# Deciding Differential Privacy of Online Algorithms with Multiple Variables

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### ABSTRACT

We consider the problem of checking the differential privacy of *online* randomized algorithms that process a stream of inputs and produce outputs corresponding to each input. This paper generalizes an automaton model called DiP automata [10] to describe such algorithms by allowing multiple real-valued storage variables. A DiP automaton is a parametric automaton whose behavior depends on the privacy budget  $\epsilon$ . An automaton  $\mathcal{A}$  will be said to be differentially private if, for some  $\mathfrak{D}$ , the automaton is  $\mathfrak{D}\epsilon$ -differentially private for all values of  $\epsilon > 0$ . We identify a precise characterization of the class of all differentially private DiP automata. We show that the problem of determining if a given DiP automaton belongs to this class is PSPACE-complete. Our PSPACE algorithm also computes a value for  $\mathfrak{D}$  when the given automaton is differentially private. The algorithm has been implemented, and experiments demonstrating its effectiveness are presented.

### CCS CONCEPTS

• Security and privacy  $\rightarrow$  Logic and verification; Formal security models.

# **KEYWORDS**

Differential Privacy, Verification, Automata, Decision procedure

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# **1** INTRODUCTION

Differential privacy [18, 20] is a popular requirement that is demanded of algorithms that analyze data containing sensitive personal information of individuals. A data analysis that meets the high bar of differential privacy guarantees the privacy of individuals. However, ensuring differential privacy is difficult, subtle and errorprone — relatively minor tweaks to correct algorithms can lead to the loss of privacy as demonstrated by the examples in [19, 26]. Though the problem of checking the differential privacy of a program is in general undecidable [2], the importance of the problem has led to extensive investigation in the last 15 years; see Section 8 for a short overview of work in this space.

In this paper, we look at the problem of verifying the differential privacy of online algorithms. An online algorithm is one that processes an unbounded (but finite) stream of inputs, samples from distributions, and produces outputs in response to the inputs. The stream of inputs is a sequence of real numbers that are answers to queries to a database. A novel approach using automata to describe and study such algorithms was proposed in [10]. It was shown that checking differential privacy of algorithms described by such automaton is in linear time. Remarkably the verification procedure in [10] checks some properties of the underlying graph of the automaton and does not explicitly reason about probabilities. However, the automaton model in [10] has one serious limitation — only one storage variable is available, and hence only one previously sampled value can be remembered.

*Contributions.* We extend the line of research initiated in [10] by generalizing the automata model in [10] to allow for multiple real-valued storage variables. A DiP automaton (DiPA for short)<sup>1</sup> is a parametric automaton (depending on privacy budget  $\epsilon$ ) with finitely many control states that process an unbounded (but finite) stream of real values that represent answers to queries asked of a database. A DiPA can sample real values from Laplace distributions whose mean may depend on the value read, and DiPA has finitely many real-valued variables in which they can store values they sample in each step (which in turn depend on the input read). Transitions depend on the current control state, the values stored, and the input read, which influences the values sampled. In response to an input,

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<sup>&</sup>lt;sup>1</sup>Even though the automata model in this paper has the same name as the one in [10], the generalization significantly extends the expressive power of the model.

they produce an output that is either a symbol from a finite set or a real number.

We show that, even in the case of automata with multiple storage variables, the problem of determining whether a given DiPA  $\mathcal{A}$  is  $\mathfrak{D}\epsilon$ -differentially private for some constant  $\mathfrak{D} > 0$  (independent of  $\epsilon$ ) and all  $\epsilon > 0$ , can be reduced to checking graph-theoretic conditions. These conditions demand the absence of certain paths, cycles, and interactions among them. However, unlike the single variable automata case [10], these paths and cycles cannot be captured only by considering the underlying graph of the automata. Instead, we use an auxiliary graph to capture these undesirable paths and cycles precisely. This is a non-trivial extension of [10]; for a more detailed comparison with [10], see Section 8. An automaton  $\mathcal{A}$  is said to be well-formed if it does not have any of these undesirable paths or cycles. We show that a well-formed DiPA is differentially private; thus, well-formedness is a sufficient condition to guarantee privacy. Conversely, we show that if additionally, for every state of  $\mathcal{A}$ , the transitions of  $\mathcal{A}$  from that state have distinct outputs (called output distinct), then well-formedness is also necessary to guarantee differential privacy. In other words, a DiPA  $\mathcal{A}$ , having distinct outputs on transitions from any state, that is differentially private is well-formed. These proofs of necessity and sufficiency require novel ideas that are a significant extension of the techniques presented in [10]; once again see Section 8 for more details.

Next, we show that there is a PSPACE algorithm that checks if a DiPA  $\mathcal{A}$  is well-formed. This algorithm additionally computes a value for  $\mathfrak{D}$  that shows that  $\mathcal{A}$  is  $\mathfrak{D}\epsilon$ -differentially private for all  $\epsilon$ . We also show that checking differential privacy of outputdistinct DiPA is PSPACE-hard, thus establishing the optimality of our verification algorithm.

We have implemented our algorithm in a tool called DiPAut. Our experiments show that the approach scales and that our algorithm produces known estimates for  $\mathfrak{D}$ . It successfully proves differential privacy and identifies violations of privacy in various examples. The tool is evaluated for scalability with respect to both the number of states and variables. Despite the PSPACE-hardness, the tool is able to perform well in our experiments. We compare DiPAut with CheckDP [29], a state-of-the-art tool to check differential privacy. DiPAut significantly outperforms CheckDP in all our experiments. The tool DiPAut is available to download at [7].

*Organization.* The rest of the paper is organized as follows. Section 2 introduces basic notation and definitions used in the paper. Our model of DiP automaton extended with multiple variables is introduced in Section 3. Section 4 defines well-formed DiPA, which is a (almost) precise characterization of differentially private automata. We show that well-formed automata are differentially private in Section 5; and show that checking well-formedness is PSPACE-complete. Section 6 shows that differentially private automata that have distinct outputs on transitions are well-formed. PSPACE-hardness of checking differential privacy is also presented in this section. Experimental results are presented in Section 7. Closely related work is discussed in Section 8. We discuss on the restrictions placed on the automata and the adjacency relations used in the paper. Finally we present our conclusions (Section 10).

The missing proofs are in the accompanying Appendix. This is the author's version of the paper. It is posted here for your personal use. Not for redistribution. The definitive version will be published in the Proceedings of the annual ACM Computer and Communications Security Conference (CCS' 2023).

#### 2 PRELIMINARIES

The definitions and notations in this section are borrowed from [10]. Let  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^{\geq 0}, \mathbb{R}, \mathbb{R}^{>0}$  denote the set of natural numbers, integers, rational numbers, non-negative rationals, real numbers, and positive real numbers, respectively. In addition,  $\mathbb{R}_{\infty}$  will denote the set  $\mathbb{R} \cup \{-\infty, \infty\}$ , where  $-\infty$  is the smallest and  $\infty$  is the largest element in  $\mathbb{R}_{\infty}$ . For a real number  $x \in \mathbb{R}, |x|$  denotes its absolute value.

Sequences. For a set  $\Sigma$ ,  $\Sigma^*$  denotes the set of all finite sequences/strings over  $\Sigma$ . We use  $\lambda$  to denote the empty sequence/string over  $\Sigma$ . For two sequences/strings  $\rho, \sigma \in \Sigma^*$ , we use their juxtaposition  $\rho\sigma$  to indicate the sequence/string obtained by concatenating them in order. Consider  $\sigma = a_0a_1 \cdots a_{n-1} \in \Sigma^*$  (where  $a_i \in \Sigma$ ). We use  $|\sigma|$  to denote its length *n* and use  $\sigma[i]$  to denote its *i*th symbol  $a_i$ . The substring  $a_ia_{i+1} \cdots a_{j-1}$  from position *i* (inclusive) to *j* (not inclusive) will be denoted as  $\sigma[i : j]$ ; if  $j \leq i$  then  $\sigma[i : j] = \lambda$ . Thus,  $\sigma[0 : |\sigma|] = \sigma$ . The suffix starting at position *j* will be denoted as  $\sigma[j :]$ , i.e.,  $\sigma[j :] = \sigma[j : |\sigma|]$ . For any partial function  $f : A \hookrightarrow B$ , where *A*, *B* are some sets, we let dom(*f*) be the set of  $x \in A$  such that f(x) is defined.

*Laplace Distribution.* Differential privacy mechanisms often add noise by sampling values from the *Laplace distribution*. The distribution, denoted Lap( $k, \mu$ ), is parameterized by two values:  $k \ge 0$  which is called the scaling parameter, and  $\mu$  which is the mean. The probability density function of Lap( $k, \mu$ ), denoted  $f_{k,\mu}$ , is given by  $f_{k,\mu}(x) = \frac{k}{2}e^{-k|x-\mu|}$ , where e is the Euler constant.

Differential Privacy. Differential privacy [18] is a framework that enables statistical analysis of databases containing sensitive, personal information of individuals while ensuring that the privacy of individuals is not adversely affected by the results of the analysis. In the differential privacy framework, a randomized algorithm, M, called the *differential privacy mechanism*, mediates the interaction between a (possibly dishonest) data analyst asking queries and a database D responding with answers. Queries are deterministic functions and typically include aggregate questions about the data, like the mean etc. In response to such a sequence of queries, Mresponds with a series of answers computed using the actual answers from the database and random sampling, resulting in "noisy" answers. Thus, M provides privacy at the cost of accuracy. Typically, M's noisy response depends on a *privacy budget*  $\epsilon > 0$ .

Differential privacy captures the privacy guarantees for individuals whose information is in the database D. For an individual i, let  $D \setminus \{i\}$  denote the database where i's information has been removed. A secure mechanism M ensures that for any individual i in D, and any sequence of possible outputs  $\overline{o}$ , the probability that M outputs  $\overline{o}$  on a sequence of queries is approximately the same whether the interaction is with the database D or with  $D \setminus \{i\}$ . To capture this definition formally, we need to characterize the inputs on which M is required to behave similarly. Inputs to a differential privacy mechanism can be seen as answers from the database to a sequence of queries asked by the data analyst. If queries are aggregate queries, then answers to q on D and  $D \setminus \{i\}$  (for individual i) are likely to be away by at most 1. <sup>2</sup> This intuition leads to the following oftenused definition of *adjacency* that characterizes inputs on which the differential privacy mechanism M is expected to behave similarly; for example this definition is used in SVT [1, 17, 19, 20, 26] and NumericSparse [20].<sup>3</sup> We assume that at each step, the differential privacy mechanism either gets a real number as input (answer to an aggregate query) or is asked to respond without an answer from the database which is encoded as  $\tau$ .

**Definition 1.** Sequences  $\rho, \sigma \in (\mathbb{R} \cup \{\tau\})^*$  are *adjacent* if  $|\rho| = |\sigma|$  and for each  $i \leq |\rho|$  (a)  $\rho[i] \in \mathbb{R}$  iff  $\sigma[i] \in \mathbb{R}$  and (b) if  $\rho[i] \in \mathbb{R}$  then  $|\rho[i] - \sigma[i]| \leq 1$ .

We are now ready to formally define the notion of privacy which uses Definition 1. In response to a sequence of inputs, a differential privacy mechanism produces a sequence of outputs from the set (say)  $\Gamma$ . Since a differential privacy mechanism *M* is a randomized algorithm, it will induce a probability distribution on  $\Gamma^*$ .

**Definition 2** ( $\epsilon$ -differential privacy). A randomized algorithm M with input in  $(\mathbb{R} \cup \{\tau\})^*$  and output in  $\Gamma^*$  is said to be  $\epsilon$ -differentially *private* if for all measurable sets  $S \subseteq \Gamma^*$  and adjacent  $\rho, \sigma \in \mathbb{R}^*$  (Definition 1),

$$\operatorname{Prob}[M(\rho) \in S] \leq e^{\epsilon} \operatorname{Prob}[M(\sigma) \in S]$$

**Input:** *q*[1 : *N*] **Output:** *out* [1 : *N*] low  $\leftarrow Lap(\frac{\epsilon}{4}, T_{\ell})$ high  $\leftarrow \text{Lap}(\frac{\epsilon}{4}, T_u)$ for  $i \leftarrow 1$  to N do  $r \leftarrow Lap(\frac{\epsilon}{4}, q[i])$ **if**  $(r \ge low) \land (r < high)$  **then**  $out[i] \leftarrow \bot$ else if  $(r \ge low) \land (r \ge high)$  then  $out[i] \leftarrow \top_1$ exit else if  $(r < low) \land (r < high)$  then  $out[i] \leftarrow \top_2$ exit end end

Algorithm 1: Range query algorithm

**Example 1.** Consider the following problem. Given a sequence of answers to queries (array q[1:N]) and an interval  $[T_{\ell}, T_u)$  given by thresholds  $T_{\ell}$  and  $T_u$ , determine the first time a query answer lies outside this interval; indicate (through the output) whether the query answer is  $\geq T_u$  or  $\leq T_{\ell}$  at this point. A differentially

private algorithm to solve this problem is shown as Algorithm 1. The algorithm starts by adding noise to both  $T_{\ell}$  and  $T_u$  to get a perturbed interval defined by numbers low and high. After that the algorithm perturbs each query answer and stores the result in r, and then checks if r lies between low and high. If it does, the algorithm outputs  $\perp$  and processes the next query answer. Otherwise, if r is larger than both low and high it outputs  $\top_1$  and stops. On the other hand, if r is less than both low and high then it outputs  $\top_2$  and halts. The algorithm's behavior depends on the value of  $\epsilon$ . It can be shown that for each value of  $\epsilon$ , the algorithm for that value of  $\epsilon$  is  $\epsilon$ -differentially private.

# 3 DIPA

DiP (**D**ifferentially **P**rivate) automata (DiPAs for short) are an automata-based model introduced in [10] to describe some differential privacy mechanisms. They process an input string  $\sigma \in (\mathbb{R} \cup \{\tau\})^*$  by sampling values from the Laplace distribution, using real variables to store information during the computation, and producing a sequence of outputs. The model introduced in [10] had only *one* storage variable. In this paper, we generalize this model naturally to allow *multiple* real-valued storage variables. However, as discussed in Section 8, both the characterization of differentially private algorithms described by them and the proofs of decidability are a non-trivial extension of the single variable model.

# 3.1 Syntax

A DiP automaton is a *parametric* automaton whose behavior depends on a parameter  $\epsilon$  (the privacy budget). It has finitely many control states and finitely many real-valued variables  $x_1, x_2, \ldots x_k$  that are used to store information during the computation. At each step, the automaton freshly samples two real values from Laplace distributions whose parameters depend on  $\epsilon$ , and these sampled values are stored in the (additional) variables insample and insample'. Given an input  $\sigma \in (\mathbb{R} \cup {\tau})^*$ , a DiPA does the following in each step.

- Two values are drawn from the distributions Lap(dε, μ) and Lap(d'ε, μ') and stored in the variables insample and insample', respectively. The scaling factors d, d' and means μ, μ' of these distributions depend on the current state.
- (2) The states of the automaton are partitioned into *input* states and *non-input* states. At a non-input state, the automaton expects to read τ from the input. On the other hand, at an input state, it expects to read a real number, say *a*, and it updates insample and insample' by adding *a* to them. The properties of the Laplacian distribution imply that the distribution of insample + *a* (insample' + *a*) is the same as the distribution of Lap(*d*ε, μ + *a*) (Lap(*d*ε, μ' + *a*) respectively).
- (3) A transition changes the control state and outputs a value. The value output could either be a symbol from a finite set or one of the two real numbers insample and insample' that are sampled in this step. At an input state, the transition is guarded by a Boolean condition that depends on the result of comparing the sampled value insample with the stored values  $x_i$  ( $1 \le i \le k$ ). It is possible that for certain values of  $x_i$  ( $1 \le i \le k$ ) and insample, no transition is enabled from the current state. In such a case, the computation ends.

 $<sup>^2</sup> The difference in general can be bounded by a constant <math display="inline">\Delta.$ 

<sup>&</sup>lt;sup>3</sup>Please see the discussion of SVT on pages 56 and 57 of [20] and its description on pages 58, 62, and 64. For simplicity, it is assumed that these queries are 1-sensitive. So, by considering SVT as an algorithm that works directly on the sequence of the outputs of queries, we get naturally the adjacency relation used here.

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(4) Finally, the automaton may choose to store the sampled value insample in any of the variables x<sub>i</sub> (1 ≤ i ≤ k).

We now formally define DiP automaton capturing the above intuition. First, some necessary notation. Let  $\mathcal{G}'$  be the set of constraints defined as  $\mathcal{G}' = \{\text{insample} \ge x_i \mid 1 \le i \le k\} \cup \{\text{insample} < x_i \mid 1 \le i \le k\}$ . Let  $\mathcal{G}''$  be the set of conditions formed by taking conjunctions of two or more constraints in  $\mathcal{G}'$  such that both insample  $\ge x_i$  and insample  $< x_i$  don't appear for any  $1 \le i \le k$ . Finally, let  $\mathcal{G} = \{\text{true}\} \cup \mathcal{G}' \cup \mathcal{G}''$ ; these are the constraints that *guard* transitions in a DiPA.<sup>4</sup>

**Definition 3** (DiPA). A *DiP automaton*  $\mathcal{A} = (Q, \Gamma, q_{init}, X, P, \delta)$  where

- *Q* is a finite set of states partitioned into two sets: the set of input states *Q*<sub>in</sub> and the set of non-input states *Q*<sub>non</sub>,
- $\Gamma$  is a finite output alphabet,
- $q_{\text{init}} \in Q$  is the initial state,
- X = {insample, insample'} ∪ {x<sub>i</sub> | 1 ≤ i ≤ k} is the set of variables; we will use stor = {x<sub>i</sub> | 1 ≤ i ≤ k} to denote the *storage* variables,
- $P: Q \to \mathbb{Q}^{\geq 0} \times \mathbb{Q} \times \mathbb{Q}^{\geq 0} \times \mathbb{Q}$  is the parameter function that assigns to each state a 4-tuple  $(d, \mu, d', \mu')$ , where insample is sampled from Lap $(d\epsilon, \mu)$  and insample' is sampled from Lap $(d'\epsilon, \mu')$ ,
- and  $\delta : (Q \times G) \hookrightarrow (Q \times (\Gamma \cup \{\text{insample, insample'}\}) \times \{\text{true, false}^k\}$  is the transition (partial) function that given a current state and the result of comparing each  $x_i$   $(1 \le i \le k)$  with insample, determines the next state, the output, and whether the variables  $x_i$  should be updated to store insample. The output could either be a symbol from  $\Gamma$  or the values insample and insample' that were sampled.

In addition, the transition function  $\delta$  satisfies the following two conditions.

**Determinism:** For any state  $q \in Q$ , if  $\delta(q, c)$  and  $\delta(q, c')$  are defined for  $c, c' \in G$  then either c = c' or  $c \wedge c'$  is unsatisfiable. That is, from any state, at most one transition is enabled at any time.

**Non-input transitions:** From any  $q \in Q_{non}$ , if  $\delta(q, c)$  is defined, then c = true; that is, there is at most one transition from a non-input state which is always enabled.

*Remark.* Although insample' is never used in comparisons, it is nevertheless needed to model examples such as NUM-SPARSE (See [20]). insample' is often used in algorithms when we want to output the noisy input value in a differentially private fashion. Outputting insample instead of insample' can violate differential privacy, as insample may have been used in other comparisons: See the definition of privacy violating path (Definition 11 in Section 4); also [26].

Before concluding this section, it is useful to introduce some notation and terminology for transitions. A quintuple t = (q, c, q', o, b)denotes a transition of  $\mathcal{A}$  if  $\delta(q, c) = (q', o, b)$ , where  $b = (b_1, b_2, \dots b_k) \in \{\text{true}, \text{false}\}^k$ . For such a transition, src(t) = qdenotes the *source*, trg(t) = q' the *target*,  $\text{out}(t) = o \in \Gamma \cup \{\text{insample}, \text{insample}'\}$  the *output*, and guard(t) = c the guard.



Figure 1: DiPA  $\mathcal{A}_{range}$  modeling Algorithm 1. Threshold  $T_{\ell}$  is set to 0 (sampling mean of insample in  $q_0$ ) and  $T_u$  is set to 1 (sampling mean of insample in  $q_1$ ). The guards  $g_1 = (\text{insample } \ge x_1) \land (\text{insample } < x_2), g_2 = (\text{insample } \ge x_1) \land (\text{insample } < x_2), g_2 = (\text{insample } \ge x_1) \land (\text{insample } < x_2).$ 

Based on the guard c and the Booleans b, we can associate the following sets of variables with transition t.

 $smallv(t) = \{x \in stor \mid insample \ge x \text{ is a conjunct of } c\}$  $largev(t) = \{x \in stor \mid insample < x \text{ is a conjunct of } c\}$  $usedv(t) = smallv(t) \cup largev(t)$  $assignv(t) = \{x_i \mid b_i = true\}$  $nonassignv(t) = \{x_i \mid b_i = false\}$ 

Intuitively, smallv(*t*) (largev(*t*)) are the storage variables that lower bound (upper bound) insample if the guard is satisfied; usedv(*t*) are the storage variables that are referenced in the guard of *t*; assignv(*t*) are the variables that are set by *t*; and nonassignv(*t*) are the variables that are left unchanged by *t*. For any *i*, if  $x_i \in assignv(t)$  then *t* sets  $x_i = insample$  during the transition and hence *t* is an *assignment transition* for variable  $x_i$ . Finally, if  $src(t) = q \in Q_{in}$  then *t* is said to be *input transition* and if  $q \in Q_{non}$  then *t* is a *non-input transition*.

**Example 2.** The differential privacy mechanism in Example 1 can be modeled as a DiPA. This is shown in Figure 1. We will use these conventions when drawing DiPAs in this paper. Input states will be represented as circles, while non-input states will be drawn as rectangles. The name of each state is written above the line, while the scaling factor *d* and mean  $\mu$  of the distribution used to sample insample is written below the line. The parameters *d'* and  $\mu'$  for sampling insample' are not shown in the figures, but will be mentioned in the caption and text when they are important; they are relevant only when insample' is output on a transition. Edges will be labeled with the guard of the transition, followed by the output, and a vector of Booleans to indicate which variables insample is stored in.

The working of  $\mathcal{A}_{range}$  in Fig. 1 can be explained as follows. Since insample' is not output in any step, the parameters associated with sampling insample' are not reported. The thresholds  $T_{\ell}$  and  $T_{u}$  are hard-coded as 0 and 1, respectively, as the distribution means for the non-input states  $q_0$  and  $q_1$ . The transition from  $q_0$  to  $q_1$  perturbs  $T_{\ell}$  (= 0) and sets this to variable x<sub>1</sub>; thus, x<sub>1</sub> corresponds to the

<sup>&</sup>lt;sup>4</sup>We could also allow guards of the form insample >  $x_i$  and insample ≤  $x_i$ . However, we chose to keep the presentation simple. As all random variables in a DiPA are noisy, the equality happens with probability 0.

variable low in Algorithm 1. The transition from  $q_1$  to  $q_2$  perturbs  $T_u$  (= 1) and stores it in  $x_2$ . Thus,  $x_2$  corresponds to variable high in Algorithm 1. State  $q_2$  is an input state. Transitions from  $q_2$  perturb the query answer given as input storing it in insample, compare insample to the values stored in  $x_1$  and  $x_2$ , and output the right value accordingly. State  $q_3$  is a halting state where no transitions are enabled.

We conclude this example by illustrating the definitions associated with transitions. The transition *t* from  $q_0$  to  $q_1$  can be denoted by the quintuple  $(q_0, \text{true}, q_1, \bot, (\text{true}, \text{false}))$ . For *t*, we have  $\text{src}(t) = q_0$ ,  $\text{trg}(t) = q_1$ ,  $\text{out}(t) = \bot$ , guard(t) = true,  $\text{smallv}(t) = \text{largev}(t) = \text{usedv}(t) = \emptyset$ ,  $\text{assignv}(t) = \{x_1\}$ , and  $\text{nonassignv}(t) = \{x_2\}$ . In this case *t* is a non-input, assignment transition for variable  $x_1$ . In contrast, the transition *t'* from  $q_2$  to itself, is an input transition that is not an assignment transition for any variable. Here we have  $\text{smallv}(t') = \{x_1\}$ ,  $\text{largev}(t') = \{x_2\}$ , and  $\text{usedv}(t') = \{x_1, x_2\}$ .

#### 3.2 Semantics

An execution/run of a DiPA  $\mathcal{A} = (Q, \Gamma, q_{\text{init}}, X, P, \delta), \rho = t_0 t_1 \cdots t_{n-1}$ , is a sequence of transitions  $t_i$  such that for every 0 < i < n,  $\operatorname{trg}(t_{i-1}) = \operatorname{src}(t_i)$  (i.e., the sequence  $\rho$  corresponds to a path in the "graph" of  $\mathcal{A}$ ). We extend the notation of length, the *i*th transition, sub-sequence and suffix from (general) sequences: thus,  $|\rho| = n, \rho[i] = t_i, \rho[i:j] = t_i \cdots t_{j-1}$  and  $\rho[j:] = t_j t_{j+1} \cdots t_{n-1}$ . We also extend the notion for source and target from transitions to a run  $-\operatorname{src}(\rho) = \operatorname{src}(t_0)$  and  $\operatorname{trg}(\rho) = \operatorname{trg}(t_{n-1})$ . Using the notation developed for transitions, guard( $\rho[i]$ ) is the guard of the *i*th transition  $t_i$  of  $\rho$ . A run  $\rho$  is a cycle if  $\operatorname{src}(\rho) = \operatorname{trg}(\rho)$ , i.e., the run begins and ends in the same state. Finally, given two runs  $\rho_1$  and  $\rho_2$  such that  $\operatorname{trg}(\rho_1) = \operatorname{src}(\rho_2), \rho_1\rho_2$  is the run which is the concatenation of  $\rho_1$  followed by  $\rho_2$ .

Recall that transitions of DiPA  $\mathcal{A}$  compare values stored in the variables  $x_i$  ( $1 \le i \le k$ ) and insample. Thus, to define the semantics of the DiPA, we need to make sure that the value of variable  $x_i$  is defined before it is used in a comparison in the guard of a transition. Therefore, we make the technical assumption that on every run starting from the initial state  $q_{init}$ , a variable is assigned a value before it is referenced in a guard. We assume that all DiPA  $\mathcal{A}$  considered in this paper are *initialized* as defined formally below.

**Initialization:** We say that a DiPA  $\mathcal{A} = (Q, \Gamma, q_{\text{init}}, X, P, \delta)$  is *initialized* if for any run  $\rho$  starting from the initial state  $q_{\text{init}}$  (i.e.,  $\operatorname{src}(\rho) = q_{\text{init}}$ ), if  $\operatorname{guard}(\rho[i])$  references variable  $x_{\ell}$  (i.e.,  $x_{\ell} \in \operatorname{usedv}(\rho[i])$ ) then there is j < i such that  $\rho[j]$  is an assignment transition for  $x_{\ell}$  (i.e.,  $x_{\ell} \in \operatorname{assignv}(\rho[j])$ ).

We need to define one more concept associated with a run  $\rho$ . For any storage variable x and position  $j \in \{0, 1, ... | \rho |\}$ , the *last position* when x was assigned before *j* is the maximum index i < j such that x was assigned on transition  $\rho[i]$ . More precisely,

$$lastassign_{\rho}(\mathbf{x}, j) = max\{i \mid i < j, \mathbf{x} \in assignv(\rho[i])\}.$$
<sup>5</sup>

When the run  $\rho$  is clear from the context, we will drop the subscript and simply refer to the last assigned position before *j* for x as lastassign(x, *j*). To define the semantics of a DiPA  $\mathcal{A}$ , we need to define the probability of "executions". But runs, as defined above, do not have all the information we need. For example, the real numbers read as input determine the values of insample and insample', which in turn determine whether a transition is enabled and what is stored in the variables. Next, on transitions where either insample or insample' are output, to define a meaningful measure space, we need to associate an interval (v, w) in which the output value lies. Thus, we define when a run corresponds to a certain sequence of inputs and outputs.

**Definition 4** (Computation). Consider DiPA  $\mathcal{A} = (Q, \Gamma, q_{\text{init}}, X, P, \delta)$  and a run  $\rho$  of  $\mathcal{A}$ . Let  $\sigma \in (\mathbb{R} \cup \{\tau\})^*$  be an *input sequence* and  $\gamma \in (\Gamma \cup (\mathbb{R}_{\infty} \times \mathbb{R}_{\infty}))^*$  be an *output sequence*. We say that  $\rho$  is a *run on*  $\sigma$  *producing output*  $\gamma$  if the following conditions hold.

- (1)  $|\rho| = |\sigma| = |\gamma|$ .
- (2) For any i, σ[i] = τ iff src(ρ[i]) ∈ Q<sub>non</sub>. That is, symbol τ is only read in non-input states.
- (3) For any i, γ[i] ∈ Γ iff out(ρ[i]) ∈ Γ. Further for such i, out(ρ[i]) = γ[i]. That is, outputs in ρ "match" outputs in γ, with the only difference being that when insample or insample' is output in ρ, the corresponding position in γ is an interval (v, w) ∈ ℝ<sup>2</sup><sub>∞</sub>.

When  $\rho$  is a run on  $\sigma$  producing  $\gamma$ , the tuple  $\kappa = (\rho, \sigma, \gamma)$  will be called a *computation*.

For a computation  $\kappa = (\rho, \sigma, \gamma)$  of DiPA  $\mathcal{A}$ , the suffix starting at position *j* is  $\kappa[j:] = (\rho[j:], \sigma[j:], \gamma[j:])$ . Notice that  $\kappa[j:]$  (for any *j*) is also a computation of  $\mathcal{A}$  since  $\rho[j:]$  is a run on  $\sigma[j:]$  producing  $\gamma[j:]$ . Also, we use length of  $\kappa$ ,  $|\kappa|$  to be  $|\rho| (= |\sigma| = |\gamma|)$ , the length of the run  $\rho$ .

*Probability of Computations.* We will now define what the probability of each computation is. Recall that in each step, the automatom samples two values from Laplace distributions, and if the transition is from an input state, it adds the read input value to the sampled values and compares the result with the values stored in the variables  $x_i$ ,  $1 \le i \le k$ . The step also outputs a value, and if the value output is one of the two sampled values, the computation requires it to belong to the interval that appears in the output sequence. The probability of such a transition thus is the probability of drawing a sample that satisfies the guard of the transition and (if the output is a real value) producing a number that lies in the interval in the output label. This intuition is formalized in a precise definition.

Let us fix a computation  $\kappa = (\rho, \sigma, \gamma)$  of DiPA  $\mathcal{A} = (Q, \Gamma, q_{\text{init}}, X, P, \delta)$ . Recall that stor = { $\mathbf{x}_i \mid 1 \leq i \leq k$ }. Since the parameters of the Laplace distribution that is used to sample insample and insample' depend on the privacy budget  $\epsilon$ , the probability of  $\kappa$  will also depend on  $\epsilon$ . In addition, the values stored in the variables  $\mathbf{x}_i \in \text{stor at the start of the computation also influence the behavior of <math>\mathcal{A}$ . Let  $\eta : \text{stor} \to \mathbb{R}$  be the *evaluation* that defines the values of  $\mathbf{x}_i, 1 \leq i \leq k$ , initially. The probability of  $\kappa$  depends on both  $\epsilon$  and  $\eta$  and is denoted as  $\Pr[\epsilon, \eta, \kappa]$ . We define this inductively on  $|\kappa|$ . For any  $\epsilon$  and any computation  $\kappa$  with  $|\kappa| = 0$ ,  $\Pr[\epsilon, \eta, \kappa] = 1$ .

Let us now consider the case when  $|\kappa| > 0$ . Before defining the probability in this case, we define the parameters that we will need. Let  $P(\operatorname{src}(\kappa[0])) = (d, \mu, d', \mu')$ . Define the value  $a_0$  as follows –

<sup>&</sup>lt;sup>5</sup>As always max  $\emptyset = -\infty$  and min  $\emptyset = \infty$ .

if  $\sigma[0] \in \mathbb{R}$  then  $a_0 = \sigma[0]$ , and if  $\sigma[0] = \tau$  then  $a_0 = 0$ . Next, let us define the values  $\ell$  and u. If  $\gamma[0] \in \Gamma$  then  $\ell = -\infty$  and  $u = \infty$ . Otherwise, if  $\gamma[0] = (v, w)$  then  $\ell = v$  and u = w. Finally, for a parameter z, let  $\eta_z$  be the evaluation that modifies  $\eta$  by setting all the variables assigned by  $\rho[0]$  to z. In other words,

$$\eta_{z}(\mathbf{x}) = \begin{cases} \eta(\mathbf{x}) & \text{if } \mathbf{x} \in \text{nonassignv}(\rho[0]) \\ z & \text{if } \mathbf{x} \in \text{assignv}(\rho[0]) \end{cases}$$

We are now ready to define  $Pr[\epsilon, \eta, \kappa]$  based on whether  $out(\rho[0]) = insample'$  or not.

**Case** out( $\rho[0]$ ) = insample': Set  $\ell' = \max\{\eta(x)|x \in \text{smallv}(\rho[0])\}$ and  $u' = \min\{\eta(x) | x \in \text{largev}(\rho[0])\}$ . Also define p to be the probability that insample'  $\in (\ell, u) = (v, w) = \gamma[0]$ , i.e.,

$$p = \int_{\ell}^{u} \frac{d'\epsilon}{2} e^{-d'\epsilon|z-\mu'-a_0|} dz$$

Then,

$$\Pr[\epsilon, \eta, \kappa] = p \int_{\ell'}^{u'} \left( \frac{d\epsilon}{2} e^{-d\epsilon |z-\mu-a_0|} \right) \Pr[\epsilon, \eta_z, \kappa[1:]] dz$$

**Case** out( $\rho[0]$ )  $\neq$  insample': In other words, either out( $\rho[0]$ )  $\in \Gamma$  or out( $\rho[0]$ ) = insample. In this case set  $\ell' = \max(\{\eta(x) \mid x \in \text{smallv}(\rho[0])\} \cup \{\ell\})$  and  $u' = \min(\{\eta(x) \mid x \in \text{largev}(\rho[0])\} \cup \{u\})$ .

$$\Pr[\epsilon, \eta, \kappa] = \int_{\ell'}^{u'} \left( \frac{d\epsilon}{2} e^{-d\epsilon |z-\mu-a_0|} \right) \Pr[\epsilon, \eta_z, \kappa[1:]] dz.$$

In the special case when  $\operatorname{assignv}(\rho[0]) = \emptyset$  (i.e., the first transition of the run does not change the assignment to any variable), observe that  $\eta_z = \eta$ . Hence,  $\Pr[\epsilon, \eta_z, \kappa[1:]]$ -term on the right hand side of both equations can be pulled out of the integral, and the expression can be simplified. We will abuse notation and use  $\Pr[\cdot]$  to also refer to the function  $\Pr[\eta, \kappa] := \epsilon \mapsto \Pr[\epsilon, \eta, \kappa]$ . Notice that when  $\rho$  starts from  $q_{\text{init}}$ , because of the initialization condition of DiPA, the value of  $\Pr[\cdot]$  does not depend on the valuation  $\eta$ . For such computations, we may drop the valuation  $\eta$  from the argument list of  $\Pr[\cdot]$  to reduce notational overhead. Even though we plan to use the same function name, the number of arguments to  $\Pr[\cdot]$  will disambiguate what we mean.

In this paper we study the computational problem of checking differential privacy for DiPAs. We conclude with a precise definition of this problem. We start by specializing the definition of differential privacy to the setting of DiPA. For a DiPA  $\mathcal{A}$ , an input sequence  $\sigma \in (\mathbb{R} \cup {\tau})^*$  and an output sequence  $\gamma \in (\Gamma \cup (\mathbb{R}_{\infty} \times \mathbb{R}_{\infty}))^*$ , let  $\operatorname{Runs}(\sigma, \gamma)$  be the set of all runs  $\rho$  of  $\mathcal{A}$  starting from the initial state  $q_{\text{init}}$  such that  $\rho$  is a run on  $\sigma$  producing  $\gamma$ .

**Definition 5.** A DiPA  $\mathcal{A}$  is  $\mathfrak{D}\epsilon$ -differentially private (for  $\mathfrak{D} > 0$ ,  $\epsilon > 0$ ) iff for every  $\sigma_1, \sigma_2 \in (\mathbb{R} \cup \{\tau\})^*$  and  $\gamma \in (\Gamma \cup (\mathbb{R}_{\infty} \times \mathbb{R}_{\infty}))^*$  such that  $\sigma_1$  and  $\sigma_2$  are *adjacent*<sup>6</sup>,

$$\sum_{\rho \in \operatorname{Runs}(\sigma_1, \gamma)} \Pr[\epsilon, (\rho, \sigma_1, \gamma)] \le e^{\mathfrak{D}\epsilon} \sum_{\rho \in \operatorname{Runs}(\sigma_2, \gamma)} \Pr[\epsilon, (\rho, \sigma_2, \gamma)].$$

**Differential Privacy Problem:** A DiPA  $\mathcal{A}$  is said to be *differentially private* if there exists a constant  $\mathfrak{D} > 0$  (independent of  $\epsilon$ ) such that  $\mathcal{A}$  is  $\mathfrak{D}\epsilon$ -differentially private,  $\forall \epsilon > 0$ . The differential

privacy problem is the problem of determining if a given DiPA  $\mathcal{A}$  is differentially private.

*Remark.* A DiPA  $\mathcal{A}$  is a parametric automaton (with parameter  $\epsilon$ ), and the probability of each of its executions on a sequence of input varies with  $\epsilon$ . Thus, considering its semantics, using  $\mathcal{A}(\epsilon)$  to refer to the automaton may be more appropriate. However, we shall use  $\mathcal{A}$  to reduce the notational overhead.

## 4 WELL FORMED DIPA

The main goal of the paper is to solve the differential privacy problem described in Section 3: Given a DiPA  $\mathcal{A}$  determine if there is a  $\mathfrak{D} > 0$  such that for all  $\epsilon > 0$ ,  $\mathcal{A}$  is  $\mathfrak{D}\epsilon$ -differentially private. In this section, we define the sub-class of *well-formed* DiPA that help characterize precisely the class of DiPA that are differentially private. Well-formed DiPA are automata that don't have four properties that lead to the violation of privacy: (a) *leaking cycles*, (b) *leaking pairs*, (c) *disclosing cycles*, and (d) *privacy violating paths*. We will define what these types of cycles and paths are in this section.

Dependency Graph of a Run. Consider a run  $\rho$  of a DiPA  $\mathcal{A}$ . Guards on transitions and decisions to store insample in storage variables, demand that if  $\mathcal{A}$  follows the run  $\rho$ , then the values sampled as insample at different steps must be ordered in a certain way to ensure that guards are satisfied. This partial order on the sampled values demanded by a run is conveniently captured as a directed graph that we call the *dependency graph*.

**Definition 6** (Dependency Graph). Let  $\mathcal{A} = (Q, \Gamma, q_{\text{init}}, X, P, \delta)$  be a DiPA and let  $\rho = t_0 t_1 \cdots t_{n-1}$  be a run of  $\mathcal{A}$ . The *dependency graph* of  $\rho$  is the directed graph  $G_{\rho} = (V, E)$  where

- $V = \{i \mid 0 \le i < n\}$ , and
- *E* is defined as  $E' \cap (V \times V)$  where

$$E' = \{(j, \text{lastassign}_{\rho}(\mathbf{x}, j)) \mid j \in V, \ \mathbf{x} \in \text{largev}(t_j)\} \\ \cup \{(\text{lastassign}_{\rho}(\mathbf{x}, j), j) \mid j \in V, \ \mathbf{x} \in \text{smallv}(t_j)\}.$$

Notice that  $E = E' \cap (V \times V)$  ensures that an edge  $(j, \text{lastassign}_{\rho}(\mathbf{x}, j))$  (or  $(\text{lastassign}_{\rho}(\mathbf{x}, j), j)$ ) is present only when lastassign<sub> $\rho$ </sub> $(\mathbf{x}, j) \neq -\infty$  (i.e., when x is assigned before position *j*). Also observe that an edge (i, j) in  $G_{\rho}$  means that, to satisfy the guards, insample at position *i* in the run  $\rho$  must be less than insample at position *j*.

Given the intuition that the dependency graph  $G_{\rho}$  captures the ordering constraints imposed by the guards in  $\rho$ , one can conclude that a cycle in  $G_{\rho}$  means that  $\rho$  places contradictory demands on the values sampled and is therefore not a valid execution of the DiPA. We define a run  $\rho$  of DiPA  $\mathcal{A}$  to be *feasible* iff  $G_{\rho}$  is acyclic. Feasibility is consistent with our semantic intuitions – if  $\rho$  is feasible then there is some evaluation  $\eta$  such that for any  $\epsilon > 0$ , any input sequence  $\sigma$  and any output sequence  $\gamma$  in which all output intervals are given by the interval  $(-\infty, \infty)$ , for which  $\rho$  is a run on  $\sigma$  that produces  $\gamma$ ,  $\Pr[\epsilon, \eta, (\rho, \sigma, \gamma)] > 0$ .

Let us consider a feasible run  $\rho = t_0t_1\cdots t_{n-1}$  of DiPA  $\mathcal{A}$ . Let  $q_i = \operatorname{src}(t_i)$  and let  $P(\operatorname{src}(t_i)) = (d_i, \mu_i, d'_i, \mu'_i)$ . We say that  $\rho$  is strongly feasible if in addition whenever there is a path from *i* to *j* in  $G_\rho$  and  $q_i, q_j \in Q_{\text{non}}$  then  $\mu_i < \mu_j$ . Thus,  $\rho$  is strongly feasible if whenever guards require two insample values on non-input transitions to be ordered, the corresponding means of the

<sup>&</sup>lt;sup>6</sup>See Definition 1 on Page 3.

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Figure 2: Dependency graphs for runs  $\rho_1$  and  $\rho_2$  from Example 3.  $G_{\rho_1}$  is on the left and  $G_{\rho_2}$  is on the right.

Laplace distributions are ordered in the same way. We only consider DiPA that satisfy the following *strong feasibility assumption*.

**Strong Feasibility:** All feasible runs from the initial state  $q_{init}$  are strongly feasible.

#### Example 3. Let us look at two example runs of length 3.

- $\begin{aligned} \rho_1 = &(q_0, \mathsf{true}, q_1, \bot, (\mathsf{true}, \mathsf{false}))(q_1, \mathsf{insample} < \mathsf{x}_1, q_2, \bot, (\mathsf{false}, \mathsf{true}))\\ &(q_2, \mathsf{insample} \geq \mathsf{x}_1 \land \mathsf{insample} < \mathsf{x}_2, q_3, \bot, (\mathsf{false}, \mathsf{false})) \end{aligned}$
- $\begin{aligned} \rho_2 = &(q_0, \mathsf{true}, q_1, \bot, (\mathsf{true}, \mathsf{false}))(q_1, \mathsf{insample} \ge \mathsf{x}_1, q_2, \bot, (\mathsf{false}, \mathsf{true})) \\ &(q_2, \mathsf{insample} \ge \mathsf{x}_1 \land \mathsf{insample} < \mathsf{x}_2, q_3, \bot, (\mathsf{false}, \mathsf{false})) \end{aligned}$

The only difference between  $\rho_1$  and  $\rho_2$  is the guard on the second transition, which goes from state  $q_1$  to  $q_2$ . Their dependency graphs are shown in Figure 2.  $G_{\rho_1}$  is on the left and can be explained as follows. Transition 0 sets variable  $x_1$  and transition 1 sets variable  $x_2$ . The guard insample  $< x_1$  in transition 1 results in the edge from 1 to 0. The conjunct insample  $\ge x_1$  in transition 2 results in an edge from 0 to 2, and the conjunct insample  $< x_2$  results in the edge from 2 to 1.  $G_{\rho_1}$  is cyclic which means that  $\rho_1$  is not feasible. Graph  $G_{\rho_2}$  on the right in Figure 2 is similar but the guard insample  $\ge x_1$  in transition 1 results in an edge from 1 to 0 in  $G_{\rho_2}$ ) which removes the cycle. Thus,  $\rho_2$  is feasible.

*Leaking cycle.* We are now ready to present the first graph theoretic condition on DiPA that demonstrates a violation of differential privacy.

**Definition 7** (Leaking cycle). A run  $\rho$  of  $\mathcal{A} = (Q, \Gamma, q_{init}, X, P, \delta)$  from the initial state  $q_{init}$  (i.e.,  $\operatorname{src}(\rho) = q_{init}$ ) is said to be a *leaking cycle* if there is an index  $0 \le j < |\rho|$  and a storage variable  $x \in$  stor such that the following conditions hold.

**Cycle:**  $C = \rho[j:]$  is a cycle.

**Leak:** There are indices  $i_1$  and  $i_2$  in *C* (i.e.,  $j \le i_1, i_2$ ) such that  $x \in assignv(\rho[i_1])$  and  $x \in usedv(\rho[i_2])$ .

**Repeatability:** *C* can be repeated arbitrarily many times. That is, for every  $m \ge 0$ , the run  $\rho C^m$  is feasible.<sup>7</sup>

Intuitively, the condition Leak in Definition 7 is to ensure that variable x is assigned a value in the cycle C that is later tested against in a guard. <sup>8</sup> The main effect of the 3 conditions in Definition 7, is to identify two transitions (namely, those corresponding to assignment and test) that can be taken arbitrarily many times (since they are on a repeatable cycle) such that the insample values sampled in the two transitions are ordered in the same way each time the transitions are taken. This property leads to a "leaking" of the privacy budget, as shall be explained when we sketch the proof.

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A cycle *C* that does not satisfy the condition Leak will be said to be *non-leaking*.

**Definition 8** (Non-leaking cycle). A run *C* is a *non-leaking cycle* if *C* is a cycle and for every  $x \in$  stor and *i*, if  $x \in \text{usedv}(C^2[i])$  then lastassign<sub>*C*<sup>2</sup></sub>(x, i) =  $-\infty$ , i.e, x is not assigned a value in *C*. Here  $C^2$  is the concatenation of *C* with itself.

In Definition 8, we use the run  $C^2$  to ensure that we also account for the case when x is assigned *after* it is used in C. One important property about non-leaking cycle is that it is always repeatable; this is the content of the next proposition. Thus repeatability is a non-trivial requirement only for cycles that have a leak.

**Proposition 1.** Let  $\rho$  be a feasible run of  $\mathcal{A} = (Q, \Gamma, q_{init}, X, P, \delta)$ from the initial state  $q_{init}$  such that  $C = \rho[i : j]$  (for some  $0 \le i < j \le |\rho|$ ) is a non-leaking cycle. Then for every m > 0,  $\rho[0 : i](\rho[i : j])^m \rho[j :]$  is feasible.

*Leaking pair.* Recall that the key property of a leaking cycle that leads to the violation of differential privacy is finding two transitions that can be repeated arbitrarily many times such that the insample value sampled in the two transitions is ordered every time they are taken. Leaking cycles achieve this by finding both transitions on a cycle that can be repeated. However, that is not the only way such a pair of transitions can arise — the two transitions could be on two different cycles that can each be repeated. This leads to the definition of a *leaking pair*. The definition of a leaking pair is subtle and we will discuss its details after presenting it formally.

**Definition 9** (Leaking pair). A feasible run  $\rho$  of  $\mathcal{A} = (Q, \Gamma, q_{\text{init}}, X, P, \delta)$  from the initial state  $q_{\text{init}}$  is a *leaking pair* if there are indices  $0 \le i_1 < j_1 \le |\rho|$  and  $0 \le i_2 < j_2 \le |\rho|$  such that the following conditions hold.

**Cycles:**  $C_1 = \rho[i_1 : j_1]$  and  $C_2 = \rho[i_2 : j_2]$  are both non-leaking cycles.

**Disjointness:** Either  $j_1 \le i_2$  or  $j_2 \le i_1$ . That is,  $C_1$  and  $C_2$  are non-overlapping subsequences of  $\rho$ .

**Order:** There is a path  $k_1, k_2, \ldots, k_m$  in the dependency graph  $G_\rho$  such that  $i_1 \le k_1 < j_1$  ( $k_1$  is on  $C_1$ ),  $i_2 \le k_m < j_2$  ( $k_m$  is on  $C_2$ ),  $k_2 < k_1$  and  $k_{m-1} < k_m$ .

As mentioned before Definition 9, the motivation behind leaking pairs is to identify a pair of transitions t and t' that can be executed multiple times and such that the insample value each time t is taken is smaller than the insample value each time t' is taken. Such a pair of transitions represents a "leak" of the privacy budget that can be exploited to prove that DiPA is not differentially private. Definition 9 achieves this goal in the following manner. The desired transitions t and t' are  $\rho[k_1]$  and  $\rho[k_m]$ , respectively. The fact that t and t' are on cycles  $C_1$  and  $C_2$  which are disjoint (in  $\rho$ ) and nonleaking, ensures that they can be repeated thanks to Proposition 1. The condition Order in Definition 9 is the most subtle. The fact that  $k_2 < k_1$  and  $(k_1, k_2)$  is an edge in  $G_\rho$  means that there is a storage variable  $x \in$  stor such that x is assigned in  $\rho[k_2]$  and insample < x is one of the conjuncts in guard( $\rho[k_1]$ ). Further since  $C_1$  is nonleaking, x is not updated within  $C_1$  and hence  $\rho[k_2]$  is taken before  $C_1$ . Similar conclusions can be drawn about  $k_{m-1}$  and  $k_m$  – there is a variable  $y \in$  stor that is assigned in  $\rho[k_{m-1}]$  which is taken before  $C_2$ , and insample  $\geq$  y is a conjunct in guard( $\rho[k_m]$ ). Finally,

 $<sup>{}^{7}</sup>C^{m}$  denotes the *m*-fold concatenation of *C* with  $C^{0} = \lambda$ .

<sup>&</sup>lt;sup>8</sup>Definition 7 does not require  $i_1 < i_2$ . Therefore, strictly speaking the assignment in  $i_1$  may not be before the test in  $i_2$ . But this can be easily addressed by taking  $C^2$ instead of *C* as the cycle.

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Figure 3: DiPA  $\mathcal{A}_{\text{leakp}}$  from Example 4.  $\mathcal{A}_{\text{leakp}}$  has two variables,  $x_1$  and  $x_2$ , assigned in the first and the second transition, respectively. The guards  $g_1 = (\text{insample} \ge x_1), g_2 = (\text{insample} < x_1), g_3 = (\text{insample} < x_2), g_4 = (\text{insample} < x_1) \land (\text{insample} \ge x_2).$ 

the path from  $k_1$  to  $k_m$  means that the insample value sampled in  $\rho[k_1]$  is less than the value assigned to x in  $\rho[k_2]$ , which in turn is less than the value assigned to y in  $\rho[k_{m-1}]$  and that is less than the insample value sampled in  $\rho[k_m]$ .  $\rho[k_2]$  is before  $C_1$  which means that the value assigned to x in  $\rho[k_2]$  does not change no matter how many times  $C_1$  and  $C_2$  are repeated. Next,  $\rho[k_{m-1}]$  is before  $C_2$ . It is possible that  $\rho[k_{m-1}]$  is on  $C_1$ , in which case the value assigned to y changes when  $C_1$  is repeated. However, one can show by induction, that the presence of a path in the dependency graph from  $\rho[k_2]$  to  $\rho[k_{m-1}]$  and an edge from  $\rho[k_{m-1}]$  to  $\rho[k_m]$ means that when  $C_1$  and  $C_2$  are repeated, there will be a path from  $\rho[k_2]$  and the last instance of  $\rho[k_{m-1}]$  and the last value assigned to y in  $\rho[k_{m-1}]$  will be less than every insample value sampled in  $\rho[k_m]$ . Thus, every insample value sampled in  $\rho[k_1]$  will be less than every insample value sampled in  $\rho[k_m]$ , no matter how many times  $C_1$  and  $C_2$  are repeated.

**Example 4.** Consider the automaton  $\mathcal{A}_{\text{leakp}}$  in Figure 3. The automaton is drawn following the convention outlined in Example 2. The automaton has two real variables  $x_1$  and  $x_2$ , assigned in the first and the second transition, respectively. For states  $q_i, q_j$  of  $\mathcal{A}_{\text{leakp}}$ , let  $t_{ij}$  denote the unique transition of  $\mathcal{A}_{\text{leakp}}$  from state  $q_i$  to  $q_j$ . Observe that  $t_{22}$  and  $t_{33}$  are cycles. Consider the run  $\rho_1 = t_{01}t_{12}t_{22}t_{23}t_{33}$  that visits both the cycles  $t_{22}$  and  $t_{33}$  and its extension  $\rho_2 = \rho_1 t_{34}$ . Their dependency graphs for these runs are shown in Figure 4. The nodes 2 and 4 correspond to the cycle transitions  $t_{22}$  and  $t_{33}$  respectively. Considering just the run  $\rho_1$ , these cycles do not constitute a leaking pair. However, when we consider the extended run,  $\rho_2$ , we see that these cycles form a leaking pair via the path  $4 \rightarrow 1 \rightarrow 5 \rightarrow 0 \rightarrow 2$ .

Before moving onto the other two properties needed to define well-formed DiPA, it is useful to remark that the cycles  $C_1$  and  $C_2$  in Definition 9 may be the "same cycle", i.e.,  $C_1$  and  $C_2$  could, respectively, be the first and second iterations of the same sequence of  $\mathcal{A}$  transitions.

*Disclosing cycle.* Real valued outputs present another avenue through which private information in the input can be leaked. The condition identified by leaking cycles and leaking pairs do not account for such violations because they are agnostic to the type of

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Figure 4: Dependency graphs for runs  $\rho_1$  and  $\rho_2$  from Example 4.  $G_{\rho_1}$  is on the top and  $G_{\rho_2}$  is on the bottom. The nodes are numbered according to the order in which the corresponding transition appears in the run.

output produced by the DiPA. Our next condition *disclosing cycle*, identifies a transition that can be executed repeatedly, and which outputs a pertubed input.

**Definition 10** (Disclosing cycle). A feasible run  $\rho$  of  $\mathcal{A} = (Q, \Gamma, q_{\text{init}}, X, P, \delta)$  from the initial state  $q_{\text{init}}$  is a *disclosing cycle* if there are indices  $0 \le j \le i < |\rho|$  such that the following conditions hold.

**Cycle:**  $C = \rho[j:]$  is a non-leaking cycle.

**Disclosing:**  $\rho[i]$  is an input transition that outputs a real value, i.e., src( $\rho[i]$ )  $\in Q_{in}$  with out( $\rho[i]$ )  $\in$  {insample, insample'}.

Observe that in Definition 10,  $\rho[i]$  is a transition that is on cycle *C*. Moreover, since *C* is non-leaking cycle, by Proposition 1, the run  $\rho C^m$  is feasible for every  $m \ge 0$ . Thus, the transition  $\rho[i]$  can be executed repeatedly. Since  $\rho[i]$  is an input transition that outputs a real-value, each time it is executed it reveals some information about the input which results in a loss of privacy.

*Privacy violating path.* We now present the last property needed to define well formed DiPA. This last property also concerns privacy violations that arise from real valued outputs. Leaking cycles and leaking pairs identify a transition that is executed arbitrarily many times where the sampled insample value is bounded by values sampled in another transition (that is also executed many times) on the same run. However, with real valued outputs, we could have a situation where this bound is revealed once, explicitly in an output. This is captured in our next definition.

**Definition 11** (Privacy violating path). A feasible run  $\rho$  of  $\mathcal{A} = (Q, \Gamma, q_{\text{init}}, X, P, \delta)$  from the initial state  $q_{\text{init}}$  is a *privacy violating path* if there are indices  $0 \le i \le j \le |\rho|$  such that the following conditions hold.

**Cycle:**  $C = \rho[i : j]$  is a non-leaking cycle.

**Privacy Violation:** There is a path  $k_1, k_2, ..., k_m$  in the dependency graph  $G_{\rho}$  such that either (a)  $\operatorname{out}(\rho[k_1]) = \operatorname{insample}$ ,  $k_{m-1} < k_m$ , and  $i \le k_m < j$ , i.e.,  $\rho[k_m]$  is on cycle *C*, or (b)  $i \le k_1 < j$  ( $\rho[k_1]$  is on cycle *C*),  $k_2 < k_1$ , and  $\operatorname{out}(\rho[k_m]) = \operatorname{insample}$ .

It is useful to see how Definition 11 captures the intuitions laid out before. The path from  $k_1$  to  $k_m$  in  $G_\rho$  ensures that the insample value sampled in  $\rho[k_1]$  is less than the insample value sampled in  $\rho[k_m]$ . Moreover, since C is non-leaking, by Proposition 1, it is repeatable. Condition (a) in (Privacy Violation) says that  $\rho[k_m]$ is a transition on C, and the edge  $(k_{m-1}, k_m)$  in  $G_\rho$  along with  $k_{m-1} < k_m$  means that there is a variable  $x \in$  stor that is set in  $\rho[k_{m-1}]$  and insample  $\geq x$  is in guard( $\rho[k_m]$ ). Moreover, since *C* is non-leaking, x is not updated in *C* and hence  $k_{m-1}$  is before C. Thus, the presence of the path means that the value output in  $\rho[k_1]$  is less than the insample value sampled in  $\rho[k_{m-1}]$  which in turn is less than the insample value sampled in  $\rho[k_m]$  every time *C* is repeated. Therefore, there is a lower bound, which is output in  $\rho[k_1]$ , for arbitrary many insample values that are generated in  $\rho[k_m]$ . Condition (b) in (Privacy Violation) is similar but *dual*. Here  $\rho[k_1]$  is on *C*,  $\rho[k_2]$  is before *C* and sets a variable x that is an upper bound on the values sampled in  $\rho[k_1]$ , and finally,  $\rho[k_m]$ outputs a value that upper bounds all these values, no matter how many times  $\rho[k_1]$  is executed by repeating C.

*Well-formed DiPA.* The properties defined in this section identify witnesses for the violation of privacy. The class of *well-formed* automata are those that do not suffer from these deficiencies.

**Definition 12** (Well-formed DiPA). A DiPA  $\mathcal{A}$  is said to be *well-formed* if  $\mathcal{A}$  does not have any leaking cycles, leaking pairs, disclosing cycles, and privacy violating paths.

Our main results are: (i) a well-formed DiPA is differentially private; (ii) if a DiPA satisfying the output distinction property (see Definition 13) is differentially private then it must be well-formed. We will also show that there is an effective procedure for checking if a DiPA is well-formed. These observations together will provide a decidability result for solving the differential privacy problem for DiPA that satisfy output distinction property.

# 5 WELL-FORMED DIPA ARE DIFFERENTIALLY PRIVATE

One of our main results, which we call the *sufficiency theorem*, is that well-formed DiPAs are differentially private. The proof of this Theorem is involved and carried out in Appendix A.

**Theorem 2.** Let  $\mathcal{A}$  be a DiPA. If  $\mathcal{A}$  is well-formed then there is a  $\mathfrak{D} > 0$  such that for every  $\epsilon > 0$ ,  $\mathcal{A}$  is  $\mathfrak{D}\epsilon$ -differentially private. Further, such a  $\mathfrak{D}$  can be computed in time exponential in the size of the automaton  $\mathcal{A}$ .

PROOF SKETCH. Let  $\mathcal{A}$  be a well-formed DiPA. Given a feasible run  $\rho = t_0 \cdots t_n$  of  $\mathcal{A}$  from the initial state, fix computations  $\kappa_i = (\rho, \sigma_i, \gamma)$  for i = 1, 2 such that  $\sigma_1$  and  $\sigma_2$  are adjacent. For each j, let  $I_j$  be the "less than" relation on stor imposed by the prefix  $\rho[0: j-1] - (\mathbf{x}, \mathbf{x}') \in I_j$  if there is a path of non-zero length from lastassign  $\rho(\mathbf{x}, j)$  to lastassign  $\rho(\mathbf{x}', j)$ . Similarly, eq<sub>j</sub> is the "equality" relation on stor imposed by the prefix  $\rho[0: j-1] - (\mathbf{x}, \mathbf{x}') \in eq_j$  if lastassign  $\rho(\mathbf{x}, j) = lastassign_{\rho}(\mathbf{x}', j)$ .

We can show that there are numbers  $wt_j$  and functions  $m_j$ : stor  $\rightarrow \{-1, 0, 1\}$  such that



Figure 5: DiPA  $\mathcal{A}_{nwf}$  with one variable x is not well-formed but differentially private. The guards  $g_1 = (\text{insample} \ge x)$  and  $g_2 = (\text{insample} < x)$ .

(1) For any valuations  $\eta_1, \eta_2$  such that  $\eta_2 = \eta_1 + m_i$ , <sup>9</sup>

 $\operatorname{Prob}[\epsilon, \eta_2, \kappa_2[j:]] \le e^{\sum_{\ell=j}^{n-1} \operatorname{wt}_{\ell}} \operatorname{Prob}[\epsilon, \eta_1, \kappa_1[j:]].$ 

- (2) If  $t_{j_1} = t_{j_2}$ ,  $|t_{j_1} = |t_{j_2}|$  and  $eq_{j_1} = eq_{j_2}$  for  $j_1 \le j_2$  then  $wt_{j_1} = 0$
- (3) wt<sub>j</sub>  $\leq 2d_j + d'_j$  where  $d_j$  and  $d'_j$  are such that  $P(\operatorname{src}(t_j)) = (d_j, \mu_j, d'_j, \mu'_j)$ .

Observe that the last two conditions imply that there is a number  $\mathfrak{D}$  independent of  $\rho$  such that  $\sum_{\ell=j}^{n-1} \operatorname{wt}_{\ell} < \mathfrak{D}$ . Note that as  $\rho$  is a run from initial state then  $\operatorname{Prob}[\epsilon, \eta_i, \kappa_i]$  is independent of  $\eta_i$ . The above observations imply that

$$\operatorname{Prob}[\epsilon, \kappa_2] \leq e^{\mathfrak{V}} \operatorname{Prob}[\epsilon, \kappa_1].$$

This shows that  $\mathcal{A}$  is  $\mathfrak{D}\epsilon$ -differentially private. To carry out the formal proof, we construct an augmented automaton  $\operatorname{aug}(\mathcal{A})$ , whose states are triples of the form  $(q, \mathsf{lt}, \mathsf{eq})$  where q is a state of  $\mathcal{A}$ ,  $\mathsf{lt}$ , and eq are strict partial orders and equivalence relations on stor. The value for  $\mathfrak{D}$  is also computed using the augmented automaton.  $\Box$ 

The problem of checking well-formedness can be shown to be in PSPACE. The proof is in Appendix A.

**Theorem 3.** The problem of checking whether a DiPA is well-formed is in PSPACE. When the number of variables is taken to be a constant k, then the problem of checking whether a DiPA is well-formed is decidable in polynomial time.

# 6 DIFFERENTIALLY PRIVATE DIPA ARE WELL-FORMED

While well-formedness is sufficient for ensuring differential privacy, it is not a necessary condition for differential privacy as illustrated by the following example.

**Example 5.** Consider the DiPA  $\mathcal{A}_{nwf}$  with one variable insample given in Figure 5. The automaton is drawn following the convention outlined in Example 2. As each transition outputs  $\top$ ,  $\mathcal{A}_{nwf}$ , on any input of length *n*, outputs the string  $\top^n$  with probability 1. Thus,  $\mathcal{A}_{nwf}$  is trivially differentially private. However,  $\mathcal{A}_{nwf}$  is not well-formed as it has a leaking cycle,  $t_a t_b$  where  $t_a$  is the transition from  $q_0$  to  $q_1$  and  $t_b$  is the transition from  $q_1$  to  $q_0$ .

We show, however, that differentially private DiPA that satisfy an additional technical property of *output distinction* are well-formed. Thus, for DiPA satisfying this property, well-formedness is a precise

<sup>&</sup>lt;sup>9</sup>For functions  $f, g : A \to \mathbb{R}$ , f + g is the function that adds the result of f and g for each argument, i.e., (f + g)(a) = f(a) + g(a).

characterization of when they are differentially private. Before presenting this *restricted necessity* theorem and proof sketch, let us define what it means for a DiPA to satisfy the condition of output distinction.

**Definition 13** (Output Distinction). A DiPA  $\mathcal{A} = (Q, \Gamma, q_{\text{init}}, X, P, \delta)$  satisfies *output distinction* if the following holds: If  $t_1$  and  $t_2$  are distinct transitions of  $\mathcal{A}$  such that  $\operatorname{src}(t_1) = \operatorname{src}(t_2)$  then  $\operatorname{out}(t_1) \neq \operatorname{out}(t_2)$  and  $\{\operatorname{out}(t_1), \operatorname{out}(t_2)\} \cap \Gamma \neq \emptyset$ .

Output Distinction demands that distinct outgoing transitions from a state have different outputs and at most one of the outgoing transitions outputs a real value. In particular, there cannot be two transitions out of a state q that output insample and insample'. Distinct outputs on transitions ensure that given a starting state qand an output sequence  $\gamma$ , there is at most one run  $\rho$  starting from qthat can produce *y*. Observe that the automaton of Figure 5 does not satisfy output distinction property. The necessity proof proceeds by showing that if  $\mathcal A$  is not well-formed, then given  $\mathfrak D$ , there are computations  $(\rho, \sigma_1, \gamma)$  and  $(\rho, \sigma_2, \gamma)$  with the same run  $\rho$  such that  $\rho$  outputs  $\gamma$ ,  $\sigma_1$ ,  $\sigma_2$  are adjacent and the ratio of the probability measures of these computations is >  $e^{\mathfrak{D}\epsilon}$  for sufficiently large  $\epsilon$ . Output distinction guarantees that  $\rho$  is the *only* run on  $\sigma_1, \sigma_2$ that outputs  $\gamma$ , allowing us to conclude that  $\mathcal{A}$  is not differentially private for non-well formed A. Without output distinction, the deficit in probability measures of  $\gamma$  can be made up by other paths. The output distinction property is also needed in [10] for the case of a single variable. We are now ready to present the main result of this section.

# **Theorem 4.** Let $\mathcal{A}$ be a DiPA that satisfies the output distinction property. If $\mathcal{A}$ is not well-formed, then it is not differentially private.

PROOF SKETCH. Let us fix a DiPA  $\mathcal{A} = (Q, \Gamma, q_{init}, X, P, \delta)$  that satisfies the output distinction property. Recall that the output distinction property ensures that for any input sequence  $\sigma$  and output sequence  $\gamma$ ,  $|\text{Runs}(\sigma, \gamma)| \leq 1$ . We sketch the main ideas behind the proof; the full details can be found in Appendix B. Assume that  $\mathcal{A}$  is not well-formed. Now, for each value of  $\mathfrak{D}$  and  $\epsilon$ , the proof identifies a run  $\rho$ , an output sequence  $\gamma$ , and a pair of adjacent input sequences  $\alpha$  and  $\beta$  such that the computations  $(\rho, \alpha, \gamma)$  and  $(\rho, \beta, \gamma)$ demonstrate a violation of differential privacy (Definition 5). The construction of witnesses is based on the following sequence of observations.

(1) Let us fix a run  $\rho$  from  $q_{init}$  and an output sequence  $\gamma$  consistent with  $\rho$ . Observe that the number read in an input transition determines the mean of the distributions from which insample and insample' are drawn in that step. Let us call an input sequence  $\sigma$ *strongly compliant* with  $\rho$  and  $\gamma$ , if the sampling means satisfy the constraints imposed by  $\rho$  and  $\gamma$ . This has two requirements. First, whenever there is a path from *i* to *j* in  $G_{\rho}$ , the sample mean at step *i* is less than the sample mean at step *j*. Notice that strong feasibility ensures this when *i* and *j* are non-input transitions, and here we are requiring this to hold when either *i* or *j* is an input transition in which case the mean is determined by  $\sigma$ . Second, if  $out(\rho[i]) \in \{\text{insample, insample'}\}$  (real outputs), the sample mean at step *i* is in the interval  $\gamma[i]$ . Intuitively, for a strongly compliant input sequence  $\sigma$ , the probability of computation ( $\rho, \sigma, \gamma$ ) is likely to be "high". On the flip side, let us call an input sequence  $\sigma$  *non-compliant* at *i*, if the sample mean set by  $\sigma$  at step *i* either violates an order constraint or an output constraint. Again intuitively, one can imagine that, as the number of non-compliant transitions increase in  $\sigma$ , the probability of the computation ( $\rho$ ,  $\sigma$ ,  $\gamma$ ) decreases. Now one can prove that if we consider two input sequences  $\sigma_1$ , which is strongly compliant, and  $\sigma_2$ , which has non-compliant transitions, then the ratio of the probabilities of ( $\rho$ ,  $\sigma_1$ ,  $\gamma$ ) and ( $\rho$ ,  $\sigma_2$ ,  $\gamma$ ) grows as the number of non-compliant transitions in ( $\rho$ ,  $\sigma_2$ ,  $\gamma$ ) increases.

- (2) Observations in (1) above provide a template for how to identify witnesses for differential privacy violation: the presence of a leaking cycle, leaking pair, disclosing cycle, or privacy violating path help identify a run, and we then construct two input sequences  $\alpha$ , which is strongly compliant, and  $\beta$  which has many non-compliant steps. Observe that each witness to non-well-formedness is a run containing a cycle that can be repeated arbitrarily many times and contains a transition that will be made non-compliant in the input sequence  $\beta$ . The intuitions laid out in Section 4 for defining well-formed DiPA will be used and we spell this out in each case. A leaking cycle has a transition with index  $i_1$  (see Definition 7) that sets a variable which is then used later in the transition indexed  $i_2$ . Since the guard of  $i_2$  is not true, it is an input transition. We will construct the run  $\rho$  by repeating the cycle as many times as needed (based on  $\mathfrak{D}$  and  $\epsilon$ ), and in  $\beta$  the sample mean at step  $i_2$  will be in the wrong order with respect to  $i_1$  in each repetition, making it non-compliant. In a leaking pair (Definition 9) there is a pair of transitions indexed  $k_1$  and  $k_m$  on cycles that can be repeated, and whose sampled values need to be ordered each time they are executed. Moreover, transitions  $k_1$  and  $k_m$  are input transitions because their guards are not true (see discussion after Definition 9). Thus, in  $\beta$  we will flip the order of the sample means at these steps to create an arbitrary number of non-compliant steps. The transition indexed i in a disclosing cycle (Definition 10) is an input transition on a cycle that can be repeated. To create non-compliant steps in  $\beta$  we will set the mean of these transitions to not be in the output interval given for this step. Finally, in a privacy violating path (Definition 11) there is an input transition with index  $k_m$  for case (a) (or  $k_1$  for case (b)) that is on a repeatable cycle whose sampled value is required to be larger than (smaller than in case (b)) the value output in step  $k_1$  (step  $k_m$  for case (b)). To construct the input sequence  $\beta$ , we set the input for each time  $k_m$  ( $k_1$  in case (b)) is taken to be smaller than the value output in  $k_1$ , and thereby creating arbitrarily many non-compliant steps.
- (3) The general principles behind constructing the input sequences  $\alpha$  and  $\beta$  are laid out in (2). However, one key requirement for  $\alpha$  and  $\beta$  to constitute a witness to privacy violation is that they be *adjacent* (Definition 1) which demands that the values in  $\alpha$  and  $\beta$  be not too far apart. One challenge is carrying this out is the presence of non input transitions, where the sample means are *fixed*. This can be overcome by carefully analyzing the dependency graph  $G_{\rho}$  and the parameters decorating the states appearing in the run  $\rho$ .

The precise proof based on the above ideas is long and deferred to Appendix B.

In Section 5, we showed that there is a PSPACE algorithm to determine if an output-distinct DiPA  $\mathcal{A}$  is well-formed (Theorem 3).

This complexity bound is tight; we show that the problem of determining if a DiPA is differentially private is PSPACE-hard. (See Appendix C for the proof.)

**Theorem 5.** Given an output-distinct DiPA  $\mathcal{A}$ , the problem of determining if there is a  $\mathfrak{D} > 0$  such that for all  $\epsilon$ ,  $\mathcal{A}$  is  $\mathfrak{D}\epsilon$ -differentially private, is PSPACE-hard.

# 7 EXPERIMENTS

We implemented the algorithm that checks whether a DiPA  $\mathcal{A}$  is well-formed. In case  $\mathcal{A}$  is well-formed, it computes a bound  $\mathfrak{D}$ , which we call the *weight* of the automaton, such that  $\mathcal{A}$  is  $\mathfrak{D}\epsilon$ differentially private for all  $\epsilon$ . The software tool, DiPAut, is built in Python 3.9.5 and is available for download at [7]. It uses the PLY package [6] for parsing the program and the IGRAPH package [14] to store the input automaton as a graph. The IGRAPH package is also used to perform graph-theoretic operations on the input automaton.

DiPAut has three major components. The first component, called core, tokenizes and parses the input using PLY. The second component, builders, constructs the augmentation of the input automaton. The augmentation is built using a breadth-first-search of the (implicit) graph of the augmentation. The relations It and eq are stored as dictionaries during augmentation. To prepare for checking of leaking pair and privacy violating path, the automaton also builds an "enhanced" augmentation. For example, it also builds the graphs that include assignments to the variables  $V_1$  and  $V_2$  in the algorithm for checking leaking pair (See the proof of Theorem 3). The third component DP tests, implements the finals checks for leaking cycle, leaking pair, privacy violating path and disclosing cycle from the augmentations. If the automaton is well-formed, it also computes the weight of the automaton. If it is not well-formed, it further checks if it is output-distinct. In that case, we report that the automaton is not differentially private.

DiPAut was evaluated against a suite of examples (See Table 1), which we describe briefly.

#### 7.1 Description of Examples

The first examples we consider are the standard Sparse Vector Technique (SVT) [19] and the Numeric Sparse (NUM-SPARSE) [20]. These algorithms use one variable. Detailed discussion of these algorithms can be found in [19, 20]. Apart from SVT and NUM-SPARSE, all other examples use more than one variable. The details of these algorithms are also located in Appendix D.

We also designed new examples, described below. The first set of examples was designed to ensure that the tests of well-formedness were implemented correctly. A second set of examples were designed to evaluate the scalability of our tool. They include k-MIN-MAX (for each k > 0) and m-RANGE (for each m > 0). The 1-RANGE is the range query algorithm given in Example 1.

*Examples LC-EXAMPLE and DC-EXAMPLE.* The algorithm LC-EXAMPLE and DC-EXAMPLE are variants of 1-RANGE. The algorithm LC-EXAMPLE is designed to have a leaking cycle and DC-EXAMPLE is designed to have a disclosing cycle. A detailed description of the algorithms can be found in Appendix D.

*Examples Num-Range-1 and Num-Range-2.* The algorithm Num-Range-1 is the variant of 1-Range which outputs insample (instead of  $\top$ ) when the sampled value q[i] is greater than high. The algorithm Num-Range-2 on the other hand outputs insample'. Num-Range-2 is well-formed, output-distinct and hence differentially private but Num-Range-1 has a privacy-violating path. A detailed description of the algorithms can be found in Appendix D.

Examples Two-RANGE-1 and Two-RANGE-2. Two-RANGE-1 is a variant of 1-RANGE. In both algorithms, at the beginning, three thresholds,  $T_{\ell}$ ,  $T_m$ , and  $T_u$ , are perturbed by adding noise sampled from the Laplace distribution. The algorithms then proceed to process the queries, checking if the remaining noisy queries are between the noisy  $T_{\ell}$  and  $T_m$ . If at some point the input noisy query exceeds the noisy  $T_m$ , Two-RANGE-1 checks that the remaining queries are in between the noisy  $T_m$  and the noisy  $T_u$ . In contrast, the algorithm Two-RANGE-2 resamples  $T_m$  before checking that the remaining queries are in between the noisy  $T_m$  and the noisy  $T_u$ . Two-RANGE-1 has a leaking pair and is not differentially privacy. Two-RANGE-2, on the other hand, is well-formed, output distinct, and hence differentially private Two-RANGE-1 and Two-RANGE-2 are described in detail in Appendix D.

<b>Input:</b> <i>q</i> [1 : <i>N</i> ]
<b>Output:</b> <i>out</i> [1 : <i>N</i> ]
$\min, \max \leftarrow Lap(\frac{\epsilon}{4k}, q[1]))$
for $i \leftarrow 2$ to $k$ do
$r \leftarrow Lap(\frac{\epsilon}{4k}, q[i])$
if $(r > max) \land (r > min)$ then
$\max \leftarrow r$
else if $(r < min) \land (r < max)$ then
min $\leftarrow$ r
end
$out[i] \leftarrow read$
end
<b>for</b> $i \leftarrow k + 1$ <b>to</b> N <b>do</b>
$r \leftarrow Lap(\frac{\epsilon}{4}, q[i])$
if $(r \ge min) \land (r < max)$ then
$  out[i] \leftarrow \bot$
else if $(r \ge min) \land (r \ge max)$ then
$out[i] \leftarrow \top$
exit
else if $(r < min) \land (r < max)$ then
$out[i] \leftarrow \bot$
<b>:</b> t
end

**Algorithm 2:** *k*-MIN-MAX algorithm. *k*-MIN-MAX is differentially private.

*Example k-MIN-MAX.* One set of examples designed to check scalability of our algorithm is k-MIN-MAX ( $k \ge 2$ ). Initially, k-MIN-MAX reads k-queries, adds noise from the Laplace distribution at each step, remembering the maximum and minimum amongst the perturbed queries. During this phase, the outputs do not inform the observer whether the noisy query being processed updates the maximum or minimum.

Benchmarks				DiPAut			CheckDP [29]		
Example	υ	S	trans	wt calc	total	differentially	D	time (s)	Counterexample
				time (s)	time (s)	private?			Validated?
SVT	1	3	3	0.00046	0.238	$\checkmark$	5/4	29.92	N.A.
NUM-SPARSE	1	3	3	0.00045	0.249	$\checkmark$	7/4	52.43	N.A.
DC-Example	2	4	5	N.A.	0.237	×, DC	N.A.	43.59	T.O.
NUM-RANGE-1	2	4	4	N.A.	0.234	×, PV	N.A.	316.05	T.O.
NUM-RANGE-2	2	4	4	0.00078	0.231	$\checkmark$	5/4	1909.43	T.O.
LC-Example	2	4	4	N.A.	0.231	×, LC	N.A.	T.O.	
Two-Range-1	3	6	10	N.A.	0.239	×, LP	N.A.	T.O.	
Two-Range-2	3	7	11	0.00258	0.277	$\checkmark$	2	T.O.	
2-Min-Max	2	4	7	0.00065	0.220	$\checkmark$	1	T.O.	
10-Min-Max	2	12	31	0.00221	0.230	$\checkmark$	1	M.E.	
20-Min-Max	2	22	61	0.00434	0.248	$\checkmark$	1	M.E.	
100-Min-Max	2	102	301	0.0291	0.409	$\checkmark$	1	M.E.	
200-Min-Max	2	202	601	0.0803	0.643	$\checkmark$	1	M.E.	
1-Range	2	4	5	0.00083	0.227	$\checkmark$	1	T.O.	
10-Range	20	31	50	0.00797	0.611	$\checkmark$	1	M.E.	
20-Range	40	61	100	0.0212	3.469	$\checkmark$	1	M.E.	
40-Range	80	121	200	0.06242	35.89	$\checkmark$	1	M.E.	
80-Range	160	241	400	0.25867	506.3	$\checkmark$	1	M.E.	

Table 1: Summary of experimental results for DiPAut and comparison with CheckDP. The columns in the table are as follows. v is the number of variables in the automaton. s is the number of states in the automaton. trans is the number of transitions in the automaton. The weight calculation time and total time taken by DiPAut averaged over six executions are reported next, and are measured in seconds. Differentially private indicates if the automaton is differentially private or not. In case, it is not, we report the reason detected by the tool: DC/PV/LC/LP means that disclosing cycle/privacy-violating path/leaking cycle/leaking pair, respectively is detected. D is the weight of the automaton computed by the algorithm in case it is differentially private. For CheckDP, the time column indicates the running time for CheckDP measured in seconds. The last column indicates the time taken for counterexample validation by PSI in case a counterexample is generated. T.O. denotes that the tool did not finish in 30 minutes. M.E. indicates that CheckDP reported a memory error.

After reading the first k-queries, each subsequent query is perturbed by adding noise, and the algorithm checks if the noisy query is between the maximum and minimum found in the first k-noisy queries. It continues processing the queries as long as it is between those two. Otherwise, it quits. Observe that k-MIN-MAX is a parametric set of examples, one for each value of k. For each k, the DiPA modeling k-MIN-MAX has two variables, has k + 2 states and 3k + 1transitions. Further, k-MIN-MAX does not satisfy output distinction for any k as the outputs do not distinguish whether maximum or minimum is being updated in the first phase. However, it is wellformed and  $\epsilon$ -differentially private. Psuedocode for k-MIN-MAX is shown as Algorithm 2.

Examples m-Range. Another set of examples for scalability is m-RANGE (for each m). m-RANGE is the m-dimensional version of RANGE. It repeatedly checks whether a sequence of points in the *m*-dimensional space is contained in a *m*-dimensional rectangle. The rectangle is specified by giving the upper and lower threshold for each coordinate of the rectangle. The algorithm initially adds Laplacian noise to each of these 2*m* thresholds, then processes the points by adding noise to each coordinate and checking that each noisy coordinate is within the noisy thresholds for that coordinate. Observe that *m*-RANGE is a set of examples, one for each *m*. For each *m*, the DiPA modeling *m*-RANGE has 2m variables, has 3m + 1states and 5m transitions. For each m, m-RANGE satisfies output distinction, is well-formed, and is  $\epsilon$ -differentially private. *m*-RANGE is given in Algorithm 3. Here the arrays  $T_1$  and  $T_2$  store the *m*lower and *m*-upper thresholds, respectively. The arrays low and high store the noisy version of the lower and upper thresholds. In the experiments,  $T_1$  is taken to be all 0s, and  $T_2$  is taken to be all 1s.

#### 7.2 Summary of experimental results

The experimental results are summarized in Table 1. All experiments were run on a macOS computer with a 1.4 GHz Quad-Core

Input: 
$$q[1:m]$$
  
Output:  $out[1:Nm]$   
for  $j \leftarrow 1$  to  $m$  do  
 $| low[j] \leftarrow Lap(\frac{\epsilon}{4m}, T_1[j])$   
 $high[j] \leftarrow Lap(\frac{\epsilon}{4m}, T_2[j])$   
 $out[j] \leftarrow cont$   
end  
for  $i \leftarrow 1$  to  $N$  do  
 $| r \leftarrow Lap(\frac{\epsilon}{4}, q[m(i-1)+j])$   
 $| if (r \ge low[j]) \land (r < high[j])$  then  
 $| out[m(i-1)+j] \leftarrow cont$   
else if  $((r \ge low[j]) \land (r > high[j]))$  then  
 $| out[m(i-1)+j] \leftarrow \top$   
 $| exit$   
end  
else if  $((r < low[j]) \land (r < high[j]))$  then  
 $| out[m(i-1)+j] \leftarrow \perp$   
 $| exit$   
end  
end  
end

**Algorithm 3:** *m*-RANGE algorithm. *m*-Range is differentially private.

Intel Core i5 CPU processor with 8GB RAM. The running time is benchmarked using PYPERF [21], which runs each example 6 times and takes the average over the 6 instances. Figure 6 plots the running time of our implementation for k-MIN-MAX. As predicted, the tool confirms that k-MIN-MAX is  $\epsilon$ -differentially private. A close examination of the algorithm for checking well-formedness reveals



Figure 6: Running time for k-MIN-MAX. The y-axis gives the running time measured in seconds, while the x-axis gives k. The size of the DiPA is linear in k. k-MIN-MAX is differentially private with weight 1.

that the algorithm can check the well-formedness of k-MIN-MAX in time that is linear in k. This observation is confirmed by the experimental results. Note that the size of the DiPA modeling k-MIN-MAX is linear in k, and hence the running time is also linear in the size of DiPA. In contrast, a careful analysis reveals that the algorithm checking well-formedness takes time that is cubic in m for m-RANGE. This observation is also confirmed by the experimental results. (See Figure 7). As predicted, the tool confirms that m-RANGE is  $\epsilon$ -differentially private. Note that the number of variables in m-RANGE is 2m, implying a quartic dependence on the number of variables as well. Data used to generate the graphs is given in Appendix D.

Salient observations about our tool are as follows:

- DiPAut is able to check whether the algorithm described by a DiPA is well-formed in reasonable time.
- (2) In case the automaton A is well-formed, it is able to compute a weight D that A is Dε-differentially private. The computed values match the theoretical values. Further, the computation of weight has little overhead.
- (3) As predicted by the theory, the number of variables plays a crucial role in performance. While the theory predicts that this dependence is exponential (since the augmentation can be of exponential size), nevertheless, there are interesting examples in which the dependence is polynomial and not exponential.
- (4) DiPAut is not only able to verify differential privacy for examples but also find violations of privacy in a reasonable time, as shown in Table 1.

Comparison with CheckDP. We compare the performance of our tool, DiPAut with CheckDP [29]. CheckDP employs the randomness alignment technique and attempts to prove differential privacy. If it fails to prove differential privacy, it generates a potential counterexample that must be validated using the PSI probabilistic model checker [24]. The key differences between CheckDP and DiPAut are as follows: (1) CheckDP supports other arithmetic operations besides comparison operators. (2) However, CheckDP is sound but incomplete and may fail to prove or disprove differential privacy. (3) CheckDP checks if a program is  $\mathfrak{D}\epsilon$  differentially private for a



Figure 7: Running time for *m*-RANGE. The *y*-axis gives the running time measured in seconds, while the *x*-axis gives *m*. The size of the DiPA is linear in *m*. *m*-RANGE is differentially private with weight 1.

given  $\mathfrak{D}$ . DiPAut, on the other hand, computes a  $\mathfrak{D}$  for which the program is  $\mathfrak{D}\epsilon$  differentially private. (4) DiPAut operates as a standalone tool, assessing the differential privacy of a given mechanism. The results of the comparison are summarized in Table 1. Apart from SVT and NUM-SPARSE, CheckDP times out on all other examples. For those two examples, DiPAut significantly outperforms CheckDP.

#### 8 RELATED WORK

Online Programs and Comparison with [10]. The results in this paper are an extension of those presented in [10]. However, the automaton model proposed in [10] has only one storage variable, whereas we consider the generalization where the automaton has finitely many real-valued storage variables. Even though we use the same name for the automata model and for the conditions characterizing wellformed DiPA, the generalization to handle multiple real-valued storage variables is a significant extension. Defining leaking cycles, leaking pairs, privacy violating paths and disclosing cycles, requires a careful analysis of the ordering constraints imposed on values sampled in a run based on what gets stored in variables and the Boolean constraints that guard transitions. These concepts cannot be defined using just the underlying graph of the DiPA as in [10]; they require introducing the notion of a dependency graph of a run. Even with dependency graphs, the definition of these graphtheoretic conditions is subtle. For example, two cycles contained in a run may not form a leaking pair. However, they may become a leaking pair in an extension of the run as the additional transitions in the extension introduce new dependencies in the dependency graph (see Example 4 on Page 8). In the case of a single variable [10], such a situation does not arise.

Next, even though the proof showing that well-formedness is necessary for an output-distinct DiPA to be differentially private uses a strategy similar to the case for one variable [10], it is significantly more involved. For example, in showing that a leaking cycle is a witness to privacy violation, complications arise due to the need to track the dependency between multiple storage variables and the presence of non-input transitions. When constructing a pair of adjacent inputs that witness the violation of privacy, intervals of real numbers called *bands* need to be carefully identified, where the input of certain transitions is restricted to lie (see Appendix B ). The proof that a leaking pair is a witness to privacy violation uses new ideas. In [10], the proof constructs, given  $\mathfrak{D}$ , two adjacent computations whose ratio is  $> e^{\mathfrak{D}\epsilon}$  for each  $\epsilon > 0$ . In this paper, the adjacent computations have a ratio  $> e^{\mathfrak{D}\epsilon}$  only for sufficiently large  $\epsilon$ .

The proof showing that a well-formed DiPA is differentially private is also innovative. In [10], the proof is by induction on the number of assignments to the stored variable in a run. In contrast, here the induction is on the number of transitions in a run, and the induction hypothesis is constructed by classifying the dependency graph nodes as gcycle\_node or lcycle\_node. (See Appendix A).

*Privacy proof construction.* Techniques based on type systems have been proposed in many papers [15, 16, 22, 27, 29, 31] for generating proofs of differential privacy. Some of these methods such as [15, 16, 22, 27] employ linear dependent types, for which the type-checking and type-inference may be challenging. In [1, 3–5] methods based on probabilistic couplings and random alignment arguments have been employed for proving differential privacy. Shadow executionbased method was introduced in [30]. Probabilistic I/O automata are used in [28] to model interactive differential privacy algorithms and simulation-based methods are used to verify differential privacy, but these methods have not been shown to be complete.

*Counterexample generation.* Automated techniques to search for privacy violations by generating counter examples have been proposed in [8, 17, 29]. Techniques include the use of statistical hypothesis testing [17], optimization techniques and symbolic differentiation [8] and program analysis [29]. These methods search over a bounded number of inputs.

*Model-checking/Markov Chain approaches.* Probabilistic model checking approach for verifying  $\epsilon$ -differential privacy is employed in [12, 13, 25], where it is assumed that the program is given as a finite Markov Chain. These approaches do not allow for sampling from continuous random variables.

Decision Procedures. The decision problem of checking whether a randomized program is differentially private is studied in [2], where it is shown to be undecidable for programs with a single input and single output, assuming that the program can sample from Laplacian distributions. A decidable sub-class is identified where the inputs and outputs are constrained to be from a finite domain and have bounded length.

*Complexity.* Gaboardi et. al [23] study the complexity of deciding differential privacy for randomized Boolean circuits, and show that the problem is **coNP**<sup>#P</sup>-complete. The results are extended to Boolean programs [9] for which the verification problem is PSPACE-complete. In this line of work, programs have a finite number of inputs, the only probabilistic choices are fair coin tosses, and  $e^{\epsilon}$  is taken to be a fixed rational number.

# 9 DISCUSSION

We discuss the restrictions used in various definitions in this paper.

Strong feasability. From the theoretical point of view, strong feasibility is used only to prove the necessity of well-formedness (Theorem 4). The sufficiency proof (Theorem 2) does not require the condition of strong feasibility. Nevertheless, we believe that all differential privacy mechanisms are strongly feasible. We have not encountered examples that violate the strong feasibility condition. Our intuition for this belief is as follows. First, any DiPA that does not have any non-input states is, by definition, strongly feasible. For DiPA with non-input states, the condition implies that the mean of the distribution at any two non-input states respects the order given by the dependency graph of a run. Let us consider the "deterministic" version of the automaton in which no noise is added. Intuitively, the "deterministic" version captures the behavior of the automaton in the limit as the privacy budget  $\epsilon$  tends to infinity, i.e., becomes unlimited. A strongly feasible run implies that we can choose inputs such that the probability of that run tends to 1 as  $\epsilon$ tends to  $\infty$  and is executable in the "deterministic" version. A path that is not strongly feasible implies that the probability of this path tends to 0 as  $\epsilon$  tends to  $\infty$ , irrespective of the choice of inputs, and will never be executed in the "deterministic version" because the insample values stored at the non-input states do not follow the order given by the dependency graph. The deterministic version of the automaton is relevant as a differentially private algorithm is often the noisy version of a deterministic algorithm (with noise added to make the automaton differentially private).

Output-distinction. Some examples do not meet the condition output distinction. For example, the k-MIN-MAX (See Section 7.1) and NOISYMAX [20] are *not* output distinct. However, other examples (m-Range, SVT, NumericSparse) are output distinct. The output distinction condition is only needed to establish necessity but not for sufficiency. In other words, if an automaton is well-formed, it is differentially private, *even if it is not output distinct*. This is true for the k-MIN-MAX examples. However, the traditional NOISYMAX is neither well-formed nor output distinct, and hence our technique does not establish its differential privacy. Some variants of NOISY-MAX (like checking if the kth input is maximum) are well-formed and hence can be handled by our techniques.

Adjaceny Relations. For algorithms working on a sequence of answers to queries on a database like SVT and NUM-SPARSE (see [20], pages 56 and 57), the assumption that queries are *1-sensitive* is common; here 1-sensitive means that adding or removing a member from a database can cause a difference of at most 1 in the output of each query. This assumption is satisfied by all counting queries and can be found in Algorithms 1, 2, 3 in [20] on pages 58, 62, 64, first paragraph on page 5 of [1] and third paragraph of Section 4 in [17].

More generally, our results also apply to a sequence of queries each of which is  $\Delta$ -sensitive. The computation of  $\mathfrak{D}$  will change, but the theorems of the sufficiency of well-formedness and necessity for well-formedness for output distinct DiPA remain true.

Boolean Guards on transitions in leaking cycle. In the definition of a leaking cycle (see Definition 8), it is possible that the constraint involving x in the guard of  $\rho[i_2]$  is superfluous. When this happens, there have to be other variables in the guard of  $\rho[i_2]$ . However, we can show that after removing all superfluous checks from  $\rho[i_2]$ , either the original cycle will be a leaking cycle for some (possibly different) variable, or the leaking cycle gives rise to a leaking pair when repeated twice. Therefore, in principle, even a superfluous test does leak information (though indirectly).

The expressiveness of multi-variable DiPA vs one-variable DiPA. We can prove that multi-variable DiPA are strictly more expressive than one-variable DiPA. For example, we can formally show that the DiPA  $\mathcal{A}_{RANGE}$  (See Figure 1) cannot be modeled using single-variable DiPA.

# **10 CONCLUSIONS**

We extended the DiP automaton model introduced in [10] for modeling online algorithms that process a stream of unbounded real values representing answers to queries and, in response, produce a sequence of real or discrete output values. In the extended model, a DiPA  $\mathcal{A}$  may use *multiple* storage variables to store noisy input values when executing transitions that are used in Boolean conditions that guard transitions. Our main contribution is a precise characterization of when DiPAs are differentially private using the notion of well-formed automata. The definition of well-formed automata is subtle and complicated, and requires the use of new graph structures associated with the runs of the automata, called dependency graphs. Well-formed DiPAs are shown to be differentially private and DiPAs satisfying the condition of output distinction that are differentially private are necessarily well-formed. The problem of checking well-formedness is PSPACE-complete. The algorithm for checking differential privacy has been implemented in a tool called DiPAut, and our experimental results demonstrate its promise.

As future work, it will be interesting to identify necessary conditions for classes of automata that do not satisfy the output distinction property. Extending DiPAs to allow a richer class of comparisons in the guards and a richer class of assignments, like using expressions involving additions of storage variables and/or constants in the guard conditions, is left for future exploration. Computing the optimal weight  $\mathfrak{D}$  is another open problem.

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### A SUFFICIENCY OF WELL-FORMEDNESS

# Well-formed automata are differentially private

We shall now show that a well-formed DiPA is also differentially private. We start by constructing for each DiP automaton  $\mathcal{A}$ , another automaton  $\operatorname{aug}(\mathcal{A})$ , which we shall call the *augmentation* of  $\mathcal{A}$ . Intuitively,  $\operatorname{aug}(\mathcal{A})$  captures all the paths of  $\mathcal{A}$  that occur with non-zero probability (See Proposition 6).

In the augmented automaton,  $\operatorname{aug}(\mathcal{A})$ , each state will carry additional information regarding the relationships that must hold amongst the values stored in the real variables  $x_i$ ,  $1 \le i \le k$ . In particular, each state will carry two binary relations over the set of real variables stor =  $\{x_i, 1 \le i \le k\}$ ; the first relation will capture the "less-than" relation and the second relation shall capture the "equals-to" relation.

**Definition 14** (Augmentation of a DiPA). The augmentation of a *DiP automaton*  $\mathcal{A} = (Q, \Gamma, q_{\text{init}}, X, P, \delta)$  is the automaton  $\operatorname{aug}(\mathcal{A}) = (\operatorname{aug}(Q), \Gamma, \operatorname{aug}(q_{\text{init}}), X, \operatorname{aug}(P), \operatorname{aug}(\delta))$  defined as follows. Let stor =  $X \setminus \{\text{insample, insample'}\}.$ 

- The set aug(Q) is the set of states (q, lt, eq) such that q ∈ Q, lt ⊆ stor × stor is a strict partial order, eq ⊆ stor × stor is an equivalence relation, lt ∩ eq = Ø and lt ∪ eq is transitive. The state (q, lt, eq) is an input state if and only if q ∈ Q<sub>in</sub>.
- $\operatorname{aug}(q_{\text{init}}) = (q_{\text{init}}, \emptyset, \operatorname{id}_{\operatorname{stor}})$ . where  $\operatorname{id}_{\operatorname{stor}} = \{(x, x) \mid x \in \operatorname{stor}\}$ .
- $\operatorname{aug}(P)((q, \mathsf{lt}, \mathsf{eq})) = P(q)$  for each  $(q, \mathsf{lt}, \mathsf{eq}) \in \operatorname{aug}(Q)$ .
- aug(δ) : (aug(Q) × G) → (aug(Q) × (Γ ∪ {insample, insample'})×{true, false}<sup>k</sup>) is defined as follows.
   If δ((q, c)) is undefined, then so is aug(δ)((q, lt, eq), c) for each possible lt and eq.
  - Otherwise, assume that  $\delta(q, c) = (q_1, o, \vec{b})$ . Let It be a strict partial order, and let eq be an equivalence relation such that  $\text{It} \cap \text{eq} = \emptyset$  and  $\text{It} \cup \text{eq}$  is transitive. Consider the following definitions:

$$\begin{split} \text{sm\_vars} &= \{x_i \in \text{stor} \mid \exists x_j.(x_i, x_j) \in \text{It} \cup \text{eq}, \\ & \text{insample} \geq x_j \text{ is a conjunct of } c\} \\ \text{Ig\_vars} &= \{x_i \in \text{stor} \mid \exists x_j.(x_j, x_i) \in \text{It} \cup \text{eq}, \\ & \text{insample} < x_j \text{ is a conjunct of } c\} \\ \text{It_{before}} &= \text{It} \cup \{(x_i, x_j) \mid x_i \in \text{sm\_vars}, x_j \in \text{Ig\_vars}\}. \end{split}$$

Now,  $\delta((q, c))$  is defined only if  $|t_{before} \cap eq = \emptyset$ . In this case,  $aug(\delta)((q, |t, eq), c) = ((q_1, |t_{after}, eq_{after}), o, \vec{b})$  where  $|t_{after}$  and  $eq_{after}$  are defined as follows:

assignv =  $\{x_i | b[i] = true\}$ nonassignv =  $\{x_i | b[i] = false\}$ 

 $lt_{after} = (lt_{before} \cap (nonassignv \times nonassignv))$ 

 $\cup\{(\mathbf{x}_i, \mathbf{x}_j) \mid \mathbf{x}_i \in \text{sm\_vars} \cap \text{nonassignv}, \mathbf{x}_j \in \text{assignv}\} \\ \cup\{(\mathbf{x}_i, \mathbf{x}_i) \mid \mathbf{x}_i \in \text{assignv}, \mathbf{x}_j \in \text{Ig vars} \cap \text{nonassignv}\}$ 

 $eq_{after} = (eq \cap (nonassignv \times nonassignv)) \cup$ 

 $\{(\mathbf{x}_i, \mathbf{x}_j) \mid \mathbf{x}_i, \mathbf{x}_j \in \text{assignv}\}.$ 

**Definition 15.** We shall say that a valuation  $\eta$  is compatible with an augmented state  $(q, \mathsf{lt}, \mathsf{eq})$  if  $\eta(\mathsf{x}_i) < \eta(\mathsf{x}_j)$  whenever  $(\mathsf{x}_i, \mathsf{x}_j) \in \mathsf{lt}$  and  $\eta(\mathsf{x}_i) = \eta(\mathsf{x}_j)$  whenever  $(\mathsf{x}_i, \mathsf{x}_j) \in \mathsf{eq}$ .

For each transition t = ((q, |t, eq), c, (q', |t', eq'), o, b), of aug( $\mathcal{A}$ ), there is a unique transition proj(t) = (q, c, q', o, b) of  $\mathcal{A}$ . If  $\rho = t_0t_1 \cdots t_{n-1}$  is an execution of aug( $\mathcal{A}$ ), the execution proj( $\rho$ ) = proj( $t_0$ )proj( $t_1$ )  $\cdots$  proj( $t_{n-1}$ ) is said to be the projection of  $\rho$ . Observe that if  $\rho$  is a run of aug( $\mathcal{A}$ ) on  $\sigma$  outputting  $\gamma$ then proj( $\rho$ ) is a run of  $\mathcal{A}$  on  $\sigma$  outputting  $\gamma$ . For a computation  $\kappa = (\rho, \sigma, \gamma)$  of aug( $\mathcal{A}$ ) we write proj( $\kappa$ ) = (proj( $\rho$ ),  $\sigma, \gamma$ ). As always, a run  $\rho$  is said to be a run from the initial state if src( $\rho$ ) is the initial state of aug( $\mathcal{A}$ ).

We have the following proposition:

**Proposition 6.** Let  $\mathcal{A}$  be a DiP automaton and  $aug(\mathcal{A})$  be its augmentation.

- (1) Let  $\rho = t_0 \cdots t_{n-1}$  be a run of  $\operatorname{aug}(\mathcal{A})$  from the initial state of  $\mathcal{A}$ . For each  $0 < i \le n$ , let  $q_i$ ,  $\operatorname{lt}_i$ ,  $\operatorname{eq}_i$  be such that  $\operatorname{trg}(t_{i-1}) = (q_i, \operatorname{lt}_i, \operatorname{eq}_i)$ . For each  $0 < i \le n, x_1, x_2 \in \operatorname{stor}$ ,
  - $x_1 | t_i x_2$  if and only if there is a path of non-zero length from lastassign<sub>o</sub>( $x_1$ , i) to lastassign<sub>o</sub>( $x_2$ , i) in  $G_{o[0:i]}$ .
  - $x_1 eq_i x_2$  if and only lastassign<sub> $\rho$ </sub> $(x_1, i) = lastassign<sub><math>\rho$ </sub> $(x_2, i)$ .
  - G<sub>ρ</sub> is acyclic. Hence, every path of aug(A) from the initial state is feasible.
- (2) If κ = (ρ, σ, γ) is a computation of aug(A) and η a valuation compatible with first(ρ), then
  - $\Pr[\epsilon, \eta, \kappa] = \Pr[\epsilon, \eta, \operatorname{proj}(\kappa)].$
  - The dependency graph of  $G_{\rho}$  is the same as the dependency graph of  $G_{\text{proj}(\rho)}$ .
- (3) If ρ is a feasible run of A from the initial state of A, then there is a unique run ρ<sup>†</sup> of aug(A) from the initial state of aug(A) such that proj(ρ<sup>†</sup>) = ρ.
- (4)  $\operatorname{aug}(\mathcal{A})$  is well-formed if and only if  $\mathcal{A}$  is well-formed.
- (5) aug(A) is (d, ε)-differentially private if and only if A is (d, ε)differentially private.

Thanks to Proposition 6, if we can show that if the augmentation of a well-formed automaton  $\mathcal{A}$  is  $(d, \epsilon)$ -differentially private, then so is  $\mathcal{A}$ . For the rest of the section, without loss of generality, we shall assume that all states of an augmented automaton,  $\operatorname{aug}(\mathcal{A})$  are reachable from the initial state of the  $\operatorname{aug}(\mathcal{A})$ . We start with some useful definitions.

**Definition 16.** Let  $aug(\mathcal{A})$  be the augmented automaton of  $\mathcal{A}$ .

- A transition *t* of  $aug(\mathcal{A})$  is said to be a cycle transition if there is a cycle  $C = t_0 \cdots t_{n-1}$  of  $aug(\mathcal{A})$  such that  $t = t_i$  for some  $0 \le i < n$ .
- Given a transition t of aug( $\mathcal{A}),$  Let  $P(\mathrm{src}(t))=(d,\mu,d',\mu').$  We define

$$d(t) = d \quad \mu(t) = \mu \quad d'(t) = d' \quad \mu'(t) = \mu'$$

**Definition 17.** Let  $\mathcal{A}$  be a DiP automaton and  $\operatorname{aug}(\mathcal{A})$  be its augmentation. Let  $\rho = t_0 \cdots t_{n-1}$  be a run from the initial state of  $\operatorname{aug}(\mathcal{A})$ . Let  $G_{\rho}$  be the dependency graph of  $\rho$ . For  $0 \le j \le n$ , let  $\rho_j = \rho[j:]^{.10}$ 

 $<sup>^{10}\</sup>text{Recall,}$  by convention (See Section 2),  $\rho_n$  is the empty string.

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- (1) The vertex *j* of  $G_{\rho}$  is said to be a gcycle\_node (lcycle\_node) if there is a path  $i_1, \ldots, i_k$  in  $G_{\rho}$  such that  $i_1 = j$  ( $i_k = j$  resp.),  $i_{k-1} < i_k$  ( $i_2 < i_1$  resp.) and  $t_{i_k}$  ( $t_{i_1}$  resp.) is a cycle transition.
- (2) The sets usedv(ρ<sub>j</sub>) for 0 ≤ j < n are defined by backward induction on j as follows:

 $usedv(\rho_n) = \emptyset$   $usedv(\rho_j) = smallv(t_j) \cup largev(t_j)$  $\cup (nonassignv(t_i) \cap usedv(\rho_{i+1}))$ 

- (3) The transition t<sub>j</sub> is said to be a *quasi-cycle* transition if the following hold:
  - (a)  $t_i$  does not output insample, ie, insample  $\notin$  out $(t_i)$ .
  - (b) assignv $(t_i) \cap$  usedv $(\rho_{i+1}) = \emptyset$ ,
  - (c) if  $x \in \text{smallv}(t_j)$  then  $\text{lastassign}_{\rho}(j, x)$  is a gcycle\_node and
  - (d) if  $x \in \text{largev}(t_j)$  then  $\text{lastassign}_{\rho}(j, x)$  is a lcycle\_node.
- (4) The weight of transition  $t_j$  at position j, wt $(t_j) \stackrel{\text{def}}{=} f_j(a_j + b_j)d(t_j) + c_jd'(t_j)$  where

$f_j$	_	∫1	if $t_j$ is <i>not</i> quasi-cyclic		
	=	)0	otherwise		
		(1	if <i>j</i> is a lcycle_node or a		
aj	=	{	gcycle_node of $G_{ ho}$		
		(o	otherwise		
h .	_	∫1	if $t_j$ is an input transition		
$v_j =$	-	- ]0	otherwise		
		(1	if insample' $\in \operatorname{out}(t_j)$		
c <sub>j</sub>	=	{	and $t_j$ is an input transition		
		(o	otherwise		

(5) The weight the run  $\rho$ , wt( $\rho$ ) =  $\sum_{j < n} \text{wt}(t_j)$ .

**Proposition 7.** Let  $\mathcal{A}$  be a well-formed DiP automaton and  $\operatorname{aug}(\mathcal{A})$  be its augmentation. Let  $\rho = t_0 \cdots t_{n-1}$  be a run from the initial state of  $\operatorname{aug}(\mathcal{A})$  and  $G_{\rho}$  be the dependency graph of  $\rho$ . For each j, j' We consider each part of the Proposition.

- (1) If  $x \in \text{smallv}(t_j)$  ( $x \in \text{largev}(t_j)$  resp.) is such that  $\text{lastassign}_{\rho}(j, x)$  is a lcycle\_node (gcycle\_node resp.) then *j* is a lcycle\_node (gcycle\_node resp.) also.
- (2) A gcycle\_node of  $G_{\rho}$  cannot be a lcycle\_node.
- (3) If j is a gcycle\_node or a lcycle\_node then the transition t<sub>j</sub> cannot output insample.
- (4) If j < j' and  $t_j = t_{j'}$ , then wt $(t_j) = 0$ .

**PROOF.** Thanks to Proposition 6  $\operatorname{aug}(\mathcal{A})$  is well-formed as  $\mathcal{A}$  is.

- (1) Immediate from the definitions of lcycle\_node and gcycle\_node.
- (2) Immediate from the fact that aug(A) is a well-formed DiP automaton, and hence has no leaking pairs.
- (3) Immediate from the fact that aug(A) is a well-formed DiP automaton, and hence has no privacy-violating path.
- (4) Follows from well-formedness: If t<sub>j</sub> = t<sub>j'</sub>, then t<sub>j</sub>t<sub>j+1</sub>...t<sub>j'</sub> is a cycle. As aug(A) is well-formed, t<sub>j</sub> does not output insample' and is easily seen as a quasi-cycle transition. □

**Proposition 8.** For i = 1, 2, let  $\kappa = (\rho, \sigma, \gamma)$  be a computation of  $aug(\mathcal{A})$ . Let  $\eta^1, \eta^2$  be valuations compatible with  $src(\rho)$ . If  $\eta^1(x) = \eta^2(x)$  for each  $x \in usedv(\rho)$ , then

$$\Pr[\epsilon, \eta^1, \kappa] = \Pr[\epsilon, \eta^2, \kappa]$$

**Proof.** By induction on  $|\rho|$ .

**Theorem 9.** Let  $\mathcal{A}$  be a well-formed DiPA. For i = 1, 2, let  $\kappa_i = (\rho, \sigma_i, \gamma)$  be computations of aug( $\mathcal{A}$ ) such that  $\sigma_1$  and  $\sigma_2$  are adjacent and  $\rho$  starts from the initial state of aug( $\mathcal{A}$ ). Then

$$\Pr[\epsilon, \kappa_2] \leq e^{\operatorname{wt}(\rho)\epsilon} \Pr[\epsilon, \kappa_1]$$

PROOF. Let  $\rho = t_0 \cdots t_{n-1}$ . Let  $G_{\rho}$  be the dependency graph of  $\rho$ . For each  $0 \le j < n$ , define  $m_j$ : stor  $\rightarrow \{-1, 0, 1\}$  as follows:

$$m_j(\mathbf{x}) = \begin{cases} 1 & \text{if lastassign}(\mathbf{x}, j) \text{ is a gcycle_node} \\ -1 & \text{if lastassign}(\mathbf{x}, j) \text{ is a lcycle_node} \\ 0 & \text{otherwise} \end{cases}$$

For  $0 \le j \le n$ ,  $\rho_j = \rho[j:]$ . For  $i = 1, 2, 0 \le j \le n$  let  $\kappa_{i,j} = \kappa_i[j:]$ . <sup>11</sup> For  $0 \le j \le n$ , we shall say that a valuation  $\eta$  is compatible with  $\rho_j$  if  $\eta$  is compatible src $(\rho_j)$  if j < n and with trg $(\rho)$  otherwise. The theorem follows from Proposition 8 and the following claim.

**Claim 1.** Let  $0 \le j \le n$ , and let  $\eta^1, \eta^2$  be valuations such that  $\eta^1, \eta^2$  are compatible with  $\rho$ . If

$$\eta^2|_{\mathsf{usedv}(\rho)} = (\eta^1 + m_j)|_{\mathsf{usedv}(\rho)},$$

then

$$\Pr[\epsilon, \eta^1, \kappa_{1,j}] \ge e^{-\mathsf{wt}_j \epsilon} \Pr[\epsilon, \eta^2, \kappa_{2,j}]$$

where

$$\operatorname{wt}_j = \sum_{u=j}^{n-1} \operatorname{wt}(t_u).$$

The proof is by induction on n - j.

*Base Case:* j = n. The claim follows from the definitions.

Induction Hypothesis: j < n. Assume that the claim is true for j + 1. Fix  $\eta^1, \eta^2$ .

For  $i = 1, 2, \epsilon > 0$ , and  $z \in \mathbb{R}$  let

 $<sup>^{11}</sup>$ By convention, (See Section 2),  $\rho_n$  and  $\kappa_n$  is the empty string  $\lambda.$ 

Now if  $\ell_2 \ge u_2$  then  $\Pr[\epsilon, \eta^2, \rho_{2,j}] = 0$  and the claim is trivially true. Hence, without loss of generality, we assume that  $\ell_2 < u_2$ . We will shortly argue that if  $\ell_2 < u_2$  then  $\ell_1 < u_1$  also. Assuming that this is case, for i = 1, 2 and  $\epsilon > 0$ , let

$$q_{i}(\epsilon) = \begin{cases} \frac{d'(t_{j})\epsilon}{2} \int_{r}^{s} e^{-d'(t_{j})\epsilon} |z-v_{i}'| dz & \text{if } \gamma[j] = (\text{insample}', r, s) \\ 1 & \text{otherwise} \end{cases}$$
$$p_{i}(\epsilon) = \frac{d(t_{j})\epsilon}{2} \int_{\ell_{i}}^{u_{i}} e^{-d(t_{j})\epsilon} |z-v_{i}| \Pr[\epsilon, \eta_{z}^{i}, \kappa_{i,j+1}] dz$$

By definition, we will then have

$$\Pr[\epsilon, \eta^{i}, \kappa_{i, j}] = q_{i}(\epsilon) p_{i}(\epsilon)$$

Let

$$\Delta = v_2 - v_1 = v_2' - v_1'$$

We have that  $-1 \le \Delta \le 1$  and  $\Delta = 0$  if  $t_j$  is a non-input transition. Let  $f_i, a_j, b_j, c_j$  be as in the Definition 17. It is easy to see that

$$q_1(\epsilon) \ge e^{-c_j d'(t_j)\epsilon} q_2(\epsilon).$$

Thus, we shall be done if we can show that

$$\ell_1 < u_1 \text{ and } p_1(\epsilon) \ge e^{\epsilon(-f_j(a_j+b_j)d(t_j)-\mathsf{wt}_{j+1})}p_2(\epsilon)$$

We consider three mutually exclusive but exhaustive cases, depending on the values of  $f_i$  and  $a_i$ .

(a) Let us consider the case when  $f_j = 0$ . Thus, the transition  $t_j$  is a quasi-cycle transition.

By definition of a quasi-cycle transition, if  $x \in \text{smallv}(t_j)$ then lastassign(x) is a gcycle\_node and if  $x \in \text{largev}(t_j)$ then lastassign(x) is a lcycle\_node. Thus, we must have  $m_j(x) = 1$  for each  $x \in \text{smallv}(t_j)$  and  $m_j(x) = -1$  for each  $x \in \text{largev}(t_j)$ . Note that  $t_j$  does not output insample. Thus,

$$\ell_2 = \ell_1 + 1 \qquad u_2 = u_1 - 1.$$

Thus, from the assumption that  $\ell_2 < u_2$ , it is easy to see that  $\ell_1 < u_1$ .

Now, by definition of a quasi-cycle transition,

assignv
$$(t_j) \cap$$
 usedv $(\rho_{j+1}) = \emptyset$ .

Fix *b* such that  $\ell_2 < b < u_2$  be some number. In this case, we can write using Proposition 8

$$p_{i}(\epsilon) = \frac{d(t_{j})\epsilon}{2} \int_{\ell_{i}}^{u_{i}} e^{-d(t_{j})\epsilon |z-v_{i}|} \Pr[\epsilon, \eta_{z}^{i}, \kappa_{i,j+1}] dz$$
$$= \Pr[\epsilon, \eta_{b}^{i}, \kappa_{i,j+1}] \left(\frac{d(t_{j})\epsilon}{2} \int_{\ell_{i}}^{u_{i}} e^{-d(t_{j})\epsilon |z-v_{i}|} dz\right)$$

We have by construction, for all  $x \in usedv(\rho_{j+1})$ ,

$$\eta_{h}^{2}(\mathbf{x}) = \eta_{h}^{1}(\mathbf{x}) + m_{j+1}(\mathbf{x})$$

It is also easy to see that  $\eta_b^i$  is compatible with  $\rho_{j+1}$  for each i = 1, 2. Thus, by the induction hypothesis, we have that

$$\Pr[\epsilon, \eta_b^1, \kappa_{1,j+1}] \ge e^{-\epsilon \operatorname{wt}_{j+1}} \Pr[\epsilon, \eta_b^2, \kappa_{2,j+1}].$$

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The lemma, now follows in this case from the following observation:

$$\int_{\ell_1}^{u_1} e^{-d(t_j)\epsilon |z-v_1|} dz = \int_{\ell_1}^{u_1} e^{-d(t_j)\epsilon |z-v_2+\Delta|} dz$$
$$= \int_{\ell_1+\Delta}^{u_1+\Delta} e^{-d(t_j)\epsilon |z-v_2|} dz$$
$$\ge \int_{\ell_1+1}^{u_1-1} e^{-d(t_j)\epsilon |z-v_2|} dz$$
$$= \int_{\ell_2}^{u_2} e^{-d(t_j)\epsilon |z-v_2|} dz$$

(b) Let us consider the case when  $f_i = 1$  and  $a_i = 0$ .

Please note that if  $a_j = 0$  then j is neither a lcycle\_node nor a gcycle\_node. Thanks to Proposition 7, it follows that  $m_j(x) \neq -1$  for each  $x \in \text{smallv}(t_j)$  and  $m_j(x) \neq 1$  for each  $x \in \text{largev}(t_j)$ . Thus  $\ell_1 \leq \ell_2$  and  $u_2 \leq u_1$ , and hence  $\ell_1 < u_1$ . Also, observe that we have by definition  $m_{j+1}(x) = 0$  for each  $x \in \text{assignv}(t_j)$  and  $m_{j+1}(x) = m_j(x)$  for each  $x \in$ nonassignv $(t_j)$ . From this it is easy to see that for each  $x \in$ usedv $(\rho_{j+1})$ ,

$$\eta_z^2(\mathbf{x}) = \eta_z^1(\mathbf{x}) + m_{j+1}(\mathbf{x}).$$

Furthermore, it is easy to see that for each  $\ell_i < z < u_i$ ,  $\eta_z^k$  is compatible with  $\eta_{j+1}$ . As  $\ell_1 \le \ell_2$  and  $u_2 \le u_1$ , we get that for each  $\ell_2 < z < u_2$ , and  $k = 1, 2 \eta_z^k$  is compatible with  $\eta_{j+1}$ . By induction hypothesis, we get that for each  $\ell_2 < z < u_2$ ,

$$\Pr[\epsilon, \eta_z^1, \kappa_{1,j+1}] \ge e^{-\epsilon \operatorname{wt}_{j+1}} \Pr[\epsilon, \eta_z^2, \kappa_{2,j+1}]$$

Thus, we have

$$p_{1}(\epsilon) = \frac{d(t_{j})\epsilon}{2} \int_{\ell_{1}}^{u_{1}} e^{-d(t_{j})\epsilon |z-\nu_{1}|} \Pr[\epsilon, \eta_{z}^{1}, \kappa_{1,j+1}] dz$$

$$\geq \frac{d(t_{j})\epsilon}{2} \int_{\ell_{2}}^{u_{2}} e^{-d(t_{j})\epsilon |z-\nu_{1}|} \Pr[\epsilon, \eta_{z}^{1}, \kappa_{1,j+1}] dz$$

$$\geq e^{-\epsilon \operatorname{wt}_{j+1}} \frac{d(t_{j})\epsilon}{2}$$

$$\int_{\ell_{2}}^{u_{2}} e^{-d(t_{j})\epsilon |z-\nu_{1}|} \Pr[\epsilon, \eta_{z}^{2}, \kappa_{2,j+1}] dz$$

$$= e^{-\epsilon \operatorname{wt}_{j+1}} \frac{d(t_{j})\epsilon}{2}$$

$$\int_{\ell_{2}}^{u_{2}} e^{-d(t_{j})\epsilon |z-\nu_{2}+\Delta|} \Pr[\epsilon, \eta_{z}^{2}, \kappa_{2,j+1}] dz$$

Now, in case  $t_j$  is a non-input transition,  $\Delta = 0$  and  $b_j = 0$ . Hence we get

$$p_1(\epsilon) \ge e^{-\epsilon \operatorname{wt}_{j+1}} p_2(\epsilon) = e^{\epsilon(-f_j(a_j+b_j) - \operatorname{wt}_{j+1})} p_2(\epsilon)$$

as required.

Otherwise,  $b_j = 1$ . Since  $\Delta \in [-1, 1]$ , we have that

$$e^{-d(t_j)\epsilon} |z-v_2+\Delta| > e^{-d(t_j)\epsilon} e^{-d(t_j)\epsilon} |z-v_2|$$

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Thus,

$$p_{1}(\epsilon) \geq e^{-\epsilon \operatorname{wt}_{j+1}} \frac{d(t_{j})\epsilon}{2} e^{-d(t_{j})\epsilon}$$
$$\int_{\ell_{2}}^{u_{2}} e^{-d(t_{j})\epsilon |z-\nu_{2}|} \Pr[\epsilon, \eta_{z}^{2}, \kappa_{2,j+1}] dz$$
$$= e^{\epsilon(-f_{j}(a_{j}+b_{j})d(t_{j}) - \operatorname{wt}_{j+1})} p_{2}(\epsilon)$$

as required.

(c) Let us consider the case when  $f_i = 1$  and  $a_i = 1$ . Thus *j* is a lcycle node or a gcycle node. We consider the case when *j* is a lcycle\_node. The case when *j* is a gcycle\_node is similar and left out.

Since j is a lcycle\_node,  $t_j$  does not output insample. Thus, we have that  $\ell_i = \log_k^i$  and  $u_i = up_k^i$  for each i = 1, 2. Further, we must have by definition for each  $x \in largev(t_j)$ , lastassign(x, j) is also a lcycle\_node. Thus,  $m_i(x) = -1$  for each  $x \in largev(t_i)$ . From these observations, we get that

$$u_2 = u_1 - 1$$
  $\ell_2 - 1 \le \ell_1 \le \ell_2 + 1.$ 

Now, as we have assumed that  $\ell_2 < u_2$ , we get

$$\ell_1 < u_2 + 1 = u_1$$

as desired.

It is also easy to see that  $\eta_z^1$  and  $\eta_{z-1}^2$  is compatible with  $\rho_{j+1}$ for each  $\ell_2 < z - 1 < u_2$ , ie, for each  $\ell_2 + 1 < z < u_1$ . Also, by definition as *j* is a lcycle\_node,  $m_{j+1}(x) = -1$  for each  $x \in assignv(t_i)$ . Thus, we have  $x \in usedv(\rho_{i+1})$ ,

$$\eta_{z-1}^2(\mathbf{x}) = \eta_z^1(\mathbf{x}) + m_{j+1}(\mathbf{x}).$$

By induction hypothesis, we get that for each  $\ell_2 + 1 < z < u_1$ ,

$$\Pr[\epsilon, \eta_z^1, \kappa_{1,j+1}] \ge e^{-\epsilon \operatorname{wt}_{j+1}} \Pr[\epsilon, \eta_{z-1}^2, \kappa_{2,j+1}]$$

Observe that 
$$\ell_2 + 1 \ge \ell_1$$
. We thus have

$$\begin{split} &) = \frac{d(t_{j})\epsilon}{2} \int_{\ell_{1}}^{u_{1}} e^{-d(t_{j})\epsilon |z-v_{1}|} \Pr[\epsilon, \eta_{z}^{1}, \kappa_{1,j+1}] dz \\ &\geq \frac{d(t_{j})\epsilon}{2} \int_{\ell_{2}+1}^{u_{1}} e^{-d(t_{j})\epsilon |z-v_{1}|} \Pr[\epsilon, \eta_{z}^{1}, \kappa_{1,j+1}] dz \\ &\geq \frac{d(t_{j})\epsilon e^{-\epsilon w t_{j+1}}}{2} \\ &\int_{\ell_{2}+1}^{u_{1}} e^{-d(t_{j})\epsilon |z-v_{1}|} \Pr[\epsilon, \eta_{z-1}^{2}, \kappa_{2,j+1}] dz \\ &= \frac{d(t_{j})\epsilon e^{-\epsilon w t_{j+1}}}{2} \\ &\int_{\ell_{2}+1}^{u_{1}} e^{-d(t_{j})\epsilon |z-v_{1}|} \Pr[\epsilon, \eta_{z-1}^{2}, \kappa_{2,j+1}] dz \\ &= \frac{d(t_{j})\epsilon e^{-\epsilon w t_{j+1}}}{2} \\ &\int_{\ell_{2}}^{u_{1}-1} e^{-d(t_{j})\epsilon |z+1-v_{1}|} \Pr[\epsilon, \eta_{z}^{2}, \kappa_{2,j+1}] dz \\ &= \frac{d(t_{j})\epsilon e^{-\epsilon w t_{j+1}}}{2} \\ &\int_{\epsilon}^{u_{2}} e^{-d(t_{j})\epsilon |z+1-v_{2}+\Delta|} \Pr[\epsilon, \eta_{z}^{2}, \kappa_{2,j+1}] dz \end{split}$$

 $p_1(\epsilon)$ 

=

$$\geq \frac{d(t_j)\epsilon e^{-\epsilon \operatorname{wt}_{j+1}-\epsilon d(t_j)(1+\Delta)}}{2}$$
$$\int_{\ell_2}^{u_2} e^{-d(t_j)\epsilon |z-v_2|} \Pr[\epsilon, \eta_z^2, \kappa_{2,j+1}] dz$$
$$= e^{-\epsilon \operatorname{wt}_{j+1}-\epsilon d(t_j)(a_j+\Delta)} p_2(\epsilon)$$

The result follows by observing that  $\Delta \leq b_i$  as  $\Delta = 0$  if  $b_i = 0$  and in the interval [-1, 1] otherwise. 

**Corollary 10.** If the DiPA  $\mathcal{A}$  is well-formed then there is a number wt( $\mathcal{A}$ ) such that  $\mathcal{A}$  is wt( $\mathcal{A}$ ) $\epsilon$ -differentially private. Further, the number  $wt(\mathcal{A})$  can be computed from  $aug(\mathcal{A})$  in polynomial time, and hence from  $\mathcal{A}$  is exponential time.

PROOF. Thanks to Proposition 6 and Theorem 9, it suffices to show that there is a  $\mathfrak{D}$  such that  $wt(\rho) \leq \mathfrak{D}$  for every run  $\rho$  of  $\operatorname{aug}(\mathcal{A})$ . Now, from that fact that if a transition of the automaton can contribute to the weight of a run at most once (See Proposition 7), it is immediate to see such a  $\mathfrak{D}$  exists. We detail below a better bound on  $\mathfrak{D}$ .

Consider the underlying labeled graph G constructed from the augmented automaton  $aug(\mathcal{A})$  as follows. Its vertices are states of  $aug(\mathcal{A})$  and there is an edge from  $q_1$  to  $q_2$  if and only if there is a transition t from  $q_1$  and  $q_2$ . The label of the edge is t. We shall also assign weights to the edge as follows. We assign the edge weight  $ew_1 + w_2$  where e is 2 if the transition is an input transition and 1 otherwise.  $w_1$  is d(t) if either t is not a cycle transition or if there is a variable  $x \in assignv(t)$  and a run  $\rho$  from starting from  $q_2$  in which x is accessed without being assigned. Otherwise,  $w_1$  is 0.  $w_2$ is d'(t) if insample' is output in *t* and 0 otherwise.

Once the graph G has been constructed, we can construct its component graph G' and assign weights to each node and transition of this graph. We shall take the weight of a component in C to be the sum of the weights of all transitions in *C*. The weight of the edge from component  $C_1$  to  $C_2$  labeled *t* is taken to be the weight of the edge *t*. Note that *G'* is a DAG and can be computed in time polynomial in the size of aug( $\mathcal{A}$ ). Now, each path in *G'* has a weight which is the sum of weights of transitions and nodes. Let  $\mathfrak{D}$  be the maximum value amongst the weights of paths in *G'* starting from the component containing the initial state of aug( $\mathcal{A}$ ). It is easy to see that weight of any run of  $\mathcal{A}$  is bounded by  $\mathfrak{D}$  and that  $\mathfrak{D}$  can be computed from  $\mathcal{A}'$  in polynomial time.

A better approximation to  $\mathfrak{D}$  can be constructed by taking the bisimulation quotient of  $\operatorname{aug}(\mathcal{A})$  before running the above algorithm.

# Checking Well-formedness is in PSPACE.

**Theorem 11.** The problem of checking whether a DiPA is wellformed is decidable in PSPACE. When the number of variables is taken to be a constant k, then the problem of checking whether a DiPA is well-formed is decidable in polynomial time.

PROOF. Recall that  $\mathcal{A}$  is well-formed iff  $aug(\mathcal{A})$  is well-formed. Our PSPACE algorithm will first non-deterministically check if  $aug(\mathcal{A})$  has a leaking cycle without needing to construct the whole automaton. This will allow us to conclude that the problem of checking whether  $aug(\mathcal{A})$  has a leaking cycle is in PSPACE, thanks to Savitch's theorem.

The non-deterministic algorithm Alg for checking whether aug( $\mathcal{A}$ ) has a leaking cycle guesses a variable  $x \in$  stor and a run  $\rho C$  of aug( $\mathcal{A}$ ) incrementally such that (i) C is a cycle, and (ii) there are indices  $i_1$  and  $i_2$  such that x is assigned in the transition  $t_{i_1}$  and used in transition  $t_{i_2}$ . Note that as all runs of aug( $\mathcal{A}$ ) are feasible, the algorithm does not need to check the repeatability of the cycle C.

The algorithm Alg performs the above by guessing the variable x and the run  $\rho C = t_0 \cdots t_{n-1}$  one-by-one from the initial state of aug( $\mathcal{A}$ ), and at each steo

- checks that the source of the current guessed transition is exactly the target of the last guessed transition,
- checks that the current guesses transition is a valid transition of aug(A),
- if has not guessed as yet, Alg guesses if the current guessed transition is the first transition of *C*; if it guesses that it indeed is, then it remembers src(*t<sub>i</sub>*) in the memory, and that fact that it guessed cycle *C* has begun,
- if the cycle *C* begins at position *i* or has already begun then it additionally checks if
- (1) x is assigned in the current guessed transition
- (2) x is used in the current guessed transition.
- Alg declares that aug( $\mathcal{A}$ ) has a leaking cycle if the target of the last transition is exactly the source of the cycle *C* it guessed, and if x was assigned and used in its guessed cycle *C*.

It is easy to see that the path  $\rho C$  can be guessed without explicitly constructing  $\operatorname{aug}(\mathcal{A})$  and that the above checks require only space polynomial in the size of  $\mathcal{A}$ . If  $\operatorname{aug}(\mathcal{A})$  has a leaking cycle; then

we can declare that  $\mathcal{A}$  is not well-formed. Otherwise, we check if  $aug(\mathcal{A})$  has a leaking pair.

To check for a leaking pair of  $aug(\mathcal{A})$ , we have to search for a run  $\rho$  of  $aug(\mathcal{A})$  from the initial state, such that there are indices  $0 \le i_1 < j_1 \le |\rho|$  and  $0 \le i_2 < j_2 \le |\rho|$  such that following conditions hold.

- C<sub>1</sub> = ρ[i<sub>1</sub> : j<sub>1</sub>] and C<sub>2</sub> = ρ[i<sub>2</sub> : j<sub>2</sub>] are cycles. (Note that since aug(A) does not have leaking cycles by assumption, all cycles of aug(A) are non-leaking cycles).
- (2)  $C_1$  and  $C_2$  are non-overlapping.
- (3) There is a path  $k_1, k_2, \ldots k_m$  in the dependency graph  $G_\rho$  such that  $i_1 \le k_1 < j_1$  ( $k_1$  is on  $C_1$ ),  $i_2 \le k_m < j_2$  ( $k_m$  is on  $C_2$ ),  $k_2 < k_1$  and  $k_{m-1} < k_m$ .

Now, it is easy to see that a non-deterministic algorithm that runs in space polynomial in the size of  $\mathcal{A}$  can check for a run  $\rho$ that satisfies the first two conditions above, as in the case of a leaking cycle. The challenge is to check for the third condition, as maintaining the dependency graph for the entire run may not be possible in polynomial space. However, we will exploit the relations lt and eq in an augmented state. Let  $\operatorname{trg}(\rho[i]) = (q, \operatorname{lt}, \operatorname{eq})$ . Recall that

- (1)  $(\mathbf{x}_1, \mathbf{x}_2) \in \mathsf{It}$  if and only if there is a path from  $\mathsf{lastassign}_{\rho}(i, \mathbf{x}_1)$  to  $\mathsf{lastassign}_{\rho}(i, \mathbf{x}_2)$  in the graph  $G_{\rho[0:i]}$ .
- (2) and  $(x_1 x_2) \in eq$  if and only if  $astassign_{\rho}(i, x_1) = astassign_{\rho}(i, x_2)$ .

To exploit the relations It and eq, the algorithm shall pretend that there are two additional real variables,  $V_1$  and  $V_2$  that are assigned exactly once each during the run  $\rho$ . The variable  $V_1$  is assigned when the algorithm guesses that the current index is the index  $k_2$ and the  $V_2$  is assigned when the algorithm guesses that the current index is the index  $k_{m-1}$ . The non-deterministic algorithm, Alg<sub>1</sub>, for checking the existence of a leaking cycle proceeds as follows. It guesses the run  $\rho = t_0 \cdots t_{n-1}$  incrementally. At each step,

- checks that the source of the current guessed transition is exactly the target of the last guessed transition,
- checks that the current guessed transition is a valid transition of aug(A),
- if Alg<sub>1</sub> has not guessed as yet that the index k<sub>2</sub> has been encountered, Alg<sub>1</sub> guesses if the current transition is the desired transition t<sub>k<sub>2</sub></sub> or not. If it guesses that the current transition is the desired transition t<sub>k<sub>2</sub></sub>, then it treats the variable V<sub>1</sub> as being assigned in the current transition.
- if  $Alg_1$  has not guessed as yet that the index  $k_{m-1}$  has been encountered,  $Alg_1$  guesses if the current transition is the desired transition  $t_{k_{m-1}}$  or not, if it guesses that the current transition is the desired transition  $t_{k_{m-1}}$ , then it treats the variable  $V_2$  as being assigned in the current transition,
- if Alg<sub>1</sub> has yet to guess the cycle C<sub>1</sub>, then it guesses if the current transition is the first transition of cycle C<sub>1</sub>; if it guesses that it indeed does, then it remembers the source of the current transition in the memory, and the fact that it guessed cycle C<sub>1</sub> has begun,
- it Alg<sub>1</sub> has yet to guess the cycle C<sub>2</sub>, then it guesses if the cycle C<sub>2</sub> begins at the current transition; if it guesses that it indeed does, then it remembers the source of the current

transition in the memory, and that fact that it guessed cycle  $C_2$  has begun,

- makes sure that it is not guessing that it is in cycle *C*<sub>1</sub> and *C*<sub>2</sub> simultaneously,
- if Alg<sub>1</sub> is guessing that the cycle C<sub>1</sub> is being processed, then it guesses if the current transition is the transition t<sub>k<sub>1</sub></sub>; and checks the guess by checking if there is a variable x ∈ largev(t) such that (V<sub>1</sub>, x) ∈ eq where t is the current transition and eq is such that src(t) = (q, lt, eq);
- if Alg<sub>1</sub> is guessing that the cycle C<sub>2</sub> is being processed, then it guesses if the current transition is the transition t<sub>k<sub>m</sub></sub>; and checks the guess by checking if there is a variable x ∈ smallv(t) such that (V<sub>2</sub>, x) ∈ eq where t is the current transition and eq is such that src(t) = (q, lt, eq);
- if it is guessing that the current transition is in the cycle C<sub>i</sub>, for i = 1, 2, it guesses if the current guessed transition is the last transition of C<sub>i</sub>; if that is the case, then it checks that the target of the current transition is exactly the source of the cycle C<sub>i</sub> it has stored in its memory, and
- once the algorithm guesses that both cycles C<sub>1</sub> and C<sub>2</sub> are completed, it guesses if the current transition is the final transition of *ρ*. If it guesses that the current transition is indeed the final transition and the target of the transition is the triple (*q*, lt, eq), then it declares that aug(*A*) has a leaking pair if all the above checks passed, and if either (V<sub>1</sub>, V<sub>2</sub>) ∈ lt or (V<sub>1</sub>, V<sub>2</sub>) ∈ eq.

It is easy to show that the above algorithm runs in space polynomial in the size of  $\operatorname{aug}(\mathcal{A})$  and that the algorithm declares that  $\operatorname{aug}(\mathcal{A})$  has a leaking pair iff  $\operatorname{aug}(\mathcal{A})$  has a leaking pair thanks to the properties of lt and eq.

Now, if  $\operatorname{aug}(\mathcal{A})$  does not have a leaking cycle or a leaking pair, then the algorithm for well-formedness will check for disclosing cycle and privacy violating path next. The PSPACE algorithm for checking disclosing cycle can be designed along the same lines as the algorithm for checking for leaking cycle, and the PSPACE algorithm for checking for privacy violating path can be designed along the same lines as the algorithm for checking for check for leaking pair.

# B NECESSITY OF WELL-FORMEDNESS FOR OUTPUT-DISTINCT DIPA

In this section, we give the proof showing that if DiPA  $\mathcal{A}$  satisfying Output Distinction property is not well-formed then  $\mathcal{A}$  is not differentially private. The proof will be broken into four Lemmas 12, 14, 15 and 16, given in this section.

## Leaking cycles implies no privacy

**Lemma 12.** A DiPA  $\mathcal{A}$ , satisfying Output Distinction property, is not differentially private if it has a leaking cycle.

Let  $\mathcal{A} = (Q, \Gamma, q_{init}, X, P, \delta)$ . Assume that  $\mathcal{A}$  satisfies Output Distinction property and has a leaking cycle.

Let  $\eta' = t_0, t_1, ..., t_{m'-1}$  be a run of  $\mathcal{A}$ , starting from the initial state  $q_{\text{init}}$ , that is a leaking cycle. For  $0 \le u < m'$ , let  $c_u$  be the guard of transition  $t_u$ . From the definition of a leaking cycle, we see that there exists an integer  $m \le m' - 1$  such that the suffix  $C' = t_m, ..., t_{m'-1}$  is a cycle that is repeatable, and there exist distinct integers *i*, *j* such that  $m \le i, j < m'$  and a variable  $x_{i'}$   $(1 \le i' \le k)$ 

such that  $t_i$  is an assignment transition for the variable  $x_{i'}$  and  $c_j$ references  $x_{i'}$ . Let n' = m' - m. Now, we extend  $\eta'$  by repeating the cycle C' to get the run  $\eta = \eta'C'$ . We let  $\eta = t_0, ..., t_{m+n-1}$  where n = 2n'. For  $0 \le u < m + n$ , let  $q_u = \operatorname{src}(t_u)$  and  $\gamma_u = \operatorname{out}(t_u)$ . Also, let  $q_{m+n} = \operatorname{trg}(t_{m+n-1})$ . Note that  $q_u = q_{u+n'}$ ,  $\gamma_u = \gamma_{u+n'}$  for  $m \le u < m + n'$ . Now, let  $C = t_m, ..., t_{m+n-1}$ . Note that C is a cycle. It is not difficult to see the following property is satisfied by  $\eta$ : the variable  $x_{i'}$  is referenced in the condition  $c_{j+n'}$  and the transition  $t_i$  is an assignment transition for  $x_{i'}$ .

As before, let  $t_u$  be the *u*-th transition of  $\eta$  and  $c_u$  be the guard of the *u*-th transition. Further, let  $d_u$  and  $\mu_u$  be such that  $P(q_u) = (d_u, \mu_u)$  for each *u*.

From our discussion above, we see that there exist at least one triple (u', v', w') of integers such that  $m \leq u' < v' < m + n$ ,  $1 \le w' \le k$  and the following properties are satisfied: (i)  $c_{v'}$  references  $x_{w'}$  and (ii)  $u' = \text{lastassign}_n(x_{w'}, v')$ . We call such a triple as an assign\_refer triple. Now, we give the definitions and proof assuming that there exists at least one triple (u', v', w'), as given above, such that the condition insample  $\geq x_{w'}$  is a conjunct in the guard  $c_{v'}$  (the case when for all assign\_refer triples (u', v', w'), the condition insample  $< x_{w'}$  is a conjunct in the guard  $c_{v'}$ , is handled similarly in a symmetric fashion as outlined later). Now we fix a triple of integers (i, j, i') as follows. If there exists at least one assign\_refer triple (u', v', w') such that  $q_{u'} \in Q_{non}$  then we take (i, j, i') to be any such triple so that  $\mu_i$  is the maximum among all such triples; otherwise, we take (i, j, i') be any assign\_refer triple. In the remainder of our proof, we fix the triple of integers i, j, i' as specified above.

Consider any integer  $\ell > 0$ . We define a run  $\eta_{\ell}$  starting from  $q_{\text{init}}$  by repeating the cycle  $C = t_m, \ldots t_{m+n-1}$ ,  $\ell$  times. Formally,  $\eta_{\ell} = t_0, t_1, \ldots, t_{m+\ell n-1}$  such that  $q_u = q_{u-n}$  and  $\gamma_u = \gamma_{u-n}$  for  $m + n \leq u < m + \ell n$ . Let  $\gamma(\ell) = o_0 \cdots o_{m+\ell n-1}$  be the output sequence of length  $m + \ell n$  such that  $o_u = \gamma_u$  if  $\sigma_u \in \Gamma$ , otherwise  $o_u = (\gamma_u, -\infty, \infty)$ . Once again, we let  $c_u$  be the guard of the *u*-th transition  $t_u$ . Now, for the given  $\ell > 0$ , we define two neighboring input sequences  $\alpha(\ell) = a_0 \cdots a_{m+\ell n-1}$  and  $\beta(\ell) = b_0 \cdots b_{m+\ell n-1}$  each of length  $m + \ell n$ .

Let  $Z' = \{\frac{1}{2}\} \cup \{|\mu_u - \mu_{u'}| : 0 \le u, u' < m + n\ell, q_u, q_{u'} \in Q_{\text{non}}, \mu_u \ne \mu_{u'}\}$ , and  $\Delta = \frac{\min(Z')}{m+n\ell}$ . Observe that  $\Delta > 0$ . Let  $Z = \{\mu_u : m \le u < m + n, q_u \in Q_{\text{non}}\}$ . Now, we define a constant z as follows. If  $Z \ne \emptyset$  then  $z = \min(Z) - \frac{1}{2}$ , otherwise  $z = -\frac{1}{2}$ . Let  $U = \{u : q_u \in Q_{\text{non}}, m \le u < m + n\ell\}$  and  $U' = \{u : q_u \in Q_{\text{non}}, 0 \le u < m\}$ .

Recall that  $G_{\eta_{\ell}} = (V, E)$  is the dependence graph of the run  $\eta_{\ell}$ . Note that  $V = \{u : 0 \le u < m + n\ell\}$ . A source node in  $G_{\eta_{\ell}}$  is a node that has no incoming edges and a sink node is a node that has no outgoing edges. The length of a path in  $G_{\eta_{\ell}}$  is the number of edges on the path. Note that if the path is a single node, then it's length is zero. Observe that the length of any path is less than  $m + n\ell$ . We say that a path  $p = (u_0, ..., u_r)$  in  $G_{\eta_{\ell}}$ , is a maximal path iff either  $u_0$  is a source node or  $u_0 \in U$ , and  $\forall k_1, 0 < k_1 \le r, u_{k_1} \notin U$ . For a maximal path p, as given above, we define weight(p) to be the value  $z' + r\Delta$  where  $z' = \mu_{u_0}$  if  $u_0 \in U$ , otherwise z' = z. Now, we define a function  $\psi$  that associates a real value with each node in V as follows. For  $u \in V$ ,  $\psi(u)$  is as given below: if  $u \in U$ ,  $\psi(u) = \mu_u$ ; if  $u \notin U$  and is a source node then  $\psi(u) = z$ ; in all other cases,  $\psi(u)$  is the maximum weight of a maximal path ending in u.

From our assumption about  $\mathcal{A}$ , we observe that  $\eta_{\ell}$  is a strongly feasible run. Using this fact, we establish that, if  $(u, u') \in E$  then  $\psi(u') \geq \psi(u) + \Delta$ . This is shown as follows. If  $u' \notin U$  then  $\psi(u')$ is the maximum weight of a maximal path ending in u'. If u'' is the node just before u' on such a maximal path, then by definition  $\psi(u') = \psi(u'') + \Delta$ , and further more  $\psi(u'') \geq \psi(u)$ , and the desired result follows. Now, consider the case, when  $u' \in U$ . Now, consider a predecessor u'' of u', i.e.,  $(u'', u') \in E$ , such that  $\psi(u'')$ is maximum. Clearly  $u'' \notin U$ . Consider the maximal path ending in u'' whose weight is maximum. If this path starts from a node which is a source node, then by definition, we see that the weight of the path is less than  $\min(\{\mu_u : u \in U\})$  and the result follows from this. On the other, if the above maximal path ending u'' starts from a node  $w \in U$ , we see that  $\mu_w < \mu_{u'}$  (because every feasible execution in  $\mathcal{A}$  is strongly feasible); the required result follows from this observation and the fact the length of the above maximal path is less than  $m + n\ell$ .

For each  $u, 0 \le u < m + n\ell$ , if  $u \in U' \cup U$  then let  $a_u := \tau$ , otherwise let  $a_u = \psi(u)$ . For any such u, let  $X_u$  be the random variable with distribution  $Lap(d_u \epsilon, \mu_u)$  or  $Lap(d_u \epsilon, a_u)$ , respectively, depending on whether  $u \in U' \cup U$  or not. Consider any  $u, 0 \leq u < m + n\ell$ , such that  $q_u \notin Q_{\text{non}}$ . (Note that if  $q_u \in Q_{\text{non}}$ then  $c_u$  is the condition true.) The guard  $c_u$  is a conjunction of atomic conditions of the form insample  $\geq x_{k_1}$  or of the form insample  $\langle x_{k_1}$  for some  $k_1$ ,  $1 \leq k_1 \leq k$ . Let  $u_1 < u$  be the maximum integer such that the transition  $t_{u_1}$  is an assignment transition for the variable  $x_{k_1}$ . Now, in  $c_u$ , we replace insample by the random variable  $X_u$  and replace  $x_{k_1}$  by  $X_{u_1}$ . Let  $c'_u$  be the condition obtained by modifying every atomic condition in  $c_u$ as specified above. Now, let  $X(\ell) = \{X_u : 0 \le u < m + n\ell\},\$  $C(\ell) = \{c'_u : 0 \le u < m + n\ell\}$ . Let  $\rho_{\alpha}(\ell)$  denote the computation given by the triple  $(\eta_{\ell}, \alpha(\ell), \gamma(\ell))$ . Now,  $\Pr[\epsilon, \rho_{\alpha}(\ell)]$  is the probability that the random variables in  $X(\ell)$  satisfy all the guard conditions in  $C(\ell)$ . Let  $RPr[\epsilon, \rho_{\alpha}(\ell)]$  be the probability that the random variables in  $X(\ell)$  satisfy all the guard conditions in  $C(\ell)$  and  $\forall u \in U', X_u \in [\psi(u) - \frac{\Delta}{2}, \psi(u) + \frac{\Delta}{2}]$ . For all  $u \in U'$ , we call the intervals  $[\psi(u) - \frac{\Delta}{2}, \psi(u) + \frac{\Delta}{2}]$  as bands. Clearly,  $\Pr[\epsilon, \rho_{\alpha}(\ell)] \ge R\Pr[\epsilon, \rho_{\alpha}(\ell)]$ . Let  $C\Pr[\epsilon, \rho_{\alpha}(\ell)]$  be the conditional probability that the random variables in  $X(\ell)$  satisfy all the guard conditions in  $C(\ell)$  given that  $\forall u \in U', X_u \in [\psi(u) - \frac{\Lambda}{2}, \psi(u) + \frac{\Lambda}{2}]$ . Now, we see that  $RPr[\epsilon, \rho_{\alpha}(\ell)] = (CPr[\epsilon, \rho_{\alpha}(\ell)] \cdot Prob[\forall u \in$  $U', X_u \in [\psi(u) - \frac{\Lambda}{2}, \psi(u) + \frac{\Lambda}{2}])$ . It can be easily shown that there exists  $\epsilon' > 0$ , such that  $\forall \epsilon > \epsilon'$ , for every  $u \in U'$ ,  $\operatorname{Prob}[X_u \in [\psi(u) - \frac{\Lambda}{2}, \psi(u) + \frac{\Lambda}{2}]] \geq \frac{1}{4}e^{-\epsilon d_u |\mu_u - \psi(u) - \frac{\Lambda}{2}|}.$  From this, we see that, there exists constants  $c_1, c_2, \epsilon' > 0$ , such that  $\forall \epsilon > \epsilon'$ ,  $\operatorname{Prob}[\forall u \in U', X_u \in [\psi(u) - \frac{\Lambda}{2}, \psi(u) + \frac{\Lambda}{2}]] \geq c_1 e^{-c_2 \epsilon}.$ 

Now, we give a lower bound for  $CPr[\epsilon, \rho_{\alpha}(\ell)]$ , for large values of  $\epsilon$ . Recall that, each conjunct in  $c'_{u} \in C(\ell)$ , for  $u \notin U \cup U'$ , involves two random variables, say  $X_{u}, X_{u'} \in X(\ell)$ . We replace each such conjunct in  $c'_{u}$  as follows, if  $u' \in U'$ ; if the conjunct is the atomic condition  $X_{u'} \ge X_{u}$  we replace it by  $\psi(u') - \frac{\Lambda}{2} \ge X_{u}$ , otherwise the conjunct is the atomic condition  $X_{u'} < X_{u}$  and we replace it by  $\psi(u') + \frac{\Lambda}{2} < X_{u}$ . The condition  $c'_{u}$  is unchanged if  $u' \notin U'$ . Let the resulting set of conditions be denoted by  $c''_{u}$ . Now, let  $C''(\ell) = \{c''_u : 0 \le u < m + n\ell\}$ . Now, it is easily seen that  $\operatorname{CPr}[\epsilon, \rho_{\alpha}(\ell)]$  is greater than or equal to  $p_{\ell}$ , where  $p_{\ell}$  is the probability that the random variables in  $X(\ell)$  satisfy all the conditions in  $C''(\ell)$ . Now, using similar proof technique as given in [10, 11], we can show that there exists a constant  $\epsilon'_{\ell}$ , such that  $\forall \epsilon > \epsilon'_{\ell}, p_{\ell} > \frac{1}{2}$ . (This is because, for every conjunct of  $c''_u$  of the form  $X_{u_1} \ge X_{u_2}$  or of the form  $X_{u_2} < X_{u_1}$ , it is the case that  $a_{u_1} > a_{u_2}$ . For every conjunct of the form  $X_{u_1} \le c'$ , we have  $a_{u_1} < c'$  and for conjuncts of the form  $c' < X_{u_1}$ , it is the case that  $a_{u_1} > c'$ , where c' is a constant). Now putting all the above observations together, by taking  $\epsilon_{\ell} = \max(\epsilon', \epsilon'_{\ell})$ , we see that  $\forall \epsilon > \epsilon_{\ell}$ ,  $\Pr[\epsilon, \rho_{\alpha}(\ell)] > \frac{c_1}{2}e^{-c_2\epsilon}$ .

Now, we define  $\hat{\beta}(\ell) = b_0 \cdots b_{m+\ell n-1}$ . To do this, we prove some properties of  $\alpha(\ell)$ . For each u,  $0 \le u < m + n\ell$ , we define the real value  $\mathbf{a}_u$  as follows: if  $u \in U' \cup U$  then  $\mathbf{a}_u = \mu_u$ , otherwise  $\mathbf{a}_u = a_u$ . from the way we chose the integers i, j, i' and our assumption that the condition insample  $\ge x_{i'}$  is a conjunct of the guard  $c_j$ , we see that, for every  $\ell'$  such that  $0 \le \ell' < \ell$  the following properties are satisfied:  $t_{i+n\ell'}$  is an assignment transition for  $\mathbf{x}_{i'}$ ; the condition insample  $\ge \mathbf{x}_{i'}$  is not an assignment transition for  $\mathbf{x}_{i'}$ ; the condition  $t_i + n\ell' < k_1 < j + n\ell'$ ,  $t_{k_1}$  is not an assignment transition for  $\mathbf{x}_{i'}$ . Now, we fix  $\ell'$  to be any integer such that  $0 \le \ell' < \ell$ . We show below that  $\mathbf{a}_{i+n\ell'} < \mathbf{a}_{j+n\ell'} \le \mathbf{a}_{i+n\ell'} + \frac{1}{2}$ .

We proceed as follows. First, observe that  $j + n\ell' \notin U$ , since  $c_{j+n\ell'}$  is not the condition true. From our definition, we see that  $\mathbf{a}_{j+n\ell'}$  is the maximum of the weights of maximal paths in  $G_{\eta_{\ell}}$  that end at the node  $j + n\ell'$ . Let  $(i_0, i_1, ..., i_{\ell_2})$  be the maximal path in  $G_{\eta_\ell}$ that ends in the node  $j + n\ell'$  (i.e.,  $i_{\ell_2} = j + n\ell'$ ) having maximum weight among all such paths. Clearly the length of this path is  $\ell_2$ and  $\ell_2 < m + n\ell$ . If the node  $i + n\ell'$  lies on the above path then the desired result follows from the definition of  $\psi(i + n\ell')$ . Now assume that the node  $i + n\ell'$  does not lie on the above path. Now, we have two cases. In the first case,  $i \in U$ . Clearly  $(i, j) \in E$ . From the way we chose *i*, *j*, *i'*, it is the case that  $\mu_i$  is the maximum of all  $\mu_u$  such that there is an assign\_refer triple (u, v, w) where  $u \in U$ and insample  $\geq x_w$  is a conjunct of  $c_v$ . It should be easy to see that  $\mu_{i_0} = \mu_i$  and  $\mathbf{a}_{i+n\ell'} < \mathbf{a}_{i+n\ell'} < \mathbf{a}_{i+n\ell'} + \frac{1}{2}$ . Now, consider the case when  $i \notin U$  and hence  $i_0 \notin U$ . In this case also, we see, from the definition of  $\psi$ , that  $\mathbf{a}_{j+n\ell'} \leq z + \frac{1}{2}$  and  $\mathbf{a}_{i+n\ell'} \geq z$  and the desired result follows.

Now, we give the values of  $b_u$ , for  $0 \le u < m + n\ell$ . For every  $\ell'$ , such that  $0 \le \ell' < \ell$ ,  $b_{j+n\ell'} = a_{j+n\ell'} - 1$  and for all other values of u,  $b_u = a_u$ . For each u,  $0 \le u < m + n\ell$ , let  $\mathbf{b}_u$  be defined as follows: if  $u \in U' \cup U$  then  $\mathbf{b}_u = \mu_u$ , otherwise  $\mathbf{b}_u = b_u$ . Since  $\mathbf{a}_{i+n\ell'} < \mathbf{a}_{j+n\ell'} \le \mathbf{a}_{i+n\ell'} + \frac{1}{2}$ , we see that  $\mathbf{b}_{j+n\ell'} \le \mathbf{b}_{i+n\ell'} - \frac{1}{2}$ . Clearly, the input sequence  $\beta(\ell)$  is a neighbor of  $\alpha(\ell)$ . Now, using the same analysis, as in [10, 11], it is easily shown that the input sequences  $\alpha(\ell)$  and  $\beta(\ell)$  are witnesses for violation of the differential privacy property by  $\mathcal{A}$ .

In the above definition of the input sequences  $\alpha(\ell)$  and  $\beta(\ell)$ , we assumed that there exists an assign\_refer triple (u, v, w) such that the condition insample  $\geq x_w$  is a conjunct in the guard  $c_v$ . Now, we give the construction for the other case when no such assign\_refer triple exists, that is, for every assign\_refer triple (u, v, w), the condition insample  $< x_w$  is a conjunct of  $c_v$ . Now, we chose an assign\_refer triple (i, j, i') as follows. If there exists at least one assign\_refer triple (u, v, w) such that  $q_u \in Q_{non}$ , then we chose (i, j, i') to be one such triple so that  $\mu_i$  is the minimum among all such triples; otherwise, we chose (i, j, i') to be any of the assign\_refer triples. Now, the proof is similar to the earlier case with the following changes. Basically, we change the definitions (in a symmetric way) of the function  $\psi$ , of maximal paths and their weights in  $G_{\eta_\ell} = (V, E)$ .

The constant  $\Delta$  is same as before, i.e.,  $\Delta = \min(Z')$  where  $Z' = \{\frac{1}{2}\} \cup \{|\mu_u - \mu_{u'}| : 0 \le u, u' < m + n\ell, q_u, q_{u'} \in Q_{\text{non}}, \mu_u \neq \mu_{u'}\}$ . However, the constant *z* is given as follows. If  $Z \neq \emptyset$  then  $z := \max(Z) + \frac{1}{2}$ , otherwise,  $z = \frac{1}{2}$ ; here  $Z = \{\mu_u : m \le u < m + n, q_u \in Q_{\text{non}}\}$ . As before,  $U = \{u : q_u \in Q_{\text{non}}, m \le u < m + n\ell\}$  and  $U' = \{u : q_u \in Q_{\text{non}}, 0 \le u < m\}$ .

We say that a path  $p = (u_0, ..., u_r)$  in  $G_{\eta_\ell}$ , is a maximal path iff either  $u_r$  is the sink node or  $u_r \in U$ , and  $\forall k_1, 0 \le k_1 < r, u_{k_1} \notin U$ . For a maximal path p, as given above, we define weight(p) to be the value  $z' - r\Delta$  where  $z' = \mu_{u_r}$  if  $u_r \in U$ , otherwise z' = z. Now, we define the function  $\psi$  that associates a real value with each node in V as follows. For  $u \in V, \psi(u)$  is as given below: if  $u \in U, \psi(u) = \mu_u$ ; if  $u \notin U$  and is a sink node then  $\psi(u) = z$ ; in all other cases,  $\psi(u)$  is the minimum weight of a maximal path starting with u.

The definition of  $\alpha(\ell)$  is same as before with the modified definition of  $\psi$ . To define  $\beta(\ell)$ , we modify the earlier approach as follows. First, for each integer  $\ell'$ , such that  $0 \le \ell' < \ell$ , we show that  $\mathbf{a}_{j+n\ell'} < \mathbf{a}_{i+n\ell'} \le \mathbf{a}_{j+n\ell'} + \frac{1}{2}$ . Now, the values of  $b_u$ , for  $0 \le u < m + n\ell$  are given as follows. For every  $\ell'$ , such that  $0 \le \ell' < \ell$ ,  $b_{j+n\ell'} = a_{j+n\ell'} + 1$  and for all other values of u,  $b_u = a_u$ . Now, for each u,  $0 \le u < m + n\ell$ , we define  $\mathbf{b}_u$  to be as in the previous case. Since  $\mathbf{a}_{j+n\ell'} < \mathbf{a}_{i+n\ell'} \le \mathbf{a}_{j+n\ell'} + \frac{1}{2}$ . The proof remains the same as in the previous case with the above changes.

#### Leaking pair implies no privacy

The following technical lemma states the properties of the dependency graph of a path that has a non-leaking cycle repeated many times.

**Lemma 13.** Let *C* be a non-leaking cycle of DiPA  $\mathcal{A}$  of length *m* and  $\rho = \rho' C \rho''$  be a run of  $\mathcal{A}$  starting from the initial state and  $\rho_L = \rho' C^L \rho''$  be the run in which the cycle *C* is repeated *L* times, for some L > 0. If (i, j) is an edge in  $G_\rho$  (respectively, (j, i) is an edge) then the following properties hold.

- At most one of the two indices i, j is on the cycle C (i.e., corresponds to a position on C).
- (2) If i is before C and j is on C then, in G<sub>ρL</sub>, there are edges from i to the node j + km (respectively, an edge from j + km to i), for every k, 0 ≤ k < L.</p>
- (3) If *i* is after *C* and *j* is on *C* then, in  $G_{\rho_L}$ , there is an edge from i + m(L 1) to j + m(L 1) (respectively, from j + m(L 1) to i + m(L 1)).

**PROOF.** We make the following observations. An edge (i, j) (or (j, i)) in  $G_{\rho}$  indicates that the transition in  $\rho$  at the position given by min(i, j), is an assignment transition for some variable  $x_{\ell}$  which is referenced in the guard of the transition in  $\rho$  at the position given by max(i, j). Property (1) of the lemma follows from this observation and the fact that *C* is non-leaking cycle. Property (2)

of the lemma follows from the fact that the transition of  $\rho$  at the position *i* is an assignment transition for some variable  $x_{\ell}$  which is referenced in the guard of the transition at the position *j* which is on *C*, and none of the transitions on *C* is an assignment transition for  $x_{\ell}$ . (Note that j + km is the position in the  $k^{th}$  iteration of *C* in  $\rho_L$  that corresponds to position *j* in  $\rho$ .) Property (3) follows from the observation that the transition in  $\rho$  at position *j* is an assignment transition for some variable  $x_{\ell}$  which is referenced in the guard of the transition at position *i*.

# **Lemma 14.** A DiPA $\mathcal{A}$ is not differentially private if it has a leaking pair.

PROOF. Let  $\mathcal{A} = (Q, \Gamma, q_{\text{init}}, X, P, \delta)$ . Assume that  $\mathcal{A}$  has a leaking pair. From the definition 9, we see that there exists a feasible run  $\rho$  of  $\mathcal{A}$  from the initial state  $q_{\text{init}}$  and there are indices  $\ell_1, \ell_2, \ell_3$  and  $\ell_4$  such that  $0 \leq \ell_1 < \ell_2 \leq |\rho|$  and  $0 \leq \ell_3 < \ell_4 \leq |\rho|$  such that the three conditions specified in the definition are satisfied. These conditions state the following. The sub-runs  $C = \rho[\ell_1 : \ell_2]$  and  $C' = \rho[\ell_3 : \ell_4]$  are both non-leaking cycles. The cycles C, C' are non-overlapping and may appear in either order, i.e., either  $\ell_2 \leq \ell_3$  or  $\ell_4 \leq \ell_1$ . Further more, there exists a path  $i_1, i_2, ..., i_{m-1}, i_m$  in  $G_\rho$  such that  $i_1$  is on C (i.e.,  $\ell_1 \leq i_1 < \ell_2$ ),  $t_{i_m}$  is on C' (i.e.,  $\ell_3 \leq i_m < \ell_4$ ),  $i_2 < i_1$  and  $i_{m-1} < i_m$ .

Let the lengths of *C*, *C*' be  $n_1, n_2$ , respectively; that is  $n_1 = \ell_2 - \ell_1$ and  $n_2 = \ell_4 - \ell_3$ . We give the proof assuming C appears before C', i.e.,  $\ell_2 \leq \ell_3$ . (The proof for the case when C' appears before *C*, i.e.,  $\ell_4 \leq \ell_1$  is similar and is left out). Now, we can write the run  $\rho$  as  $\rho = \rho_1 C \rho_2 C' \rho_3$  where  $\rho_1 = \rho[0 : \ell_1], \rho_2 = \rho[\ell_2 : \ell_3]$ and  $\rho_3 = \rho[\ell_4 : |\rho|]$ . Now, for any L > 0, consider the run  $\rho_L$ in  $\mathcal{A}$  starting from  $q_{init}$  in which the cycles C, C' are repeated L times each. That is,  $\rho_L = \rho_1(C)^L \rho_2(C')^L \rho_3$ . Using Lemma 13, the following are easily observed. For every k',  $0 \le k' < L$ , we have the following edges in  $G_{\rho_L}$ : there is an edge from the node  $i_1 + k' n_1$  to  $i_2$  (note that the former node is a copy of node  $i_1$  in iteration k' of *C*); if  $i_{m-1}$  is before *C* then there is an edge from  $i_{m-1}$  to the node  $(i_m + n_1(L-1) + n_2k')$  (note the later node is a copy of node  $i_m$  in iteration k' of C'); if  $i_{m-1}$  is on or after C then there is an edge from the node  $(i_{m-1} + (L-1)n_1)$  to  $(i_m + n_1(L-1) + n_2k')$ . Further more, using Lemma 13, the following can be shown. If  $i_{m-1}$  is before C in  $\rho$  then there is a path in  $G_{\rho_L}$  from  $i_2$  to  $i_{m-1}$ , otherwise there is a path in  $G_{\rho_L}$  from  $i_2$  to  $(i_{m-1} + n_1(L-1))$ . (Roughly speaking, such a path in  $G_{\rho_L}$  can be obtained by taking the path from  $i_2$  to  $i_{m-1}$  in  $G_{\rho}$ , and replacing every node in the path that is on C or on C' by the copy of the same node in the last iteration of that cycle in  $\rho_L$ ). Putting all the above observations together, it is easily seen that, for every k', k'' such that  $0 \le k', k'' < L$ , there is a path in  $G_{\rho_{I}}$  from the node  $(i_{1} + k'n_{1})$  to the node  $(i_{m} + n_{1}(L-1) + k''n_{2})$ ; essentially this states that from each node which is a copy of node  $i_1$  in every iteration of *C*, there is a path to the copy of node  $i_m$  in every iteration of C'.

Now, for each k, let  $t_k$  be the k-th transition of  $\rho_L$ ,  $c_k$  be the guard of the k-th transition,  $q_k = \operatorname{src}(t_k)$  and  $\sigma_k = \operatorname{out}(t_k)$ . Note that  $\rho = \rho_L$  when L = 1.

Let  $\gamma(L) = o_0 \cdots o_{n-1+(n_1+n_2)(L-1)}$  be the output sequence of length  $n + (n_1 + n_2)(L-1)$  such that  $o_k = \sigma_k$  if  $\sigma_k \in \Gamma$ , otherwise  $o_k = (\sigma_k, -\infty, \infty)$ .

Now, we define two adjacent input sequences  $\alpha(L) = a_0 \cdots a_{n-1+(n_1+n_2)(L-1)}$  and  $\beta(L) = b_0 \cdots b_{n-1+(n_1+n_2)(L-1)}$  as follows. We define  $\alpha(L)$  similar to the way we defined the corresponding input sequence in the proof of the Lemma 12. To do this, we consider the dependence graph  $G_{\rho_L} = (V, E)$  where  $V = \{u' : 0 \le u' < n+(n_1+n_2)(L-1)\}$ . For any  $u' \in V$ , let  $\mu_{u'}$  be the constant as defined in the proof of Lemma 12. Let  $Z' = \{\frac{1}{2}\} \cup \{|\mu_{u'} - \mu_{u''}| : u', u'' \in V, q_{u'}, q_{u''} \in Q_{\text{non}}, \mu_{u'} \neq \mu_{u''}\}$  and  $\Delta = \frac{\min(Z')}{n+(n_1+n_2)(L-1)}$ . Let  $Z = \{\mu_{u'} : \ell_1 \le u' < \ell_2 \text{ or } \ell_3 \le u' < \ell_4, q_{u'} \in Q_{\text{non}}\}$ . Now, we define the constant z as follows. If  $Z \neq \emptyset$  then  $z = \min(Z) - \frac{1}{2}$ , otherwise  $z = -\frac{1}{2}$ . Let  $U = \{u' : q_{u'} \in Q_{\text{non}}, \ell_1 \le u' < \ell_2 + n_1(L-2) \text{ or } \ell_3 + n_1(L-1) \le u' < \ell_4 + (n_1+n_2)(L-2)\}$  and  $U' = \{u' : q_{u'} \in Q_{\text{non}}, 0 \le u' < n + (n_1+n_2)(L-1), u' \notin U\}$ . We define maximal paths in  $G_{\eta_L}$  and weights of the maximal paths as in the proof of lemma 12.

First, observe that, the set U is the set of all indices corresponding to the non-input states in the first L - 1 iterations of C and C'; the corresponding indices in the  $L^{th}$  iteration (last iteration) of C and C' are included in the set U'. We make the following additional observations using the fact that C and C' are non-leaking cycles. Any variable that is assigned a value in C is not referenced by another transition in C; the same holds for C'. Also, if a variable that is assigned a value in C in the run  $\rho$  is referenced in a transition in C', then in  $\rho_L$  only the value assigned in the last iteration of C (i.e.,  $L^{th}$  iteration ) is referenced in a transition of any of the iterations of C'.

Based on all these observations, we see that every maximal path in  $G_{\rho_L}$  starts with node  $u_0 \notin U$ . (Maximal paths of  $G_{\rho_L}$  and their weights are as defined in the proof of lemma 12). As a consequence, weights of any two maximal paths differ by at most  $\frac{1}{2}$ .

Now, we define the function  $\psi$  that associates a value with each  $u' \in V$ , exactly as in the proof of lemma 12 using the above values of  $\Delta$ , *z*, *U* and *U'*.

From the way we defined the graph  $G_{\rho_L}$  and the above observations, the following properties are easily shown to hold.

- For every  $u', u'', 0 \le u', u'' < L$ , there is a path, in  $G_{\rho_L}$ , from the node  $i_1 + u'n_1$  to the node  $i_m + n_1(L-1) + u''n_2$ .
- For every  $u', 0 \le u' < L$ , every maximal path, in  $G_{\rho_L}$ , that ends in the node  $i_1 + u'n_1$  or in the node  $i_m + n_1(L-1) + u'n_2$ , starts from a node that is not in U.

Putting all the above observations together, we see that, for every  $u', 0 \le u' < L-1, \psi(i_1+u'n_1) \le \frac{1}{2}$  and  $\psi(i_m+n_1(L-1)+u'n_2) \le \frac{1}{2}$ . Further more, for every u', u'', such that  $0 \le u', u'' < L$ , there is a path, in  $G_{\rho_L}$ , from the node  $i_1+u'n_1$  to the node  $i_m+n_1(L-1)+u''n_2$ , and  $\psi(i_1+u'n_1) < \psi(i_m+n_1(L-1)+u''n_2) \le \psi(i_1+u'n_1) + \frac{1}{2}$ .

Now using proof similar to that of lemma 12, it is easily shown that there exist constants  $\epsilon_{\ell}, c_1, c_2 > 0$  such that  $\forall \epsilon > \epsilon_{\ell}$ ,  $\Pr[\epsilon, \rho_{\alpha}(L)] > \frac{c_1}{2}e^{-c_2\epsilon}$ .

Now, let  $\beta(L) = b_0 \cdots b_{n-1+(n_1+n_2)(L-1)}$  be such that, for  $0 \le u_1 < n + (n_1 + n_2)(L-1)$ ,  $b_{u_1}$  is defined as follows: if  $u_1 = i_m + n_1(L-1) + u'n_2$  for some u',  $0 \le u' < L-1$ ,  $b_{u_1} = a_{u_1} - 1$ , otherwise  $b_{u_1} = a_{u_1}$ . It is easily seen that the input sequences  $\alpha(L), \beta(L)$  are neighboring sequences. For each  $u_1, 0 \le u_1 < n + (n_1 + n_2)L$ , let  $X_{u_1}$  be the random variable with the distribution  $Lap(d_{u_1}\epsilon, b_{u_1})$ . Now, for each u',  $0 \le u' < L - 1$ , consider the

pair of random variables  $X_{i_1+u'n_1}$  and  $X_{i_m+n_1(L-1)+u'n_2}$ . The guard conditions of all the transitions of the execution of  $\eta_L$ , with input sequence  $\beta(L)$ , imply that for each u',  $0 \le u' < L - 1$ ,  $X_{i_1+u'n_1} < X_{i_m+n_1(L-1)+u'n_2}$ . However, from the definition of  $\beta(L)$ , we see that  $b_{i_m+n_1(L-1)+u'n_2} \le b_{i_1+u'n_1} - \frac{1}{2}$ . Now, using the same analysis, as in [10, 11], it is easily shown that the input sequences  $\alpha(\ell)$  and  $\beta(\ell)$  are witnesses for violation of the differential privacy property by  $\mathcal{A}$ .

#### Disclosing cycles implies no privacy

**Lemma 15.** A DiPA  $\mathcal{A}$  is not differentially private if it has a disclosing cycle.

PROOF. Thanks to Lemma 12 and Lemma 14, we can assume  $\mathcal{A}$  does not have leaking cycles or leaking pairs. Assume that  $\mathcal{A}$  has a disclosing cycle *C*. By definition, there is a feasible run in  $\mathcal{A}$  starting from the initial state, having a suffix which is a non-leaking cycle, say cycle *C*, such that *C* has a transition whose output is insample or insample'. We consider the case when *C* has a transition whose output is insample. The proof for the case when *C* has a transition whose output is insample' is simpler and is left out. Now, if the transition of *C* whose output is insample has the guard true, then it can be shown easily that repeating the cycle  $\ell$  times incurs a privacy cost linear in  $\ell \epsilon$ , and hence  $\mathcal{A}$  cannot be  $\mathfrak{D}\epsilon$ -differentially private for any  $\mathfrak{D} > 0$ . Thus, we consider more interesting case when the guard is a condition other than true.

As indicated above, we consider the case when *C* has a transition with output insample. Let  $\rho$  be a run of  $\mathcal{A}$  such that  $|\rho| = j + m$ where  $m > 0, j \ge 0, \operatorname{src}(\rho) = q_{init}$ , and the following conditions are satisfied:  $\rho[j : |\rho|] = C$  and and for each  $\ell > 0$ , the run  $\rho[0 : j]C^{\ell}$ (i.e., the run obtained by repeating the cycle *C*,  $\ell$  times) is feasible. Fix  $0 \le r < m$  be such that the output of the (j + r)-th transition of  $\rho$  is insample. Let the guard of the (j + r)-th transition of  $\rho$  be the condition  $c_r$ . Let *hset* =  $\{x_j | \text{insample} < x_j \text{ is a conjunct of } c_r\}$  and *lset* =  $\{x_j | \text{insample} \ge x_j$  is a conjunct of  $c_r\}$ . Observe that, since *C* is non-leaking cycle, it has no assignment transitions for any of the variables in *lset*  $\cup$  *hset*.

Fix  $\ell > 0$ . We define the run  $\rho_{\ell}$  starting from  $q_{\text{init}}$  by repeating the cycle C in  $\rho$ ,  $\ell$  times. Formally,  $\rho_{\ell} = \rho[0: j]C^{\ell}$ . Observe that  $|\rho_{\ell}| = j + \ell m$ . For each k',  $0 \le k' < j + \ell m$ , let  $t_{k'}, c_{k'}$  be the k'-th transition and it's guard in  $\rho_{\ell}, q_{k'} = \operatorname{src}(t_{k'}), \sigma_{k'} = \operatorname{out}(t_{k'})$  and  $P(q_{k'}) = (d_{k'}, \mu_{k'}, d'_{k'}, \mu'_{k'})$ . Observe that  $t_{k'} = t_{k'-m}$  and  $\sigma_{k'} = \sigma_{k'-m}$  for  $j + m \le k' < j + \ell m$ . Observe that  $\sigma_{j+nm+r} = \text{insample}$ , for all n such that  $0 \le n < \ell$ .

Now we construct two input sequences  $\alpha(\ell) = a_0 \cdots a_{j+\ell m-1}$ and  $\beta(\ell) = b_0 \cdots b_{j+\ell m-1}$  as follows. We take  $a_{k'} = -\mu_{k'}$ , for all  $k', 0 \le k' < j + \ell m$  such that  $t_{k'}$  is an input transition, otherwise we take  $a_{k'} = \tau$ . We take  $b_{k'} = -\mu_{k'} - 1$  if k' = j + nm + r for some  $0 \le n < \ell$  and  $b_{k'} = a_{k'}$  otherwise. We also construct an output sequence  $O(\ell) = o_0 \cdots o_{j+\ell m-1}$  as follows: for all k',  $0 \le k' < j + \ell m$ , i)  $o_{k'} = \sigma_{k'}$  if  $\sigma_{k'} \in \Gamma$ , ii)  $o_{k'} = (\sigma_{k'}, 0, \infty)$  if k' = j + nm + r for some  $0 \le n < \ell$ , and iii)  $o_{k'} = (\sigma_{k'}, -\infty, \infty)$ otherwise. Let  $\kappa(\ell)$  and  $\kappa'(\ell)$  respectively be the computations given by the triples  $(\rho_\ell, \alpha(\ell), O(\ell))$  and  $(\rho_\ell, \beta(\ell), O(\ell))$ .

Let  $\kappa(\ell)[k':] = (\rho(\ell)[k':], \alpha(\ell)[k':], O(\ell)[k':])$  and  $\kappa'(\ell)[k':] = (\rho(\ell)[k':], \beta(\ell)[k':], O(\ell)[k':])$ , respectively, be the suffixes of the computations  $\kappa(\ell)$  and  $\kappa'(\ell)$  from the position k'.

Using backward induction on k', i.e., in decreasing values of k', we can easily show that for each evaluation  $\psi$ , for the set of variables  $\{\mathbf{x}_{j'}|1 \leq j' \leq k\}$ , either both  $\Pr[\epsilon, \psi, \kappa(\ell)[k':]], \Pr[\epsilon, \psi, \kappa'(\ell)[k':]]$ are zero (intuitively speaking, this happens when the interval  $[u, v] \cap (0, \infty) = \emptyset$  where u is the maximum value of a variable in *lset* and v is the minimum value of a variable in *hset* for a given evaluation), or both the probabilities are non-zero and

$$\Pr[\epsilon, \psi, \kappa(\ell)[k':]] = e^{\#(k')d_{j+r}\epsilon}\Pr[\epsilon, \psi, \kappa'(\ell)[k':]]$$

where #(k') is the number of indices  $k_1$  such that  $k' \le k_1 < j+m\ell-1$ and  $k_1 = j + nm + r$  for some  $0 \le n < \ell$ . Thus,

$$\Pr[\epsilon, \kappa(\ell)] = e^{\ell d_{j+r}\epsilon} \Pr[\epsilon, \kappa'(\ell)].$$

Now,  $\ell$  is arbitrary and hence for every  $\mathfrak{D} > 0$ , there is an  $\ell$  such that  $\Pr[\epsilon, \kappa(\ell)] > e^{\mathfrak{D}\epsilon} \Pr[\epsilon, \kappa'(\ell)]$ . Hence  $\mathcal{A}$  is not differentially private.  $\Box$ 

### Privacy violating paths implies no privacy

**Lemma 16.** A DiPA  $\mathcal{A}$  is not differentially private if it has a privacy violating path.

PROOF. Thanks to Lemma 12, Lemma 15 and Lemma 14, we can assume  $\mathcal{A}$  does not have leaking cycles, disclosing cycles or leaking pairs. We give the proof for one of the two cases of a privacy violating path. (The proof for the other case of the privacy violating path is similar and is leftout.) Specifically, we give the proof when condition (b) of the definition 11 is satisfied; that is, the privacy violating path  $\rho$  which starts from the the initial state  $q_{\text{init}}$  has a non-leaking cycle *C*, and the dependency graph  $G_{\rho}$  has a path of the form  $i_1, ..., i_m$  where the index  $i_1$  (i.e. the transition trans( $\rho[i_1]$ )) is on *C*,  $i_2 < i_1$  and the transition trans( $\rho[i_m]$ ) outputs insample. Since *C* is not a leaking cycle and  $i_2 < i_1$ , it follows that there is a variable  $x_{k'}$  such that trans( $\rho[i_2]$ ) is an assignment transition for  $x_{k'}$  and the condition insample  $< x_{k'}$  is a conjunct of the guard of trans( $\rho[i_1]$ ).

Fix  $\ell > 0$ . Consider the run  $\rho(\ell)$  of length *n* from the initial state  $q_{\text{init}}$  such that  $\rho(\ell)$  is obtained from  $\rho$  by repeating the cycle *C*,  $\ell$  times. Let  $k_1, k_2, \ldots, k_\ell$  be the indices where the transition trans( $\rho[i_1]$ ) of  $\rho$  (which is on the cycle *C*) occurs in  $\rho(\ell)$ . Similarly, let *r* be the index where the transition trans( $\rho[i_m]$ ) of  $\rho$  occurs in  $\rho(\ell)$ . Let  $P(\operatorname{src}(\operatorname{trans}(\rho(\ell)[i])) = (d_i, \mu_i, d'_i, \mu'_i)$ , for all  $i, 0 \le i \le n$ . Next, we construct two input sequences  $\alpha(\ell) = a_0 \cdots a_{n-1}$  and  $\beta(\ell) = b_0 \cdots b_{n-1}$  of length *n* as follows. If the *i*-th transition of  $\rho(\ell)$  is a non-input transition then  $a_i = b_i = \tau$ . If  $i \in \{k_1, k_2, \ldots, k_\ell\}$  then  $a_i = -\mu_i$  and  $b_i = -\mu_i + 1$ . For all other values of  $i, a_i = b_i = -\mu_i$ . For  $0 \le i < n$ , let  $\sigma_i = \operatorname{out}(\operatorname{trans}(\rho(\ell)[i])$ . Let  $O(\ell) = o_0, \ldots, o_{n-1}$  be the output sequence defined as follows: for all  $i, 0 \le i < n$ , i)  $o_i = \sigma_i$  if  $\sigma_i \in \Gamma$ , ii)  $o_i = (\sigma_i, -\infty, 0)$  if i = r, and iii)  $o_i = (\sigma_i, -\infty, \infty)$  otherwise.

Let  $\kappa(\ell)$  and  $\kappa'(\ell)$  be the computations given by the triples  $(\rho(\ell), \alpha(\ell), O(\ell))$  and  $(\rho(\ell), \beta(\ell), O(\ell))$  respectively. Please note that in  $\kappa(\ell), \kappa'(\ell)$ , the *r*-th output (i.e., the value output in  $o_r$ ) is a non-positive number. From the construction of  $\kappa(\ell), \kappa'(\ell)$ , it can be shown that

$$\Pr[\epsilon, \kappa(\ell)] = e^{\ell d_{k_1} \epsilon} \Pr[\epsilon, \kappa'(\ell)].$$

As in the case of disclosing cycle (See Lemma 15), we can conclude that  $\mathcal A$  is not differentially private.

# C PSPACE-HARDNESS OF CHECKING WELL-FORMEDNESS

In this sub-section we show that the problem of checking well-formedness of a DiPA.

# **Lemma 17.** The problem of checking if a given output-distinct DiPA is well-formed is PSPACE-hard.

**PROOF.** We prove the lemma by giving a polynomial time reduction from the problem of checking whether a polynomial space bounded single tape Turing Machine (TM) halts on a given input. More specifically, given a single tape TM M that is polynomial space bounded and an input u, we give a polynomial time algorithm that outputs a DiPA  $\mathcal{A}$  so that M halts on the input u iff  $\mathcal{A}$  has no leaking cycle.

Let *M* be the given TM. Without loss of generality, we assume that the input alphabet of *M* is given by  $\Sigma = \{0, 1\}$  and it's tape alphabet  $\Upsilon = \Sigma \cup \{B\}$  where *B* is the blank symbol. Let *M* be given by the 4-tuple  $(R, \delta', r_{init}, r_{halt})$  where *R* is the set of it's control states;  $r_{init}, r_{halt} \in R$  are the initial and halting states respectively and

$$\delta' : (R - \{r_{halt}\}) \times \Upsilon \to R \times \Sigma \times \{Left, Right\}$$

is the transition function. Intuitively, if  $\delta'(r, a) = (r', b, d)$ , where  $d \in \{Left, Right\}$  then M when in the control state r scanning the symbol a in the cell pointed by it's head, it writes the symbol b into the cell and moves it's head in the direction given by d. We assume that if M tries to move it's head further left of the left most cell then it stays in the same position. Without loss of generality, we assume that in each transition, M always writes a value 0 or 1 into the current cell its scans; the value it writes may be the same value it read or is a different value. Notice that when in the control state  $r_{halt}$ . We assume that M uses at most p(n) space on any input of length n, where p(n) is a polynomial in n.

Let  $u = u_0, ..., u_{n-1}$  be the given input to M. Let N = p(n). Now, we give the construction of the automaton  $\mathcal{A} = (Q, \Gamma, q_{init}, X, P, \delta)$ as follows. The set of store variables  $S = \{x\} \cup \{y_i, z_i \ 0 \le i < N\}$ . The set Q =

$$\{q_j, q'_j : 0 \le j \le N\} \cup \{(r, i), (r', i), (r'', i) : 0 \le i < N, r \in R\}.$$

Further more,  $q_{init} = q_0$ ,  $\Gamma = \{\top, \bot\}, P(s) = (1, 0, 1, 0)$  for all  $s \in Q$  and  $\delta$ , which defines the transitions of  $\mathcal{A}$  is defined as follows. First we define the intuition into the definition of  $\delta$  and the working of  $\mathcal{A}$ . The variables  $y_i, z_i$  together with x are used to denote the contents of the *i*-th tape cell of M. More specifically, the satisfaction of the conditions  $z_i < x, z_i \ge x$ , denote that the cell *i* has blank symbol (i.e., symbol B) and non-blank symbol, respectively; similarly, satisfaction of the conditions  $y_i < x, y_i \ge x$ , denote that cell *i* contains 0 and 1, respectively. We have a transition from  $q_0$  to  $q_1$  with guard true that assigns insample to x and outputs  $\bot$ ; this transition initializes x.

We have the following transitions that capture the fact that the first *n* cells of the tape contain non-blank input symbols, and the remaining cells contain blank symbols. For each  $j, 1 \le j \le n$ , we have two transitions: (i) from  $q_j$  to  $q'_j$  with guard insample  $\ge x$  and with assignment to variable  $z_{j-1}$ ; (ii) from  $q'_j$  to  $q_{j+1}$  with guard insample < x (resp., with guard insample  $\ge x$ ) when  $u_{j-1} = 0$ 

(resp., when  $u_{j-1} = 1$ ) with assignment to variable  $y_{j-1}$ . For each j,  $n + 1 \le j < N$ , we have a transition from  $q_j$  to  $q_{j+1}$  with guard insample < x and with assignment to  $z_{j-1}$ . We also have a transition from  $q_N$  to the state  $(r_{init}, 0)$  with guard insample < x and with assignment to  $z_{N-1}$ . All these transitions output  $\bot$ .

When the automaton  $\mathcal{A}$  is in the states of the form (r, i)  $(r \in R, 0 \le i < N)$ , it simulates M. For  $r \in R$  and for each  $i, 0 \le i < N$ ,  $\mathcal{A}$  has the following transitions.

- If  $\delta'(r, B) = (s, b, d)$  then we have the following three transitions in  $\mathcal{A}$ : (i) there is a transition from (r, i) to (r', i) with guard  $z_i < \text{insample} \land \text{insample} \le x$  that outputs  $\bot$ ; (ii) there is a transition from (r', i) to (r'', i) with guard insample  $\ge x$  that assigns insample to  $z_i$  and outputs  $\top$  (iii) there is a transition from (r'', i) to (s, j) that assigns insample to  $y_i$ , and where j = i + 1 if d = Right and j = i 1 if d = Left; if b = 0 then the guard of the transition is insample < x, otherwise the guard is insample  $\ge x$ .
- If δ'(r, a) = (s, b, d), where a ≠ B, then we have the following three transitions in A: (i) there is a transition from (r, i) to (r', i), if a = 0 then the guard of the transition is insample ≥ y<sub>i</sub> ∧ insample < x and it's output is ⊥, otherwise it's guard is insample ≥ x ∧ insample < y<sub>i</sub> and it outputs ⊤; (ii) there is a transition from (r', i) to (s, j) that assigns insample to y<sub>i</sub>, if b = 0 then the guard of the transition is insample < x and it outputs ⊥, otherwise the guard is insample ≥ x and it outputs ⊥, otherwise the guard is insample ≥ x and it outputs ⊥, otherwise the guard is insample ≥ x and it outputs ⊤; further more, j = i + 1 if d = Right and j = i 1 if d = Left.</li>

It can easily be shown that *M* halts on the input *u* iff  $\mathcal{A}$  is well-formed.  $\Box$ 

# D DETAILS OF EXAMPLES USED IN THE EXPERIMENTS

### **D.1** Pseuedocode of examples in experiments

We present the pseudocode of the examples described in Section 7.1.

```
Input: q[1:N]

Output: out[1:N]

threshold \leftarrow Lap(\frac{\epsilon}{4}, T)

for i \leftarrow 1 to N do

| r \leftarrow Lap(\frac{\epsilon}{2}, q[i])

if (r \ge threshold) then

| out[i] \leftarrow \top

exit

else

| out[i] \leftarrow \bot

end

end
```



```
Input: q[1:N]

Output: out[1:N]

threshold \leftarrow Lap(\frac{\epsilon}{4}, T)

for i \leftarrow 1 to N do

| r \leftarrow Lap(\frac{\epsilon}{2}, q[i])

if (r \ge threshold) then

| out[i] \leftarrow Lap(\frac{\epsilon}{2})

| exit

else

| out[i] \leftarrow \bot

end
```

end

**Algorithm 5:** NUM-SPARSE. NUM-SPARSE is differentially private. In experiments, the (non-private) threshold is set to 0.

```
Input: q[1:N]

Output: out[1:N]

x1 \leftarrow Lap(\frac{e}{2}, T_{\ell})

x2 \leftarrow Lap(\frac{e}{2}, T_{u})

for i \leftarrow 1 to N do

| r \leftarrow Lap(\frac{e}{2}, q[i])

if ((r < x1) \land (r > x2)) then

| out[i] \leftarrow \top

else if (r < x2) then

| x2 \leftarrow r

out[i] \leftarrow \bot

end
```

**Algorithm 6:** The example LC-EXAMPLE. LC-EXAMPLE is not differentially private as it has a leaking cycle. In the experiments,  $T_{\ell}$  and  $T_{u}$  are taken to be 0 and 1, respectively.

```
Input: q[1:N]

Output: out[1:N]

low \leftarrow Lap(\frac{e}{4}, T_{\ell})

high \leftarrow Lap(\frac{e}{4}, T_{u})

for i \leftarrow 1 to N do

| r \leftarrow Lap(\frac{e}{4}, q[i])

if (r \ge low) \land (r < high) then

| out[i] \leftarrow r

else if (r \ge low) \land (r \ge high) then

| out[i] \leftarrow - T

exit

else if (r < low) \land (r < high) then

| out[i] \leftarrow - T

exit

else if (r < low) \land (r < high) then

| out[i] \leftarrow - T

exit

end
```

# end

**Algorithm 7:** The example DC-EXAMPLE. DC-EXAMPLE is not differentially private as it has a disclosing cycle. In the experiments,  $T_{\ell}$  and  $T_{u}$  are taken to be 0 and 1, respectively.

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```
Input: q[1 : N]
Output: out [1 : N]
low \leftarrow Lap(\frac{\epsilon}{4}, T_{\ell})
high \leftarrow \text{Lap}(\frac{\epsilon}{4}, T_u)
for i \leftarrow 1 to N do
     r \leftarrow Lap(\frac{\epsilon}{4}, q[i])
     if (r \ge low) \land (r < high) then
         out[i] \leftarrow \bot
     else if (r \ge low) \land (r \ge high) then
           out[i] \leftarrow r
           exit
     else if (r < low) \land (r < high) then
           out[i] \leftarrow \top
           exit
      end
end
```

**Algorithm 8:** The algorithm NUM-RANGE-1. NUM-RANGE-1 is not differentially private as it has a privacy-violating path. In the experiments,  $T_{\ell}$  and  $T_{u}$  are taken to be 0 and 1, respectively.

```
Input: q[1 : N]
Output: out [1 : N]
low \leftarrow Lap(\frac{\epsilon}{4}, T_{\ell})
high \leftarrow \text{Lap}(\frac{\epsilon}{4}, T_u)
for i \leftarrow 1 to N do
     r \leftarrow Lap(\frac{\epsilon}{4}, q[i])
     if (r \ge low) \land (r < high) then
       out[i] \leftarrow \bot
      else if (r \ge low) \land (r \ge high) then
           out[i] \leftarrow Lap(\frac{\epsilon}{4}, q[i])
           exit
      else if (r < low) \land (r < high) then
           out[i] \leftarrow \top
           exit
     end
end
```

**Algorithm 9:** NUM-RANGE-2. NUM-RANGE-2 is differentially private. In the experiments,  $T_{\ell}$  and  $T_u$  are taken to be 0 and 1, respectively.

# D.2 Raw data for graph plots

Table 3 and Table 2 give the running times for k-MIN-MAX and m-RANGE in our experiments. We also report the time taken to compute the weight. As we can see, it is minuscule compared to the total running time.

```
Input: q[1 : N]
Output: out [1 : N]
u \leftarrow Lap(\frac{\epsilon}{4}, T_{\ell})
v \leftarrow Lap(\frac{\epsilon}{4}, T_m)
w \leftarrow Lap(\frac{\epsilon}{4}, T_u)
for i \leftarrow 1 to N do
     r \leftarrow Lap(\frac{\epsilon}{4}, q[i])
     if (r \ge u) \land (r < v) then
           out[i] \leftarrow cont
      else if (r < u) then
           out[i] \leftarrow \bot
           exit
      else if (r > v) \land (r < w) then
           out[i] \leftarrow \top
           break
     end
end
for i \leftarrow i + 1 to N do
     r \leftarrow Lap(\frac{\epsilon}{4}, q[i])
     if (r \ge v) \land (r < w) then
          out[i] \leftarrow cont
     else if (r < v) then
           out[i] \leftarrow \bot
           exit
     else if (r > w) then
           out[i] \leftarrow \top
           exit
     end
end
```

**Algorithm 10:** Two-RANGE-1 algorithm. Two-RANGE-1 is not differentially private as it has a leaking pair. In the experiments, the thresholds  $T_{\ell}$ ,  $T_m$  and  $T_u$  are chosen as 0, 1, and 2, respectively

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Table 2: Experimental result of *m*-RANGE examples. The columns vars, states, and transitions give the number of variables of states and transitions in the example. The column wt calculation time gives the average running time in seconds, and running time gives the average running time in seconds. The running times are averaged over 6 executions. In each case, DiPAut, returns that the automaton is differentially private with weight 1.

m	vars	# states	# transitions	wt calc. time	running time
1	2	4	5	0.001s	0.227s
2	4	7	10	0.001s	0.225s
3	6	10	15	0.002s	0.242s
4	8	13	20	0.003s	0.234s
5	10	16	25	0.003s	0.258s
6	12	19	30	0.004s	0.3s
7	14	22	35	0.005s	0.354s
8	16	25	40	0.006s	0.392s
9	18	28	45	0.007s	0.467s
10	20	31	50	0.008s	0.611s
11	22	34	55	0.009s	0.735s
12	24	37	60	0.01s	0.894s
13	26	40	65	0.012s	1.061s
14	28	43	70	0.012s	1.259s
15	30	46	75	0.014s	1.467s
16	32	49	80	0.015s	1.737s
17	34	52	85	0.017s	2.085s
18	36	55	90	0.018s	2.47s
19	38	58	95	0.02s	2.911s
20	40	61	100	0.021s	3.469s
25	50	76	125	0.029s	7.332s
30	60	91	150	0.04s	13.466s
35	70	106	175	0.049s	22.775s
40	80	121	200	0.062s	35.894s
45	90	136	225	0.077s	56.14s
50	100	151	250	0.09s	83.05s
55	110	166	275	0.12s	114.16s
60	120	181	300	0.145s	156.80s
65	130	196	325	0.167s	207.30s
70	140	211	350	0.224s	286.87s
75	150	226	375	0.284s	402.43s
80	160	241	400	0.259s	506.33s

Table 3: Experimental result of k-MIN-MAX examples. The columns vars, states, and transitions give the number of variables of states and transitions in the example. The column wt calculation time gives the average running time in seconds, and running time gives the average running time in seconds. The running times are averaged over six executions. In each case, DiPAut, returns that the automaton is differentially private with weight 1.

k	vars	# states	# transitions	wt. calc time	running time
2	2	4	7	0.001s	0.22s
3	2	5	10	0.001s	0.22s
4	2	6	13	0.001s	0.223s
5	2	7	16	0.001s	0.223s
6	2	8	19	0.001s	0.226s
7	2	9	22	0.002s	0.227s
8	2	10	25	0.002s	0.228s
9	2	11	28	0.002s	0.23s
10	2	12	31	0.002s	0.23s
11	2	13	34	0.002s	0.234s
12	2	14	37	0.003s	0.234s
13	2	15	40	0.003s	0.236s
14	2	16	43	0.003s	0.238s
15	2	17	46	0.003s	0.24s
16	2	18	49	0.004s	0.241s
17	2	19	52	0.004s	0.244s
18	2	20	55	0.004s	0.246s
19	2	21	58	0.004s	0.247s
20	2	22	61	0.004s	0.248s
30	2	32	91	0.007s	0.264s
40	2	42	121	0.01s	0.282s
50	2	52	151	0.012s	0.299s
60	2	62	181	0.015s	0.317s
70	2	72	211	0.019s	0.335s
80	2	82	241	0.022s	0.365s
90	2	92	271	0.025s	0.386s
100	2	102	301	0.029s	0.409s
110	2	112	331	0.034s	0.44s
120	2	122	361	0.038s	0.448s
130	2	132	391	0.044s	0.476s
140	2	142	421	0.047s	0.492s
150	2	152	451	0.052s	0.515s
160	2	162	481	0.057s	0.54s
170	2	172	511	0.062s	0.565s
180	2	182	541	0.068s	0.583s
190	2	192	571	0.074s	0.609s
200	2	202	601	0.08s	0.643s