A Formalized Reduction of Keller's Conjecture

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Abstract

Keller's conjecture in d dimensions states that there are no faceshare-free tilings of d-dimensional space by translates of a d-dimensional cube. In 2020, Brakensiek et al. resolved this 90-year-old conjecture by proving that the largest number of dimensions for which no faceshare-free tilings exist is 7. This result, as well as many others pertaining to Keller's conjecture, critically relies on a reduction from Keller's original conjecture to a statement about cliques in generalized Keller graphs. In this paper, we present a formalization of this reduction in the Lean 3 theorem prover. Additionally, we combine this formalized reduction with the verification of a large clique in the Keller graph G_8 to obtain the first verified end-to-end proof that Keller's conjecture is false in 8 dimensions.

CCS Concepts: • **Theory of computation** → **Logic and verification**.

Keywords: Formal Verification, Keller's Conjecture, Interactive Theorem Proving, Lean

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1 Introduction

In 1930, Keller conjectured that any tiling of d-dimensional Euclidean space by translates of a d-dimensional cube must contain two cubes that share a $(d-1)$-dimensional face [6]. Ten years later, Perron proved that this conjecture was true for $d \leq 6$ using combinatorial casework [13, 14]. Over the next fifty years, some increased understanding of the conjecture came about in the form of various reductions to group-theoretic problems [5, 15], but no concrete progress was made on determining the set of dimensions in which Keller's conjecture holds.

Then, in 1990, Corrádi and Szabó used the insights gained from these group-theoretic reductions to introduce the class of Keller graphs [3]. Each Keller graph is denoted G_d for some $d \in \mathbb{N}^+$, and each G_d is defined as follows: G_d has 4^d vertices, each of which is a vector of length d with entries from $\{0, 1, 2, 3\}$. Vectors v_1 and v_2 are adjacent in G_d if there are indices i and j such that $i \neq j$, $v_1[i] = v_2[i] \pm 2$, and $v_1[j] \neq v_2[j]$. In addition to introducing the Keller graphs, Corrádi and Szabó showed that Keller's conjecture in d dimensions is false if there is a clique of size 2^d in G_d [3].

This graph-theoretic reduction turned out to be critical for further progress on determining the set of dimensions in which Keller's conjecture holds. Just two years after the Keller graphs were introduced, Lagarias and Shor proved that Keller's conjecture is false for $d \geq 10$ by showing the existence of a 2^{10} size clique in G_{10} [10]. In 2002, Mackey found a clique of size 2^8 in G_8 to show that Keller's conjecture is false for $d \geq 8$ [11]. Between this result and Perron's result that the conjecture was true for $d \leq 6$, the only dimension left unresolved was the $d = 7$ case.

To reduce the truth of Keller's conjecture to a graph-theoretic statement in the $d = 7$ case, a more general notion of the Keller graphs is needed. Let the generalized Keller graph $G_{d,s}$ be defined as follows: $G_{d,s}$ has $(2s)^d$ vertices, each of which is a vector of length d with entries in $\{0, 1, \ldots, 2s-1\}$. Vectors v_1 and v_2 are adjacent in $G_{d,s}$ if there are indices i and j such that $i \neq j$, $v_1[i] = v_2[i] \pm s$, and $v_1[j] \neq v_2[j]$. Note that G_d is simply the special case of $G_{d,s}$ where $s = 2$.

In a series of papers from 2015 to 2017, Kisielewicz incrementally reduced the $d = 7$ case, ultimately showing that Keller's conjecture is true in seven dimensions if there is no clique of size 2^7 in $G_{7,3}$ [7–9]. Finally, in 2020, Brakensiek et al. resolved the final case of Keller's conjecture by showing that no clique of size 2^7 exists in $G_{7,3}$, $G_{7,4}$, or $G_{7,6}$ [1].

From this history, the importance of the Keller graph reduction to progress on Keller's conjecture is clear. Except for Perron's early proof that Keller's conjecture holds for $d \leq 6$, all known results about the set of dimensions in which Keller's conjecture holds critically rely on it. Additionally, the reduction itself is nontrivial. In Corrádi and Szabó's original paper introducing the Keller graphs, they claimed not only that Keller's conjecture in d dimensions is false if there exists a clique of size 2^d in G_d, but that these two statements were logically equivalent [3]. However, as Debroni et al. note in

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their paper that shows G_7 has a maximum clique size of 124, the fact that G_7 has no clique of size 2^7 only resolves Keller's conjecture in seven dimensions if all cubes are assumed to have integer or half-integer coordinates [4]. The importance and nontriviality of the Keller graph reduction makes it a good target for formalization. Furthermore, the reduction itself does not rely on a significant body of background theory, making it a viable target as well.

In this paper, we present a formalized reduction of Keller's conjecture to a statement about generalized Keller graphs. Our formalization[1] uses the Lean 3 theorem prover, building on top of Lean's library mathlib [12]. Our main contributions are as follows:

- We give the first *verified* proof that Keller's conjecture is false in d dimensions if there is a clique of size 2^d in $G_{d,s}$ for any $s \in \mathbb{N}^+$, and Keller's conjecture is true in d dimensions if there is no clique of size 2^d in $G_{d,2^{d-1}}$.
- We formally verify the clique of size 2^8 in G_8 given by Mackey [11]. Together with the formalized reduction described above, this yields a verified end-to-end proof that Keller's conjecture is false in 8 dimensions.

2 Formal Definitions

Since Keller's conjecture concerns translates of a single d-dimensional cube, we can without loss of generality take this cube to be the unit cube and orient it to be axis-aligned. To avoid counting points on partially shared faces as being contained in multiple cubes, we adopt the convention that between two opposite faces of a cube, only the face on the lower coordinate is included in the cube. So the set of points contained in the 2-dimensional square with corners $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$ is $\{(x,y) \mid 0 \le x, y < 1\}$.

In our formalization, we define cubes as sets of d-dimensional points, and we represent each unit cube by its unique corner with minimal value along each coordinate. Throughout this paper, if a cube c is represented by the corner p, then we say that p defines c. So the previously described 2-dimensional square would be defined by the point $(0,0)$. Our definitions for points and cubes in Lean are as follows:

```
def point (d : ℕ) : Type := vector ℝ d

def in_cube {d : ℕ} (corner p : point d) : Prop :=
  ∀ coord : fin d,
    vector.nth corner coord ≤ vector.nth p coord ∧
    vector.nth p coord < vector.nth corner coord + 1

def cube {d : ℕ} (corner : point d) : set (point d)
  := { p : point d | in_cube corner p }
```

Members of the type point d are d-length vectors of reals meant to encode elements of $\mathbb{R}^d$. The type fin d is the subtype of natural numbers containing exactly $\{0, 1, \ldots, d-1\}$, so the predicate in_cube corner p encodes that p is in the cube defined by the point corner. Finally, cube corner gives the set of points that are contained in the cube defined by corner.

With these definitions, we proceed to define the predicates is_tiling and tiling_faceshare_free. For both of these predicates, we represent tilings as sets of points, each point defining a cube, rather than sets of sets of points. The predicate is_tiling is given below:

```
def is_tiling {d : ℕ} (T : set (point d)) : Prop :=
  ∀ p : point d, ∃ corner ∈ T, p ∈ cube corner ∧
  (∀ alt_corner ∈ T,
    p ∈ cube alt_corner → alt_corner = corner)
```

In order for a set of cubes to form a tiling, two conditions must be met. First, every d-dimensional point must belong to at least one cube in the tiling, and second, every d-dimensional point must belong to at most one cube in the tiling. The first condition is enforced in the first line of the above predicate's definition, and the second condition is enforced in the second line. Together, these conditions entail that every d-dimensional point belongs to a unique cube in the tiling.

In order for a tiling to be faceshare-free, there must be no pair of cubes in the tiling that share a complete $(d-1)$-dimensional face. Two cubes share such a face if the points that define them agree on all coordinates except one, and the difference between the points along that coordinate is exactly one. This is equivalent to saying that two cubes are facesharing if one cube is a translation of the other by an axis-aligned unit vector. The predicate tiling_faceshare_free, and the predicate is_facesharing which is used to help define it, are as follows:

```
def is_facesharing {d : ℕ}
    (c1 c2 : point d) : Prop :=
  ∃ x : fin d,
    (vector.nth c1 x - vector.nth c2 x = 1 ∨
     vector.nth c2 x - vector.nth c1 x = 1) ∧
  ∀ y : fin d,
    x = y ∨ vector.nth c1 y = vector.nth c2 y

def tiling_faceshare_free {d : ℕ}
    (T : set (point d)) : Prop :=
  ∀ c1 c2 ∈ T, ¬is_facesharing c1 c2
```

These two predicates, is_tiling and tiling_faceshare_free, allow us to finally define Keller's conjecture, which we treat as a proposition dependent on some natural number of dimensions d.

[1] https://github.com/JOSHCLUNE/Keller_reduction/

```
def Keller_conjecture (d : ℕ) : Prop :=
  ∀ T : set (point d),
    is_tiling T → ¬tiling_faceshare_free T
```

This definition is sufficient to formalize our second verified result. We produce a proof with the type ¬Keller_conjecture 8 and take this as a verified proof that Keller's conjecture is false in eight dimensions. If there is any issue then, it must be found either in Lean's trusted core or in this encoding of Keller's conjecture. Our hope is that these formal definitions are written sufficiently clearly that there can be no question that they really do encode Keller's conjecture.

To formalize our other main result, the reduction from Keller's conjecture to a statement about generalized Keller graphs, we must first define generalized Keller graphs. For this, we use the simple_graph module from mathlib [12] which allows us to define irreflexive undirected graphs as irreflexive symmetric relations on a vertex type. The vertex type we use is vector (fin (2s)) d, where fin (2s) is the subtype of natural numbers containing exactly $\{0, 1, \ldots, 2s-1\}$, and d indicates the length of the vector. The graph we define on this vertex type is as follows:

```
def Keller_graph (d : ℕ) (s : ℕ) :
    simple_graph (vector (fin (2*s)) d) :=
  let vertex : Type := vector (fin (2*s)) d in
  let adjacent_fn : vertex → vertex → Prop :=
    λ v1, λ v2, ∃ i : fin d, ∃ j : fin d,
      (v1.nth i).val = (v2.nth i).val + s ∧
      v1.nth j ≠ v2.nth j ∧ i ≠ j
  in simple_graph.from_rel adjacent_fn
```

The careful reader may find it odd that in adjacent_fn above, we require that (v1.nth i).val = (v2.nth i).val + s, as opposed to requiring that (v1.nth i).val = (v2.nth i).val + s ∨ (v2.nth i).val = (v1.nth i).val + s. The reason we can omit this explicit disjunction is that simple_graph.from_rel automatically symmetrizes the relation it is given, making this definition equivalent to but more convenient than a definition that explicitly treats v1 and v2 symmetrically.

With a formalization of generalized Keller graphs, we proceed to define m-sized cliques on these graphs. This allows us to write the two final theorems of our reduction.

```
def has_clique {d : ℕ} {s : ℕ}
    (G : simple_graph (vector (fin (2*s)) d))
    (m : ℕ) : Prop :=
  let vertex : Type := vector (fin (2*s)) d in
  ∃ clique : finset vertex, clique.card = m ∧
  ∀ v1 : vertex, ∀ v2 : vertex,
    v1 ∈ clique → v2 ∈ clique → v1 ≠ v2 →
    G.adj v1 v2
```

```
theorem clique_existence_refutes_Keller_conjecture
    {d s : ℕ} (s_ne_zero : s ≠ 0) :
  has_clique (Keller_graph d s) (2^d) →
    ¬Keller_conjecture d := ...

theorem clique_nonexistence_implies_Keller_conjecture
    {d : ℕ} (d_gt_zero : d > 0) :
  ¬has_clique (Keller_graph d (2^(d-1))) (2^d) →
    Keller_conjecture d := ...
```

3 Reducing to Periodic Tilings

At a high level, we follow the strategy outlined in the Appendix of Brakensiek et al.'s extended version of "The Resolution of Keller's Conjecture" [1]. The proof presented here follows Brakensiek et al.'s closely, but it provides a lot more low-level detail. Our goal in doing so is to document the ways that the combinatorial and geometric intuitions were represented in terms that are precise enough to admit direct formalization.

We begin by establishing two lemmas about tilings. The first is an important structural property needed for several later results, and the second is that if there exists any faceshare-free tiling in d-dimensions, then there also exists a d-dimensional faceshare-free periodic tiling. This second lemma entails that to reduce Keller's conjecture to a statement about generalized Keller graphs, it suffices to reduce a restricted version of Keller's conjecture that only considers periodic tilings.

3.1 A Structural Property of Tilings

Our first lemma concerns an important structural property that is common to all tilings.

Lemma 1. Let T be an arbitrary tiling in d-dimensional space, and let l be a line parallel to the i-th axis of that space. If we consider just the cubes in T that intersect l, then we will find that there exists some $a \in \mathbb{R}$ such that every cube's i-th coordinate is congruent to a mod 1.

The intuition for this fact is that since T is a tiling, every point on l must be covered by exactly one cube in T. Since each point on l is covered by at most one cube, none of the cubes can intersect, and since each point on the line is covered by at least one cube, there can be no gaps between cubes. So each cube must start exactly where the previous cube ends, meaning the cubes' i coordinates must be integrally spaced and therefore congruent mod 1.

On a first pass, this intuition may seem readily formalizable. We might consider taking two arbitrary consecutive cubes that intersect l and casing on whether the distance between their defining corners along the i-th axis is less than, equal to, or greater than one unit. In the first and last case, we can show that this contradicts the assumption that T is a tiling, and in the second case, we can show that these two cubes' i-th coordinates must be congruent mod 1.

Unfortunately, the aforementioned proof strategy is infeasible, at least without some modification. The issue is that, in order to derive a contradiction in the case that the distance between our chosen cubes exceeds one unit, it is essential that the two chosen cubes are consecutive. Otherwise, we would not be able to show that there exists a point on l that is covered by zero cubes. However, implicit in this strategy is the assumption that for every cube in T that intersects l, there must be some unique cube that comes immediately after it. But this assumption is unwarranted, as there may be infinitely many cubes within a finite region, for instance at 0, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ and so on.

One way to recover this strategy is to note that, in the unfortunate case described above, there is an infinite converging sequence of terms that describes cubes' starting locations. Whenever there is an infinite converging sequence of terms, there must be some n such that all terms after the n-th are within one unit of each other. This demonstrates that if the argument fails due to this kind of series, then a contradiction can still be derived from the fact that there are intersecting cubes in T.

Although this argument could, in principle, be used to recover the previously described strategy, it would require formally proving this nontrivial fact which is otherwise unrelated to our goal. For this reason, we opt to instead adopt a slightly more complicated proof strategy that will allow us to sidestep this technicality. This proof strategy is implemented in our proof of tiling_lattice_structure.

We begin by taking an arbitrary cube $t \in T$ that intersects l, and taking a to be its i-th coordinate. Then, we consider the set of cubes in T that intersect l whose i-th coordinates are not congruent to a mod 1. If this set is empty, then we are done, as we have shown that every cube in T which intersects l has an i-th coordinate congruent to a mod 1. Otherwise, we note that every unit cube defined by some corner p contains exactly one integer point (i.e. a point whose coordinates are all integers). We call this integer point $\lceil p \rceil$ because it is the point for which at each coordinate j, $\lceil p \rceil[j]$ is the ceiling of $p[j]$.

Although there may not be a cube in T that is closest to t along the i-th axis, there must be a smallest $n \in \mathbb{N}$ such that there exists a $p \in T$ where $|\lceil t \rceil[i] - \lceil p \rceil[i]| = n$ and $p[i] \not\equiv a$ mod 1. This is guaranteed by the well-foundedness of the natural numbers, a fact conveniently already formalized in mathlib. With this n, we can consider an arbitrary point $p \in T$ where $|\lceil t \rceil[i] - \lceil p \rceil[i]| = n$ and $p[i] \not\equiv a$ mod 1. If n is 0, then t and p must intersect, contradicting the assumption that T is a tiling. If n is 1, we can use a modified version of the previously described intuition to show that either p and t intersect, there is a gap between p and t not covered by any cube, or there is some third cube that intersects either p or t. In any of these cases, a contradiction can successfully be derived. Finally, if n is greater than 1, we can case on whether there is a cube exactly one unit closer to p than where t is. If so, we proceed inductively on a strictly smaller n, and if not, then the fact that n is the smallest integral distance at which a cube not congruent to a mod 1 can be found guarantees that there is an uncovered point on l, contradicting that T is a tiling.

Given how straightforward the intuitive explanation for this lemma is, it's surprising that such a circuitous proof was necessary. In retrospect, part of of this is caused by our choice of cube definition. Another proof strategy we considered involved taking an arbitrary cube t and performing an inductive argument to show that for all $n \in \mathbb{N}$, there is no cube at integral distance n whose i-th coordinate fails to be congruent to $t[i]$ mod 1, and there is a cube at integral distance n whose i-th coordinate is congruent to $t[i]$ mod 1. The difference between this argument and the previous one is that, in the previous argument, after we identify a $p \in T$ that is not congruent to a mod 1 we can interchange p and t to ensure that without loss of generality, $t[i] < p[i]$. This ensures that in the inductive step where n is greater than 1, we can always use a cube which begins one unit higher than t along the i-th axis. In contrast, for this proof strategy which does not explicitly identify a p, it is necessary to induct out from t in both the positive and negative directions of the i-th axis.

Had we made our notion of cubes entirely symmetric, for instance by designating cubes by their centers and resolving the issue of partially shared faces differently, we could have used essentially the same proof to induct out from t in both the positive and negative directions. But with our definition, which is inherently asymmetric, it would be necessary to prove two distinct inductions: one to show there is no counterexample cube whose i-th coordinate is greater than t's, and one to show that there is no counterexample cube whose i-th coordinate is less than t's. This would double the length and casework of an already long and tedious proof.

The impact this seemingly innocuous choice of cube definition has on the length and efficiency of our proof can be viewed as part of a trade-off that sometimes arises between simplicity and efficiency. Hard-coding the asymmetric convention for which faces belong to any given cube made our formalization significantly conceptually easier to reason about (at least, in our subjective opinion). However, this same asymmetry also resulted in longer proofs at various portions in our formalization. In our proof of Lemma 1, we found a strategy that allowed us to avoid repeating casework for positive and negative directions, but in other parts of our formalization, we are forced to bite the bullet and essentially duplicate arguments up to some small changes.

We believe that at the end of the day, our choice of cube definition may still be preferable to a totally symmetrical one, solely due to the effort saved by having a simple convention determine which faces belong to a cube. But we acknowledge that it is entirely possible that committing to symmetrical cubes could yield a more concise overall formalization.

3.2 Periodic Tilings

Our second lemma states that if there exists any faceshare-free tiling in d-dimensions, then there also exists a d-dimensional faceshare-free periodic tiling. In order to understand this lemma and its corresponding proof, we must first clarify what we mean by a periodic tiling. Informally, we refer to a tiling as periodic if cube positions that appear in the tiling are guaranteed to repeat mod 1 after a certain distance. For instance, the tiling consisting of all cubes defined by points in $\mathbb{Z}^d$ could be considered periodic with period 1 because for any $t \in \mathbb{Z}^d$ and any integer vector x, $t + x \in \mathbb{Z}^d$.

In this paper, whenever we refer to a tiling as being periodic, we implicitly mean that the tiling is periodic with period 2. This is because, as seen in the example of $\mathbb{Z}^d$, tilings that are periodic with period 1 are guaranteed to have facesharing. Additionally, it turns out that if a d-dimensional faceshare-free periodic tiling exists with any period $n \in \mathbb{N}^+$, there is a d-dimensional faceshare-free periodic tiling with period 2. So it suffices to just consider periodic tilings with period 2.

In our formalization, we have two notions of what it means for a tiling to be periodic. These definitions are equivalent, which we prove in is_periodic_of_has_periodic_core and has_periodic_core_of_is_periodic, but they are convenient in different settings.

Definition 1 (*Periodic Tiling*). A d-dimensional tiling T is periodic if, for each cube-defining corner $t \in T$ and for all d-length integer vectors x, $t + 2x$ is also in T.

From this definition, if $t \in T$ and T is a periodic tiling, then every translate of t by a vector whose coordinates are all even integers must also be in T. This notion of periodic is convenient when making local claims about where cubes must be located in a periodic tiling. In particular, buried in the technical details of some later proofs, there are multiple instances in which we use the fact that if t designates a cube in a periodic tiling and e_i is the unit vector pointed along the i-th axis, then $t + 2e_i$ and $t - 2e_i$ must also be in said tiling.

Our second notion of what it means to be a periodic tiling invokes the concept of a periodic core:

Definition 2 (*Core*). Let S denote the set of d-dimensional points whose coordinates each equal 0 or 1. The core of a tiling T is the set of cubes in T that contain points in S.

Definition 3 (*Periodic Core*). A d-dimensional tiling T has a periodic core if, for each $t \in T$, there exists some t' in the core of T and some d-length integer vector x such that $t = t' + 2x$.

Definition 4 (*Periodic Tiling (II)*). A d-dimension tiling T is periodic if it has a periodic core.

This notion of periodic is convenient primarily in two settings. First, it is useful for defining tilings in such a way that easily guarantees they are periodic. Second, it is useful for establishing that all periodic tilings consist of translates of finitely many cubes. Specifically, since $|S| = 2^d$, there are 2^d cubes in the core of any tiling, so all periodic tilings consist of translates of 2^d core cubes. As it turns out, this is why Keller's conjecture reduces to showing the nonexistence of a 2^d-sized clique in the appropriate Keller graph.

To reduce Keller's conjecture to a restricted version that only considers periodic tilings, we show the following:

Lemma 2. If there exists any d-dimensional faceshare-free tiling, then there also exists a d-dimensional faceshare-free periodic tiling.

Our proof of Lemma 2, which we implement in our formalization as periodic_reduction, takes an arbitrary d-dimensional faceshare-free tiling and uses it to define a d-dimensional faceshare-free periodic tiling. The construction is as follows. Let T be an arbitrary faceshare-free tiling in d dimensions and let $\hat{T}$ denote the core of T. We construct a tiling T' such that $\hat{T}$ is the periodic core of T'. This can be done be defining T' as the set of all translates of elements in $\hat{T}$ by even integer vectors. By Definition 4, T' is clearly periodic, so all that remains is to show that T' is indeed a faceshare-free tiling.

Lemma 2.1. If T is a d-dimensional faceshare-free tiling with the core $\hat{T}$, and T' is the set of all translates of elements in $\hat{T}$ by even integer vectors, then T' is a tiling.

To show that T' is a tiling, it suffices to show that for any arbitrary point p, there exists exactly one cube in T' containing p. Let $\lfloor p \rfloor$ denote the d-length integer vector for which at each coordinate j, $\lfloor p \rfloor[j]$ is the floor of $p[j]$. Then, for each $i \in \mathbb{N}$ such that $0 \le i \le d$, let $\lfloor p \rfloor_i$ denote an approximation of $\lfloor p \rfloor$ such that for every coordinate j, if $j < i$ then $\lfloor p \rfloor_i[j] = \lfloor p \rfloor[j]$, and if $i \le j$, then $\lfloor p \rfloor_i[j] = 0$. We define the closed cube of $\lfloor p \rfloor_i$ as the set of points whose coordinates are each no lower than $\lfloor p \rfloor_i$'s and no more than one unit greater than $\lfloor p \rfloor_i$'s. In other words, the closed cube of $\lfloor p \rfloor_i$ is $\{q \in \mathbb{R}^d \mid \forall j \in \mathbb{N}, 0 \le j < d \to \lfloor p \rfloor_i[j] \le q[j] \le \lfloor p \rfloor_i[j]+1\}$. Note that $\lfloor p \rfloor_0$ is the zero-vector, and $\lfloor p \rfloor_d = \lfloor p \rfloor$, meaning p itself is in the closed cube of $\lfloor p \rfloor_d$.

Our outermost inductive strategy is to show that for all $i \in \mathbb{N}$ such that $0 \le i < d$, if each point in the closed cube of $\lfloor p \rfloor_i$ is covered by exactly one cube in T', then each point in the closed cube of $\lfloor p \rfloor_{i+1}$ is also covered by exactly one cube in T'. This, along with the base case that each point in $\lfloor p \rfloor_0$ is covered by exactly one cube in T', entails that each point in $\lfloor p \rfloor_d$ is covered by exactly one cube in T', including p.

To establish the base case, let q be an arbitrary point in the closed cube of $\lfloor p \rfloor_0$. To show that q is covered by exactly one cube in T', we note that q must be covered by exactly one point in $\hat{T}$. This is because every coordinate of q is between 0 and 1 inclusive, so any unit cube that contains q must also contain one of the points in S (the set of points whose coordinates each equal 0 or 1). From this, it follows that the unique cube in T that contains q must also be in $\hat{T}$, meaning q

is covered by exactly one cube in $\hat{T}$. Then, from our definition of T', it can be shown that no cubes in T' can cover q except those in $\hat{T}$, so the fact that exactly one cube in $\hat{T}$ covers q entails that exactly one cube in T' covers q, as desired.

For our outer inductive step, we show that if each point in the closed cube of $\lfloor p \rfloor_i$ is covered by exactly one cube in T', then each point in the closed cube of $\lfloor p \rfloor_{i+1}$ is also covered by exactly one cube in T'. We proceed by an inner induction on the absolute value of $\lfloor p \rfloor[i]$. If $\lfloor p \rfloor[i] = 0$, then $\lfloor p \rfloor_i = \lfloor p \rfloor_{i+1}$, so the outer inductive step follows immediately. Otherwise, let $\lfloor p \rfloor_{i,x}$ define a d-length integer vector that equals $\lfloor p \rfloor_i$ at all coordinates except the i-th coordinate, where it instead equals x. Note that $\lfloor p \rfloor_{i,0} = \lfloor p \rfloor_i$ and $\lfloor p \rfloor_{i,\lfloor p \rfloor[i]} = \lfloor p \rfloor_{i+1}$. Our inner inductive statement is that if each point in the closed cube of $\lfloor p \rfloor_{i,x}$ is covered by exactly one cube in T', then each point in the closed cubes of $\lfloor p \rfloor_{i,x+1}$ and $\lfloor p \rfloor_{i,x-1}$ are also covered by exactly one cube in T'. The fact that $\lfloor p \rfloor_{i,0} = \lfloor p \rfloor_i$ resolves the base case when $x = 0$, and the fact that $\lfloor p \rfloor_{i,\lfloor p \rfloor[i]} = \lfloor p \rfloor_{i+1}$ entails that proving this inner inductive step suffices to prove our outer inductive step.

To prove our inner inductive step, let q be an arbitrary point in the closed cube on $\lfloor p \rfloor_{i,x+1}$. Then, let q' and q'' be defined as points equal to q at all coordinates except the i-th coordinate and whose i-th coordinates are $\lfloor q[i] \rfloor$ and $\lfloor q[i] \rfloor - 1$ respectively. Note that both q' and q'' are in the closed cube of $\lfloor p \rfloor_{i,x}$, and so by the inductive hypothesis, there must be unique cubes t' and t'' in T' that cover q' and q'' respectively. Since q' and q'' are one unit apart along the i-th axis, t' and t'' must be distinct, and from Lemma 1, it can be shown that $t'[i] \equiv t''[i] \mod 1$. From this, it can be shown that every point on the line segment connecting q' and q'' must be covered by t' or t''.

Let $(q - 1_i)$ denote the point equal to q at all coordinates except the i-th coordinate and whose i-th coordinate is $q[i] - 1$. Since $(q - 1_i)$ lies on the line segment connecting q' and q'', $(q - 1_i)$ must uniquely be covered by either t' or t''. If $(q - 1_i)$ is uniquely covered by t'', then it can be shown that q must be uniquely covered by t'. Otherwise, $(q - 1_i)$ must be uniquely covered by t'. In this case, note that since T' is periodic, there must exist a $t \in T'$ that is equal to t'' at all coordinates except the i-th coordinate and whose i-th coordinate is $t''[i] + 2$. It can then be shown that q must be uniquely covered by t.

This argument demonstrates that if each point in the closed cube on $\lfloor p \rfloor_{i,x}$ is covered by exactly one cube in T', then each point in the closed cube of $\lfloor p \rfloor_{i,x+1}$ must also be covered by exactly one cube in T'. Although the analogous argument concerning the closed cube of $\lfloor p \rfloor_{i,x-1}$ is totally symmetric at the informal level, we note that our asymmetric in_cube definition unfortunately forces us to repeat this proof twice in our formalization with minor alterations. This is mildly inconvenient, but ultimately unproblematic.

Our outer inductive step follows directly from our inner induction, and from our outer induction, it follows that p is covered by exactly one cube in T', demonstrating that T' is a tiling, as desired.

Lemma 2.2. If T is a d-dimensional faceshare-free tiling with the core $\hat{T}$, and T' is the set of all translates of elements in $\hat{T}$ by even integer vectors, then T' is faceshare-free.

We prove Lemma 2.2 by establishing the contrapositive that if T' has facesharing cubes, then T also has facesharing cubes. Let t_1 and t_2 be points in T' that define facesharing cubes. From the definition of T', there must be points $\hat{t_1}$ and $\hat{t_2}$ in $\hat{T}$ such that t_1 is a translate of $\hat{t_1}$ by the even integer vector x_1 and t_2 is a translate of $\hat{t_2}$ by the even integer vector x_2. Since $\hat{t_1}$ and $\hat{t_2}$ are in $\hat{T}$, and $\hat{T}$ is the core of T, $\hat{t_1}$ and $\hat{t_2}$ are also in T. So to show that T has facesharing cubes, it suffices to show that $\hat{t_1}$ and $\hat{t_2}$ define facesharing cubes. Indeed, this follows from the fact that t_1 and t_2 define facesharing cubes.

The details of this proof are largely uninteresting, as most of the work goes into proving facts that are intuitively obvious, but we make two high-level remarks. First, it is not technically necessary to prove Lemma 2.2 after Lemma 2.1. However, there is a practical benefit in doing so, since the fact that T' is a tiling makes the result of Lemma 1 applicable to T'. Lemma 1's result may not seem particularly relevant at the informal level, but it provides some structure that simplifies much of the necessary casework. Second, although our formalization contains several lemmas whose proofs are made longer by our asymmetric in_cube definition, Lemma 2.2 is largely unaffected by this design decision. Of course, some of the arithmetic that is carried out is impacted, but in particular, the proof is made no longer or more tedious by our asymmetric notion of cubes.

Since Definition 4 establishes that T' is periodic, Lemmas 2.1 and 2.2 suffice to prove Lemma 2. Then, from Lemma 2 it follows that Keller's conjecture in d dimensions is equivalent to the related conjecture that there are no d-dimensional faceshare-free periodic tilings. This enables us to restrict our attention to periodic tilings for the remainder of the formalization.

4 Reducing to Keller Graphs

In the previous section, we reduced Keller's conjecture to the claim that there are no faceshare-free periodic tilings in d dimensions. In this section, we extend the reduction to generalized Keller graphs by establishing the following two theorems: First, the existence of a clique with size 2^d in $G_{d,s}$ for any $s \in \mathbb{N}^+$ entails the existence of a d-dimensional faceshare-free tiling. In Section 2, this theorem is formalized as clique_existence_refutes_Keller_conjecture. Second, the existence of a d-dimensional faceshare-free tiling entails the existence of a clique in $G_{d,2^{d-1}}$ with size 2^d. The contrapositive of this theorem is formalized in Section 2 as clique_nonexistence_implies_Keller_conjecture.

4.1 From Cliques to Tilings

Theorem 3. For any $s \in \mathbb{N}^+$, if there exists a clique with size 2^d in the graph $G_{d,s}$, then there exists a faceshare-free tiling in d dimensions.

Let K be a clique in the graph $G_{d,s}$ with size 2^d. Based on K, we construct an explicit faceshare-free periodic tiling in d dimensions. Recall that each vertex in the graph $G_{d,s}$ is a d-length vector with entries in $\{0, 1, \ldots, 2s-1\}$. Then, for each vertex v in $G_{d,s}$, let $\frac{v}{s}$ be the d-dimensional point whose i-th coordinate is $\frac{v[i]}{s}$. With this, we define the set of d-dimensional points $\hat{T}$ as $\{\frac{v}{s} \mid v \in K\}$, and we define T as the set of all translates of elements in $\hat{T}$ by even integer vectors. Technically, $\hat{T}$ may not be the periodic core of T, since the cubes defined by points in $\hat{T}$ may not cover all points in S, but T is periodic.

To prove Theorem 3, it suffices to show that T is a faceshare-free tiling.

Lemma 3.1. If K is a clique with size 2^d in $G_{d,s}$, $\hat{T}$ is the set $\{\frac{v}{s} \mid v \in K\}$, and T is the set of all translates of points in $\hat{T}$ by even integer vectors, then T is a tiling.

We begin by noting that there are no two points in $\hat{T}$ whose cubes intersect. Let p_1 and p_2 be arbitrary distinct points in $\hat{T}$. From the definition of $\hat{T}$, there must exist vertices v_1 and v_2 in K such that $p_1 = \frac{v_1}{s}$ and $p_2 = \frac{v_2}{s}$. In order for the cubes defined by p_1 and p_2 to intersect, it must be the case that for each coordinate i, $|p_1[i] - p_2[i]| < 1$. After substituting in that $p_1 = \frac{v_1}{s}$ and $p_2 = \frac{v_2}{s}$, it follows that for each index i, $|v_1[i] - v_2[i]| < s$. But since v_1 and v_2 are vertices in K, they must be adjacent in $G_{d,s}$, meaning there must be some index i such that $v_1[i] = v_2[i] \pm s$. Therefore, it is impossible for the cubes defined by p_1 and p_2 to intersect, meaning $\hat{T}$ does not contain any points defining intersecting cubes. Using the definition of T, this argument can readily be extended to show that T also does not contain any points defining intersecting cubes.

Since T does not contain any points defining intersecting cubes, Lemma 3.1 reduces to the claim that for every point p in d-dimensional space, there is some point $t \in T$ which defines a cube containing p. To show this, we introduce the notions of s-cubelets and core s-cubelets.

Definition 5 (s-Cubelet). For any $s \in \mathbb{N}^+$, an s-cubelet is an axis-aligned cube with edge length $\frac{1}{s}$. As with cubes, we define each s-cubelet by its unique corner with minimal value along each coordinate, and we adopt the convention that between two opposite faces of an s-cubelet, only the face on the lower coordinate is included as part of the s-cubelet.

Definition 6 (Core s-Cubelet). An s-cubelet is a core s-cubelet if the d-dimensional point that defines it can be expressed as $\frac{v}{s}$ where v is a vertex in $G_{d,s}$.

Note that there are exactly as many core s-cubelets as there are vertices in $G_{d,s}$, meaning there are $(2s)^d$ core s-cubelets. Additionally, for every point p, there is some s-cubelet that contains p and is an even integer translate of some core s-cubelet. This s-cubelet can be defined explicitly by taking each coordinate of p, multiplying it by s, taking the floor of the result, and dividing that floor by s. So the resulting i-th coordinate of the s-cubelet that contains p is $\frac{\lfloor p[i] \cdot s \rfloor}{s}$. In our formalization, we prove that this s-cubelet both contains p and is an even integer translate of some core s-cubelet.

Having defined s-cubelets and core s-cubelets, we note the following about translates of s-cubelets by even integer vectors. Let t be a point that defines a cube, let q be a point that defines an s-cubelet, and let x be an even integer vector. If every point in the s-cubelet defined by q is contained in the cube defined by t, then every point in the s-cubelet defined by $q + x$ is in the cube defined by $t + x$. Consequently, to show that every point p is covered by a cube in T, it suffices to show that for every core s-cubelet q, there is some $t \in T$ that defines a cube that covers every point in q's s-cubelet. In fact, we will show that for every core s-cubelet q, there is some $t \in \hat{T}$ that defines a cube that covers every point in q's s-cubelet.

To show that the cubes defined by $\hat{T}$ collectively cover every core s-cubelet, we make three observations. First, as noted previously, there are $(2s)^d$ total core s-cubelets. Second, there are 2^d points in $\hat{T}$. This follows from the definition of $\hat{T}$ and the fact that K is a clique with size 2^d. Finally, each point in $\hat{T}$ defines a cube that contains s^d distinct core s-cubelets. Notably, the truth of this fact critically relies on the convention we chose to resolve which faces belong to an s-cubelet. Although the convention we chose to resolve which faces belong to a cube was arbitrary, it is necessary that the convention for s-cubelets aligns with it. Had we chosen a different convention, then there would be s^d s-cubelets that mostly fit in each unit cube, but some nonzero number of s-cubelets that are not fully contained in the cube because of points on a shared face that are counted as part of the s-cubelet but not part of the cube.

With these three facts, we can see that the cubes defined by $\hat{T}$ collectively cover every point in every core s-cubelet. Since there are 2^d points in $\hat{T}$, and each point defines a cube that covers each point in s^d core s-cubelets, and no two cubes defined by points in $\hat{T}$ intersect, the points in $\hat{T}$ must define cubes that collectively cover $2^d s^d$ distinct core s-cubelets. This is exactly how many core s-cubelets there are in total, so every point in every core s-cubelet must be covered by a cube defined by a point in $\hat{T}$. This is sufficient to prove that T is a tiling.

Lemma 3.2. If K is a clique with size 2^d in $G_{d,s}$, $\hat{T}$ is the set $\{\frac{v}{s} \mid v \in K\}$, and T is the set of all translates of points in $\hat{T}$ by even integer vectors, then T is faceshare-free.

To show that T is faceshare-free, we begin by showing that $\hat{T}$ is faceshare-free. Let p_1 and p_2 be arbitrary distinct points in $\hat{T}$. From the definition of $\hat{T}$, there must exist vertices v_1 and v_2 in K such that $p_1 = \frac{v_1}{s}$ and $p_2 = \frac{v_2}{s}$. In order for p_1 and p_2 to faceshare, there must be some coordinate i for which $p_1[i] = p_2[i] \pm 1$ and for all coordinates j such that $j \neq i$, $p_1[j] = p_2[j]$. After substituting in that $p_1 = \frac{v_1}{s}$ and $p_2 = \frac{v_2}{s}$, it follows that $v_1[i] = v_2[i] \pm s$ and for all j such that $j \neq i$, $v_1[j] = v_2[j]$. From this, we can see that v_1 and v_2 are not adjacent in $G_{d,s}$, since adjacency requires not only that there is some i such that $v_1[i] = v_2[i] \pm s$, but also there is a j such that $j \neq i$ and $v_1[j] \neq v_2[j]$. So if p_1 and p_2 faceshare, then the vertices v_1 and v_2 that define them cannot be adjacent in $G_{d,s}$. Since $\hat{T}$ consists only of points defined from vertices in the clique K, every pair of distinct points taken from $\hat{T}$ must be defined by adjacent vertices. This demonstrates that no pair of distinct points in $\hat{T}$ can faceshare. Using the definition of T, this argument can readily be extended to show that T itself is faceshare-free.

4.2 From Tilings to Cliques

In Section 3, we showed that if there is a faceshare-free tiling in d dimensions, then there must also be a faceshare-free periodic tiling in d dimensions. In this section, we extend this reduction to show that if there is a faceshare-free tiling in d dimensions, then there must be a clique with size 2^d in $G_{d,2^{d-1}}$. But before we can complete the transformation from faceshare-free tilings to cliques in generalized Keller graphs, we must take another intermediate step by showing that if there is a faceshare-free periodic tiling in d dimensions, then there must also be a d-dimensional faceshare-free periodic tiling whose core has a particular structure.

4.2.1 s-Vertex Cores.

Definition 7 (*s-Vertex Core*). Let $\hat{T}$ be the periodic core of some d-dimensional tiling T, and let $s \in \mathbb{N}^+$. $\hat{T}$ is an s-vertex core if, for each $t \in \hat{T}$ and each coordinate i, $s \cdot t[i] + s - 1 \in \{0, 1, \ldots, 2s - 1\}$.

The intuition guiding Definition 7 is that in order to construct cliques in $G_{d,s}$ from faceshare-free periodic tilings, we will need an injective function that maps cubes in the tiling's core to vertices in $G_{d,s}$. With the right choice of function, we will be able to show that the function's image is actually a clique in $G_{d,s}$, and since there are 2^d cubes in the core of any tiling, the clique will have a size of 2^d as well. Although it would be possible to construct such a function for any faceshare-free periodic core, it is substantially more straightforward to do so if the core is an s-vertex core. This motivates our next lemma.

Definition 8 (*s-Discrete Tiling*). Let T be a d-dimensional periodic tiling, and let $s \in \mathbb{N}^+$. T is s-discrete if, for every coordinate i, the set $\{t[i] \mid t \in T\}$ has at most s distinct values mod 1.

Lemma 4. If there exists any s-discrete faceshare-free periodic tiling in d dimensions, then there also exists a d-dimensional faceshare-free periodic tiling with an s-vertex core.

We begin by defining an operation on tilings that allows us to modify particular coordinates that appear in the tiling without disrupting properties that are of interest to us. Given a d-dimensional tiling T, axis i, and values $a, b \in \mathbb{R}$, let $\text{shift}(T, i, a, b)$ be defined as $\{t \in T \mid t[i] \not\equiv a \mod 1\} \cup \{t' \mid \exists t \in T, t[i] \equiv a \mod 1 \land t'[i] = t[i] + b \land \forall j \in \mathbb{N}, 0 \leq j < d \to j \neq i \to t'[j] = t[j]\}$. So if $t \in T$ and t is not congruent to a mod 1 along the i-th axis, then t will also be in $\text{shift}(T, i, a, b)$. Simultaneously, if $t \in T$ and t is congruent to a mod 1 along the i-th axis, then t will be replaced by t' in $\text{shift}(T, i, a, b)$ where t' is identical to t except that its i-th coordinate is shifted by b. In other words, the shift operation adjusts all points in T that are congruent to a mod 1 by the value b.

Ultimately, we will use a finite number of applications of this operation to transform an arbitrary s-discrete faceshare-free periodic tiling to a faceshare-free periodic tiling with an s-vertex core. The values for a and b used throughout this process can be chosen to ensure that the resulting set has an s-vertex core, but to guarantee that the resulting set is a faceshare-free periodic tiling, we need the following lemma.

Lemma 4.1. Let T be an s-discrete faceshare-free periodic tiling in d dimensions. Then for any axis i and values $a, b \in \mathbb{R}$ such that there is no $t \in T$ with $t[i] \equiv a + b \mod 1$, $\text{shift}(T, i, a, b)$ is an s-discrete faceshare-free periodic tiling in d dimensions.

The primary observation needed to prove Lemma 4.1 is that for any point p, if the cube that contains p is congruent to a mod 1 on its i-th coordinate, then every cube that intersects the line parallel to the i-th axis drawn through p will also have an i-th coordinate which is congruent to a mod 1. This follows directly from Lemma 1. So the informal reason that $\text{shift}(T, i, a, b)$ will still be an s-discrete faceshare-free periodic tiling is that there are no instances of individual cubes being shifted to a position that causes gaps, intersections, or facesharing. For each cube in the tiling, either the cube will not shift and no cube in its column along the i-th axis will shift either, or the cube will shift and every other cube in its column along the i-th axis will also shift by the same amount. In either case, it can be proven that if there are no gaps, intersections, or facesharing before the shift, then no gaps, intersections, or facesharing will result from the shift.

With Lemma 4.1, we can iteratively apply the shift operation to an initial s-discrete faceshare-free periodic tiling T to obtain a faceshare-free periodic tiling whose coordinates in each dimension are shifted to elements of $\{\frac{0}{s}, \frac{1}{s}, \ldots, \frac{s-1}{s}\}$ mod 1. The constraint that T is s-discrete both determines the size of $\{\frac{0}{s}, \frac{1}{s}, \ldots, \frac{s-1}{s}\}$ and guarantees that shift only needs to be applied finitely many times. In particular, if T is s-discrete, then for every dimension, at most s applications of shift are necessary to fix all coordinates along the current dimension. If there are fewer than s distinct values mod 1 along some coordinate, or if T already has cubes with coordinates congruent to elements of $\{\frac{0}{s}, \frac{1}{s}, \ldots, \frac{s-1}{s}\}$ mod 1, then fewer than s applications of shift are needed, but this is no issue. Having obtained a faceshare-free periodic tiling whose coordinates in each dimension are congruent to elements of $\{\frac{0}{s}, \frac{1}{s}, \ldots, \frac{s-1}{s}\}$ mod 1, we can show that this tiling has an s-vertex core, completing the proof of Lemma 4.

4.2.2 From s-Vertex Cores to Cliques.

Lemma 5. If there exists an s-discrete faceshare-free periodic tiling in d dimensions, then there exists a clique with size 2^d in the graph $G_{d,s}$.

From Lemma 4, we know that if there exists an s-discrete faceshare-free periodic tiling in d dimensions, then there must also exist a d-dimensional faceshare-free periodic tiling with an s-vertex core. Let this tiling be T, and let its s-vertex core be $\hat{T}$. We claim that $K = \{v \in \mathbb{R}^d \mid \exists t \in \hat{T}, \forall i \in \mathbb{N}, 0 \leq i < d \to v[i] = s \cdot t[i] + s - 1\}$ is not only a set of vertices in $G_{d,s}$, but is also a clique with size 2^d.

To show that K is a set of vertices in $G_{d,s}$, it suffices to show that for any $v \in K$ and any coordinate i, $v[i] \in \{0, 1, \ldots 2s - 1\}$. This property is guaranteed by the fact that $\hat{T}$ is an s-vertex core. To show that K has size 2^d, we note that we can view K as the image of an injective function mapping points in $\hat{T}$ to vertices in $G_{d,s}$. Our presentation of K and s-vertex cores differs from Brakensiek et al.'s specifically to make this fact more explicit and easy to prove [1]. Since there are 2^d elements in $\hat{T}$, it follows that there must also be 2^d elements in K. So all that remains to prove Lemma 5 is establishing that K is a clique.

Let v_1 and v_2 be two arbitrary distinct elements of K. To show that K is a clique, it suffices to show that v_1 and v_2 are adjacent in $G_{d,s}$. Note that from the definition of K, there exists $t_1, t_2 \in \hat{T}$ such that for each coordinate i, $v_1[i] = s \cdot t_1[i] + s - 1$ and $v_2[i] = s \cdot t_2[i] + s - 1$. When we substitute these values into the definition of adjacency in $G_{d,s}$ and simplify the resulting expressions, we obtain that v_1 and v_2 are adjacent if and only if there are coordinates i and j such that $i \neq j$, $t_1[i] = t_2[i] \pm 1$, and $t_1[j] \neq t_2[j]$.

To show that there is some coordinate i such that $t_1[i] = t_2[i] \pm 1$, consider the even integer vector x defined as follows. For each coordinate i, let $x[i]$ equal 0 if $|t_1[i] - t_2[i]| < 1$, let $x[i]$ equal 2 if $t_2[i] \geq t_1[i] + 1$, and let $x[i]$ equal -2 otherwise (i.e. if $t_1[i] \geq t_2[i] + 1$). Since T is a periodic tiling and $t_1 \in T$, it follows that $t_1 + x$ is also in T. Note that since t_1 and t_2 are both part of the core of T, it is impossible that any of their coordinates differ by 2 or more. From this fact and our definition of x, we can prove that for each coordinate i, $|(t_1+x)[i] - t_2[i]| \leq 1$. If there were no coordinate i such that $t_1[i] = t_2[i] \pm 1$, then it would follow that for each coordinate i, $|(t_1 + x)[i] - t_2[i]| < 1$. In other words, the cubes defined by $t_1 + x$ and t_2 would intersect. Since T is a tiling, it cannot contain points defining intersecting cubes, so there must be some coordinate i such that $t_1[i] = t_2[i] \pm 1$.

To show that, in addition to there being some coordinate i such that $t_1[i] = t_2[i] \pm 1$, there must also be a distinct coordinate j at which t_1 and t_2 differ, assume for the sake of contradiction that t_1 and t_2 agree on all coordinates except i. Then it immediately follows that t_1 and t_2 share a $(d-1)$-dimensional face. Since T is faceshare-free, this cannot occur, so there must be some coordinate j such that $j \neq i$ and $t_1[j] \neq t_2[j]$.

Altogether, this demonstrates that there is some i and j such that $i \neq j$, $t_1[i] = t_2[i] \pm 1$, and $t_1[j] \neq t_2[j]$. When we carry these facts through the definitions of v_1 and v_2, we obtain that there is some i and j such that $i \neq j$, $v_1[i] = v_2[i] \pm s$, and $v_1[j] \neq v_2[j]$. In other words, v_1 and v_2 are adjacent in $G_{d,s}$, and therefore, K is a clique in $G_{d,s}$. This completes the proof of Lemma 5.

Both Lemmas 4 and 5 require a d-dimensional tiling that is not only faceshare-free and periodic, but is also s-discrete. Although we established in Section 3 that the existence of any d-dimensional faceshare-free tiling implies the existence of a d-dimensional faceshare-free periodic tiling, we have yet to establish any claims about whether such a tiling must be s-discrete.

Lemma 6. All d-dimensional periodic tilings are (2^{d-1})-discrete.

Let T be an arbitrary d-dimensional periodic tiling. By Definition 4, T must have a periodic core $\hat{T}$. By Definition 3, for each $t \in T$, there must exist some $t' \in \hat{T}$ and even integer vector x such that $t = t' + 2x$. Note that each coordinate of t is congruent to the corresponding coordinate of t' mod 1, meaning for each coordinate there can only be as many distinct values mod 1 in T as there are in $\hat{T}$. Already, this establishes that T is 2^d-discrete, since there are only 2^d points in $\hat{T}$, and therefore, for each coordinate there can be at most 2^d distinct values mod 1 in $\hat{T}$.

To show that T is (2^{d-1})-discrete, we note that for each coordinate i and each $t \in \hat{T}$, there must be some $t' \in \hat{T}$ such that $t[i] \equiv t'[i] \mod 1$. To see this, consider an arbitrary $t \in \hat{T}$. Since $\hat{T}$ is a core, there must be some point p whose coordinates are each 0 or 1 that is contained in the cube defined by t. Let p' be defined as the point equal to p at all coordinates except the i-th coordinate and whose i-th coordinate is $1 - p[i]$. From our construction, p' is also a point whose coordinates are each 0 or 1, so there must be some $t' \in \hat{T}$ that defines a cube containing p'. Note that both the cube defined by t and the cube defined by t' intersects the line parallel to the i-th axis connecting p and p'. By Lemma 1, it follows that $t[i] \equiv t'[i] \mod 1$. Since for each coordinate

i, every element of $\hat{T}$ can be paired with another element in $\hat{T}$ whose i-th coordinate is congruent mod 1, it follows that T is not only 2^d-discrete, but is (2^{d-1})-discrete.

With Lemma 6, we finally have the machinery required to prove that the truth of Keller's conjecture in d dimensions reduces to the nonexistence of any clique with size 2^d in $G_{d,2^{d-1}}$.

Theorem 7. If there is a faceshare-free tiling in d dimensions, then there is a clique with size 2^d in the graph $G_{d,2^{d-1}}$.

By Lemma 2, the existence of a d-dimensional faceshare-free tiling entails the existence of a d-dimensional faceshare-free periodic tiling. By Lemma 6, this tiling must be (2^{d-1})-discrete. By Lemma 5, this entails the existence of a clique with size 2^d in the graph $G_{d,2^{d-1}}$, as desired.

5 Verifying the Clique in G_8

Theorem 3 from the previous section establishes that to refute Keller's conjecture in d dimensions, it suffices to prove the existence of a clique with size 2^d in the graph $G_{d,s}$ for any $s \in \mathbb{N}^+$. When $s = 2$, $G_{d,s} = G_d$, so this shows that refuting Keller's conjecture in d dimensions only requires proving the existence of a clique with size 2^d in the graph G_d. In 2002, Mackey published an explicit example of such a clique for $d = 8$ [11]. In this section, we verify that Mackey's example is indeed a clique and use that result in conjunction with Theorem 3 to obtain an end-to-end verified proof that Keller's conjecture is false in 8 dimensions.

5.1 Defining the Clique

We begin by defining the function mk_vertex as a convenient shorthand for indicating vertices in G_8. Since vertices in G_8 are vectors of length 8 whose entries are all elements in $\{0, 1, 2, 3\}$, mk_vertex takes as input 8 elements of $\{0, 1, 2, 3\}$ to produce a vector of length 8. Using this function, we define Mackey's clique as a finite set obtained from a hardcoded list of vertices. The fact that the resulting finite set has a cardinality of 256 can be proven just by unfolding definitions and applying computational reductions. Conveniently, Lean's reflexivity tactic refl can do these things automatically, so the proof that our defined clique has the appropriate cardinality is one line long.

```
def mk_vertex (a b c d e f g h : fin (4)) :
    vector (fin 4) 8 :=
  ⟨[a, b, c, d, e, f, g, h], by refl⟩

def clique : finset (vector (fin 4) 8) :=
  list.to_finset
    [mk_vertex 3 1 1 1 0 2 1 1,
     mk_vertex 0 0 0 0 0 0 0 0, ...]

lemma clique_size : clique.card = 256 :=
  by refl
```

After providing a clique with the type finset (vector (fin 4) 8), and a proof that its cardinality is 2^8, all that remains to complete has_clique (Keller_graph 8 2) (2^8) is proving that for any two vertices v1, v2 ∈ clique, v1 ≠ v2 implies that v1 and v2 are adjacent in (Keller_graph 8 2).

5.2 Proving Distinct Vertices Are Adjacent

Given two specific vertices in G_8, it is straightforward to prove whether they are adjacent. Recall that v_1 and v_2 are adjacent in G_8 if and only if there are indices i and j such that $i \neq j$, $v_1[i] = v_2[i] \pm 2$, and $v_1[j] \neq v_2[j]$. So for the particular vertices v_1 = mk_vertex 3 1 1 1 0 2 1 1 and v_2 = mk_vertex 0 0 0 0 0 0 0 0, we can prove that v_1 and v_2 are adjacent by identifying j as coordinate 0 and i as coordinate 5. Indeed, $5 \neq 0$, $v_1[5] = v_2[5]+2$ because $2 = 0+2$, and $v_1[0] \neq v_2[0]$ because $3 \neq 0$. This argument is readily formalizable in Lean, as seen in Figure 1.

```
1  lemma case0_1 (v1 v2 : vector (fin (2 * 2)) 8)
2    (v1_in_clique : v1 = mk_vertex 3 1 1 1 0 2 1 1)
3    (v2_in_clique : v2 = mk_vertex 0 0 0 0 0 0 0 0)
4    (v1_ne_v2 : v1 ≠ v2) :
5    (∃ (i : fin 8),
6      ↑(v1.nth i) = ↑(v2.nth i) + 2 ∧
7      ∃ (j : fin 8), v1.nth j ≠ v2.nth j ∧ i ≠ j) ∨
8    (∃ (i : fin 8),
9      ↑(v2.nth i) = ↑(v1.nth i) + 2 ∧
10     ∃ (j : fin 8), v2.nth j ≠ v1.nth j ∧ i ≠ j) :=
11 begin
12   left,
13   rw [v1_in_clique, v2_in_clique],
14   repeat{rw mk_vertex},
15   use 5,
16   split,
17   { change 2 = 0 + 2,
18     exact two_eq_zero_add_two,
19   },
20   use 0,
21   split, exact three_ne_zero_fin_four,
22   exact five_ne_zero_fin_eight,
23 end
```

Figure 1. A formalized proof that mk_vertex 3 1 1 1 0 2 1 1 and mk_vertex 0 0 0 0 0 0 0 0 are adjacent in G_8.

The code in Figure 1 simply formalizes the previously described argument. On line 12, the tactic left reduces the disjunctive goal $p \vee q$ to its left proposition p. In this case, since coordinate 5 of v1 is 2 greater than coordinate 5 of v2, left reduces the goal written on lines 5 through 10 to just the portion written on lines 5 through 7. On line 15, use 5 instantiates the variable i with the value 5, and likewise, on line 20, use 0 instantiates the variable j with the value 0. The fact that $v_1[i] = v_2[i] + 2$ because $2 = 0 + 2$ is established on lines 17 and 18, the fact that $v_1[j] \neq v_2[j]$ because $3 \neq 0$ is established on line 21, and the fact that $i \neq j$ because $5 \neq 0$ is established on line 22.

From this example, we can see that the argument demonstrating two specific vertices are adjacent in G_8 is not only readily formalizable, it can be formalized in a structured and procedural manner. Had we chosen different vertices for v_1 or v_2, the resulting proof might instantiate i and j with different values, or it might use right instead of left on line 12, but the basic structure would be the same. Since this argument can be formalized so procedurally, we wrote a script that outputs a proof like the one shown in Figure 1 for every pair of distinct vertices in the G_8 clique. These procedurally generated proofs form the main bulk of our proof that G_8 has a clique of size 2^8.

5.3 Efficiency of Our Verification

Our verification of the G_8 clique is inefficient. Since the verified clique has 256 vertices, our approach requires $256 \cdot 255$ or $65,280$ distinct lemmas to prove that every pair of distinct vertices are adjacent. Even using the fact that adjacency is a symmetric relation, we can only halve this number to 32,640 lemmas. Although our approach is inefficient, it is by no means intractable. On a Mac with a 3.8 GHz processor and 16 GB of RAM, it takes under half an hour to compile and typecheck our proof. This is fast enough that a skeptical reader can easily recompile our proof if they are so inclined, though we note that our public repository[2] includes .olean files caching our results so that readers just interested in examining our proof needn't bother with recompilation.

In correspondence with Carneiro, we have been made aware of a verification of the G_8 clique in Lean 4 that is significantly shorter and faster to run than our own [2]. Compared to our verification which is 659k lines long and takes a bit under half an hour to run, the computational component of Carneiro's verification is just over 100 lines long and takes a couple of seconds to run.

The significant disparity between the efficiency of Carneiro's verification and our own is explained by the fact that Carneiro's proof better utilizes the structure of the G_8 clique being verified. Our approach is naive in the sense that the only fact needed or proven about the clique is that every pair of distinct elements are adjacent. However, there is much more structure to the G_8 clique that can be exploited to obtain a shorter proof. In particular, the clique itself is built up through a block substitution construction that more efficiently represents individual vertices in G_8 as pairs of vertices in G_4 [11]. With this representation, verifying the G_8 clique reduces to proving properties about the small set of G_4 vertices that are composed to produce the G_8 clique.

With his permission, we have included Carneiro's proof in our repository[2], though the proof that is actually used in our verification is the one described in Section 5.2. This is because, although Carneiro's approach is more efficient and elegant than our own, his implementation is written in Lean 4 rather than Lean 3. While it would be possible in principle to implement the key ideas of Carneiro's approach in Lean 3, we found that it was much easier to produce the conceptually simple brute force approach than translate Carneiro's proof.

6 Conclusions and Future Work

In this paper, we have formalized the reduction from Keller's conjecture to a statement about generalized Keller graphs. Additionally, we used this formalized reduction to produce the first verified end-to-end proof that Keller's conjecture is false in 8 dimensions.

A natural extension to this work would be to verify the exact set of dimensions in which Keller's conjecture holds. The proof that a refutation of Keller's conjecture in d dimensions can be transformed to a refutation of Keller's conjecture in $d+1$ dimensions is extremely straightforward, and therefore, likely easy to formalize [10]. So the main work that remains is verifying that Keller's conjecture holds in 7 dimensions.

There are at least three potential paths forward towards verifying the truth of Keller's conjecture in 7 dimensions. First, a verified proof that $G_{7,64}$ has no clique of size 2^7 would be sufficient to complete an end-to-end proof that Keller's conjecture holds in 7 dimensions. If the proof were to be implemented in Lean 3, then it could be immediately plugged into clique_nonexistence_implies_Keller_conjecture to obtain the desired result. Verifying that $G_{7,64}$ has no clique of size 2^7 would likely require modifying the SAT encoding described by Brakensiek et al. to produce a SAT proof that $G_{7,64}$ has no clique of size 2^7, and then formally verifying the correctness of Brakensiek et al.'s SAT encoding [1].

Alternatively, there are a series of papers by Kisielewicz showing that if there are any 7-dimensional faceshare-free periodic tilings, then there must be 7-dimensional faceshare-free periodic tilings that are 3-discrete, 4-discrete, and 6-discrete [7–9]. If any of these results were formalized, then verifying Brakensiek et al.'s SAT encoding would again suffice to verify that Keller's conjecture holds in 7 dimensions. Unlike the previously described approach, this approach does not require modifying Brakensiek et al.'s procedure, because they already provide SAT proofs that $G_{7,3}$, $G_{7,4}$, and $G_{7,6}$ have no 2^7-sized cliques. However, this approach does require formally verifying some of Kisielewicz's results, while the previous approach does not.

Finally, one could obtain an end-to-end verification of the 7-dimensional case without verifying the correctness of Brakensiek et al.'s SAT encoding by instead opting to formalize Debroni et al.'s proof that G_7 has no clique of size 2^7 [4]. The primary hurdle of this approach, besides having to formalize Debroni et al.'s proof, is that it would also require proving that the existence of a 7-dimensional faceshare-free

[2] https://github.com/JOSHCLUNE/Keller_reduction/

periodic tiling implies the existence of a 7-dimensional 2-discrete faceshare-free periodic tiling. Unfortunately, there is only one known proof of this fact, namely, Brakensiek et al.'s proof that no 7-dimensional faceshare-free periodic tiling exists. So the only way this approach might be viable is if a new and relatively simple proof is discovered that demonstrates that the existence of a 7-dimensional faceshare-free periodic tiling implies the existence of a 7-dimensional 2-discrete faceshare-free periodic tiling.

Barring the discovery of a new proof that makes the last approach feasible, it seems that the next step to take towards verifying the exact set of dimensions in which Keller's conjecture holds is formally verifying the correctness of Brakensiek et al.'s SAT encoding.

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