as a fixed constant which is independent of *n*, here it is okay for ε to depend on *n*.

Probabilistically, we will assume the following events occur in the random graph $G \sim \mathcal{G}_{n,p}$.

- 1. The norm bounds hold (Corollary D.4)
- 2. A small subgraph density bound holds (Proposition D.9)

Both of these occur with probability at least $1 - n^{-\eta}$ for a tweakable parameter η . For Theorem 1.1 we may use $\eta = 1$.

D Norm bounds

A central part of our analysis is norm bounds for graph matrices. Fortunately, these norm bounds were proven in our prior work [JPR⁺22]. For completeness and exposition we reprove them here in a simpler form which can be applied more smoothly in our setting. In particular, we simplify the lower-order dependence at the cost of having extra log(n) factors in the base of the exponent.

Theorem D.1. For all $\eta > 0$, the following statement holds with probability at least $1 - n^{-\eta}$.

For all shapes α such that $|V(\alpha)| \leq 3D_V$, $|U_{\alpha}| \leq D_{SoS}$, and $|V_{\alpha}| \leq D_{SoS}$, letting S be an SMVS of α and taking $q = \max \{10D_V^2, \lceil D_{SoS} \ln(n) \rceil, \lceil \eta \ln(n) \rceil\}$,

$$\|\mathbf{M}_{\alpha}\| \leq 5(6D_V q)^{|V(\alpha)| - \frac{|U_{\alpha}| + |V_{\alpha}|}{2}} n^{\frac{|V(\alpha)| - w(S) + \operatorname{Iso}(\alpha)}{2}}$$

Proof. It is easy to reduce the case when $Iso(\alpha) \neq \emptyset$ to the case when $Iso(\alpha) = \emptyset$ so it is sufficient to prove the theorem when $Iso(\alpha) = \emptyset$. To prove this theorem when $Iso(\alpha) = \emptyset$, we use the following lemma.

Lemma D.2. For all shapes α and all $q \in \mathbb{N}$,

$$\mathbb{E}\left[\operatorname{tr}\left((\mathbf{M}_{\alpha}\mathbf{M}_{\alpha}^{\mathsf{T}})^{q}\right)\right] \leq (2q|V(\alpha)|)^{2q(|V(\alpha)|-\frac{|U_{\alpha}|+|V_{\alpha}|}{2})} n^{q|V(\alpha)|} \left(\max_{separator S} n^{-\frac{|S|}{2}} \left(\sqrt{\frac{1-p}{p}}\right)^{|E(S)|}\right)^{2q-2}$$

To use this lemma, we observe that for all $\varepsilon > 0$ and all $q \in \mathbb{N}$, by Markov's inequality,

$$\Pr\left(\|\mathbf{M}_{\alpha}\| > \sqrt[2q]{\frac{\mathbb{E}\left[\operatorname{tr}\left((\mathbf{M}_{\alpha}\mathbf{M}_{\alpha}^{\mathsf{T}})^{q}\right)\right]}{\varepsilon}}\right) \le \Pr\left(\operatorname{tr}\left((\mathbf{M}_{\alpha}\mathbf{M}_{\alpha}^{\mathsf{T}})^{q}\right) > \frac{\mathbb{E}\left[\operatorname{tr}\left((\mathbf{M}_{\alpha}\mathbf{M}_{\alpha}^{\mathsf{T}})^{q}\right)\right]}{\varepsilon}\right) < \varepsilon$$

We want to ensure that with probability $1 - n^{-\eta}$, all of our bounds hold. To do this, we choose ε to be at most $n^{-\eta}$ times the number of shapes we are considering.

Proposition D.3. For all $D_V \in \mathbb{N}$, there are at most $2^{10D_V^2}$ shapes α such that $|V(\alpha)| \leq 3D_V$

Proof. We can specify each shape α with at most $3D_V$ vertices as follows.

- 1. For each $j \in [3D_V]$, we specify whether vertex j is in $U_{\alpha} \setminus V_{\alpha}$, $V_{\alpha} \setminus U_{\alpha}$, $U_{\alpha} \cap V_{\alpha}$, $V(\alpha) \setminus (U_{\alpha} \cup V_{\alpha})$, or does not exist at all. There are 5^{3D_V} choices for this.
- 2. For each $i, j \in [3D_V]$ such that i < j, we specify whether there is an edge between vertex *i* and vertex *j* (assuming they both exist). There are at most $2^{\binom{3D_V}{2}}$ choices for this.

This gives a total of $5^{3D_V} 2^{\binom{3D_V}{2}}$ choices. For $D_V = 1$, $5^{3D_V} 2^{\binom{3D_V}{2}} = 1000 \le 1024 = 2^{10D_V^2}$ and the ratio $\frac{2^{10D_V^2}}{5^{3D_V} 2^{\binom{3D_V}{2}}}$ grows for larger D_V .

Taking $\varepsilon = \frac{1}{2^{10D_V^2}n^{\eta}}$ and taking $q = \max\{10D_V^2, \lceil D_{SoS}\ln(n)\rceil, \lceil \eta\ln(n)\rceil\}$, we make the following observations:

1.
$$\sqrt[2q]{\frac{1}{\varepsilon}} < e$$

2. Letting S_{min} be an SMVS of α ,

$$n^{q|V(\alpha)|} \left(\max_{\text{separator } S} n^{-\frac{|S|}{2}} \left(\sqrt{\frac{1-p}{p}} \right)^{|E(S)|} \right)^{2q-2} \le n^{w(S_{min})} \left(n^{\frac{|V(\alpha)|-w(S_{min})}{2}} \right)^{2q}$$

and since $w(S_{min}) \leq D_{SoS}$ and $q \geq D_{SoS} \ln(n)$, $\sqrt[2q]{n^{w(S_{min})}} \leq \sqrt{e}$

Putting these pieces together, we have that for all shapes α such that $|V(\alpha)| \le 3D_V$, with probability at least $1 - \varepsilon$,

$$\|\mathbf{M}_{\alpha}\| \le 5(6D_V q)^{|V(\alpha)| - \frac{|U_{\alpha}| + |V_{\alpha}|}{2}} n^{\frac{|V(\alpha)| - w(S_{min})}{2}}$$

Theorem D.1 now follows by taking a union bound over all such shapes α .

In our setting, we take D_V so that $D_V \ge D_{SoS} \ln(n)$ and $\eta \ln(n) \le 10D_V^2$ so that we can bound the truncation error appropriately. Thus, in our setting we can take $q = 10D_V^2$.

Corollary D.4. For all D_V , $D_{SoS} \in \mathbb{N}$ and $\eta > 0$ such that $D_V \ge D_{SoS} \ln(n)$ and $\eta \ln(n) \le 10D_{V'}^2$, the following statement holds with probability at least $1 - n^{-\eta}$.

For all shapes α (allowing isolated vertices but not multiedges or edge indicators) such that $|V(\alpha)| \leq 3D_V$, $|U_{\alpha}| \leq D_{SoS}$, and $|V_{\alpha}| \leq D_{SoS}$, letting S_{min} be an SMVS of α ,

$$\|\mathbf{M}_{\alpha}\| \leq 5(60D_V^3)^{|V(\alpha)| - \frac{|U_{\alpha}| + |V_{\alpha}|}{2}} n^{\frac{|V(\alpha)| - w(S_{min}) + |\mathrm{Iso}(\alpha)|}{2}}$$

Based on this corollary, we make the following definition.

Definition D.5 (B_{adjust}). Given a shape α , we define

$$B_{adjust}(\alpha) = 5(60D_V^3)^{|V(\alpha)| - \frac{|U_\alpha| + |V_\alpha|}{2}}$$

Therefore for $\eta \leq D_V$, with probability at least $1 - n^{-\eta}$, for all of the shapes α which we consider, recalling the notation $\|\mathbf{M}_{\alpha}^{\approx}\|$ from Section 4,

$$\|\mathbf{M}_{\alpha}\| \le B_{adjust}(\alpha) \|\mathbf{M}_{\alpha}^{\approx}\|$$

We now prove Lemma D.2, which says that

$$\mathbb{E}\left[\operatorname{tr}\left((\mathbf{M}_{\alpha}\mathbf{M}_{\alpha}^{\mathsf{T}})^{q}\right)\right] \leq (2q|V(\alpha)|)^{2q(|V(\alpha)|-\frac{|U_{\alpha}|+|V_{\alpha}|}{2})} n^{q|V(\alpha)|} \left(\max_{\text{separator }S} n^{-\frac{|S|}{2}} \left(\sqrt{\frac{1-p}{p}}\right)^{|E(S)|}\right)^{2q-2}$$

Proof of Lemma D.2.

Definition D.6. *Define* $H(\alpha, 2q)$ *to be the graph formed as follows.*

- 1. Take the shapes $\alpha_1, \ldots, \alpha_{2q}$ where for all $j \in [q]$, α_{2i-1} is a copy of α and α_{2i} is a copy of α^{\intercal} .
- 2. For all $i \in [2q 1]$, glue α_i and α_{i+1} together by setting $V_{\alpha_i} = U_{\alpha_{i+1}}$. Then glue α_{2q} and α_1 together by setting $V_{\alpha_{2q}} = U_{\alpha_1}$.

When we expand out tr $((\mathbf{M}_{\alpha}\mathbf{M}_{\alpha}^{\mathsf{T}})^{q})$, we obtain

$$\operatorname{tr}\left((\mathbf{M}_{\alpha}\mathbf{M}_{\alpha}^{\mathsf{T}})^{q}\right) = \sum_{\substack{\pi: V(H(\alpha,2q)) \to [n]:\\ \pi \text{ is injective on each } \alpha_{i}}} \prod_{i=1}^{2q} \prod_{e \in E(\alpha_{i})} \chi_{\{\pi(e)\}}$$

where if $e = \{u, v\}$ then $\pi(e) = \{\pi(u), \pi(v)\}.$

We split the maps π into equivalence classes based on the following data.

For each vertex $v \in V(\alpha_j) \setminus U_{\alpha_j}$, we specify whether there exists an i < j and a vertex $u \in V(\alpha_i)$ such that $\pi(v) = \pi(u)$. If so, we specify such a pair (i, u). There are at most $2q|V(\alpha)|$ choices for this. We have that $\sum_{j=1}^{2q} V(\alpha_j) \setminus U_{\alpha_j} = 2q(|V(\alpha)| - \frac{|U_{\alpha}| + |V_{\alpha}|}{2})$ so the total number of equivalence classes is at most $(2q|V(\alpha)|)^{2q(|V(\alpha)| - \frac{|U_{\alpha}| + |V_{\alpha}|}{2})}$.

We now analyze the contribution to $\mathbb{E}\left[\operatorname{tr}\left(\left(\mathbf{M}_{\alpha}\mathbf{M}_{\alpha}^{\mathsf{T}}\right)^{q}\right)\right]$ from each equivalence class.

Definition D.7. For each $j \in [2, 2q - 1]$, let S_j be the set of vertices $v \in \alpha_j$ such that there exists a i < j and $u \in V(\alpha_i)$ such that $\pi(u) = \pi(v)$ and there exists a k > j and a vertex $w \in V(\alpha_k)$ such that $\pi(v) = \pi(w)$. Note that we may take u = v if $v \in U_{\alpha_j} = V_{\alpha_{j-1}}$ and we may take w = v if $v \in V_{\alpha_j} = U_{\alpha_{j+1}}$.

We now observe that for the terms with nonzero expected value, each S_j must be a vertex separator of α_j .

Proposition D.8. If S_j is not a vertex separator of α_j for some $j \in [2, 2q - 1]$ then $\mathbb{E}\left[\prod_{i=1}^{2q} \prod_{e \in E(\alpha_i)} \chi_{\{\pi(e)\}}\right] = 0$

Proof. Assume S_j is not a vertex separator of α_j for some $j \in [2, 2q - 1]$ and let *P* be a path from U_{α_i} to V_{α_i} which does not intersect S_j .

Let *v* be the last vertex in *P* such that there is an i < j and a $u \in V(\alpha_i)$ such that $\pi(u) = \pi(v)$. This vertex must exist as the first vertex in *P* is in $U_{\alpha_j} = V_{\alpha_{j-1}}$ and thus has this property.

Note that there cannot be a k > j and a vertex $w \in V(\alpha_k)$ such that $\pi(v) = \pi(w)$ as otherwise we would have that $v \in S_j$. This implies that $v \notin V_{\alpha_j}$ so v cannot be the last vertex in P.

Let v' be the vertex after v in P and consider the edge $e = \{v, v'\}$. By the way we chose v, there is no i' < j and $u' \in V(\alpha_i)$ such that $\pi(u') = \pi(v')$. As we noted above, there is no k > j and $w \in V(\alpha_k)$ such that $\pi(v) = \pi(w)$. This implies that $\chi_{\{\pi(e)\}}$ only appears once in the product $\prod_{i=1}^{2q} \prod_{e \in \mathbb{E}(\alpha_i)} \chi_{\{\pi(e)\}}$ so $\mathbb{E}\left[\prod_{i=1}^{2q} \prod_{e \in E(\alpha_i)} \chi_{\{\pi(e)\}}\right] = 0$, as needed.

We now bound the contribution to $\mathbb{E}\left[\operatorname{tr}\left((\mathbf{M}_{\alpha}\mathbf{M}_{\alpha}^{\mathsf{T}})^{q}\right)\right]$ from an equivalence class as follows. Let $S_{1} = S_{2q} = \emptyset$. For each *j*,

1. For each vertex $v \in V(\alpha_i) \setminus S_i$, we assign a factor of \sqrt{n} to v.

Note that for each distinct vertex in $\pi(V(H(\alpha, 2q)))$, this assigns a factor of \sqrt{n} to this vertex for the first and last time it appears which gives a total factor of *n*.

2. For each edge $e \in E(S_j)$, we assign a factor of $\sqrt{\frac{1-p}{p}}$ to e.

Note that for each distinct edge {*x*, *y*} in the multiset $\pi(E(H(\alpha, 2q)))$, letting $m_{\{x,y\}}$ be the number of times the edge {*x*, *y*} appears in $\pi(E(H(\alpha, 2q)))$,

$$m_{\{x,y\}} \le 2 + |\{j \in [2q] : \{x,y\} \in \pi(S_j)\}|$$

Thus, this assigns a factor of at least

$$\left(\sqrt{\frac{1-p}{p}}\right)^{m_{\{x,y\}}-2} \ge \mathbb{E}\left[\chi_{\{x,y\}}^{m_{\{x,y\}}}\right]$$

to the edge $\{x, y\}$.

This gives an upper bound of

$$n^{q|V(\alpha)|} \prod_{j=2}^{2q-1} \left(n^{-\frac{|S_j|}{2}} \left(\sqrt{\frac{1-p}{p}} \right)^{|E(S_j)|} \right) \le n^{q|V(\alpha)|} \left(\max_{\text{separator } S} n^{-\frac{|S|}{2}} \left(\sqrt{\frac{1-p}{p}} \right)^{|E(S)|} \right)^{2q-2}$$

Since there are at most $(2q|V(\alpha)|)^{2q(|V(\alpha)|-\frac{|U_{\alpha}|+|V_{\alpha}|}{2})}$ equivalence classes,

$$\mathbb{E}\left[\operatorname{tr}\left((\mathbf{M}_{\alpha}\mathbf{M}_{\alpha}^{\mathsf{T}})^{q}\right)\right] \leq (2q|V(\alpha)|)^{2q(|V(\alpha)|-\frac{|U\alpha|+|V\alpha|}{2})} n^{q|V(\alpha)|} \left(\max_{\text{separator }S} n^{-\frac{|S|}{2}} \left(\sqrt{\frac{1-p}{p}}\right)^{|E(S)|}\right)^{2q-2}$$

as needed.

D.1 Conditioning

Subgraphs of *G* which are excessively dense are highly unlikely to occur and will need to be "conditioned away". This part of the analysis is only needed for controlling the truncation error and calculating $\widetilde{\mathbb{E}}[1]$ and is not needed for analysis of the approximate PSD factorization.

For a small graph *S* on [n], w(S) is approximately equal to the logarithm of the expected number of copies of *S* in $G \sim \mathcal{G}_{n,p}$. Therefore, we expect that all subgraphs satisfy $w(S) \ge 0$. We show that this holds approximately in the following proposition.

Proposition D.9. For all $\eta > 0$ and $D \in \mathbb{N}$ such that $4\log_n(D) < \beta$, the probability that there is a subgraph H of $G \sim \mathcal{G}_{n,n^{-\beta}}$ such that $|V(H)| \leq D$ and $|E(H)| > \frac{|V(H)| + \eta}{\beta - 2\log_n(D)}$ is at most $n^{-\eta}$. Equivalently, with probability at least $1 - n^{-\eta}$ all subgraphs H of G such that $|V(H)| \leq D$ satisfy

$$w(H) \ge -\eta - 2 \left| E(H) \right| \log_n(D)$$

Proof. We use the first moment method. For all *a* and *b* such that $a \le D$, the expected number of subgraphs with exactly *a* vertices and at least *b* edges is at most

$$\binom{n}{a}\binom{\frac{a(a-1)}{2}}{b}p^b \leq \frac{n^a\left(\frac{a^2}{2}\right)^b}{a!b!}n^{-b\beta} \leq \frac{n^{-\eta}}{a!}n^{a+\eta+(2\log_n(D)-\beta)b}$$

If $b \ge \frac{a+\eta}{\beta-2\log_n(D)}$ then $n^{a+\eta+(2\log_n(D)-\beta)b} \le 1$ so the expected number of graphs with exactly a vertices and at least b edges is at most $\frac{n^{-\eta}}{a!}$. Using Markov's inequality, this implies that the probability that there is a subgraph with exactly a vertices and more than $\frac{a+\eta}{\beta-2\log_n(D)}$ edges is at most $\frac{n^{-\eta}}{a!}$. Taking a union bound over all $a \in [2, D]$ (the cases when $a \le 1$ are trivial), the probability of having a subgraph H with at most D vertices and at least $\frac{|V(H)|+\eta}{\beta-2\log_n(D)}$ edges is at most $n^{-\eta}$.

Theorem D.10. Conditioned on *G* having no subgraphs *H* such that $|V(H)| \le D$ and $|E(H)| > \frac{|V(H)|+\eta}{\beta-2\log_n(D)}$ and the norm bounds holding,

1. For all shapes α such that $|V(\alpha)| \leq D$,

$$\lambda_{\alpha} ||\mathbf{M}_{\alpha}|| \leq 2B_{adjust}(\alpha) n^{(1-\alpha)\frac{|U_{\alpha}|+|V_{\alpha}|}{2}} + \eta - (\gamma - \alpha\beta - 3\log_n(D))|E(\alpha)|$$

2. For all left shapes σ such that $|V(\sigma)| \leq D$ and $|U_{\sigma}| \leq D_{SoS}$,

$$\lambda_{\sigma} ||\mathbf{M}_{\sigma}|| \le 2B_{adjust}(\sigma) n^{(1-\alpha)D_{SoS} + \eta + (\frac{\beta}{2} - \alpha\beta)|E(V_{\sigma})| - (\gamma - \alpha\beta - 3\log_n(D))|E(\sigma)|}$$

Note that this may be stronger than the first bound when $|V_{\sigma}| > D_{SoS}$.

Proof. We start with the first statement. Given a shape α , let S^{\emptyset}_{α} be a set of vertices of α (not necessarily a separator) which minimizes $w(S^{\emptyset}_{\alpha}) = |S^{\emptyset}_{\alpha}| - \beta E(S^{\emptyset}_{\alpha})$. In other words, S^{\emptyset}_{α} determines the norm bound of the scalar $\mathbf{1}^{\mathsf{T}}\mathbf{M}_{\alpha}\mathbf{1}$, which is equivalently the shape obtained from α by setting the matrix indices U_{α} and V_{α} to be \emptyset .

If $w(S_{\alpha}^{\emptyset}) \ge 0$, letting *S* be the SMVS of α , we have that $w(\alpha) \ge w(S) \ge 0$ so we have the following bounds

1.

$$\lambda_{\alpha} = n^{(\alpha-1)(|V(\alpha)| - \frac{|U_{\alpha}| + |V_{\alpha}|}{2}) + (\frac{\beta}{2} - \gamma)|E(\alpha)|}$$

= $n^{(1-\alpha)\frac{|U_{\alpha}| + |V_{\alpha}|}{2} - \frac{|V(\alpha)|}{2} - (\frac{1}{2} - \alpha)w(\alpha) - (\gamma - \alpha\beta)|E(\alpha)|}$
$$\leq n^{(1-\alpha)\frac{|U_{\alpha}| + |V_{\alpha}|}{2} - \frac{|V(\alpha)|}{2} - (\gamma - \alpha\beta)|E(\alpha)|}$$

2.
$$\|\mathbf{M}_{\alpha}\| \le B_{adjust}(\alpha) n^{\frac{|V(\alpha)|-w(S)|}{2}} \le B_{adjust}(\alpha) n^{\frac{|V(\alpha)|}{2}}$$

Putting these bounds together,

$$\lambda_{\alpha} \|\mathbf{M}_{\alpha}\| \le B_{adjust}(\alpha) n^{(1-\alpha)\frac{|U_{\alpha}| + |V_{\alpha}|}{2} - (\gamma - \alpha\beta)|E(\alpha)|}$$

We now analyze the case when $w(S_{\alpha}^{\emptyset}) < 0$. If $|E(S_{\alpha}^{\emptyset})| > \frac{|S_{\alpha}^{\emptyset}| + \eta}{\beta - 2\log_n(D)}$, we apply the conditioning argument (Lemma 6.30) from [JPR⁺22] which allows us to reduce \mathbf{M}_{α} to a linear combination of shapes with fewer edges before applying our norm bounds.

Lemma D.11. *Given a set of edges* E*, if we know that not all of the edges of* E *are in* E(G) *then*

$$\chi_E = -\sum_{E'\subseteq E: E'\neq E} \left(-\sqrt{\frac{p}{1-p}}\right)^{|E|-|E'|} \chi_{E'}$$

If $|E(S_{\alpha}^{\emptyset})| > \frac{|S_{\alpha}^{\emptyset}| + \eta}{\beta - 2\log_n(D)}$, we apply Lemma D.11 to S_{α}^{\emptyset} . In particular, we apply the following step repeatedly:

1. Specify an edge which is removed from $E(S_{\alpha}^{\emptyset})$ and specify whether there is at least one more edge which is removed from this application of Lemma D.11 or we are done with this application of Lemma D.11. Note that there are at most $2\binom{D}{2} \leq D^2$ possibilities for this.

If there is still a subset $S \subseteq S^{\emptyset}_{\alpha}$ such that $|E(S)| > \frac{|S|+\eta}{\beta-2\log_n(D)}$, then we apply Lemma D.11 to *S* and repeat this process. Otherwise, we stop.

For each $j \in \mathbb{N} \cup \{0\}$, this gives a total of at most D^{2j} terms where j edges were removed. Each such term comes with a factor of $\left(\sqrt{\frac{p}{1-p}}\right)^j$ for the removed edges.

We have the following bounds:

1.
$$\lambda_{\alpha} = n^{(\alpha-1)(|V(\alpha)| - \frac{|U_{\alpha}| + |V_{\alpha}|}{2}) + (\frac{\beta}{2} - \gamma)|E(\alpha)|} = n^{(1-\alpha)\frac{|U_{\alpha}| + |V_{\alpha}|}{2} - \frac{|V(\alpha)|}{2} - (\frac{1}{2} - \alpha)w(\alpha) - (\gamma - \alpha\beta)|E(\alpha)|}$$

2. For each of the resulting terms, letting *j* be the number of edges which were removed and letting *I* be the set of isolated vertices after the edges are removed,

 $|I| \le (\beta - 2\log_n(D))j + w(S^{\emptyset}_{\alpha}) + 2\log_n(D)|E(S^{\emptyset}_{\alpha})| + \eta$

To see this, observe that $I \subseteq S^{\emptyset}_{\alpha}$ as only edges in S^{\emptyset}_{α} can be removed. Now consider the set *S* obtained by deleting the removed edges and all vertices in *I* from S^{\emptyset}_{α} . We have that

$$|E(S^{\emptyset}_{\alpha})| - j = |E(S)| \le \frac{|S| + \eta}{\beta - 2\log_n(D)} = \frac{|S^{\emptyset}_{\alpha}| + \eta - |I|}{\beta - 2\log_n(D)}$$

which implies that

$$\begin{split} |I| &\leq |S^{\emptyset}_{\alpha}| + \eta - (\beta - 2\log_n(D))(|E(S^{\emptyset}_{\alpha})| - j) \\ &= w(S^{\emptyset}_{\alpha}) + 2\log_n(D)|E(S^{\emptyset}_{\alpha})| + \eta + (\beta - 2\log_n(D))j \end{split}$$

3. For each of the resulting terms β , letting S' be the SMVS after the edges are removed, $w(S') \ge -2\log_n(D)|E(S^{\emptyset}_{\alpha})| - \eta$. We can show this as follows. Let $S'' = S' \cap S^{\emptyset}_{\alpha}$ and observe that $w(S'') \le w(S')$. To see this, observe that since none of the edges in $E(S') \setminus E(S'')$ are removed, w(S') - w(S'') is unaffected by removing the edges. Now note that before the edges are removed, $w(S') - w(S'') \ge w(S^{\emptyset}_{\alpha} \cup (S' \setminus S'')) - w(S^{\emptyset}_{\alpha}) \ge 0$.

Since $S'' \subseteq S^{\emptyset}_{\alpha}$, we have that after the edges are removed, $|E(S'')| \leq \frac{|S''|+\eta}{\beta-2\log_n(D)}$ which implies that $w(S'') \geq -2\log_n(D)|E(S'')| - \eta \geq -2\log_n(D)|E(S^{\emptyset}_{\alpha})| - \eta$.

Putting these bounds together, using the bounds that $|E(S_{\alpha}^{\emptyset})| \leq |E(\alpha)|$ and $w(S_{\alpha}^{\emptyset}) \leq \min\{0, w(\alpha)\}$, and observing that removing vertices and/or edges from a shape α can only reduce $B_{adjust}(\alpha)$, we have that

$$\begin{split} \frac{\lambda_{\alpha} \|\mathbf{M}_{\alpha}\|}{B_{adjust}(\alpha)} &\leq \lambda_{\alpha} \sum_{j=0}^{|E(\alpha)|} \left(D^{2} \sqrt{\frac{p}{1-p}} \right)^{j} n^{\frac{|V(\alpha)|+|I|-w(S')}{2}} \\ &\leq n^{(1-\alpha)\frac{|U_{\alpha}|+|V_{\alpha}|}{2} - \frac{|V(\alpha)|}{2} - (\frac{1}{2} - \alpha)w(\alpha) - (\gamma - \alpha\beta)|E(\alpha)|} n^{\frac{|V(\alpha)|+2\log_{n}(D)|E(S_{\alpha}^{\emptyset})|+\eta}{2}} \\ &\sum_{j=0}^{|E(\alpha)|} \left(D^{2} \sqrt{\frac{p}{1-p}} \right)^{j} n^{\frac{(\beta - 2\log_{n}(D))j+w(S_{\alpha}^{\emptyset})+2\log_{n}(D)|E(S_{\alpha}^{\emptyset})|+\eta}{2}} \\ &\leq n^{(1-\alpha)\frac{|U_{\alpha}|+|V_{\alpha}|}{2} - (\gamma - \alpha\beta)|E(\alpha)|} \sum_{j=0}^{|E(\alpha)|} \left(D^{2} \sqrt{\frac{p}{1-p}} \right)^{j} n^{\frac{(\beta - 2\log_{n}(D))j}{2} + 2\log_{n}(D)|E(\alpha)|+\eta} \\ &\leq 2n^{(1-\alpha)\frac{|U_{\alpha}|+|V_{\alpha}|}{2} + \eta - (\gamma - \alpha\beta - 3\log_{n}(D))|E(\alpha)|} \end{split}$$

To prove the second statement, we split into cases based on whether $w(V_{\sigma})$ is non-negative. For the case where $w(V_{\sigma}) < 0$, we follow the same procedure as before. We let S_{σ}^{\emptyset} be a set of vertices of σ which minimizes $w(S_{\sigma}^{\emptyset})$. As long as there is a subset $S \subseteq S_{\sigma}^{\emptyset}$ such that $|E(S)| > \frac{|S|+\eta}{\beta-2\log_n(D)}$, we apply Lemma D.11 to S. For each $j \in \mathbb{N} \cup \{0\}$, this gives a total of at most D^{2j} terms where j edges were removed where each such term comes with a factor of $\left(\sqrt{\frac{p}{1-p}}\right)^j$ for the removed edges. For each such term, letting I be the set of isolated vertices and letting S' be the new SMVS, we have the same bounds as before:

- 1. $|I| \le (\beta 2\log_n(D))j + w(S_{\sigma}^{\emptyset}) + 2\log_n(D)|E(S_{\sigma}^{\emptyset})| + \eta$
- 2. $w(S') \ge -2\log_n(D)|E(S^{\emptyset}_{\alpha})| \eta$

To bound λ_{σ} , we observe that

$$\lambda_{\sigma} = n^{(1-\alpha)\frac{|U_{\sigma}| + |V_{\sigma}|}{2} - \frac{|V(\sigma)|}{2} - (\frac{1}{2} - \alpha)w(\sigma) - (\gamma - \alpha\beta)|E(\sigma)|}$$

= $n^{(1-\alpha)\frac{|U_{\sigma}|}{2} - \frac{|V(\sigma)|}{2} + \frac{1-\alpha}{2}w(V_{\sigma}) + (\frac{\beta}{2} - \alpha\beta)|E(V_{\sigma})| - (\frac{1}{2} - \alpha)w(\sigma) - (\gamma - \alpha\beta)|E(\sigma)|}$

Using the bound that $w(V_{\sigma}) \leq |U_{\sigma}| + w(S_{\sigma}^{\emptyset}) \leq D_{SoS} + w(S_{\sigma}^{\emptyset})$, we have that

$$\lambda_{\sigma} \leq n^{(1-\alpha)D_{\text{SoS}} - \frac{|V(\sigma)|}{2} + \frac{1-\alpha}{2}w(S_{\sigma}^{\emptyset}) + (\frac{\beta}{2} - \alpha\beta)|E(V_{\sigma})| - (\frac{1}{2} - \alpha)w(\sigma) - (\gamma - \alpha\beta)|E(\sigma)|}$$

Putting these bounds together, using the bounds that $|E(S_{\sigma}^{\emptyset})| \leq |E(\sigma)|$ and $w(S_{\sigma}^{\emptyset}) \leq \min\{0, w(\sigma)\}$, and observing that removing vertices and/or edges from a shape σ can only reduce $B_{adjust}(\sigma)$, we have that

$$\begin{split} \frac{\lambda_{\sigma} ||\mathbf{M}_{\sigma}||}{B_{adjust}(\sigma)} &\leq \lambda_{\sigma} \sum_{j=0}^{|E(\sigma)|} \left(D^{2} \sqrt{\frac{p}{1-p}} \right)^{j} n^{\frac{|V(\sigma)|+|I|-w(S')}{2}} \\ &\leq n^{(1-\alpha)D_{\text{SoS}} - \frac{|V(\sigma)|}{2} + \frac{1-\alpha}{2} w(S_{\sigma}^{\emptyset}) + (\frac{\beta}{2} - \alpha\beta)|E(V_{\sigma})| - (\frac{1}{2} - \alpha)w(\sigma) - (\gamma - \alpha\beta)|E(\sigma)|} n^{\frac{|V(\sigma)|+2\log_{n}(D)|E(S_{\sigma}^{\emptyset})|+\eta}{2}} \\ &\sum_{j=0}^{|E(\sigma)|} \left(D^{2} \sqrt{\frac{p}{1-p}} \right)^{j} n^{\frac{(\beta - 2\log_{n}(D))j+w(S_{\sigma}^{\emptyset}) + 2\log_{n}(D)|E(S_{\sigma}^{\emptyset})|+\eta}{2}} \\ &\leq n^{(1-\alpha)D_{\text{SoS}} + (\frac{\beta}{2} - \alpha\beta)|E(V_{\sigma})| - (\gamma - \alpha\beta)|E(\sigma)|} \sum_{j=0}^{|E(\sigma)|} \left(D^{2} \sqrt{\frac{p}{1-p}} \right)^{j} n^{\frac{(\beta - 2\log_{n}(D))j}{2} + 2\log_{n}(D)|E(\sigma)| + \eta} \\ &\leq 2n^{(1-\alpha)D_{\text{SoS}} + \eta + (\frac{\beta}{2} - \alpha\beta)|E(V_{\sigma})| - (\gamma - \alpha\beta - 3\log_{n}(D))|E(\sigma)|} \end{split}$$

For the case when $w(V_{\sigma}) \ge 0$, we again observe that

$$\lambda_{\sigma} = n^{(1-\alpha)\frac{|U_{\sigma}|}{2} - \frac{|V(\sigma)|}{2} + \frac{1-\alpha}{2}w(V_{\sigma}) + (\frac{\beta}{2} - \alpha\beta)|E(V_{\sigma})| - (\frac{1}{2} - \alpha)w(\sigma) - (\gamma - \alpha\beta)|E(\sigma)|$$

Since $w(\sigma) \ge w(V_{\sigma}) \ge 0$ and $w(V_{\sigma}) \le |U_{\sigma}| \le D_{SoS}$,

$$\lambda_{\sigma} \leq n^{(1-\alpha)D_{\text{SoS}} - \frac{|V(\sigma)|}{2} + (\frac{\beta}{2} - \alpha\beta)|E(V_{\sigma})| - (\gamma - \alpha\beta)|E(\sigma)|}$$

Since $\|\mathbf{M}_{\sigma}\| \leq B_{adjust}(\sigma)n^{\frac{|V(\sigma)|-w(S)|}{2}} \leq B_{adjust}(\sigma)n^{\frac{|V(\sigma)|}{2}}$, we have that

$$\lambda_{\sigma} \|\mathbf{M}_{\sigma}\| \le B_{adjust}(\sigma) n^{(1-\alpha)D_{SoS} + \eta + (\frac{p}{2} - \alpha\beta)|E(V_{\sigma})| - (\gamma - \alpha\beta)|E(\sigma)|}.$$

Using the forthcoming notation of Appendix E.1, we deduce the following norm bounds for left shapes.

Corollary D.12. With the conditioning, for all $U \in I_{mid}$ and all $\sigma \in \mathcal{L}_{U,\leq D_V}$,

$$\lambda_{\sigma^{-}} \|\mathbf{M}_{\sigma^{-}}\| \leq 2B_{adjust}(\sigma) n^{(1-\alpha)D_{SoS} + \eta - (\frac{\beta}{2} - 3\log_n(D_V))|E(U)| - (\gamma - \alpha\beta - 3\log_n(D_V))|E(\sigma) \setminus E(U)|}$$

Proof. By Theorem D.10,

$$\lambda_{\sigma} ||\mathbf{M}_{\sigma}|| \le 2B_{adjust}(\sigma) n^{(1-\alpha)D_{SoS} + \eta + (\frac{\beta}{2} - \alpha\beta)|E(U)| - (\gamma - \alpha\beta - 3\log_n(D))|E(\sigma)|}$$

We now observe that

$$\lambda_{\sigma^{-}} \|\mathbf{M}_{\sigma^{-}}\| \leq \left(\sqrt{\frac{p}{1-p}}\right)^{|E(U)|} n^{(\gamma - \frac{\beta}{2})|E(U)|} \lambda_{\sigma} \|\mathbf{M}_{\sigma}\| = n^{(\gamma - \beta)|E(U)|} \lambda_{\sigma} \|\mathbf{M}_{\sigma}\|$$

where we remove the indicators from V_{σ} for the purpose of bounding $||\mathbf{M}_{\sigma}||$. Combining these bounds gives the result.

Corollary D.13. With the conditioning, for all $U \in I_{mid}$ and all $\sigma \in \mathcal{L}_{U,\leq D_V}$,

$$\lambda_{\sigma^{-}} \|\mathbf{M}_{\sigma^{-}}\| \sqrt{\lambda_{U}} \|\mathbf{M}_{U}\| \leq 2B_{adjust}(\sigma) n^{D_{\text{SoS}} + \eta - \frac{\varepsilon}{2}D_{\text{SoS}} - \frac{\varepsilon}{8}|E(\sigma)| - \frac{\varepsilon}{8}|V(\sigma)|}$$

Proof. By Corollary D.12,

$$\lambda_{\sigma^{-}} \|\mathbf{M}_{\sigma^{-}}\| \leq 2B_{adjust}(\sigma) n^{(1-\alpha)D_{SoS} + \eta - (\frac{\beta}{2} - 3\log_n(D_V))|E(U)| - (\gamma - \alpha\beta - 3\log_n(D_V))|E(\sigma) \setminus E(U)|}$$

Since $\sqrt{\lambda_U ||\mathbf{M}_U||} = n^{(\frac{\beta}{2} - \frac{\gamma}{2})|E(U)|}$, using the fact that $|E(\sigma)| \ge |V(\sigma) \setminus U_{\sigma}|$, we have that

$$\begin{split} \lambda_{\sigma^{-}} \|\mathbf{M}_{\sigma^{-}}\| \sqrt{\lambda_{U}} \|\mathbf{M}_{U}\| &\leq 2B_{adjust}(\sigma) n^{(1-\alpha)D_{\text{SoS}} + \eta - (\frac{\gamma}{2} - 3\log_{n}(D_{V}))|E(U)| - (\gamma - \alpha\beta - 3\log_{n}(D_{V}))|E(\sigma) \setminus E(U)|} \\ &\leq 2B_{adjust}(\sigma) n^{D_{\text{SoS}} + \eta - \frac{\varepsilon}{2}D_{\text{SoS}} - \frac{\varepsilon}{8}|E(\sigma)| - \frac{\varepsilon}{8}|V(\sigma)|} \end{split}$$

as needed.

E Formal Approximate PSD Decomposition

E.1 Starting point for the approximate PSD decomposition

In this section, we show that with high probability, the pseudo-calibrated moment matrix Λ (formally defined in Section 2.9) is PSD. We do this by giving an approximate PSD factorization for Λ . We will then show that the error is PSD dominated by the terms in this approximate PSD factorization. For this approximate PSD factorization, we use much of the technical framework of [PR20] (although we cannot formally apply the machinery there because it does not work well for *sparse* random graphs, nor does it incorporate the PMVS idea).

Definition E.1 (I_{mid}). Let I_{mid} be the set of shapes of separators of S. In other words, I_{mid} is the set of diagonal shapes α such that $V(\alpha) = U_{\alpha} = V_{\alpha}$ and $|V(\alpha)| \le D_V$.

Definition E.2 (\mathcal{L} and \mathcal{L}_U and $\mathcal{L}_{U,\leq D}$). Let \mathcal{L} be the set of left shapes in \mathcal{S} . Given $U \in I_{mid}$ and $D \in \mathbb{N}$, we define \mathcal{L}_U to be the set of all left shapes $\sigma \in \mathcal{S}$ such that $V_{\sigma} = U$. The set $\mathcal{L}_{U,\leq D}$ also requires $|V(\sigma)| \leq D$.

Definition E.3 (\mathcal{M} and $\mathcal{M}_{U,V}$ and $\mathcal{M}_{U,V,\leq D}$). Let \mathcal{M} be the set of middle shapes in \mathcal{S} . Given $D \in \mathbb{N}$ and $U, V \in I_{mid}$ (which may intersect), we define $\mathcal{M}_{U,V}$ to be the set of middle shapes $\tau \in \mathcal{S}$ such that $U_{\tau} = U$ and $V_{\tau} = V$. The set $\mathcal{M}_{U,V,\leq D}$ also requires $|V(\tau)| \leq D$.

Remark E.4. Due to the size constraints on shapes in S, we only have shapes with at most D_V vertices. We will sometimes write $\mathcal{L}_{U,\leq D_V}$ instead of the equivalent \mathcal{L}_U when it is relevant to the current section. Since all of the shapes we will consider have an SMVS with weight at most D_{SoS} , we only consider $U \in I_{mid}$ such that $w(U) \leq D_{SoS}$.

Definition E.5 (\mathcal{T}_{τ} and $\mathcal{T}_{\tau,\leq D_1,\leq D_2}$). *Given a shape* τ (which may or may not be proper), we define \mathcal{T}_{τ} to be the set of triples of ribbons (R_1, R_2, R_3) such that

- (i) R_2 has shape τ .
- (ii) R_1 is a left ribbon and R_3 is a right ribbon.
- (iii) R_1, R_2, R_3 are properly composable.
- (iv) The edges and edge indicators agree on $B_{R_1} = A_{R_2}$ and $B_{R_2} = A_{R_3}$.
- (v) R_1, R_3 have no edge indicators outside of $B_{R_1} = A_{R_2}$ and $B_{R_2} = A_{R_3}$.

 $\mathcal{T}_{\tau,\leq D_1,\leq D_2}$ additionally requires that $|V(R_1)| \leq D_1, |V(R_3)| \leq D_2$.

Definition E.6 ($\mathcal{R}(\alpha)$). *Given a shape* α *, we define* $\mathcal{R}(\alpha)$ *to be the set of ribbons* R *which have shape* α *.*

Definition E.7 (Minus abbreviation). *Given a left ribbon* L, *let* $L^- = L \setminus E(B_L)$. *Given a right ribbon* R, *let* $R^- = R \setminus E(A_R)$. *The notation is defined similarly for shapes.*

Definition E.8 $(U \sim V)$. Given $U, V \in I_{mid}$, we write $U \sim V$ if |U| = |V|, |E(U)| = |E(V)|, U and V have the same edges on the vertex set $U \cap V$, and U and V have the same order on $U \cap V$.

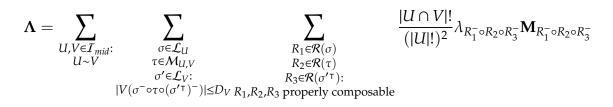
Starting from the pseudocalibrated formula for Λ and incorporating this notation, we have the following lemma.

Lemma E.9.

$$\mathbf{\Lambda} = \sum_{\substack{U, V \in \mathcal{I}_{mid}: \\ U \sim V}} \frac{|U \cap V|!}{(|U|!)^2} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{R_1, R_2, R_3 \in \mathcal{T}_{\tau, \leq D_V, \leq D_V}} \lambda_{R_1^- \circ R_2 \circ R_3^-} \mathbf{M}_{R_1^- \circ R_3^-} \mathbf{M}_$$

where truncation₁ is defined in Definition E.10.

Proof. Starting from $\Lambda = \sum_{R \in S} \lambda_R \mathbf{M}_R$, we apply Proposition 2.33 to factor R into a left, middle, and right ribbon. We symmetrize over all choices of the order for the leftmost SMVS U and the rightmost SMVS V such that the permutation on $U \cap V$ is the identity permutation. There are exactly $\frac{|U|!|V|!}{|U \cap V|!}$ such choices for the orders. By Proposition A.4, $U \sim V$ and $\frac{|U|!|V|!}{|U \cap V|!} = \frac{(|U|!)^2}{|U \cap V|!}$. Therefore,

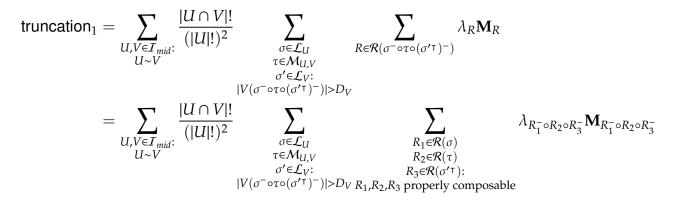


The condition $|V(\sigma^- \circ \tau \circ (\sigma'^{\intercal})^-)| \le D_V$ arises because the size of the entire ribbon $R = R_1^- \circ R_2 \circ R_3^-$ is bounded by D_V . We would like instead that the sizes of the individual pieces R_1, R_2, R_3 are bounded separately by D_V . The difference between these two consists only of very large ribbons, which we will bound as a truncation error.

Definition E.10 (truncation₁).

$$\operatorname{truncation}_{1} = \sum_{\substack{U, V \in \mathcal{I}_{mid}: \\ U \sim V}} \frac{|U \cap V|!}{(|U|!)^{2}} \sum_{\substack{\sigma \in \mathcal{L}_{U, \leq D_{V}} \\ \tau \in \mathcal{M}_{U, V, \leq D_{V}} \\ \sigma' \in \mathcal{L}_{V, \leq D_{V}}: \\ |V(\sigma^{-}\circ\tau\circ(\sigma'^{\intercal})^{-})| > D_{V}}} \frac{\lambda_{\sigma^{-}\circ\tau\circ(\sigma'^{\intercal})^{-}}\mathbf{M}_{\sigma^{-}\circ\tau\circ(\sigma'^{\intercal})^{-}}}{|\operatorname{Aut}(\sigma^{-}\circ\tau\circ(\sigma'^{\intercal})^{-})|}$$

We have grouped truncation₁ into shapes. In terms of ribbons, by Proposition A.3,



Continuing the calculation,

$$\begin{split} \mathbf{\Lambda} + \operatorname{truncation}_{1} &= \sum_{\substack{U, V \in \mathcal{I}_{mid}: \\ U \sim V}} \frac{|U \cap V|!}{(|U|!)^{2}} \sum_{\substack{\sigma \in \mathcal{L}_{U} \\ \tau \in \mathcal{M}_{U,V} \\ \sigma' \in \mathcal{L}_{V}}} \sum_{\substack{R_{1} \in \mathcal{R}(\sigma) \\ R_{2} \in \mathcal{R}(\tau) \\ R_{3} \in \mathcal{R}(\sigma'^{\tau}): \\ R_{1}, R_{2}, R_{3} \text{ properly composable}} \end{split}$$
$$= \sum_{\substack{U, V \in \mathcal{I}_{mid}: \\ U \sim V}} \frac{|U \cap V|!}{(|U|!)^{2}} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{\substack{R_{1}, R_{2}, R_{3} \in \mathcal{T}_{\tau, \leq D_{V}, \leq D_{V}}} \lambda_{R_{1}^{-} \circ R_{2} \circ R_{3}^{-}} \mathbf{M}_{R_{1}^{-} \circ R_{2} \circ R_{3}^{-}} \end{split}$$

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* *! 4

as desired.

E.2 Interaction patterns

In order to analyze Λ , we use the procedure in Section 3 as described by *interaction patterns*. These generalize intersection patterns (Definition 2.36) to account for the additional ways that shapes can interact in the recursion.

Definition E.11 (PMVS interaction, implicit definition). *Given composable shapes* $\gamma, \tau, \gamma'^{\tau}$ such that $E(V_{\gamma}) = E(U_{\tau})$ and $E(V_{\tau}) = E(U_{\gamma'^{\tau}})$, let $\mathcal{P}_{\gamma,\tau,\gamma'^{\tau}}^{PMVS}$ be the set of possible choices for one iteration of the **Finding PMVS subroutine** (Section 3.2, note that this includes the **Removing middle edge indicators operation**) run on ribbons R_1, R_2, R_3 of shapes $\gamma, \tau, \gamma'^{\tau}$.

Definition E.12 (Intersection term interaction, implicit definition). *Given composable shapes* $\gamma, \tau, \gamma'^{\intercal}$ such that $E(V_{\gamma}) = E(U_{\tau})$ and $E(V_{\tau}) = E(U_{\gamma'^{\intercal}})$, let $\mathcal{P}_{\gamma,\tau,\gamma'^{\intercal}}^{intersect}$ be the set of possible choices for the Intersection term decomposition operation followed by the Removing middle edge indicators operation (Section 3.3) run on ribbons R_1, R_2, R_3 of shapes $\gamma, \tau, \gamma'^{\intercal}$.

Definition E.13 (Interaction pattern). Let $\mathcal{P}_{\gamma,\tau,\gamma'\tau}^{interact} = \mathcal{P}_{\gamma,\tau,\gamma'\tau}^{PMVS} \cup \mathcal{P}_{\gamma,\tau,\gamma'\tau}^{intersect}$.

Definition E.14 (PMVS interaction, explicit definition). *Given composable shapes* $\gamma, \tau, \gamma' \tau$ such that $E(V_{\gamma}) = E(U_{\tau})$ and $E(V_{\tau}) = E(U_{\gamma'\tau})$, a PMVS interaction pattern $P \in \mathcal{P}_{\gamma,\tau,\gamma'\tau}^{PMVS}$ consists of:

- (*i*) For each edge in $E(U_{\tau}) \cup E(V_{\tau})$ which does not yet have an edge indicator, we specify whether the edge is given an indicator or is removed,
- (ii) For each edge in $E(U_{\tau}) \cup E(V_{\tau})$ which is now in the middle of $\gamma^{-} \circ \tau \circ (\gamma'^{\tau})^{-}$, we specify whether the edge is kept or removed when the indicator is removed.

We furthermore have the structural property of P with $\gamma, \tau, \gamma'^{\intercal}$ that after the edges have been removed from $V_{\gamma} = U_{\tau}$ and $V_{\tau} = U_{\gamma'^{\intercal}}$ in the first step, U_{γ} is the leftmost SMVS for γ and $V_{\gamma'^{\intercal}}$ is the rightmost SMVS for γ'^{\intercal} .

Definition E.15 (Intersection term interaction, explicit definition). *Given composable shapes* $\gamma, \tau, \gamma'^{\tau}$ such that $E(V_{\gamma}) = E(U_{\tau})$ and $E(V_{\tau}) = E(U_{\gamma'\tau})$, an intersection term interaction pattern $P \in \mathcal{P}_{\gamma,\tau,\gamma'^{\tau}}^{intersect}$ consists of:

- (i) An intersection pattern between $\gamma^-, \tau, (\gamma'^{\intercal})^-$
- *(ii) After these intersections, some edges appear with multiplicity greater than* 1*. For each such edge, we linearize it and specify whether we are taking the term with an edge, or the constant term.*
- (iii) For each edge indicator in $E(U_{\tau}) \cup E(V_{\tau})$ which is now in the middle of $\gamma^{-} \circ \tau \circ (\gamma'^{\tau})^{-}$, we specify whether the edge is kept or removed when the indicator is removed.

We furthermore have the structural property of P with $\gamma, \tau, \gamma'^{\intercal}$ that U_{γ} is the leftmost SMVS in γ of U_{γ} and $V_{\gamma} \cup V_{intersected}(\gamma)$, and $V_{\gamma'^{\intercal}}$ is the rightmost SMVS in γ'^{\intercal} of $U_{\gamma'^{\intercal}} \cup V_{intersected}(\gamma')$ and $V_{\gamma'^{\intercal}}$.

Definition E.16 (τ_P). *Given an interaction pattern* $P \in \mathcal{P}_{\gamma,\tau,\gamma'\tau}^{interact}$, let τ_P be the resulting shape.

Definition E.17 (*N*(*P*), implicit definition). Given $\gamma, \tau, \gamma'^{\intercal}$ and an interaction pattern $P \in \mathcal{P}_{\gamma,\tau,\gamma'^{\intercal}}^{interact}$, we define *N*(*P*) so that for each ribbon *R* of shape τ_P , *N*(*P*) is the number of triples of ribbons R_1 , R_2 , R_3 of shapes $\gamma, \tau, \gamma'^{\intercal}$ which result in the ribbon *R* through interaction pattern *P*.

N(P) > 1 holds when there is an increase in symmetry in an interaction term.

Definition E.18 (c_P , implicit definition). *Given an interaction pattern* $P \in \mathcal{P}_{\gamma,\tau,\gamma'\tau}^{interact}$, we define c_P such that $N(P)c_P\lambda_{\tau_P}$ is the coefficient on a resulting ribbon R of shape τ_P .

Following the analysis in Claim 4.17, Claim 4.23, we have the following explicit formulas for c_P .

Definition E.19 (*c*_{*P*}, explicit definition). Given an interaction pattern $P \in \mathcal{P}_{\gamma,\tau_{\gamma}\gamma^{\prime}\tau}^{interact}$

1. If P is a PMVS interaction,

$$c_P = (n^{-\gamma})^{(\text{total # of removed edges})} (-1)^{\text{# of edges removed from } U_{\tau} \cup V_{\tau} \text{ for not being in } E(G)} \\ \left(\frac{1}{1-p}\right)^{(\text{# of new edge indicators})} (1-p)^{\text{# of edges and edge indicators removed from the middle}}$$

2. If P is an intersection term interaction,

$$c_{P} = \left(\frac{k}{n}\right)^{(\# \text{ of intersections})} \left(\prod_{e \in E_{tot}(\tau_{P})} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{\operatorname{mult}(e)-1+\mathbf{1}_{e} \text{ vanishes}} \left(\frac{1-2p}{\sqrt{p(1-p)}}\right)^{(\operatorname{mult}(e)-2)\mathbf{1}_{e} \text{ vanishes}} \left(\frac{1-2p}{\sqrt{p(1-p)}}\right)^{1} \left(\frac{1-2p}{\sqrt{p(1-$$

Lemma E.20.

$$|c_{P}| \leq (n^{\alpha-1})^{(\# \text{ of intersections})} \left(\prod_{e \in E_{tot}(\tau_{P})} (n^{\frac{\beta}{2}-\gamma})^{\text{mult}(e)-1+\mathbf{1}_{e \text{ vanishes}}} (n^{\frac{\beta}{2}})^{\text{mult}(e)-1-\mathbf{1}_{e \text{ vanishes}}} \right)$$
$$\cdot (1-p)^{\# \text{ of edges and edge indicators removed from the middle}} (n^{-\gamma})^{(\# \text{ of edges removed from the middle})}$$

Proof. This may be proven either directly from the exact formula above, or by the analysis already presented in Claim 4.23. From the above, note that $\frac{1-2p}{\sqrt{p(1-p)}} \leq \sqrt{\frac{1-p}{p}} = n^{-\frac{\beta}{2}}$. Note that $\frac{1-3p+3p^2}{p(1-p)} \leq \frac{1-p}{p} = n^{-\frac{\beta}{2}}$ as $p \leq \frac{1}{2}$.

Definition E.21 (c_p^{\approx}). Define c_p^{\approx} to be the polynomial factors in c_P , namely

$$c_{P}^{\approx} = \left(n^{\alpha-1}\right)^{(\# of intersections)} \left(\prod_{e \in E_{tot}(\tau_{P})} \left(n^{\frac{\beta}{2}-\gamma}\right)^{\operatorname{mult}(e)-1+\mathbf{1}_{e \text{ vanishes}}} \left(n^{\frac{\beta}{2}}\right)^{\operatorname{mult}(e)-1-\mathbf{1}_{e \text{ vanishes}}}\right)$$
$$\cdot (n^{-\gamma})^{(\# of edges removed from the middle)}$$

E.3 The approximate PSD decomposition

Applying the recursion as described in Section 3 with the definitions from the previous section, we obtain the approximate PSD decomposition as follows. The proofs essentially follow from the definitions and are delayed to Appendix E.3.1.

Definition E.22 (Plus abbreviation). *Given a shape* τ *, let* τ^+ *denote the improper shape with left and right edge indicators added to* U_{τ} *and* V_{τ} *.*

Definition E.23 (Terminal interaction). $P \in \mathcal{P}_{\gamma,\tau,\gamma'^{\dagger}}^{PMVS}$ is a terminal PMVS interaction *if all edges* are given indicators (none are removed). $P \in \mathcal{P}_{\gamma,\tau,\gamma'^{\dagger}}^{intersection}$ is a terminal intersection interaction *if* there are no intersections.

Lemma E.24 (One iteration, PMVS operation). *For all shapes* τ *and* $D_1, D_2 \in \mathbb{N}$ *,*

$$=\sum_{\substack{(R_1,R_2,R_3)\in\mathcal{T}_{\tau,\leq D_1,\leq D_2}}}\lambda_{R_1^-\circ R_2\circ R_3^-}\mathbf{M}_{R_1^-\circ R_2\circ R_3^-} \\ +\sum_{\substack{(R_1,R_2,R_3)\in\mathcal{T}_{\tau,\leq D_1,\leq D_2}}}\left(\frac{1}{1-p}\right)^{(\#\ of\ new\ edge\ indicators)}\lambda_{R_1^-\circ R_2\circ R_3^-}\mathbf{M}_{R_1^-\circ R_2^+\circ R_3^-} \\ +\sum_{\substack{\gamma\in\mathcal{L}_{U_{\tau,\leq D_1},\gamma'\in\mathcal{L}_{V_{\tau,\leq D_2}}}}\sum_{\substack{non-terminal\\P\in\mathcal{P}_{\gamma,\tau,\gamma'\tau}}}\frac{N(P)c_P}{|U_\gamma|!|V_{\gamma'\tau}|!}\left(\sum_{\substack{(R_1',R_2',R_3')\in\mathcal{T}_{\tau_P,\leq D_1',\leq D_2'}}\lambda_{R_1'^-\circ R_2'\circ R_3'^-}\mathbf{M}_{R_1'^-\circ R_2'\circ R_3'^-}\right)$$

where $D'_1 = D_1 - |V(\gamma) \setminus U_{\gamma}|$ and $D'_2 = D_2 - |V(\gamma'^{\intercal}) \setminus V_{\gamma'^{\intercal}}|$

Definition E.25 (L and L_U and L_{U,≤D}). Let $\mathbf{L} = \sum_{L \in \mathcal{L}} \lambda_{L^-} \mathbf{M}_{L^-}$ and $\mathbf{L}_U = \sum_{L \in \mathcal{L}_U} \lambda_{L^-} \mathbf{M}_{L^-}$ and $\mathbf{L}_{U,\leq D} = \sum_{L \in \mathcal{L}_{U,\leq D}} \lambda_{L^-} \mathbf{M}_{L^-}$.

Lemma E.26 (One iteration, intersection term operation). *For all shapes* τ *and* $D_1, D_2 \in \mathbb{N}$ *,*

$$\sum_{(R_1,R_2,R_3)\in\mathcal{T}_{\tau,\leq D_1,\leq D_2}} \lambda_{R_1^-\circ R_2\circ R_3^-} \mathbf{M}_{R_1^-\circ R_2^+\circ R_3^-} = \mathbf{L}_{U_\tau,\leq D_1} \left(\frac{\lambda_\tau \mathbf{M}_{\tau^+}}{|\operatorname{Aut}(\tau)|}\right) \mathbf{L}_{V_\tau,\leq D_2}^{\mathsf{T}}$$
$$-\sum_{\gamma\in\mathcal{L}_{U_\tau,\leq D_1,\gamma'}\in\mathcal{L}_{V_\tau,\leq D_2}} \sum_{\substack{\text{non-terminal}\\ P\in\mathcal{P}_{intersect}\\ \gamma,\tau^+,\gamma'^{\mathsf{T}}}} \frac{N(P)c_P}{|U_\gamma|!|V_{\gamma'^{\mathsf{T}}}|!} \left(\sum_{(R_1',R_2',R_3')\in\mathcal{T}_{\tau_P,\leq D_1',\leq D_2'}} \lambda_{R_1'^-\circ R_2'\circ R_3'^-} \mathbf{M}_{R_1'^-\circ R_2'\circ R_3'^-}\right)$$

where $D'_1 = D_1 - |V(\gamma) \setminus U_{\gamma}|$ and $D'_2 = D_2 - |V(\gamma'^{\intercal}) \setminus V_{\gamma'^{\intercal}}|$

We repeatedly apply Lemma E.24 until we have only terms with PMVS identified (i.e. having both left and right indicators), then we apply Lemma E.26, then we repeat

Lemma E.24, and so forth. The next lemma gives the formal statement of the final result of the iteration on Λ .

We define iterated interaction patterns, which are the combinatorial objects describing the branches of the full recursion.

Definition E.27 (Iterated interaction pattern, $\mathcal{P}_j(\tau)$). *Given a shape* τ *and* $j \in \mathbb{N}$, *define* $\mathcal{P}_j(\tau)$ *to be the set of tuples* $(\Gamma, \Gamma'^{\tau}, P)$ *such that*

- 1. Γ is a tuple of j composable left shapes $(\gamma_j, \ldots, \gamma_1)$. Let $\gamma = \gamma_j^- \circ \cdots \circ \gamma_2^- \circ \gamma_1$.
- 2. Γ'^{T} is a tuple of j composable right shapes $(\gamma'_1^{\mathsf{T}}, \ldots, \gamma'_j^{\mathsf{T}})$. Let $\gamma'^{\mathsf{T}} = \gamma'_1^{\mathsf{T}} \circ (\gamma'_2^{\mathsf{T}})^- \cdots \circ (\gamma'_j^{\mathsf{T}})^-$.
- 3. *P* is a tuple of *j* interaction patterns (P_1, \ldots, P_j) such that for each $i \in [j]$, $P_i \in \mathcal{P}_{\gamma_i, \tau_{P_{i-1}}, \gamma'_i}^{interact}$. For i = 1, we take $\tau_{P_0} = \tau$.
- 4. P consists of sequences of non-terminal PMVS interactions ending with a terminal PMVS interaction (there is at least one sequence), with a non-terminal intersection interaction in between consecutive sequences, then finally ending with a terminal intersection interaction.
- 5. *P* has at least one non-terminal interaction (we will explicitly separate the middle shapes i.e. the terms with a terminal PMVS interaction followed by a terminal intersection interaction because they are a good warm-up for the analysis of the other terms).

We use $\tau_P = \tau_{P_i}$ to denote the final resulting shape.

Definition E.28 $(D_L(P) \text{ and } D_R(P))$. *Define* $D_L(P) = D_V - |V(\gamma) \setminus U_{\gamma}|$ *and* $D_R(P) = D_V - |V(\gamma'^{\intercal}) \setminus V_{\gamma'^{\intercal}}|$.

Lemma E.29 (Result of full iteration).

$$\begin{split} \mathbf{\Lambda} &= \sum_{U,V \in \mathcal{I}_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^2} \mathbf{L}_{U,\leq D_V} \left(\sum_{\tau \in \mathcal{M}_{U,V}} \left(\frac{1}{1-p} \right)^{|E(U_{\tau}) \cup E(V_{\tau})|} \frac{\lambda_{\tau} \mathbf{M}_{\tau^+}}{|\operatorname{Aut}(\tau)|} \right) \mathbf{L}_{V,\leq D_V}^{\mathsf{T}} \\ &+ \sum_{U,V \in \mathcal{I}_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^2} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{j=1}^{\infty} \sum_{(\Gamma,\Gamma'^{\mathsf{T}},P) \in \mathcal{P}_j(\tau)} \frac{(-1)^{\# of intersection indices in [j]}}{\prod_{i=1}^{j} |U_{\gamma_i}|! |V_{\gamma_i'^{\mathsf{T}}}|!} \\ &\quad \mathbf{L}_{U_{\tau_P},\leq D_L(P)} \left(\frac{\left(\prod_{i=1}^{j} c_{P_i} N(P_i) \right) \lambda_{\tau_P} \mathbf{M}_{\tau_P^+}}{|\operatorname{Aut}(\tau_P)|} \right) \mathbf{L}_{V_{\tau_P},\leq D_R(P)}^{\mathsf{T}} \end{split}$$

Remark E.30. The dominant terms in the decomposition are

$$\sum_{U,V \in I_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^2} \mathbf{L}_{U,\leq D_V} \left(\sum_{\substack{\tau \in \mathcal{M}_{U,V}: \\ \tau \text{ diagonal}}} \left(\frac{1}{1-p} \right)^{|E(U_{\tau}) \cup E(V_{\tau})|} \frac{\lambda_{\tau} \mathbf{M}_{\tau^+}}{|\operatorname{Aut}(\tau)|} \right) \mathbf{L}_{V,\leq D_V}^{\mathsf{T}}$$
$$= \sum_{U \in I_{mid}} \left(\frac{1}{1-p} \right)^{|E(U)|} \frac{\lambda_U}{|U|!} \mathbf{L}_{U,\leq D_V} \mathbf{M}_{U^+} \mathbf{L}_{U,\leq D_V}^{\mathsf{T}}$$
$$\geq \sum_{U \in I_{mid}} \frac{\lambda_U}{|U|!} \mathbf{L}_{U,\leq D_V} \mathbf{M}_{U^+} \mathbf{L}_{U,\leq D_V}^{\mathsf{T}}.$$

We will show that all of the remaining terms are PSD dominated by these terms.

E.3.1 Proofs of Lemma E.24 and Lemma E.26

Proof of Lemma E.24. For a ribbon $R_1^- \circ R_2 \circ R_3^-$ on the left-hand side, let $E_{missing}(R_2)$ be the set of edges in $E(A_{R_2}) \cup E(B_{R_2})$ which do not yet have indicators. We update the single nonzero entry of $\mathbf{M}_{R_1^- \circ R_2 \circ R_3^-}$ using the identity

$$\prod_{e \in E_{missing}(R_2)} \left(\frac{q-p}{\sqrt{p(1-p)}} \chi_e \right) = \prod_{e \in E_{missing}(R_2)} \left(\frac{1}{1-p} \mathbf{1}_{e \in E(G)} \frac{q-p}{\sqrt{p(1-p)}} \chi_e - \frac{q-p}{1-p} \right).$$

Note that $n^{-\gamma} = \frac{q-p}{1-p}$. Expanding this identity,

$$\sum_{(R_1,R_2,R_3)\in\mathcal{T}_{\tau,\leq D_1,\leq D_2}} \lambda_{R_1^-\circ R_2\circ R_3^-} \mathbf{M}_{R_1^-\circ R_2\circ R_3^-}$$

$$= \sum_{(R_1,R_2,R_3)\in\mathcal{T}_{\tau,\leq D_1,\leq D_2}} \sum_{E_1\subseteq E_{missing}(R_2)} \left(\frac{1}{1-p}\right)^{|E_{missing}(R_2)|-|E_1|} (-n^{-\gamma})^{|E_1|} \lambda_{R_1^-\circ(R_2\setminus E_1)^+\circ R_3^-} \mathbf{M}_{R_1^-\circ(R_2\setminus E_1)^+\circ R_3^-}$$

The term choosing $\frac{1}{1-p} \mathbf{1}_{e \in E(G)} \chi_e$ for all e (i.e. $E_1 = \emptyset$) is the terminal interaction and yields $\mathbf{M}_{R_1^- \circ R_2^+ \circ R_3^-}$.

For the other E_1 terms, we refactor using the new SMVS,

$$R_1^- \circ (R_2 \setminus S)^+ \circ R_3^- = R_1^{\prime -} \circ R_2^{\prime} \circ R_3^{\prime -}$$

Define G_1 and G_3^{T} by $R_1 = R_1^{\prime-} \circ G_1$ and $R_3 = G_3^{\mathsf{T}} \circ R_3^{\prime-}$. Recall that we need to specify the orders of A_{G_1} and $B_{G_3^{\mathsf{T}}}$. We symmetrize over all $|A_{G_1}|!|B_{G_3^{\mathsf{T}}}|!$ possible choices.

At this point, we may have edge indicators in the middle of $G_1^- \circ (R_2 \setminus E_1)^+ \circ (G_3^{\mathsf{T}})^-$. Letting E_{extra} be the set of edges in the middle of $G_1^- \circ (R_2 \setminus E_1)^+ \circ (G_3^{\mathsf{T}})^-$,

$$\prod_{e \in E_{extra}} \left(1_{e \in E(G)} \frac{q-p}{\sqrt{p(1-p)}} \chi_e \right) = \prod_{e \in E_{extra}} \left((1-p) \frac{q-p}{\sqrt{p(1-p)}} \chi_e + (q-p) \right)$$

Note that $(q - p) = (1 - p)n^{-\gamma}$. Expanding this product out, we let E_2 be the set of edges where we choose the $(1 - p)n^{-\gamma}$ term. After doing this, we set

$$R_2' = G_1^- \circ (R_2 \setminus (E_1 \cup E_2))^+ \circ (G_3^{\mathsf{T}})^-$$

We replace the summation over $(R_1, R_2, R_3) \in \mathcal{T}_{\tau, \leq D_1, \leq D_2}$ by a summation over $(R'_1, G_1, R_2, G^{\mathsf{T}}_3, R'_3)$. We crucially have that R'_2 does not depend on R'_1 or R'_3 , Remark 3.7. Therefore, we first sum over $(G_1, R_2, G^{\mathsf{T}}_3), E_1 \subseteq E_{missing}(R_2)$, and $E_2 \subseteq E_{extra}(G^{\mathsf{T}}_1 \circ (R_2 \setminus E_1)^+ \circ (G^{\mathsf{T}}_3)^-)$ and then sum over R'_1 and R'_3 . R_2 is a ribbon of shape τ , G_1 and G^{T}_3 are arbitrary left and right ribbons which match the left and right sides of R_2 , and together there is the additional structural property that

$$A_{G_1}$$
 is the leftmost SMVS of G_1 ,
 $B_{G_3^{\mathsf{T}}}$ is the rightmost SMVS of G_3^{T} , (*)

In summary,

$$\begin{split} \sum_{(R_1,R_2,R_3)\in\mathcal{T}_{\tau,\leq D_1,\leq D_2}} \sum_{E_1\subseteq E_{missing}(R_2)} \left(\frac{1}{1-p}\right)^{|E_{missing}|-|E_1|} (-n^{-\gamma})^{|E_1|} \lambda_{R_1^-\circ(R_2\setminus E_1)^+\circ R_3^-} \mathbf{M}_{R_1^-\circ(R_2\setminus E_1)^+\circ R_3^-} \\ &= \sum_{R_2\in\mathcal{R}_{\tau}} \sum_{S\subseteq E_{missing}(R_2)} \sum_{\substack{G_1\in\mathcal{L}_{A_{R_2},\leq D_1'}}} \sum_{E_2\subseteq E_{extra}} \sum_{\substack{(G_1^-\circ(R_2\setminus E_1)^+\circ(G_3^-)^-)\\G_3\in\mathcal{L}_{B_{R_2},\leq D_2}:\\ (*) \text{ holds}}} \sum_{\substack{(1-p)^{|E_{extra}|}(-1)^{|E_1|} (n^{-\gamma})^{|E_1|+|E_2|}} \sum_{\substack{R_1'\in\mathcal{L}_{A_{G_1'}\leq D_1-|V(G_1)\setminus A_{G_1}|'\\R_3'\in\mathcal{L}_{B_{G_3^-},\leq D_2-|V(G_3^-)\setminus B_{G_3^-}|}} \lambda_{R_1'^-\circ R_2'\circ R_3'^-} \mathbf{M}_{R_1'^-\circ R_2'\circ R_3'^-} \end{split}$$

Finally, we prepare to shift from ribbons to shapes. Fixing the shape γ of G_1 and γ'^{\intercal} of G_3^{\intercal} , the sum over $S \subseteq E_{missing}(R_2)$ and the condition (*) is equivalent to summing over $P \in \mathcal{P}_{\gamma,\tau,\gamma'^{\intercal}}^{PMVS}$. Summing over R'_2 with the final shape τ_P specified by P, the sum over R'_1, R'_2, R'_3 is equivalent to summing over $\mathcal{T}_{\tau_P, D'_1, D'_2}$. The coefficients are gathered into $N(P)c_P$. In summary, the above is equal to

$$\sum_{\substack{\gamma \in \mathcal{L}_{U_{\tau,\leq D_{1}'}} \sum_{P \in \mathcal{P}_{\gamma,\tau,\gamma'\tau}^{PMVS}} \frac{N(P)c_{P}}{|U_{\gamma}|!|V_{\gamma'\tau}|!} \left(\sum_{\substack{(R_{1}',R_{2}',R_{3}') \in \mathcal{T}_{\tau_{P,\leq D_{1}}-|V(\gamma)\setminus U_{\gamma'}| \leq D_{2}-|V(\gamma'\tau)\setminus V_{\gamma'\tau}|} \lambda_{R_{1}'^{-}\circ R_{2}'\circ R_{3}'^{-}} \mathbf{M}_{R_{1}'^{-}\circ R_{2}'\circ R_{3}'^{-}} \right)$$

`

as needed.

Proof sketch of Lemma E.26. This follows in the same way as the previous lemma.

We have that $\mathbf{L}_{U_{\tau},\leq D_1}\left(\frac{\lambda_{\tau}\mathbf{M}_{\tau}}{|\operatorname{Aut}(\tau)|}\right)\mathbf{L}_{V_{\tau},\leq D_2}^{\mathsf{T}}$ is a sum over left, middle, and right ribbons respectively, where the sum over middle ribbons is over distinct ribbons due to normalization by $|\operatorname{Aut}(\tau)|$, Proposition A.3. We will argue that each $R = R_1^- \circ R_2 \circ R_3^-$ has the same coefficient on both sides of the equality.

Suppose $R = R_1^- \circ R_2 \circ R_3^-$ where R_1, R_2, R_3 are properly composable. This term appears on the left-hand side with coefficient $\lambda_{R_1^- \circ R_2 \circ R_3^-}$. The coefficient factors into $\lambda_{R_1^-}\lambda_{R_2}\lambda_{R_3^-}$ by Lemma 2.48. Hence these terms match up.

We now consider $R_1^- \circ R_2 \circ R_3^-$ which is an improper composition. These don't occur on the left, hence we would like to show that they cancel on the right. R_1, R_2, R_3 give rise to an intersection pattern $P \in \mathcal{P}_{\gamma,\tau,\gamma'^{\dagger}}^{intersect}$ where $\gamma, \gamma'^{\dagger}$ are defined in Definition 4.21.

In order to specify R_1, R_2, R_3 and the interaction pattern P, the latter sum instead specifies R'_1, R'_2, R'_3 in $\mathcal{T}_{\tau_P, \leq D'_1, \leq D_2}$ along with the orderings of $U_{\gamma}, V_{\gamma'}^{\mathsf{T}}$. There are $|U_{\gamma}|!$ choices for the order of U_{γ} and $|V_{\gamma'^{\mathsf{T}}}|!$ choices for the order of $V_{\gamma'^{\mathsf{T}}}$, and we symmetrize over all choices. There may be multiple ribbons R_1, R_2, R_3 leading to the same ribbon $R_1^- \circ R_2 \circ R_3^-$ even with the same intersection pattern, and this is accounted for by N(P).

The change in coefficient can be analyzed as follows:

- 1. For each intersection, there is a factor of $n^{(\alpha-1)}$ coming from the λ coefficients which needs to be shifted to c_P .
- 2. For each edge $e \in E_{tot}(\tau_P)$, there is a factor of $\left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{\text{mult}(e)-1+1_{e \text{ vanishes}}}$ coming from the λ coefficients which needs to be shifted to c_P .

3.
$$\chi_e^2 = 1 + \frac{1-2p}{\sqrt{p(1-p)}}\chi_e$$
.

- 4. $\chi_e^3 = \left(1 + \frac{1-2p}{\sqrt{p(1-p)}}\chi_e\right)\chi_e = \frac{1-2p}{\sqrt{p(1-p)}} + \frac{1-3p+3p^2}{p(1-p)}\chi_e$. Note that because only three ribbons are being composed, the maximum multiplicity of a multiedge is 3.
- 5. The edge indicators and edges which are removed from the middle can be analyzed in the same way as before.

This change is accounted for by c_P . This completes the proof.

E.4 Analyzing Λ

First we factor out the truncation error from Λ .

Lemma E.31.

$$\begin{split} \mathbf{\Lambda} &= \sum_{U,V \in \mathcal{I}_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^2} \mathbf{L}_{U} \left(\sum_{\tau \in \mathcal{M}_{U,V}} \left(\frac{1}{1-p} \right)^{|E(U_{\tau}) \cup E(V_{\tau})|} \frac{\lambda_{\tau} \mathbf{M}_{\tau^{+}}}{|\operatorname{Aut}(\tau)|} \right) \mathbf{L}_{V}^{\mathsf{T}} \\ &+ \sum_{U,V \in \mathcal{I}_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^2} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{j=1}^{\infty} \sum_{(\Gamma,\Gamma'^{\tau},P) \in \mathcal{P}_{j}(\tau)} \frac{(-1)^{\# of intersection indices in [j]}}{\prod_{i=1}^{j} |U_{\gamma_{i}}|! |V_{\gamma_{i}'^{\tau}}|!} \\ & \mathbf{L}_{U_{\tau_{p}}} \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i}) \right) \lambda_{\tau_{P}} \mathbf{M}_{\tau_{P}^{+}}}{|\operatorname{Aut}(\tau_{P})|} \right) \mathbf{L}_{V_{\tau_{P}}}^{\mathsf{T}} \\ &- \operatorname{truncation}_{1} + \operatorname{truncation}_{2} \end{split}$$

where truncation₂ *is defined in Definition E.32*.

Proof. Starting from Lemma E.29, for each interaction term τ , *j*, (Γ , Γ'^{τ} , *P*), we use

$$\begin{split} \mathbf{L}_{U_{\tau_{p}} \leq D_{L}(P)} & \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i}) \right) \lambda_{\tau_{P}} \mathbf{M}_{\tau_{p}^{+}}}{|\operatorname{Aut}(\tau_{P})|} \right) \mathbf{L}_{V_{\tau_{p}} \leq D_{R}(P)}^{\mathsf{T}} \\ &= \mathbf{L}_{U_{\tau_{p}} \leq D_{V}} \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i}) \right) \lambda_{\tau_{P}} \mathbf{M}_{\tau_{p}^{+}}}{|\operatorname{Aut}(\tau_{P})|} \right) \mathbf{L}_{V_{\tau_{p}} \leq D_{V}}^{\mathsf{T}} \\ &+ \left(\mathbf{L}_{U_{\tau_{p}} \leq D_{L}(P)} - \mathbf{L}_{U_{\tau_{p}} \leq D_{V}} \right) \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i}) \right) \lambda_{\tau_{P}} \mathbf{M}_{\tau_{p}^{+}}}{|\operatorname{Aut}(\tau_{P})|} \right) \mathbf{L}_{V_{\tau_{p}} \leq D_{V}}^{\mathsf{T}} \\ &+ \mathbf{L}_{U_{\tau_{p}} \leq D_{L}(P)} \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i}) \right) \lambda_{\tau_{P}} \mathbf{M}_{\tau_{p}^{+}}}{|\operatorname{Aut}(\tau_{P})|} \right) \left(\mathbf{L}_{V_{\tau_{p}} \leq D_{R}(P)}^{\mathsf{T}} - \mathbf{L}_{V_{\tau_{p}} \leq D_{V}}^{\mathsf{T}} \right) \right) \end{split}$$

The first term has $\mathbf{L}_{U_{\tau_p},\leq D_V} = \mathbf{L}_{U_{\tau_p}}$ and $\mathbf{L}_{V_{\tau_p},\leq D_V} = \mathbf{L}_{V_{\tau_p}}$ as needed. The second and third terms are collected into truncation₂. The following definition completes the proof of the lemma.

Definition E.32 (truncation₂).

$$\operatorname{truncation}_{2} = \sum_{U,V \in \mathcal{I}_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^{2}} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{j=1}^{\infty} \sum_{(\Gamma,\Gamma'^{\tau},P) \in \mathcal{P}_{j}(\tau)} \frac{(-1)^{\# \ of \ intersection \ indices \ in \ [j]}}{\prod_{i=1}^{j} |U_{\gamma_{i}}|! |V_{\gamma_{i}'^{\tau}}|!} \left(\left(\mathbf{L}_{U_{\tau_{p}}, \leq D_{L}(P)} - \mathbf{L}_{U_{\tau_{p}}, \leq D_{V}} \right) \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i})\right) \lambda_{\tau_{P}} \mathbf{M}_{\tau_{P}^{+}}}{|\operatorname{Aut}(\tau_{P})|} \right) \mathbf{L}_{V_{\tau_{P}}, \leq D_{V}}^{\mathsf{T}}$$

+
$$\mathbf{L}_{U_{\tau_{p}},\leq D_{L}(P)}\left(\frac{\left(\prod_{i=1}^{j}c_{P_{i}}N(P_{i})\right)\lambda_{\tau_{P}}\mathbf{M}_{\tau_{P}^{+}}}{|\operatorname{Aut}(\tau_{P})|}\right)\left(\mathbf{L}_{V_{\tau_{p}},\leq D_{R}(P)}^{\mathsf{T}}-\mathbf{L}_{V_{\tau_{p}},\leq D_{V}}^{\mathsf{T}}\right)\right)$$

We would now like to analyze the non-truncation terms. As we proved in Section 4, the norm of each individual shape $\|\mathbf{M}_{\tau}\|$ and $\|\mathbf{M}_{\tau_p}\|$ is under control. In order to sum over all the shapes, we use combinatorial functions to convert the sum into a max.

The idea is as follows. If we have a sum of the form $\sum_{\alpha} B(\alpha)$ where $B(\alpha)$ is non-negative then instead of bounding the sum directly, we can choose a relatively simple function $c(\alpha)$ such that $\sum_{\alpha} \frac{1}{c(\alpha)} \leq 1$ and observe that

$$\sum_{\alpha} B(\alpha) = \sum_{\alpha} \frac{1}{c(\alpha)} c(\alpha) B(\alpha) \le \sum_{\alpha} \frac{1}{c(\alpha)} \max_{\alpha} \{c(\alpha) B(\alpha)\} \le \max_{\alpha}$$

This allows us to use our bound on the individual terms.

Definition E.33 ($c(\alpha)$ and c(P), informal version of Definition E.42).

- 1. For shapes α , $c(\alpha)$ is used to control the number of shapes we are summing over. In particular, we have that for all $U \in I_{mid}$, $\sum_{shapes \alpha: U_{\alpha} = U, \alpha \text{ is non-trivial } \frac{1}{|U_{\alpha} \cap V_{\alpha}|! c(\alpha)} \leq 1$. Similarly, for all $V \in I_{mid}$, $\sum_{shapes \alpha: V_{\alpha} = V, \alpha \text{ is non-trivial } \frac{1}{|U_{\alpha} \cap V_{\alpha}|! c(\alpha)} \leq 1$.
- 2. For intersection patterns P, c(P) is used to control the number of interaction patterns we are summing over. In particular, for all γ , τ , and γ'^{τ} , $\sum_{P \in \mathcal{P}_{\gamma,\tau\gamma'}^{interact}} \frac{1}{c(P)} \leq 1$.

The formal definitions are given in Appendix E.5, where we will verify that they satisfy the stated summation property and also Bound C.1.

With the combinatorial functions in hand, we can bound non-square terms by the square terms $\{\mathbf{L}_{U}\mathbf{M}_{U^{+}}\mathbf{L}_{U}^{\mathsf{T}}: U \in \mathcal{I}_{mid}\}$ as follows. Corollary E.34 applies to middle shapes and Corollary E.35 applies to intersection terms. The proofs are delayed to the next subsection.

Corollary E.34. For n sufficiently large,

$$\sum_{\substack{U,V \in \mathcal{I}_{mid}: U \sim V \\ (|U|!)^2}} \frac{|U \cap V|!}{(|U|!)^2} \mathbf{L}_U \left(\sum_{\tau \in \mathcal{M}_{U,V}: \tau \text{ is non-diagonal}} \left(\frac{1}{1-p} \right)^{|E(U_{\tau}) \cup E(V_{\tau})|} \frac{\lambda_{\tau} \mathbf{M}_{\tau^+}}{|\operatorname{Aut}(\tau)|} \right) \mathbf{L}_V^{\mathsf{T}}$$

$$\geq -\frac{1}{4} \sum_{U \in \mathcal{I}_{mid}} \frac{\lambda_U}{|U|!} \mathbf{L}_U \mathbf{M}_{U^+} \mathbf{L}_U^{\mathsf{T}}$$

Corollary E.35. For n sufficiently large,

$$\sum_{\substack{U,V \in I_{mid}: U \sim V \\ (|U|!)^2}} \frac{|U \cap V|!}{(|U|!)^2} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{j=1}^{\infty} \sum_{\substack{(\Gamma, \Gamma'^{\tau}, P) \in \mathcal{P}_j(\tau) \\ (\Gamma, \Gamma'^{\tau}, P) \in \mathcal{P}_j(\tau)}} \frac{(-1)^{\# of intersection indices in [j]}}{\prod_{i=1}^{j} |U_{\gamma_i}|! |V_{\gamma_i'^{\tau}}|!}$$
$$\mathbf{L}_{U_{\tau_p}} \left(\frac{\left(\prod_{i=1}^{j} c_{P_i} N(P_i)\right) \lambda_{\tau_p} \mathbf{M}_{\tau_p}}{|\operatorname{Aut}(\tau_P)|} \right) \mathbf{L}_{V_{\tau_p}}^{\mathsf{T}}$$
$$\geq -\frac{1}{4} \sum_{U \in I_{mid}} \frac{\lambda_U}{|U|!} \mathbf{L}_U \mathbf{M}_{U^+} \mathbf{L}_U^{\mathsf{T}}$$

Putting together Corollary E.34 and Corollary E.35 with Lemma E.31, we have

Lemma E.36. For n sufficiently large,

$$\Lambda \geq \frac{1}{2} \sum_{U \in \mathcal{I}_{mid}} \frac{1}{|U|!} \lambda_U \mathbf{L}_U \mathbf{M}_{U^+} \mathbf{L}_U^{\mathsf{T}} - \text{truncation}_1 + \text{truncation}_2.$$

The remaining tasks to prove $\Lambda \geq 0$ are to verify Bound C.1, and to analyze the truncation error, which we carry out in the following sections.

E.4.1 Proofs of Corollary E.34 and Corollary E.35

The building block that allows us to formally charge these shapes is the following lemma, which lower bounds the negative impact of each individual term by "square terms".

Lemma E.37. For all shapes τ , all $D_1, D_2 \in \mathbb{N}$, and all $b \in \{-1, 1\}$,

$$b(\mathbf{L}_{U_{\tau},\leq D_{1}}\lambda_{\tau}\mathbf{M}_{\tau^{+}}\mathbf{L}_{V_{\tau},\leq D_{2}}^{\mathsf{T}}+\mathbf{L}_{V_{\tau},\leq D_{2}}\lambda_{\tau}\mathbf{M}_{\tau^{+}}^{\mathsf{T}}\mathbf{L}_{U_{\tau},\leq D_{1}}^{\mathsf{T}})$$

$$\geq \frac{-\lambda_{\tau}\|\mathbf{M}_{\tau}\|}{\sqrt{\lambda_{U_{\tau}}}\|\mathbf{M}_{U_{\tau}}\|\lambda_{V_{\tau}}\|\mathbf{M}_{V_{\tau}}\|}(\mathbf{L}_{U_{\tau},\leq D_{1}}\lambda_{U_{\tau}}\mathbf{M}_{U_{\tau}^{+}}\mathbf{L}_{U_{\tau},\leq D_{1}}^{\mathsf{T}}+\mathbf{L}_{V_{\tau},\leq D_{2}}\lambda_{V_{\tau}}\mathbf{M}_{V_{\tau}^{+}}\mathbf{L}_{V_{\tau},\leq D_{2}}^{\mathsf{T}})$$

Proof. We claim that for all s > 0,

$$b(\mathbf{L}_{U_{\tau},\leq D_{1}}\mathbf{M}_{\tau^{+}}\mathbf{L}_{V_{\tau},\leq D_{2}}^{\intercal}+\mathbf{L}_{V_{\tau},\leq D_{2}}\mathbf{M}_{\tau^{+}}^{\intercal}\mathbf{L}_{U_{\tau},\leq D_{1}}^{\intercal})$$

$$\geq -s\mathbf{L}_{U_{\tau},\leq D_{1}}(\mathbf{M}_{\tau^{+}}\mathbf{M}_{\tau^{+}}^{\intercal})^{1/2}\mathbf{L}_{U_{\tau},\leq D_{1}}^{\intercal}-\frac{1}{s}\mathbf{L}_{V_{\tau},\leq D_{2}}(\mathbf{M}_{\tau^{+}}^{\intercal}\mathbf{M}_{\tau^{+}})^{1/2}\mathbf{L}_{V_{\tau},\leq D_{2}}^{\intercal}$$

Writing $\mathbf{M}_{\tau^+} = \mathbf{X} \mathbf{\Sigma} \mathbf{Y}^{\intercal}$ for the singular value decomposition of \mathbf{M}_{τ^+} , observe that

$$0 \leq \left(\sqrt{s}\mathbf{L}_{U_{\tau},\leq D_{1}}\mathbf{X}\mathbf{\Sigma}^{1/2} + \frac{b}{\sqrt{s}}\mathbf{L}_{V_{\tau},\leq D_{2}}\mathbf{Y}\mathbf{\Sigma}^{1/2}\right) \left(\sqrt{s}\mathbf{\Sigma}^{1/2}\mathbf{X}^{\mathsf{T}}\mathbf{L}_{U_{\tau},\leq D_{1}}^{\mathsf{T}} + \frac{b}{\sqrt{s}}\mathbf{\Sigma}^{1/2}\mathbf{Y}^{\mathsf{T}}\mathbf{L}_{V_{\tau},\leq D_{2}}^{\mathsf{T}}\right)$$

$$= s \mathbf{L}_{U_{\tau}, \leq D_{1}} \mathbf{X} \mathbf{\Sigma} \mathbf{X}^{\mathsf{T}} \mathbf{L}_{U_{\tau}, \leq D_{1}}^{\mathsf{T}} + \frac{1}{s} \mathbf{L}_{V_{\tau}, \leq D_{2}} \mathbf{Y} \mathbf{\Sigma} \mathbf{Y}^{\mathsf{T}} \mathbf{L}_{V_{\tau}, \leq D_{2}}^{\mathsf{T}} \\ + b (\mathbf{L}_{U_{\tau}, \leq D_{1}} \mathbf{X} \mathbf{\Sigma} \mathbf{Y}^{\mathsf{T}} \mathbf{L}_{V_{\tau}, \leq D_{2}}^{\mathsf{T}} + \mathbf{L}_{V_{\tau}, \leq D_{2}} \mathbf{Y} \mathbf{\Sigma} \mathbf{X}^{\mathsf{T}} \mathbf{L}_{U_{\tau}, \leq D_{1}}^{\mathsf{T}}) \\ = s \mathbf{L}_{U_{\tau}, \leq D_{1}} (\mathbf{M}_{\tau^{+}} \mathbf{M}_{\tau^{+}}^{\mathsf{T}})^{1/2} \mathbf{L}_{U_{\tau}, \leq D_{1}}^{\mathsf{T}} + \frac{1}{s} \mathbf{L}_{V_{\tau}, \leq D_{2}} (\mathbf{M}_{\tau^{+}}^{\mathsf{T}} \mathbf{M}_{\tau^{+}})^{1/2} \mathbf{L}_{V_{\tau}, \leq D_{2}}^{\mathsf{T}} \\ + b (\mathbf{L}_{U_{\tau}, \leq D_{1}} \mathbf{M}_{\tau^{+}} \mathbf{L}_{V_{\tau}, \leq D_{2}}^{\mathsf{T}} + \mathbf{L}_{V_{\tau}, \leq D_{2}} \mathbf{M}_{\tau^{+}}^{\mathsf{T}} \mathbf{L}_{U_{\tau}, \leq D_{1}}^{\mathsf{T}}) \\ \end{cases}$$

which implies the claim.

We claim $(\mathbf{M}_{\tau^+}\mathbf{M}_{\tau^+}^{\mathsf{T}})^{1/2} \leq \|\mathbf{M}_{\tau}\| \frac{\mathbf{M}_{U_{\tau}^+}}{\|\mathbf{M}_{U_{\tau}}\|}$. To see this, note that $\mathbf{M}_{U_{\tau}^+}$ is a diagonal matrix with nonnegative entries, therefore $\frac{\mathbf{M}_{U_{\tau}^+}}{\|\mathbf{M}_{U_{\tau}}\|}$ has diagonal entries which are 0 and 1. The supported rows are the same as $\mathbf{M}_{\tau^+}\mathbf{M}_{\tau^+}^{\mathsf{T}}$ hence the claim.

Similarly, $(\mathbf{M}_{\tau^+}^{\mathsf{T}} \mathbf{M}_{\tau^+})^{1/2} \leq \|\mathbf{M}_{\tau}\| \frac{\mathbf{M}_{V_{\tau}^+}}{\|\mathbf{M}_{V_{\tau}}\|}$. Using these claims with $s = \sqrt{\frac{\lambda_{U_{\tau}} \|\mathbf{M}_{U_{\tau}}\|}{\lambda_{V_{\tau}} \|\mathbf{M}_{V_{\tau}}\|}}$ completes the proof.

Lemma E.38.

$$\sum_{U,V\in\mathcal{I}_{mid}:U\sim V} \frac{|U\cap V|!}{(|U|!)^{2}} \mathbf{L}_{U} \left(\sum_{\tau\in\mathcal{M}_{U,V}:\tau \text{ is nontrivial}} \frac{\lambda_{\tau}\mathbf{M}_{\tau^{+}}}{|\operatorname{Aut}(\tau)|} \right) \mathbf{L}_{V}^{\mathsf{T}}$$

$$\geq -\sum_{U\in\mathcal{I}_{mid}} \left(\max_{V,\tau:U\sim V,\tau\in\mathcal{M}_{U,V},\tau \text{ is nontrivial}} \left\{ c(\tau) \frac{\lambda_{\tau}||\mathbf{M}_{\tau}||}{\lambda_{U}||\mathbf{M}_{U}||} \right\} \right) \frac{\lambda_{U}}{|U|!} \mathbf{L}_{U} \mathbf{M}_{U^{+}} \mathbf{L}_{U}^{\mathsf{T}}$$

Proof. Applying Lemma E.37 and using the trivial bound $|\operatorname{Aut}(\tau)| \ge 1$,

$$\begin{split} &\sum_{U,V \in I_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^2} \mathbf{L}_{U} \left(\sum_{\text{nontrivial } \tau \in \mathcal{M}_{U,V}} \frac{\lambda_{\tau} \mathbf{M}_{\tau}}{|\operatorname{Aut}(\tau)|} \right) \mathbf{L}_{V}^{\mathsf{T}} \\ &\geq -\sum_{U,V \in I_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^2} \left(\sum_{\text{nontrivial } \tau \in \mathcal{M}_{U,V}} \frac{\lambda_{\tau} ||\mathbf{M}_{\tau}||}{\lambda_{U} ||\mathbf{M}_{U}||} \left(\frac{1}{2} \mathbf{L}_{U} \mathbf{M}_{U^{+}} \mathbf{L}_{U}^{\mathsf{T}} + \frac{1}{2} \mathbf{L}_{V} \mathbf{M}_{V^{+}} \mathbf{L}_{V}^{\mathsf{T}} \right) \right) \\ &= -\sum_{U \in I_{mid}} \frac{1}{2|U|!} \left(\sum_{V,\tau: U \sim V, \tau \in \mathcal{M}_{U,V}, \tau \text{ is nontrivial } \frac{|U \cap V|!}{c(\tau)|U|!} c(\tau) \frac{\lambda_{\tau} ||\mathbf{M}_{\tau}||}{\lambda_{U} ||\mathbf{M}_{U}||} \right) \mathbf{L}_{U} \mathbf{M}_{U^{+}} \mathbf{L}_{U}^{\mathsf{T}} \\ &- \sum_{V \in I_{mid}} \frac{1}{2|V|!} \left(\sum_{U,\tau: U \sim V, \tau \in \mathcal{M}_{U,V}, \tau \text{ is nontrivial } \frac{|U \cap V|!}{c(\tau)|U|!} c(\tau) \frac{\lambda_{\tau} ||\mathbf{M}_{\tau}||}{\lambda_{U} ||\mathbf{M}_{U}||} \right) \mathbf{L}_{V} \mathbf{M}_{V^{+}} \mathbf{L}_{V}^{\mathsf{T}} \\ &\geq -\sum_{U \in I_{mid}} \left(\sum_{V,\tau: U \sim V, \tau \in \mathcal{M}_{U,V}, \tau \text{ is nontrivial } \left\{ c(\tau) \frac{\lambda_{\tau} ||\mathbf{M}_{\tau}||}{\lambda_{U} ||\mathbf{M}_{U}||} \right\} \right) \frac{1}{|U|!} \mathbf{L}_{U} \mathbf{M}_{U^{+}} \mathbf{L}_{U}^{\mathsf{T}} \end{split}$$

where the last line uses the following facts:

- 1. For all $U \in \mathcal{I}_{mid}$, $\sum_{V,\tau: U \sim V, \tau \in \mathcal{M}_{U,V}, \tau \text{ is nontrivial } \frac{|U \cap V|!}{c(\tau)|U|!} \leq 1$
- 2. For all $V \in \mathcal{I}_{mid}$, $\sum_{U,\tau:U \sim V, \tau \in \mathcal{M}_{U,V,\tau} \text{ is nontrivial } \frac{|U \cap V|!}{c(\tau)|U|!} \leq 1$.

Corollary E.34. For n sufficiently large,

$$\sum_{U,V\in\mathcal{I}_{mid}:U\sim V} \frac{|U\cap V|!}{(|U|!)^2} \mathbf{L}_{U} \left(\sum_{\tau\in\mathcal{M}_{U,V}:\tau \text{ is non-diagonal}} \left(\frac{1}{1-p}\right)^{|E(U_{\tau})\cup E(V_{\tau})|} \frac{\lambda_{\tau}\mathbf{M}_{\tau^{+}}}{|\operatorname{Aut}(\tau)|} \right) \mathbf{L}_{V}^{\mathsf{T}}$$

$$\geq -\frac{1}{4} \sum_{U\in\mathcal{I}_{mid}} \frac{\lambda_{U}}{|U|!} \mathbf{L}_{U} \mathbf{M}_{U^{+}} \mathbf{L}_{U}^{\mathsf{T}}$$

Proof. We observe that

$$\max_{V,\tau:U\sim V,\tau\in\mathcal{M}_{U,V,\tau} \text{ is nontrivial}} \left\{ c(\tau) \frac{\lambda_{\tau} \|\mathbf{M}_{\tau}\|}{\lambda_{U} \|\mathbf{M}_{U}\|} \right\}$$

$$\leq \max_{V,\tau:U\sim V,\tau\in\mathcal{M}_{U,V,\tau} \text{ is nontrivial}} \left\{ c(\tau) B_{adjust}(\tau) n^{-\operatorname{slack}(\tau)} \right\} \leq \frac{1}{4}$$

where the last inequality follows from the facts that for all τ ,

$$1. \operatorname{slack}(\tau) \geq \frac{\varepsilon}{4} (|E(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} + |V(\tau)| - \frac{|U_{\tau}| + |V_{\tau}|}{2})$$

$$2. c(\tau) \leq n^{\frac{\varepsilon}{16}(|E(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} + |V(\tau)| - \frac{|U_{\tau}| + |V_{\tau}|}{2})}$$

$$3. B_{adjust}(\tau) \leq n^{\frac{\varepsilon}{16}(|E(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} + |V(\tau)| - \frac{|U_{\tau}| + |V_{\tau}|}{2})}$$

Lemma E.39.

$$\sum_{\substack{U,V \in \mathcal{I}_{mid}: U \sim V \\ (|U|!)^{2}}} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{j=1}^{\infty} \sum_{\substack{(\Gamma,\Gamma'\tau,P) \in \mathcal{P}_{j}(\tau)}} \frac{(-1)^{\# of intersection indices in [j]}}{\prod_{i=1}^{j} |U_{\gamma_{i}}|! |V_{\gamma_{i}'\tau}|!}$$

$$\mathbf{L}_{U_{\tau_{p}}} \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}}N(P_{i})\right) \lambda_{\tau_{p}} \mathbf{M}_{\tau_{p}}}{|\operatorname{Aut}(\tau_{p})|} \right) \mathbf{L}_{V_{\tau_{p}}}^{\tau}$$

$$\geq -\sum_{\substack{U \in \mathcal{I}_{mid} \\ j \in \mathbb{N}^{+} \\ (\Gamma,\Gamma',P) \in \mathcal{P}_{j}(\tau): \\ U_{\tau_{p}} = U}} \left\{ 100^{j} c(\tau) \left(\prod_{i=1}^{j} c(\gamma_{i}) c(\gamma_{i}') c(P_{i}) c_{P_{i}} N(P_{i})\right) \frac{\lambda_{\tau_{p}} \left\|\mathbf{M}_{\tau_{p}}\right\|}{\sqrt{\lambda_{U_{\tau_{p}}} \left\|\mathbf{M}_{U_{\tau_{p}}}\right\|} \mathbf{M}_{V_{\tau_{p}}}\right\|} \right\}$$

$$\frac{\lambda_U}{|U|!} \mathbf{L}_U \mathbf{M}_{U^+} \mathbf{L}_U^{\mathsf{T}}$$

Proof. Applying Lemma E.37 and using the trivial bound $|\operatorname{Aut}(\tau_P)| \geq 1$,

$$\sum_{U,V \in I_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^2} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{j=1}^{\infty} \sum_{(\Gamma,\Gamma'^{\tau},P) \in \mathcal{P}_{j}(\tau)} \frac{(-1)^{\# \text{ of intersection indices in } [j]}}{\prod_{i=1}^{j} |U_{\gamma_{i}}|!|V_{\gamma_{i}'^{\tau}}|!}$$

$$\mathbf{L}_{U_{\tau_{p}}} \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i})\right) \lambda_{\tau_{p}} \mathbf{M}_{\tau_{p}}}{|\operatorname{Aut}(\tau_{p})|} \right) \mathbf{L}_{V_{\tau_{p}}}^{\tau}$$

$$\geq -\sum_{U,V \in I_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^{2}} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{j=1}^{\infty} \sum_{(\Gamma,\Gamma'^{\tau},P) \in \mathcal{P}_{j}(\tau)} \frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i})\right) \frac{\lambda_{\tau_{p}} \|\mathbf{M}_{\tau_{p}}\|}{\sqrt{\lambda_{U_{\tau_{p}}} \|\mathbf{M}_{U_{\tau_{p}}}\|}}{\prod_{i=1}^{j} |U_{\gamma_{i}}|!|V_{\gamma_{i}'^{\tau}}|!}$$

$$\left(\frac{1}{2} \mathbf{L}_{U_{\tau_{p}}} \lambda_{U_{\tau_{p}}} \mathbf{M}_{U_{\tau_{p}}} \mathbf{L}_{U_{\tau_{p}}}^{\tau} + \frac{1}{2} \mathbf{L}_{V_{\tau_{p}}} \lambda_{V_{\tau_{p}}} \mathbf{M}_{V_{\tau_{p}}}^{+} \mathbf{L}_{V_{\tau_{p}}}^{\tau} \right)$$

We now show how to bound the $\mathbf{L}_{U_{\tau_p}}\lambda_{U_{\tau_p}}\mathbf{M}_{U_{\tau_p}^+}\mathbf{L}_{U_{\tau_p}}^{\mathsf{T}}$ terms. The $\mathbf{L}_{V_{\tau_p}}\lambda_{V_{\tau_p}}\mathbf{M}_{V_{\tau_p}^+}\mathbf{L}_{V_{\tau_p}}^{\mathsf{T}}$ terms can be bounded by a symmetrical argument.

Grouping all of the terms where $U_{\tau_P} = U$ together, we obtain that

$$\begin{split} &\sum_{U,V \in I_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^2} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{j=1}^{\infty} \sum_{(\Gamma,\Gamma'\tau,P) \in \mathcal{P}_{j}(\tau)} \frac{1}{\prod_{i=1}^{j} |U_{\gamma_{i}}|! |V_{\gamma_{i}'\tau}|!} \\ &\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i})\right) \frac{\lambda_{\tau_{P}} \left\|\mathbf{M}_{\tau_{P}}\right\|}{\sqrt{\lambda_{U_{\tau_{P}}} \left\|\mathbf{M}_{U_{\tau_{P}}}\right\|} \mathbf{L}_{U_{\tau_{P}}} \mathbf{L}_{U_{\tau_{P}}} \lambda_{U_{\tau_{P}}} \mathbf{M}_{U_{\tau_{P}}} \mathbf{L}_{U_{\tau_{P}}} \mathbf{L}_{U_{\tau_{P}}} \\ &= \sum_{U \in I_{mid}} \frac{1}{|U|!} \sum_{\tau \in \mathcal{M}} \sum_{j=1}^{\infty} \sum_{(\Gamma,\Gamma'\tau,P) \in \mathcal{P}_{j}(\tau):} \frac{|U_{\tau} \cap V_{\tau}|!}{|V_{\tau}|!} \frac{1}{\prod_{i=1}^{j} |V_{\gamma_{i}}|! |V_{\gamma_{i}'\tau}|!} \\ &\left(2^{i}c(\tau) \left(\prod_{i=1}^{j} c(\gamma_{i})c(\gamma_{i}')c(P_{i})c_{P_{i}}N(P_{i})\right) \frac{\lambda_{\tau_{P}} \left\|\mathbf{M}_{U_{\tau_{P}}}\right\|}{\sqrt{\lambda_{U_{\tau_{P}}} \left\|\mathbf{M}_{U_{\tau_{P}}}\right\|}} \frac{1}{\sqrt{\lambda_{U_{\tau_{P}}} \left\|\mathbf{M}_{U_{\tau_{P}}}\right\|}} \right) \frac{\mathbf{L}_{U\lambda U} \mathbf{M}_{U} + \mathbf{L}_{U}^{\tau}}{2^{j}c(\tau) \prod_{i=1}^{j} c(\gamma_{i})c(\gamma_{i}')c(P_{i})} \\ &\leq \sum_{U \in I_{mid}} \left(\max_{\substack{i \in \mathcal{M} \\ j \in \mathbb{N}^{+} \\ (\Gamma,\Gamma',P) \in \mathcal{P}_{j}(\tau):} \\ U_{\tau_{P}} = U} \left\{100^{j}c(\tau) \left(\prod_{i=1}^{j} c(\gamma_{i})c(\gamma_{i}')c(P_{i})c_{P_{i}}N(P_{i})\right) \frac{\lambda_{\tau_{P}} \left\|\mathbf{M}_{U_{\tau_{P}}}\right\|}{\sqrt{\lambda_{U_{\tau_{P}}} \left\|\mathbf{M}_{U_{\tau_{P}}}\right\|}} \right\} \right) \end{aligned}$$

$$\frac{\lambda_U}{|U|!} \mathbf{L}_U \mathbf{M}_{U^+} \mathbf{L}_U^{\mathsf{T}}$$

where the last inequality uses the following facts to convert the sums into a maximization:

- 1. For all $i \in [j]$, $\sum_{\gamma_i \in \mathcal{L}: U_{\gamma_i} = V_{\gamma_{i+1}}} \frac{1}{|V_{\gamma_i}|! c(\gamma_i)} \leq 2$ where we set $V_{\gamma_{j+1}} = U$. Across all *i*, this multiplies the total by 2^j .
- 2. $\sum_{\tau \in \mathcal{M}: U_{\tau} = V_{\gamma_1}} \frac{|U_{\tau} \cap V_{\tau}|!}{|V_{\tau}|!c(\tau)} \leq 2$
- 3. For all $i \in [j]$, $\sum_{\gamma'_i \in \mathcal{L}_{V_{\gamma'_{i-1}}}} \frac{1}{|V_{\gamma'_i}|!c(\gamma'_i)|} \leq 2$. Across all *i*, this multiplies the total by 2^j .
- 4. For all $i \in [j]$, $\sum_{P_i \in \mathcal{P}_{j_i, \tau_{P_{i-1}}, \gamma'_i}^{interact}} \frac{1}{c(P_i)} \leq 1$.

5.
$$\sum_{j=1}^{\infty} \frac{1}{2^j} \le 1.$$

Corollary E.35. For n sufficiently large,

$$\sum_{U,V \in \mathcal{I}_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^2} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{j=1}^{\infty} \sum_{(\Gamma, \Gamma'^{\tau}, P) \in \mathcal{P}_{j}(\tau)} \frac{(-1)^{\# of intersection indices in [j]}}{\prod_{i=1}^{j} |U_{\gamma_{i}}|!|V_{\gamma_{i}'^{\tau}}|!}$$
$$\mathbf{L}_{U_{\tau_{p}}} \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i})\right) \lambda_{\tau_{p}} \mathbf{M}_{\tau_{p}}}{|\operatorname{Aut}(\tau_{p})|} \right) \mathbf{L}_{V_{\tau_{p}}}^{\mathsf{T}}$$
$$\geq -\frac{1}{4} \sum_{U \in \mathcal{I}_{mid}} \frac{\lambda_{U}}{|U|!} \mathbf{L}_{U} \mathbf{M}_{U^{+}} \mathbf{L}_{U}^{\mathsf{T}}$$

Proof. We need to show that

$$\max_{\substack{\tau \in \mathcal{M} \\ j \in \mathbb{N}^+ \\ (\Gamma, \Gamma', P) \in \mathcal{P}_j(\tau): \\ U_{\tau_p} = U}} \left\{ 100^j c(\tau) \left(\prod_{i=1}^j c(\gamma_i) c(\gamma'_i) c(P_i) c_{P_i} N(P_i) \right) \frac{\lambda_{\tau_P} \left\| \mathbf{M}_{U_{\tau_P}} \right\|}{\sqrt{\lambda_{U_{\tau_P}} \left\| \mathbf{M}_{U_{\tau_P}} \right\|}} \right\} \le \frac{1}{4}$$

This follows from the following observations:

1.
$$\frac{\prod_{i=1}^{j} c_{P_{i}}^{\approx} \lambda_{\tau_{P}} \|\mathbf{M}_{\tau_{P}}\|}{\sqrt{\lambda_{U_{\tau_{P}}} \|\mathbf{M}_{U_{\tau_{P}}}\| \lambda_{V_{\tau_{P}}} \|\mathbf{M}_{V_{\tau_{P}}}\|}} = n^{-\operatorname{slack}(\tau_{P})}$$

2. By the slack lower bound in Bound C.1,

$$\begin{aligned} \operatorname{slack}(\tau_P) &\geq \varepsilon \left(E_{tot}(\tau_P) - \frac{|E(U_{\tau_P})| + |E(V_{\tau_P})|}{2} + |V_{tot}(\tau_P)| - \frac{|U_{\tau_P}| + |V_{\tau_P}|}{2} \right) \\ &= \varepsilon \left(|E(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} + |V(\tau)| - \frac{|U_{\tau}| + |V_{\tau}|}{2} \right) + \\ &\varepsilon \sum_{i \in [j]} \left(|E(\gamma_i)| - \frac{|E(U_{\gamma_i})| + |E(V_{\gamma_i})|}{2} + (\text{\# of edges removed from } \gamma_i) \\ &+ |V(\gamma_i)| - \frac{|U_{\gamma_i}| + |V_{\gamma_i}|}{2} + |E(\gamma_i^{\top})| - \frac{|E(U_{\gamma_i^{\top}})| + |E(U_{\gamma_i^{\top}})|}{2} \\ &+ (\text{\# of edges removed from } \gamma_i^{\top}) + |V(\gamma_i^{\top})| - \frac{|U_{\gamma_i^{\top}}| + |V_{\gamma_i^{\top}}|}{2} \right) \end{aligned}$$

- $\begin{aligned} 3. \ c(\tau) &\leq n^{\frac{\varepsilon}{32}(|E(\tau)| \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} + |V(\tau)| \frac{|U_{\tau}| + |V_{\tau}|}{2})} \\ 4. \ B_{adjust}(\tau_P) &\leq n^{\frac{\varepsilon}{32}(|E(\tau_P)| \frac{|E(U_{\tau_P})| + |E(V_{\tau_P})|}{2} + |V(\tau_P)| \frac{|U_{\tau_P}| + |V_{\tau_P}|}{2})} \end{aligned}$
- 5. For all $i \in [j]$, $c(\gamma_i)c(\gamma'_i)$, $c(P_i)$, and $N(P_i)$ are all at most *n* raised to the power

$$\frac{\varepsilon}{32} \left(|E(\gamma_i)| - \frac{|E(U_{\gamma_i})| + |E(V_{\gamma_i})|}{2} + (\text{# of edges removed from } \gamma_i) + |V(\gamma_i)| - \frac{|U_{\gamma_i}| + |V_{\gamma_i}|}{2} + |E(\gamma_i^{\mathsf{T}})| - \frac{|E(U_{\gamma_i^{\mathsf{T}}})| + |E(U_{\gamma_i^{\mathsf{T}}})|}{2} + (\text{# of edges removed from } \gamma_i^{\mathsf{T}}) + |V(\gamma_i^{\mathsf{T}})| - \frac{|U_{\gamma_i^{\mathsf{T}}}| + |V_{\gamma_i^{\mathsf{T}}}|}{2} \right)$$

$$6. \ \left|\frac{c_p}{c_p^{\approx}}\right| \le 2$$

E.5 *c*-function bounds

In this section we bound the various combinatorial functions.

Definition E.40 ($N^{shape}(U, e)$). Given a diagonal shape U and $V \in \mathbb{N}$, let $N^{shape}(U, e)$ be the number of shapes α with $U_{\alpha} = U$ and e edges outside of U_{α} .

Remark E.41. Since the permutation of V_{α} can be arbitrary, $N^{shape}(U, e)$ is a multiple of $|V_{\alpha}|!$.

Definition E.42 ($c(\alpha)$, formal).

$$c(\alpha) = 2^{|E(\alpha) \setminus E(U_{\alpha} \cap V_{\alpha})|} \cdot \frac{1}{|U_{\alpha} \cap V_{\alpha}|!} \max\left\{ N^{shape}(U_{\alpha}, |E(\alpha) \setminus E(U_{\alpha})|), N^{shape}(V_{\alpha}, |E(\alpha) \setminus E(V_{\alpha})|) \right\}$$

Lemma E.43. For all diagonal shapes U,

shapes
$$\alpha: U_{\alpha} = U_{,\alpha} \text{ non-trivial} \frac{1}{|U \cap V_{\alpha}|! c(\alpha)} \le 1$$

$$\sum_{\text{shapes } \alpha: U_{\alpha} = U} \frac{1}{|U \cap V_{\alpha}|! c(\alpha)} \le 2$$

By symmetry the same holds for the sum over α : $V_{\alpha} = V$.

Proof.

$$\sum_{\text{shapes } \alpha: U_{\alpha} = U} \frac{1}{|U \cap V_{\alpha}|! c(\alpha)}$$
$$= \sum_{e=0}^{\infty} \sum_{\text{shapes } W} \sum_{\substack{\text{shapes } \alpha: U_{\alpha} = U, \\ |E(\alpha) \setminus E(U_{\alpha})| = e}} \frac{1}{2^{e} N^{shape}(U, e)}$$
$$= \sum_{e=0}^{\infty} \frac{1}{2^{e}}$$
$$= 2.$$

To derive the first statement, note that the trivial shapes contribute exactly 1 to the sum.

Lemma E.44 (Bound for $c(\alpha)$). For all shapes α with at most D_V vertices,

$$c(\alpha) \leq 2(4D_V)^{2|E(\alpha)\setminus E(U_{\alpha}\cap V_{\alpha})|}(2D_V)^{2|(U_{\alpha}\cup V_{\alpha})\setminus (U_{\alpha}\cap V_{\alpha})|}$$

Proof. The shapes counted by $N^{shape}(U, e)$ can be generated by the following process.

- 1. Start from $V(\alpha) = U_{\alpha} = U$.
- 2. Run the following process to select a subset of U_{α} to be in $U_{\alpha} \cap V_{\alpha}$. Use a label in [2] to decide whether or not at least one vertex is in $U_{\alpha} \cap V_{\alpha}$. If so, use a label in $|U_{\alpha}|$ to choose the vertex, and then use a label in [2] to decide whether or not another vertex is in $U_{\alpha} \cap V_{\alpha}$, and so forth.
- 3. For each edge outside $E(U_{\alpha})$, identify each endpoint using a label in $[D_V]$, and additionally use a label in [2] to identify whether each endpoint is in V_{α} .

4. Specify the permutation of V_{α} in $|V_{\alpha}|!$ ways.

In total, this is at most $2(2D_V)^{2|E(\alpha)\setminus E(U_\alpha)|}(2|U_\alpha|)^{|(U_\alpha\cup V_\alpha)\setminus (U_\alpha\cap V_\alpha)|}|V_\alpha|!$. A symmetric bound applies to $N^{shape}(V, e)$. Therefore,

$$\begin{split} c(\alpha) &\leq 2^{|E(\alpha)\setminus E(U_{\alpha}\cap V_{\alpha})|} \cdot \frac{\max\{|U_{\alpha}|!, |V_{\alpha}|!\}}{|U_{\alpha} \cap V_{\alpha}|!} \cdot 2(2D_{V})^{2|E(\alpha)\setminus E(U_{\alpha}\cap V_{\alpha})|} (2D_{V})^{|(U_{\alpha}\cup V_{\alpha})\setminus (U_{\alpha}\cap V_{\alpha})|} \\ &\leq 2^{|E(\alpha)\setminus E(U_{\alpha}\cap V_{\alpha})|} \cdot D_{V}^{|(U_{\alpha}\cup V_{\alpha})\setminus (U_{\alpha}\cap V_{\alpha})|} \cdot 2(2D_{V})^{2|E(\alpha)\setminus E(U_{\alpha}\cap V_{\alpha})|} (2D_{V})^{|(U_{\alpha}\cup V_{\alpha})\setminus (U_{\alpha}\cap V_{\alpha})|} \\ &= 2(4D_{V})^{2|E(\alpha)\setminus E(U_{\alpha}\cap V_{\alpha})|} (2D_{V})^{2|(U_{\alpha}\cup V_{\alpha})\setminus (U_{\alpha}\cap V_{\alpha})|} \end{split}$$

as needed.

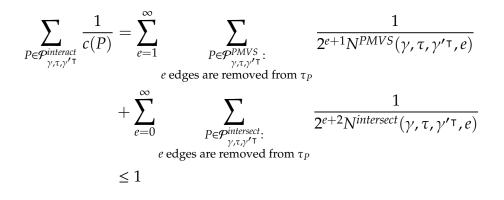
Definition E.45 ($N^{PMVS}(\gamma, \tau, \gamma'^{\intercal}, e)$). Given shapes $\gamma, \tau, \gamma'^{\intercal}$ and $e \in \mathbb{N}$, let $N^{PMVS}(\gamma, \tau, \gamma'^{\intercal}, e)$ be the number of PMVS interaction patterns such that e edges are removed from τ_P , either because of the adding indicators step or the removing middle edge indicators step.

Similarly, let $N^{intersect}(\gamma, \tau, \gamma'^{\tau}, e)$ be the number of intersection interaction patterns such that e edges are removed from τ_P in the removing middle edge indicators step.

Definition E.46 (*c*(*P*), formal). For an interaction pattern $P \in \mathcal{P}_{\gamma,\tau,\gamma'\tau}^{interact}$, let $c(P) = 2^{e+1}N^{PMVS}(\gamma,\tau,\gamma'\tau,e)$ if *P* is a PMVS interaction pattern and $c(P) = 2^{e+2}N^{intersect}(\gamma,\tau,\gamma'\tau,e)$ if *P* is an intersection interaction pattern, where *e* is the number of edges removed from τ_P .

Lemma E.47. For all $\gamma, \tau, \gamma', \sum_{P \in \mathcal{P}_{\gamma,\tau,\gamma'^{\tau}}^{interact}} \frac{1}{c(P)} \leq 1.$

Proof.



Lemma E.48 (Bound for c(P)). For all shapes $\gamma, \tau, \gamma'^{\tau}$ such that $|V(\gamma)| \leq D_V$, $|V(\tau)| \leq 3D_V$, and $|V(\gamma'^{\tau})| \leq D_V$,

1. For all PMVS interaction patterns such that e edges are removed from τ_P , $c(P) \leq 2(4D_V^2)^e$

2. For all intersection interaction patterns such that e edges are removed from τ_P ,

 $c(P) \leq 4(3D_V)^{|V(\gamma) \setminus V_{\gamma}| + |V(\gamma'^{\intercal}) \setminus U_{\gamma'^{\intercal}}|} 2^{e + |E(\gamma) \setminus E(U_{\gamma} \cap V_{\gamma})| + |E(\gamma'^{\intercal}) \setminus E(U_{\gamma'^{\intercal}} \cap V_{\gamma'^{\intercal}})|}$

Proof. For the case of a PMVS interaction pattern, we do the following.

- 1. We know that at least one edge must be removed in the adding edge indicators step for non-terminal *P*. For each such edge, we specify the endpoints for a cost of at most D_V^2 .
- 2. For the removing middle edge indicators step, we can specify each edge which is removed by specifying its two endpoints at a cost of D_v^2 per edge.

For the case of an intersection interaction pattern, we do the following.

- 1. Go through each vertex in $V(\gamma) \setminus V_{\gamma}$ and $V(\gamma'^{\intercal}) \setminus U_{\gamma'^{\intercal}}$ and indicate which vertex they intersect with, if any. This has a cost of $(3D_V)^{|V(\gamma) \setminus V_{\gamma}| + |V(\gamma'^{\intercal}) \setminus U_{\gamma'^{\intercal}}|}$
- 2. For each edge that intersected, use a label in [2] to denote its multiplicity after linearization. This has a cost of at most $2^{|E(\gamma)\setminus E(V_{\gamma})|+|E(\gamma'^{\intercal})\setminus E(U_{\gamma'^{\intercal}})|}$.
- 3. For each edge in $V_{\gamma} \cup U_{\gamma'^{\intercal}}$ which is not in $U_{\gamma} \cup V_{\gamma'^{\intercal}}$, use a label in [2] to decide whether it is removed in the **Remove middle edge indicators operation**. This has a cost of at most $2^{|E(V_{\gamma})\setminus E(U_{\gamma})|+|E(U_{\gamma'^{\intercal}})\setminus E(V_{\gamma'^{\intercal}})|}$.

Lemma E.49 (Bound for c_P). The excess in c_P over what goes into the slack is

$$\left|\frac{c_P}{c_P^{\approx}}\right| \le 2.$$

Proof. By Lemma E.20,

$$\left|\frac{c_p}{c_p^{\approx}}\right| \le \left(\frac{1}{1-p}\right)^{\# \text{ of indicators}} \le (1+O(p))^{D_V^2} \le 2$$

provided *n* is sufficiently large.

Lemma E.50 (Bound for N(P)). For all $\gamma, \tau, \gamma'^{\tau}$ with size at most D_V and $P \in \mathcal{P}_{\gamma,\tau,\gamma'^{\tau}}^{interact}$

$$N(P) \le (3D_V)^{|V(\gamma) \setminus U_{\gamma}| + |V(\gamma'^{\intercal}) \setminus V_{\gamma'^{\intercal}}|}$$

Proof. We are given γ , τ , γ'^{\intercal} , the interaction pattern *P*, and the resulting ribbon R'_2 and we need to specify the ribbons *G*, R_2 , G'^{\intercal} which have shapes γ , τ , γ'^{\intercal} , have interaction pattern *P*, and result in the ribbon R'_2 .

Suppose that *P* is a PMVS interaction. To do this, it is sufficient to specify how the vertices in γ and γ'^{\intercal} are mapped to in R'_2 . This specifies the ribbons *G* and G'^{\intercal} . Either $A_{R_2} = B_G$ and $B_{R_2} = A_{G'^{\intercal}}$ together with the remaining unmapped vertices of R'_2 have shape τ , in which case this is a possible ribbon R_2 , or they do not, in which case this is merely overcounting.

We automatically have that $A_G = A_{R'_2}$ and $B_{G'^{\intercal}} = B_{R'_2}$ so we do not need to specify where the vertices U_{γ} and $V_{\gamma'^{\intercal}}$ are mapped to. For each of the remaining vertices, the number of choices is at most $3D_V$ so the total number of choices is $(3D_V)^{|V(\gamma) \setminus U_{\gamma}| + |V(\gamma'^{\intercal}) \setminus V_{\gamma'^{\intercal}}|}$, as needed.

For an intersection term interaction, the same analysis goes through, with the added constraint that the intersection pattern along with the mappings of γ and γ'^{\dagger} fix additional labels of R_2 .

E.6 Truncation error

Definition E.51 (\mathbf{Id}_{sym}).

$$\mathbf{Id}_{sym}[I,J] = egin{cases} 1 & I = J \ as \ unordered \ sets \ 0 & otherwise \end{cases}$$

Lemma E.52. truncation₁ $\leq n^{D_{SoS}+\eta-\frac{\varepsilon}{16}(D_V-2D_{SoS})}$ **Id**_{*sum*}

Proof. Applying Theorem D.10 with $D = 3D_V$, for all shapes α such that $D_V \le |V(\alpha)| \le 3D_V$, $|U_{\alpha}| \le D_{SoS}$, $|V_{\alpha}| \le D_{SoS}$, and α has no isolated vertices outside of $U_{\alpha} \cup V_{\alpha}$,

$$\begin{aligned} \lambda_{\alpha} \|\mathbf{M}_{\alpha}\| &\leq 2B_{adjust}(\alpha) n^{(1-\alpha)\frac{|U_{\alpha}|+|V_{\alpha}|}{2}} + \eta - (\gamma - \alpha\beta - 3\log_{n}(3D_{V}))|E(\alpha)| \\ &\leq 2B_{adjust}(\alpha) n^{(1-\varepsilon)D_{SoS} - \frac{\varepsilon}{8}(D_{V} - 2D_{SoS}) - \frac{\varepsilon}{4}(|E(U_{\alpha})| + |E(V_{\alpha})|)} \end{aligned}$$

as $|E(\alpha)| \ge |E(U_{\alpha})| + |E(V_{\alpha})| + \frac{|V(\alpha)| - |U_{\alpha}| - |V_{\alpha}|}{2} \ge |E(U_{\alpha})| + |E(V_{\alpha})| + \frac{D_V - 2D_{SoS}}{2}$ and $\gamma - \alpha\beta - 3\log_n(3D_V) \ge \frac{\varepsilon}{4}$.

We now observe that

$$\begin{aligned} \operatorname{truncation}_{1} &= \sum_{\substack{U,V \in \mathcal{I}_{mid}: \\ U \sim V}} \frac{|U \cap V|!}{(|U|!)^{2}} \sum_{\substack{\sigma \in \mathcal{L}_{U, \leq D_{V}} \\ \tau \in \mathcal{M}_{U,V, \leq D_{V}} \\ \sigma' \in \mathcal{L}_{V, \leq D_{V}}: \\ |V(\sigma^{-} \circ \tau \circ (\sigma'^{\top})^{-})| > D_{V}} \end{aligned}$$

$$\leq \left(\sum_{\substack{\text{shape } \alpha: \\ D_{V} < |V(\alpha)| \leq 3D_{V}, \\ |U_{\alpha}| \leq D_{\text{SoS}}, |V_{\alpha}| \leq D_{\text{SoS}}}} \lambda_{\alpha} ||\mathbf{M}_{\alpha}|| \right) \mathbf{Id}_{Sym}$$

$$\leq \left(\sum_{\substack{\text{shape } \alpha:\\ D_V < |V(\alpha)| \le 3D_V,\\ |U_\alpha| \le D_{SOS}}} \lambda_\alpha ||\mathbf{M}_\alpha||\right) \mathbf{Id}_{Sym}$$

$$\leq \sum_{U \in I_{mid}: |U| \le D_{SOS}} \left(\sum_{\substack{\text{shape } \alpha:\\ D_V < |V(\alpha)| \le 3D_V,\\ U_\alpha = U}} \frac{D_{SOS}!}{|U_\alpha|!c(\alpha)} c(\alpha) \lambda_\alpha ||\mathbf{M}_\alpha||\right) \mathbf{Id}_{Sym}$$

$$\leq \sum_{U \in I_{mid}: |U| \le D_{SOS}} \left(\max_{\substack{\text{shape } \alpha:\\ D_V < |V(\alpha)| \le 3D_V,\\ U_\alpha = U}} \left\{c(\alpha)B_{adjust}(\alpha)\right\}\right)$$

$$D_{SOS}! n^{D_{SOS} + \eta - \frac{\varepsilon}{8}(D_V - 2D_{SOS}) - \frac{\varepsilon}{4}|E(U)|} \mathbf{Id}_{Sym}$$

$$\leq n^{D_{SOS} + \eta - \frac{\varepsilon}{16}(D_V - 2D_{SOS})} \mathbf{Id}_{sym}$$

We now analyze the second part of the truncation error.

Lemma E.53. truncation₂ $\geq -n^{2D_{SoS}+2\eta-\frac{\varepsilon}{32}D_V} \mathbf{Id}_{Sym}$

Proof. Recall that

$$\begin{aligned} \operatorname{truncation}_{2} &= \sum_{U,V \in \mathcal{I}_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^{2}} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{j=1}^{\infty} \sum_{(\Gamma,\Gamma'^{\tau},P) \in \mathcal{P}_{j}(\tau)} \frac{(-1)^{\# \text{ of intersection indices in } [j]}}{\prod_{i=1}^{j} |U_{\gamma_{i}}|!|V_{\gamma_{i}'^{\tau}}|!} \\ & \left(\left(\mathbf{L}_{U_{\tau_{p}} \leq D_{L}(P)} - \mathbf{L}_{U_{\tau_{p}}, \leq D_{V}} \right) \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i}) \right) \lambda_{\tau_{P}} \mathbf{M}_{\tau_{P}^{+}}}{|\operatorname{Aut}(\tau_{P})|} \right) \mathbf{L}_{V_{\tau_{p}}, \leq D_{V}}^{\mathsf{T}} \\ &+ \mathbf{L}_{U_{\tau_{p}}, \leq D_{L}(P)} \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i}) \right) \lambda_{\tau_{P}} \mathbf{M}_{\tau_{P}^{+}}}{|\operatorname{Aut}(\tau_{P})|} \right) \left(\mathbf{L}_{V_{\tau_{p}}, \leq D_{R}(P)}^{\mathsf{T}} - \mathbf{L}_{V_{\tau_{p}}, \leq D_{V}}^{\mathsf{T}} \right) \right) \end{aligned}$$

so

$$\operatorname{truncation}_{2} \geq -\sum_{U,V \in \mathcal{I}_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^{2}} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{j=1}^{\infty} \sum_{(\Gamma, \Gamma'^{\tau}, P) \in \mathcal{P}_{j}(\tau)} \frac{1}{\prod_{i=1}^{j} |U_{\gamma_{i}}|! |V_{\gamma_{i}'^{\tau}}|!}$$

$$\left(\left\| \mathbf{L}_{U_{\tau_{p}},\leq D_{L}(P)} - \mathbf{L}_{U_{\tau_{p}},\leq D_{V}} \right\| \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i}) \right) \lambda_{\tau_{P}} \left\| \mathbf{M}_{\tau_{P}} \right\|}{|\operatorname{Aut}(\tau_{P})|} \right) \left\| \mathbf{L}_{V_{\tau_{p}},\leq D_{V}}^{\mathsf{T}} \right\|$$

+
$$\left\| \mathbf{L}_{U_{\tau_{p}},\leq D_{L}(P)} \right\| \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i}) \right) \lambda_{\tau_{P}} \left\| \mathbf{M}_{\tau_{P}} \right\|}{|\operatorname{Aut}(\tau_{P})|} \right) \left\| \mathbf{L}_{V_{\tau_{p}},\leq D_{R}(P)}^{\mathsf{T}} - \mathbf{L}_{V_{\tau_{p}},\leq D_{V}}^{\mathsf{T}} \right\| \right) \operatorname{Id}_{Sym}$$

We can analyze this using the following claims.

Claim E.54. *For all* $U \in I_{mid}$ *,*

$$\max_{D:D \le D_V} \left\{ \left\| \mathbf{L}_{U, \le D} \right\| \right\} \sqrt{\lambda_U \left\| M_U \right\|} \le n^{D_{\text{SoS}} + \eta - \frac{\varepsilon}{16} |E(U)|}$$

Proof. By Corollary D.13, for all $\sigma \in \mathbf{L}_{U,\leq D_V}$

$$\lambda_{\sigma^{-}} \|\mathbf{M}_{\sigma^{-}}\| \sqrt{\lambda_{U}} \|M_{U}\| \leq 2B_{adjust}(\sigma) n^{D_{SoS} + \eta - \frac{\varepsilon}{2}D_{SoS} - \frac{\varepsilon}{8}|E(\sigma)| - \frac{\varepsilon}{8}|V(\sigma)|}$$

We now observe that

$$\begin{split} & \max_{D:D \leq D_V} \left\{ \left\| \mathbf{L}_{U, \leq D} \right\| \right\} \sqrt{\lambda_U \left\| M_U \right\|} \leq \sum_{\sigma \in \mathbf{L}_{U, \leq D_V}} \lambda_{\sigma^-} \left\| \mathbf{M}_{\sigma^-} \right\| \sqrt{\lambda_U \left\| M_U \right\|} \\ & \leq D_{\mathrm{SoS}}! \left(\sum_{\sigma \in \mathbf{L}_{U, \leq D_V}} \frac{1}{c(\sigma) \left| U_\sigma \right|!} \right)_{\sigma \in \mathbf{L}_{U, \leq D_V}} \left\{ c(\sigma) \lambda_{\sigma^-} \left\| \mathbf{M}_{\sigma^-} \right\| \sqrt{\lambda_U \left\| M_U \right\|} \right\} \\ & \leq 4D_{\mathrm{SoS}}! \max_{\sigma \in \mathbf{L}_{U, \leq D_V}} \left\{ c(\sigma) B_{adjust}(\sigma) n^{D_{\mathrm{SoS}} + \eta - \frac{\varepsilon}{2} D_{\mathrm{SoS}} - \frac{\varepsilon}{8} \left| E(\sigma) \right| - \frac{\varepsilon}{8} \left| V(\sigma) \right|} \right\} \\ & \leq n^{D_{\mathrm{SoS}} + \eta - \frac{\varepsilon}{16} \left| E(U) \right|} \end{split}$$

Claim E.55. For all $U \in I_{mid}$ and all $D \leq D_V$,

$$\left\|\mathbf{L}_{U,\leq D} - \mathbf{L}_{D,\leq D_{V}}\right\| \sqrt{\lambda_{U} \left\|\mathbf{M}_{U}\right\|} \le n^{D_{SoS} + \eta - \frac{\varepsilon}{16}D - \frac{\varepsilon}{16}|E(U)|}$$

Proof. By Corollary D.13, for all $\sigma \in \mathbf{L}_{U, \leq D_V}$

$$\lambda_{\sigma^{-}} \|\mathbf{M}_{\sigma^{-}}\| \sqrt{\lambda_{U}} \|\mathbf{M}_{U}\| \leq 2B_{adjust}(\sigma) n^{D_{\text{SoS}} + \eta - \frac{\varepsilon}{2}D_{\text{SoS}} - \frac{\varepsilon}{8}|E(\sigma)| - \frac{\varepsilon}{8}|V(\sigma)|}$$

We now observe that

$$\begin{aligned} \left\| \mathbf{L}_{U,\leq D} - \mathbf{L}_{U,\leq D_{V}} \right\| \sqrt{\lambda_{U}} \left\| \mathbf{M}_{U} \right\| &\leq \sum_{\sigma \in \mathbf{L}_{U,\leq D_{V}} : |V(\sigma)| > D} \lambda_{\sigma^{-}} \left\| \mathbf{M}_{\sigma^{-}} \right\| \sqrt{\lambda_{U}} \left\| \mathbf{M}_{U} \right\| \\ &\leq D_{SoS}! \left(\sum_{\sigma \in \mathbf{L}_{U,\leq D_{V}}} \frac{1}{c(\sigma) |U_{\sigma}|!} \right)_{\sigma \in \mathbf{L}_{U,\leq D_{V}} : |V(\sigma)| > D} \left\{ c(\sigma) \lambda_{\sigma^{-}} \left\| \mathbf{M}_{\sigma^{-}} \right\| \sqrt{\lambda_{U}} \left\| \mathbf{M}_{U} \right\| \right\} \end{aligned}$$

$$\leq 4D_{\text{SoS}}! \max_{\sigma \in \mathbf{L}_{U, \leq D_{V}}: |V(\sigma)| > D} \left\{ c(\sigma) B_{adjust}(\sigma) n^{D_{\text{SoS}} + \eta - \frac{\varepsilon}{2}D_{\text{SoS}} - \frac{\varepsilon}{8}|E(\sigma)| - \frac{\varepsilon}{8}|V(\sigma)|} \right\}$$

$$\leq n^{D_{\text{SoS}} + \eta - \frac{\varepsilon}{16}D - \frac{\varepsilon}{16}|E(U)|}$$

Using these claims and grouping all of the terms where $U_{\tau_p} = U$ together in the same way as in the proof of Lemma E.39, we obtain that

$$\begin{split} &\sum_{U,V \in I_{mid}: U \sim V} \frac{|U \cap V|!}{(|U|!)^2} \sum_{\tau \in \mathcal{M}_{U,V}} \sum_{j=1}^{\infty} \sum_{(\Gamma,\Gamma'^{\tau},P) \in \mathcal{P}_{j}(\tau)} \frac{1}{|\Pi_{i=1}^{j} |U_{\gamma_{i}}|! |V_{\gamma_{i}}|^{\tau}|!} \\ & \left(\left\| \mathbf{L}_{U_{\tau_{P}} \leq D_{L}(P)} - \mathbf{L}_{U_{\tau_{P}} \leq D_{V}} \right\| \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i}) \right) \lambda_{\tau_{P}} \left\| \mathbf{M}_{\tau_{P}} \right\|}{|\operatorname{Aut}(\tau_{P})|} \right) \right\| \mathbf{L}_{V_{\tau_{P}} \leq D_{V}}^{\mathsf{T}} \right) \\ & + \left\| \mathbf{L}_{U_{\tau_{P}} \leq D_{L}(P)} \right\| \left(\frac{\left(\prod_{i=1}^{j} c_{P_{i}} N(P_{i}) \right) \lambda_{\tau_{P}} \left\| \mathbf{M}_{\tau_{P}} \right\|}{|\operatorname{Aut}(\tau_{P})|} \right) \right\| \mathbf{L}_{V_{\tau_{P}} \leq D_{R}(P)}^{\mathsf{T}} - \mathbf{L}_{V_{\tau_{P}} \leq D_{V}}^{\mathsf{T}} \right\| \right) \mathbf{Id}_{Sym} \\ \leq 2 \sum_{U \in I_{mid}} \frac{1}{|U|!} \sum_{\tau \in \mathcal{M}} \sum_{j=1}^{\infty} \sum_{\substack{(\Gamma, \Gamma, \Gamma', P) \in \mathcal{P}_{j}(\tau): \\ U_{\tau_{P}} = U}} \frac{|U_{\tau} \cap V_{\tau}|!}{|V_{\tau}|!} \frac{1}{|T_{\tau_{1}}|!} \frac{1}{|V_{\gamma_{i}}|! |V_{\gamma_{i}'}^{\mathsf{T}}|!} \\ & \left(2^{j} c(\tau) \left(\prod_{i=1}^{j} c(\gamma_{i}) c(\gamma_{i}') c(P_{i}) c_{P_{i}} N(P_{i}) \right) \frac{\lambda_{\tau_{P}} \left\| \mathbf{M}_{\tau_{P}} \right\| \mathbf{n}^{-\frac{\varepsilon_{1}}{\varepsilon_{1}} \min \left\{ D_{L}(P), D_{R}(P) \right\}}}{\sqrt{\lambda_{U_{\tau_{P}}} \left\| \mathbf{M}_{U_{\tau_{P}}} \right\| \mathbf{M}_{V_{\tau_{P}}} \right\|} \frac{n^{2D_{SOS} + 2\eta - \frac{\varepsilon_{i}}{\varepsilon_{0}} |E(U)|}{|\mathbf{Id}_{Sym}} \\ \leq \sum_{\substack{U \in I_{mid}} \\ \prod_{\substack{i \in \mathcal{M} \\ i \in \mathbb{N}^{+} \\ (\Gamma,\Gamma',P) \in \mathcal{P}_{j}(\tau): \\ U_{\tau_{P}} = U}}} \left\{ 100^{j} c(\tau) \left(\prod_{i=1}^{j} c(\gamma_{i}) c(\gamma_{i}') c(P_{i}) c_{P_{i}} N(P_{i}) \right) \frac{\lambda_{\tau_{P}} \left\| \mathbf{M}_{U_{\tau_{P}}} \right\| \mathbf{M}_{U_{\tau_{P}}} \left\| \mathbf{M}_{\tau_{P}} \right\| n^{-\frac{\varepsilon_{i}}{\varepsilon_{0}} \min \left\{ D_{L}(P), D_{R}(P) \right\}}}{\sqrt{\lambda_{U_{\tau_{P}}} \left\| \mathbf{M}_{U_{\tau_{P}}} \right\| \mathbf{M}_{U_{\tau_{P}}} \right\|}} \right\} \\ \leq \sum_{\substack{U \in I_{mid}} \\ \prod_{\substack{i \in \mathcal{M} \\ i \in \mathbb{N}^{+} \\ (\Gamma,\Gamma',P) \in \mathcal{P}_{j}(\tau): \\ U_{\tau_{P}} = U}}} \left\{ 100^{j} c(\tau) \left(\prod_{i=1}^{j} c(\gamma_{i}) c(\gamma_{i}') c(P_{i}) c_{P_{i}} N(P_{i}) \right) \frac{\lambda_{\tau_{P}} \left\| \mathbf{M}_{\tau_{P}} \right\| \mathbf{M}_{\tau_{P}} \left\| \mathbf{M}_{U_{\tau_{P}}} \right\|} \frac{N_{\tau_{P}} \left\| \mathbf{M}_{U_{\tau_{P}}} \right\|}{\sqrt{\lambda_{U_{\tau_{P}}} \left\| \mathbf{M}_{U_{\tau_{P}}} \right\|} \frac{N_{\tau_{P}} \left\| \mathbf{M}_{U_{\tau_{P}}} \right\|}{\sqrt{\lambda_{U_{\tau_{P}}} \left\| \mathbf{M}_{U_{\tau_{P}}} \right\|} \frac{N_{\tau_{P}} \left\| \mathbf{M}_{T_{\tau_{P}}} \right\|}{\sqrt{\lambda_{U_{\tau_{P}}} \left\| \mathbf{M}_{U_{\tau_{P}}} \right\|}} \right\}$$

We now make the same observations as before together with an observation about $D_L(P)$ and $D_R(P)$:

1.
$$\frac{\prod_{i=1}^{j} c_{p_{i}}^{\approx} \lambda_{\tau_{p}} \|\mathbf{M}_{\tau_{p}}\|}{\sqrt{\lambda_{U_{\tau_{p}}} \|\mathbf{M}_{U_{\tau_{p}}} \|\lambda_{V_{\tau_{p}}} \|\mathbf{M}_{V_{\tau_{p}}} \|}} = n^{-\operatorname{slack}(\tau_{p})}$$

2. By the slack lower bound in Bound C.1,

slack
$$(\tau_P) \ge \varepsilon \left(E_{tot}(\tau_P) - \frac{|E(U_{\tau_P})| + |E(V_{\tau_P})|}{2} + |V_{tot}(\tau_P)| - \frac{|U_{\tau_P}| + |V_{\tau_P}|}{2} \right)$$

$$= \varepsilon \left(|E(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} + |V(\tau)| - \frac{|U_{\tau}| + |V_{\tau}|}{2} \right) + \varepsilon \sum_{i \in [j]} \left(|E(\gamma_i)| - \frac{|E(U_{\gamma_i})| + |E(V_{\gamma_i})|}{2} + (\text{# of edges removed from } \gamma_i) + |V(\gamma_i)| - \frac{|U_{\gamma_i}| + |V_{\gamma_i}|}{2} + |E(\gamma_i^{\top})| - \frac{|E(U_{\gamma_i^{\top}})| + |E(U_{\gamma_i^{\top}})|}{2} + (\text{# of edges removed from } \gamma_i^{\top}) + |V(\gamma_i^{\top})| - \frac{|U_{\gamma_i^{\top}}| + |V_{\gamma_i^{\top}}|}{2} \right)$$

3. $D_L(P) \ge D_V - |V(\gamma) \setminus V_{\gamma}|$ and $D_R(P) \ge D_V - |V(\gamma'^{\mathsf{T}}) \setminus U_{V(\gamma'^{\mathsf{T}})}|$ so

$$\min\{D_L(P), D_R(P)\} \ge D_V - 2\sum_{i=1}^j \left(|V(\gamma_i)| - \frac{|U_{\gamma_i}| + |V_{\gamma_i}|}{2} + |V(\gamma_i^{\mathsf{T}})| - \frac{|U_{\gamma_i^{\mathsf{T}}}| + |V_{\gamma_i^{\mathsf{T}}}|}{2} \right)$$

4. $c(\tau) \le n^{\frac{\varepsilon}{32}(|E(\tau)| - \frac{|E(U_{\tau})| + |E(V_{\tau})|}{2} + |V(\tau)| - \frac{|U_{\tau}| + |V_{\tau}|}{2})}$

- 5. $B_{adjust}(\tau_P) \le n^{\frac{\varepsilon}{32}(|E(\tau_P)| \frac{|E(U_{\tau_P})| + |E(V_{\tau_P})|}{2} + |V(\tau_P)| \frac{|U_{\tau_P}| + |V_{\tau_P}|}{2})$
- 6. For all $i \in [j]$, $c(\gamma_i)c(\gamma'_i)$, $c(P_i)$, and $N(P_i)$ are all at most *n* raised to the power

$$\frac{\varepsilon}{32} \sum_{i \in [j]} \left\{ |E(\gamma_i)| - \frac{|E(U_{\gamma_i})| + |E(V_{\gamma_i})|}{2} + (\text{# of edges removed from } \gamma_i) + |V(\gamma_i)| - \frac{|U_{\gamma_i}| + |V_{\gamma_i}|}{2} + |E(\gamma_i^{\mathsf{T}})| - \frac{|E(U_{\gamma_i^{\mathsf{T}}})| + |E(U_{\gamma_i^{\mathsf{T}}})|}{2} + (\text{# of edges removed from } \gamma_i^{\mathsf{T}}) + |V(\gamma_i^{\mathsf{T}})| - \frac{|U_{\gamma_i^{\mathsf{T}}}| + |V_{\gamma_i^{\mathsf{T}}}|}{2} \right\}$$

7. $\left|\frac{c_P}{c_P^{\approx}}\right| \le 2$

E.7 Well-conditionedness of L

The goal of this section is to prove a lower bound on the minimum nonzero eigenvalue of **LL**^T. More specifically we will prove the following lemma:

Lemma E.56 (Well-conditionedness of L).

$$\sum_{V \in \mathcal{I}_{mid}} \frac{\lambda_V}{|V|!} \mathbf{L}_V \mathbf{M}_{V^+} \mathbf{L}_V^{\mathsf{T}} \geq \Omega(n^{-D_{SoS}}) \mathbf{Id}_{sym}$$

The approach we take to Lemma E.56 is as follows. If we can find nonnegative weights $\{w_V : V \in \mathcal{I}_{mid}\}$ such that

$$\sum_{V \in \mathcal{I}_{mid}} \frac{w_V \lambda_V}{|V|!} \mathbf{L}_V \mathbf{M}_{V^+} \mathbf{L}_V^{\mathsf{T}} \geq \mathbf{Id}_{sym}$$

then since each term is individually PSD, the left-hand side is PSD-dominated by

$$\left(\max_{V\in \mathcal{I}_{mid}}w_V\right)\cdot\sum_{V\in \mathcal{I}_{mid}}\frac{\lambda_V}{|V|!}\mathbf{L}_V\mathbf{M}_{V^+}\mathbf{L}_V^{\mathsf{T}}.$$

This implies the lower bound

$$\sum_{V \in \mathcal{I}_{mid}} \frac{\lambda_V}{|V|!} \mathbf{L}_V \mathbf{M}_{V^+} \mathbf{L}_V^{\mathsf{T}} \geq \frac{1}{\max_{V \in \mathcal{I}_{mid}} w_V} \mathbf{Id}_{sym} \,.$$

We will therefore seek an appropriate choice of w_V that does not grow too quickly.

We will choose $w_U = 0$ unless $E(U) = \emptyset$. The following lemma shows a growth condition which is sufficient. Since $|U_{\sigma}| > |V_{\sigma}|$ for all non-diagonal left shapes σ , we can use this lemma to define w_U in order of increasing size |U|.

Lemma E.57. If we have nonnegative weights $\{w_V : V \in I_{mid}, E(V) = \emptyset\}$ such that for all $U, V \in I_{mid}$ with $E(V) = \emptyset$,

$$w_V \max_{\substack{nontrivial\\ \sigma \in \mathcal{L}_V: U_\sigma = U}} \{2c(\sigma)\lambda_\sigma \|\mathbf{M}_\sigma\|\} \le \frac{w_U}{2}$$

then

$$\sum_{\substack{V \in \mathcal{I}_{mid}:\\ E(V) = \emptyset}} \frac{w_V}{|V|!} \mathbf{L}_V \mathbf{L}_V^{\mathsf{T}} \geq \frac{1}{2} \mathbf{Id}_{sym} \,.$$

Proof. L_V consists of the trivial shape with $V(\sigma) = V$, as well as larger off-diagonal shapes. We use the following definition and claim to bound the off-diagonal shapes.

Definition E.58 (Id_{Sym,V}). For $V \in I_{mid}$, let Id_{Sym,V} be the restriction of Id_{sym} to the degree |V|-by-|V| block.

Claim E.59. For all $V \in I_{mid}$ with $E(V) = \emptyset$,

$$\frac{1}{|V|!} \mathbf{L}_{V} \mathbf{L}_{V}^{\mathsf{T}} \geq \mathbf{Id}_{Sym,V} - \sum_{U \in \mathcal{I}_{mid}} \frac{1}{|U|!} \sum_{\substack{nontrivial \\ \sigma \in \mathcal{L}_{V}: U_{\sigma} = U}} 2\lambda_{\sigma} ||\mathbf{M}_{\sigma}|| \mathbf{Id}_{Sym,U}$$

Proof of claim. We have that

$$\frac{1}{|V|!}\mathbf{L}_{V}\mathbf{L}_{V}^{\mathsf{T}} = \mathbf{Id}_{Sym,V} + \sum_{\text{non-trivial } \sigma \in \mathcal{L}_{V}} \lambda_{\sigma}(\mathbf{M}_{\sigma} + \mathbf{M}_{\sigma}^{\mathsf{T}}) + \frac{1}{|V|!} \sum_{\text{non-trivial } \sigma, \sigma' \in \mathcal{L}_{V}} \lambda_{\sigma}\lambda_{\sigma'}\mathbf{M}_{\sigma}\mathbf{M}_{\sigma'}^{\mathsf{T}}$$

The last term is PSD. Hence it remains to bound the middle term.

$$\begin{split} \sum_{\text{non-trivial } \sigma \in \mathcal{L}_{V}} \lambda_{\sigma}(\mathbf{M}_{\sigma} + \mathbf{M}_{\sigma}^{\mathsf{T}}) &= \sum_{U \in \mathcal{I}_{mid}: U \neq V} \sum_{\substack{\sigma \in \mathcal{L}_{V}: \\ U_{\sigma} = U}} \lambda_{\sigma}(\mathbf{M}_{\sigma} + \mathbf{M}_{\sigma}^{\mathsf{T}})} \\ &= \sum_{U \in \mathcal{I}_{mid}} \frac{1}{|U|!} \sum_{\substack{\text{non-trivial } \sigma \in \mathcal{L}_{V}: \\ U_{\sigma} = U}} \lambda_{\sigma} \mathbf{Id}_{Sym,U}^{1/2}(\mathbf{M}_{\sigma} + \mathbf{M}_{\sigma}^{\mathsf{T}}) \mathbf{Id}_{Sym,U}^{1/2}} \\ &\geq -\sum_{U \in \mathcal{I}_{mid}} \frac{1}{|U|!} \sum_{\substack{\text{non-trivial } \sigma \in \mathcal{L}_{V}: \\ U_{\sigma} = U}} \lambda_{\sigma} \mathbf{Id}_{Sym,U}^{1/2}(||\mathbf{M}_{\sigma}|| + ||\mathbf{M}_{\sigma}^{\mathsf{T}}||) \mathbf{Id}_{Sym,U}^{1/2}} \\ &= -\sum_{U \in \mathcal{I}_{mid}} \frac{1}{|U|!} \sum_{\substack{\text{non-trivial } \sigma \in \mathcal{L}_{V}: \\ U_{\sigma} = U}} 2\lambda_{\sigma} ||\mathbf{M}_{\sigma}|| \mathbf{Id}_{Sym,U}} \end{split}$$

which completes the proof of the claim.

Using the claim,

$$\sum_{V \in \mathcal{I}_{mid}} \frac{w_V}{|V|!} \mathbf{L}_V \mathbf{L}_V^{\mathsf{T}} \geq \sum_{V \in \mathcal{I}_{mid}} w_V \mathbf{Id}_{Sym,V} - \sum_{U,V \in \mathcal{I}_{mid}} \frac{w_V}{|U|!} \sum_{\substack{\text{nontrivial} \\ \sigma \in \mathcal{L}_V : U_\sigma = U}} 2\lambda_\sigma ||\mathbf{M}_\sigma|| \mathbf{Id}_{Sym,U}$$

$$= \sum_{V \in \mathcal{I}_{mid}} w_V \mathbf{Id}_{Sym,V} - \sum_{U \in \mathcal{I}_{mid}} \left(\sum_{V \in \mathcal{I}_{mid}} \sum_{\substack{\text{nontrivial} \\ \sigma \in \mathcal{L}_V : U_\sigma = U}} \frac{w_V}{c(\sigma)|U|!} 2c(\sigma)\lambda_\sigma ||\mathbf{M}_\sigma|| \right) \mathbf{Id}_{Sym,U}$$

$$\geq \sum_{V \in \mathcal{I}_{mid}} w_V \mathbf{Id}_{Sym,V} - \sum_{U \in \mathcal{I}_{mid}} \left(\max_{V \in \mathcal{I}_{mid}} w_V \max_{\substack{\text{nontrivial} \\ \sigma \in \mathcal{L}_V : U_\sigma = U}} \{2c(\sigma)\lambda_\sigma ||\mathbf{M}_\sigma|| \} \right) \mathbf{Id}_{Sym,U}$$

$$\geq \sum_{V \in \mathcal{I}_{mid}} w_V \mathbf{Id}_{Sym,V} - \frac{1}{2} \sum_{U \in \mathcal{I}_{mid}} w_U \mathbf{Id}_{Sym,U} \quad \text{(by assumption)}$$

$$=\frac{1}{2}\sum_{V\in\mathcal{I}_{mid}}w_V\mathbf{Id}_{Sym,V}$$

Now we calculate the bound on the weights to deduce Lemma E.56. Lemma E.60. For all left shapes σ with $E(V_{\sigma}) = \emptyset$,

$$\lambda_{\sigma} \left\| \mathbf{M}_{\sigma}^{\approx} \right\| \leq n^{(1-\alpha)\left(\frac{|U_{\sigma}| - |V_{\sigma}|}{2}\right) - (\gamma - \alpha\beta)|E(\sigma)|}$$

Proof. By Lemma 4.8 we have

$$\lambda_{\sigma} \left\| \mathbf{M}_{\sigma}^{\approx} \right\| = n^{\left(1-\alpha\right) \left(\frac{|U_{\sigma}|+|V_{\sigma}|}{2}\right) - \left(\frac{1}{2}-\alpha\right) w(\sigma) - \frac{w(V_{\sigma})}{2} - (\gamma - \alpha\beta)|E(\sigma)|}$$

Substituting $w(\sigma) \ge w(V_{\sigma})$ since σ is a left shape, and $w(V_{\sigma}) = |V_{\sigma}|$,

$$\lambda_{\sigma} \left\| \mathbf{M}_{\sigma}^{\approx} \right\| \leq n^{(1-\alpha)\left(\frac{|U_{\sigma}|+|V_{\sigma}|}{2}\right) - (1-\alpha)|V_{\sigma}| - (\gamma - \alpha\beta)|E(\sigma)|}$$
$$= n^{(1-\alpha)\left(\frac{|U_{\sigma}|-|V_{\sigma}|}{2}\right) - (\gamma - \alpha\beta)|E(\sigma)|}$$

Corollary E.61. For all $U, V \in I_{mid}$ such that $E(V) = \emptyset$,

$$\max_{\substack{nontrivial\\ \sigma \in \mathcal{L}_{V}: U_{\sigma} = U}} \{ 2c(\sigma) \lambda_{\sigma} \| \mathbf{M}_{\sigma} \| \} \le n^{(1-\alpha) \left(\frac{|U_{\sigma}| - |V_{\sigma}|}{2} \right)}$$

Proof.

$$\begin{aligned} 2c(\sigma)\lambda_{\sigma} \|\mathbf{M}_{\sigma}\| &\leq 2c(\sigma)B_{adjust}(\sigma)n^{(1-\alpha)\left(\frac{|U_{\sigma}|-|V_{\sigma}|}{2}\right) - (\gamma - \alpha\beta)|E(\sigma)|} \\ &\leq (16D_{V})^{|E(\sigma)|}n^{(1-\alpha)\left(\frac{|U_{\sigma}|-|V_{\sigma}|}{2}\right) - (\gamma - \alpha\beta)|E(\sigma)|} \\ &\leq n^{(1-\alpha)\left(\frac{|U_{\sigma}|-|V_{\sigma}|}{2}\right)}. \end{aligned}$$

(17.7.1.17.7.1)

Corollary E.62. Choosing $w_U = O\left(n^{(1-\alpha)\frac{|U|}{2}}\right)$ satisfies the assumption of Lemma E.57.

Recall that because the size of the SMVS is at most D_{SoS} , then $|U| \le D_{SoS}$. The maximum of w_U is $w_{D_{SoS}} \le O(n^{D_{SoS}})$, therefore we conclude Lemma E.56.

F Computing $\widetilde{\mathbb{E}}[1]$

Proposition 2.52. With high probability, we have $\widetilde{\mathbb{E}}[1] = 1 \pm o(1)$.

Proof.

$$\left|\widetilde{\mathbb{E}}[1] - 1\right| = \left|\sum_{\substack{\alpha \in \mathcal{S}:\\ U_{\alpha} = V_{\alpha} = \emptyset,\\ E(\alpha) \neq \emptyset}} \lambda_{\alpha} \mathbf{M}_{\alpha}\right| \le \max_{\substack{\alpha \neq \emptyset: U_{\alpha} = V_{\alpha} = \emptyset\\ \alpha \neq \emptyset: U_{\alpha} = V_{\alpha} = \emptyset}} \{c(\alpha)\lambda(\alpha) \cdot \|\mathbf{M}_{\alpha}\|\}$$

Letting *S* be the SMVS for α , observe that

- 1. $\lambda_{\alpha} ||\mathbf{M}_{\alpha}|| \le n^{-(\frac{1}{2}-\alpha)w(\alpha)-\frac{w(S)}{2}-(\gamma-\alpha\beta)|E(\alpha)|} \cdot B_{adjust}(\alpha)$ 2. $w(S) \ge -\eta - 2 |E(S)| \log_{n} D_{V}$
- 3. $w(\alpha) \ge -\eta 2|E(\alpha)|\log_n D_V$
- 4. $c(\alpha) \cdot B_{adjust}(\alpha) \cdot n^{2\eta+4|E(\alpha)|\log_n D_V} \le n^{2\eta} (16D_V)^{8|E(\alpha)|}$ which is less than $n^{\frac{\gamma-\alpha\beta}{2}|E(\alpha)|}$ when $\eta \le \frac{\gamma-\alpha\beta}{4} 4\log_n(16D_V)$.

Therefore, we can bound

$$\max_{\alpha \neq \emptyset: U_{\alpha} = V_{\alpha} = \emptyset} \{ c(\alpha) \lambda(\alpha) \cdot ||\mathbf{M}_{\alpha}|| \} \le n^{\frac{\alpha\beta - \gamma}{2}} \le o(1)$$

as $|E(\alpha)| > 0$, giving us that $|\widetilde{\mathbb{E}}[1] - 1| = o(1)$ with probability at least $1 - 2n^{4\log_n(16D_V) - \frac{\gamma - \alpha\beta}{4}}$.

Proposition 2.54. With high probability,

$$\left|\sum_{i=1}^{n} \widetilde{\mathbb{E}}[\mathbf{X}_{i}] - k\right| = o(k),$$

and

$$\left|\sum_{\{i,j\}\in E(G)}\widetilde{\mathbb{E}}[\mathbf{X}_{i}\mathbf{X}_{j}] - \frac{k^{2}q}{2}\right| = o(k^{2}q)$$

Proof. This can be shown by decomposing $\sum_{i=1}^{n} \widetilde{\mathbb{E}}[\mathbf{X}_i]$ and $\sum_{\{i,j\}\in E(G)} \widetilde{\mathbb{E}}[\mathbf{X}_i\mathbf{X}_j]$ in terms of ribbons.

For each ribbon *R* with edges E(R) and $A_R = B_R = \emptyset$, this ribbon appears in $\sum_{i=1}^{n} \widetilde{\mathbb{E}}[\mathbf{X}_i]$ in two ways.

1. For each $i \in V(R)$, the ribbon R' with E(R') = E(R), V(R') = V(R), $A_{R'} = (i)$, and $B_{R'} = \emptyset$ appears in $\widetilde{\mathbb{E}}[\mathbf{X}_i]$ with coefficient $\left(\frac{k}{n}\right)^{|V(R)|} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|}$. This happens for |V(R)| different *i*.

2. For each *i* which is not in V(R), the ribbon R' with E(R') = E(R), $V(R') = V(R) \cup \{i\}$, $A_{R'} = (i)$, and $B_{R'} = \emptyset$ appears in $\widetilde{\mathbb{E}}[\mathbf{X}_i]$ with coefficient $\left(\frac{k}{n}\right)^{|V(R)|+1} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|}$. This happens for n - |V(R)| different *i*.

Thus, for each shape α with $U_{\alpha} = V_{\alpha} = \emptyset$ where $|V(\alpha)| \le D_V$, \mathbf{M}_{α} appears in $\sum_{i=1}^{n} \mathbb{E}[\mathbf{X}_i]$ with coefficient

$$\frac{1}{|\operatorname{Aut}(\alpha)|} \left(\frac{k(n-|V(\alpha)|)}{n} \mathbf{1}_{|V(\alpha)| < D_V} + |V(\alpha)| \right) \left(\frac{k}{n} \right)^{|V(R)|} \left(\frac{q-p}{\sqrt{p(1-p)}} \right)^{|E(\alpha)|}$$

The dominant term is the trivial shape α with no vertices or edges which gives a contribution of exactly k. Using a similar analysis as the analysis used to bound $|\widetilde{\mathbb{E}}[1] - 1|$, it is not hard to show that the remaining terms have magnitude o(k) with high probability.

To analyze $\sum_{\{i,j\}\in E(G)} \widetilde{\mathbb{E}}[\mathbf{X}_i \mathbf{X}_j]$, we use the identities $\mathbf{1}_{e\in E(G)}\chi_e = \sqrt{p(1-p)} + (1-p)\chi_e$ and $\mathbf{1}_{e\in E(G)} = p + \sqrt{p(1-p)}\chi_e$.

For each ribbon *R* with edges E(R) and $A_R = B_R = \emptyset$, this ribbon appears in $\sum_{i < j} \mathbf{1}_{\{i,j\} \in E(G)} \widetilde{\mathbb{E}}[X_iX_j]$ in several ways.

1. For each $i < j \in V(R)$ such that $\{i, j\} \in E(R)$, the ribbon R'_1 with $E(R'_1) = E(R)$, V(R') = V(R), $A_{R'} = (i)$, and $B_{R'} = (j)$ appears in $\widetilde{\mathbb{E}}[\mathbf{X}_i \mathbf{X}_j]$ with coefficient $\left(\frac{k}{n}\right)^{|V(R)|} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|}$. This is then multiplied by (1-p) because of the edge indicator $\mathbf{1}_{\{i,j\}\in E(G)}$.

Similarly, the ribbon R'_2 with $E(R'_2) = E(R) \setminus \{i, j\}, V(R') = V(R), A_{R'} = (i)$, and $B_{R'} = (j)$ appears in $\widetilde{\mathbb{E}}[\mathbf{X}_i \mathbf{X}_j]$ with coefficient $\left(\frac{k}{n}\right)^{|V(R)|} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|-1}$. This is then multiplied by $\sqrt{p(1-p)}$ because of the edge indicator $\mathbf{1}_{\{i,j\}\in E(G)}$.

This gives a total contribution of $\left(1 + \frac{p(1-p)}{q-p}\right)\left(\frac{k}{n}\right)^{|V(R)|} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|}$. This happens for |E(R)| different i < j.

2. For each $i < j \in V(R)$ such that $\{i, j\} \notin E(R)$, the ribbon R'_1 with $E(R'_1) = E(R)$, V(R') = V(R), $A_{R'} = (i)$, and $B_{R'} = (j)$ appears in $\widetilde{\mathbb{E}}[X_iX_j]$ with coefficient

 $\left(\frac{k}{n}\right)^{|V(R)|} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|}.$ This is then multiplied by p because of the edge indicator $\mathbf{1}_{\{i,j\}\in E(G)}.$

Similarly, the ribbon R'_2 with $E(R'_2) = E(R) \cup \{i, j\}, V(R') = V(R), A_{R'} = (i)$, and $B_{R'} = (j)$ appears in $\widetilde{\mathbb{E}}[\mathbf{X}_i \mathbf{X}_j]$ with coefficient $\left(\frac{k}{n}\right)^{|V(R)|} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|+1}$. This is then multiplied by $\sqrt{p(1-p)}$ because of the edge indicator $\mathbf{1}_{\{i,j\}\in E(G)}$.

This gives a total contribution of $q\left(\frac{k}{n}\right)^{|V(R)|} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|}$. This happens for $\binom{V(R)}{2} - |E(R)|$ different i < j.

3. For each $i < j \in V(R)$ such that $i \in V(R)$ but $j \notin V(R)$, the ribbon R'_1 with $E(R'_1) = E(R), V(R') = V(R) \cup \{j\}, A_{R'} = (i)$, and $B_{R'} = (j)$ appears in $\widetilde{\mathbb{E}}[\mathbf{X}_i \mathbf{X}_j]$ with coefficient $\left(\frac{k}{n}\right)^{|V(R)|+1} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|}$. This is then multiplied by p because of the edge indicator $\mathbf{1}_{\{i,j\}\in E(G)}$.

Similarly, the ribbon R'_2 with $E(R'_2) = E(R) \cup \{i, j\}, V(R') = V(R) \cup \{j\}, A_{R'} = (i)$, and $B_{R'} = (j)$ appears in $\widetilde{\mathbb{E}}[\mathbf{X}_i \mathbf{X}_j]$ with coefficient $\left(\frac{k}{n}\right)^{|V(R)|+1} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|+1}$. This is then multiplied by $\sqrt{p(1-p)}$ because of the edge indicator $\mathbf{1}_{\{i,j\}\in E(G)}$.

This gives a total contribution of $\frac{kq}{n} \left(\frac{k}{n}\right)^{|V(R)|} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|}$.

The same analysis holds for the case when $i \notin V(R)$ and $j \in V(R)$. These two cases happen for |V(R)|(n - |V(R)|) different i < j.

4. For each $i < j \in V(R)$ such that $i, j \notin V(R)$, the ribbon R'_1 with $E(R'_1) = E(R)$, $V(R') = V(R) \cup \{i, j\}, A_{R'} = (i)$, and $B_{R'} = (j)$ appears in $\widetilde{\mathbb{E}}[\mathbf{X}_i \mathbf{X}_j]$ with coefficient $\left(\frac{k}{n}\right)^{|V(R)|+2} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|}$. This is then multiplied by p because of the edge indicator $\mathbf{1}_{\{i,j\}\in E(G)}$.

Similarly, the ribbon R'_2 with $E(R'_2) = E(R) \cup \{i, j\}, V(R') = V(R) \cup \{i, j\}, A_{R'} = (i)$, and $B_{R'} = (j)$ appears in $\widetilde{\mathbb{E}}[\mathbf{X}_i \mathbf{X}_j]$ with coefficient $\left(\frac{k}{n}\right)^{|V(R)|+2} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|+1}$. This is then multiplied by $\sqrt{p(1-p)}$ because of the edge indicator $\mathbf{1}_{\{i,j\}\in E(G)}$.

This gives a total contribution of $\frac{k^2 q}{n^2} \left(\frac{k}{n}\right)^{|V(R)|} \left(\frac{q-p}{\sqrt{p(1-p)}}\right)^{|E(R)|}$. This happens for $\binom{(n-|V(R)|)}{2}$

different i < j.

Putting everything together, each ribbon *R* with $A_R = B_R = \emptyset$ and $|V(R)| \le D_V$ appears with coefficient

$$\left(\left(1 + \frac{p(1-p)}{q-p} \right) |E(R)| + q \left(\binom{V(R)}{2} - |E(R)| \right) + \frac{kq}{n} |V(R)| (n - |V(R)|) \mathbf{1}_{|V(R)| < D_V} + \frac{k^2q}{n^2} \binom{(n - |V(R)|)}{2} \mathbf{1}_{|V(R)| < D_V - 1} \right) \binom{k}{n}^{|V(R)|} \left(\frac{q-p}{\sqrt{p(1-p)}} \right)^{|E(R)|}$$

in $\sum_{\{i,j\}\in E(G)} \widetilde{\mathbb{E}}[\mathbf{X}_i\mathbf{X}_j]$. Thus, for each shape α with $U_{\alpha} = V_{\alpha} = \emptyset$ and $|V(\alpha)| \leq D_V$, the graph matrix M_{α} appears with coefficient

$$\frac{1}{|\operatorname{Aut}(\alpha)|} \left(\left(1 + \frac{p(1-p)}{q-p} \right) |E(\alpha)| + q \left(\binom{V(\alpha)}{2} - |E(\alpha)| \right) + \frac{kq}{n} |V(\alpha)| (n - |V(\alpha)|) \mathbf{1}_{|V(\alpha)| < D_V} + \frac{k^2q}{n^2} \binom{(n - |V(R)|)}{2} \mathbf{1}_{|V(R)| < D_V - 1} \right) \left(\frac{k}{n} \right)^{|V(\alpha)|} \left(\frac{q-p}{\sqrt{p(1-p)}} \right)^{|E(\alpha)|}$$

in $\sum_{\{i,j\}\in E(G)} \widetilde{\mathbb{E}}[\mathbf{X}_i\mathbf{X}_j]$.

The dominant term in $\sum_{\{i,j\}\in E(G)} \widetilde{\mathbb{E}}[\mathbf{X}_i\mathbf{X}_j]$ comes from the empty shape α with no vertices or edges. This gives $\frac{k^2q}{n^2}\binom{n}{2} \approx \frac{k^2q}{2}$. Using a similar analysis as the analysis used to bound $|\widetilde{\mathbb{E}}[1] - 1|$, it is not hard to show that the remaining terms have magnitude $o(k^2q)$ with high probability.