A CFMMS AND MARKET SCORING RULES

We highlight here for completeness the equivalence between market scoring rules [30] and CFMMs. Chen and Pennock [22] show that every prediction market, based on a market scoring rule, can be represented using some "cost function."

A prediction market trades n types of shares, each of which pays out 1 unit of a numeraire if a particular future event occurs. The cost function $C(q)$ of [22] is a map from the total number of issued shares of each event, $q \in \mathbb{R}^n$, to some number of units of the numeraire. To make a trade $\delta \in \mathbb{R}^n$ with the prediction market (i.e. to change the total number of issued shares to $q + \delta$), a user pays $z = C(q + \delta) - C(q)$ units of the numeraire to the market.

One discrepancy is that traditional formulations of prediction markets (e.g. [22, 31]) allow an arbitrary number of shares to be issued by the market maker, but the CFMMs described in this work trade in assets with finite supplies. Suppose for the moment, however, that a CFMM could possess a negative quantity of shares (with the trading function f defined on the entirety of $\mathbb{R}^n$, instead of just the positive orthant). This formulation of a prediction market directly gives a CFMM that trades the n shares and the numeraire, with trading function $f(r, z) = -C(-r) + z$ for $r \in \mathbb{R}^n$ the number of shares owned by the CFMM, and z the number of units of the numeraire owned by the CFMM. Observe that for any trade δ and $dz = C(-(r + \delta)) - C(-r)$, $f(r, z) = f(r + \delta, z + dz)$. This establishes the correspondence between prediction markets and CFMMs.

In our examples with the LMSR, we consider a CFMM for which $z = 0$ (i.e., it doesn't exchange shares for dollars, but only shares of one future event for shares of another future event). The cost function $C(r)$ for the LMSR is $\log(\sum_{i=1}^{n} \exp(-r_i))$. The CFMM representation with this cost function follows by setting it to a constant.

B CONTINUOUS TRADE SIZE DISTRIBUTION

Definition B.1. *Let $size(\cdot)$ be some distribution on $\mathbb{R}_{\geq 0}$ with support in a neighborhood of* 0.
A trader appears at every timestep. The trade has size k units of Y, where k is drawn from $size(\cdot)$. A trade buys or sells from the CFMM with equal probability.

This definition implicitly encodes an assumption that the amount of trading from X to Y is balanced in expectation against the amount of trading from Y to X.

An additional assumption makes this setting analytically tractable.

Assumption 3 (Strict Slippage). *Trade requests measure slippage relative to the post-trade* spot exchange rate *of the CFMM, not the overall exchange rate of the trade.*

In other words, a trade request succeeds if and only if it would move the CFMM's reserves to some state within $L_\varepsilon(\hat{p})$.

We now analyze the Markov chain over the CFMM's state, the stationary distribution of which gives us the trade failure probability under Assumption 3.

Lemma B.2. *Let M be the Markov chain defined by the state of Y in the asset reserves of the CFMM with $\mathcal{Y} \in L_\varepsilon(\hat{p})$ and transitions induced by trades drawn from the distribution in Definition B.1. Under Assumption 3, the stationary distribution of M is uniform over $L_\varepsilon(\hat{p})$.*

Proof. Let $\mu(\cdot)$ be the uniform measure on $L_\varepsilon(\hat{p})$ and let $\tau(v, A)$ be the state transition kernel induced by the trade distribution. Specifically, given the trade size distribution $size(v)$ on the probability that the CFMM sells (for $v > 0$) or buys (for $v < 0$) $|v|$ units of Y, $\tau(v, A)$ measures the probability that for any set A, the CFMM is in a state in A after attempting a trade of size v. It suffices to show that μ is an invariant measure of the Markov chain that is induced on $L_\varepsilon(\hat{(p})$, and that this invariant measure is unique.

Note that from any $y \in L_\varepsilon(\hat{p})$, after a trade of size v, the Markov chain lands in a set A if either $y + v \in A$ (that is, the trade succeeds) or if $y \in A$ and $y + v \notin L_\varepsilon(\hat{p})$ (that is, the trade fails and the initial state was in A). Note that trade success and failure are mutually exclusive events.

$$\begin{aligned}
&\int\limits_{L_\varepsilon(\hat{p})} \mu(y)\tau(y, A)dy \\
&= \int\limits_{L_\varepsilon(\hat{p})} \int\limits_{-\infty}^{\infty} size(v)(\mathbb{1}(y + v \in A) + \mathbb{1}(y \in A \wedge y + v \notin L_\varepsilon(\hat{p})))dv\, dy \\
&= \int\limits_{A} \int\limits_{-\infty}^{\infty} size(v)(\mathbb{1}(y + v \in L_\varepsilon(\hat{p})))dv\, dy + \int\limits_{A} \int\limits_{-\infty}^{\infty} \mathbb{1}(y + v \notin L_\varepsilon(\hat{p})))dv\, dy \\
&= \mu(A)
\end{aligned}$$

where the second equality follows from the symmetricity of the trade size distribution (as in Definition B.1).

Because the trade size distribution is supported on a neighborhood of 0 (and $L_\varepsilon(\hat{p})$ is a connected interval), for any set A of nonzero measure, the probability of a transition from $L_\varepsilon(\hat{p}) \setminus A$ to A is nonzero, so the Markov chain must A infinitely many times. As such, $\mu(\cdot)$ is the unique invariant measure on $L_\varepsilon(\hat{p})$ (by Theorem 1 of [32]).

□

Proposition B.3. *The probability that a trade of size k units of Y fails is approximately* $\min(1, \frac{k}{|L_\varepsilon(\hat{p})|})$, *where the approximation error is up to Assumption 3.*

Proof. The probability that a (without loss of generality) sell of size k units of Y fails is equal to the probability that a state y, drawn uniformly from the range $L_\varepsilon(p) = [y_1, y_2]$, lies in the range $[y_2 - k, y_2]$. Lemma B.2 shows this probability is $\min(1, \frac{k}{y_2 - y_1})$. □

C OMITTED PROOFS

C.1 Omitted Proofs of §2 and §3

Restatement (Observation 1). *If f is strictly quasi-concave and differentiable, then for any constant K and spot exchange rate p, the point (x, y) where $f(x, y) = K$ and p is a spot exchange rate at (x, y) is unique.*

Proof. A constant $K = f(X_0, Y_0)$ defines a set $\{x : f(x) \geq K\}$. Because f is strictly quasi-concave, this set is strictly convex. Trades against the CFMM (starting from initial reserves (X_0, Y_0)) move along the boundaries of this set. Because this set is strictly convex, no two points on the boundary can share a gradient (or subgradient). □

Restatement (Observation 2). *If f is strictly increasing in both X and Y at every point on the positive orthant, then for a given constant function value K, the amount of Y in the CFMM reserves uniquely specifies the amount of X in the reserves, and vice versa.*

Proof. If not, then f would be constant on some line with either X or Y constant. □

Restatement (Observation 3). *$\mathcal{Y}(p)$ is monotone nondecreasing.*

Proof. If $\mathcal{Y}(p)$ is decreasing, the level set of f, i.e., $\{(x, y) : f(x, y) \geq K\}$ cannot be convex. □

Restatement (Lemma 3.8). *The function $\mathcal{Y}(\cdot)$ is differentiable when the trading function f is twice-differentiable on the nonnegative orthant, f is 0 when $x = 0$ or $y = 0$, and Assumption 1 holds.*

Proof. Observation 2 implies that the amount of Y in the reserves can be represented as a function $\hat{\mathcal{Y}}(x)$ of the amount of X in the reserves. By assumption, the level sets of f (other than for $f(\cdot) = 0$) cannot touch the boundary of the nonnegative orthant.

Because f is differentiable and increasing at every point in the positive orthant, the map $g(x)$ from reserves x to spot exchange rates at $(x, \hat{\mathcal{Y}}(x))$ must be a bijection from $(0, \infty)$ to $(0, \infty)$. Because f is twice-differentiable, $g(x)$ must be differentiable, and so the map $h(p) = g^{-1}(p)$ must also be differentiable. The map $\mathcal{Y}(p)$ from spot exchange rates to reserves Y is equal to $\hat{\mathcal{Y}}(h(p))$, and so $\hat{Y}(p)$ is differentiable because $\hat{\mathcal{Y}}(\cdot)$ is differentiable and $h(\cdot)$ is differentiable. □

Restatement (Lemma 3.9). *If the function $\mathcal{Y}(\cdot)$ is differentiable, then $L(p) = \frac{d\mathcal{Y}(p)}{d\ln(p)}$.*

Proof. Follows from Definitions 3.6 and 3.7. □

C.2 Omitted Proof of Lemma 4.13

Restatement (Lemma 4.13). *(1) $\lambda_Y Y_0 = \int_0^{p_0} \frac{\varphi_\psi(\cot^{-1}(p))\sin(\cot^{-1}(p))}{L(p)} dp$ and $\lambda_X X_0 = \int_{p_0}^{\infty} \frac{\varphi_\psi(\cot^{-1}(p))\sin(\cot^{-1}(p))}{L(p)} dp$*
(2) $Y_0 > 0$ implies $\lambda_Y > 0$. Similarly, $X_0 > 0$ implies $\lambda_X > 0$.
(3) $L(p) \neq 0$ if and only if $\lambda_{L(p)} = 0$ (unless, for $p \leq p_0$, $\lambda_Y = 0$ or for $p \geq p_0$, $\lambda_X = 0$).
(4) The objective value is $\lambda_Y Y_0 + \lambda_X X_0$.
(5) $\frac{\lambda_X}{P_X} = \frac{\lambda_Y}{P_Y}$.

Proof.

(1) Multiply each side of the first KKT condition in Lemma 4.11 by $L(p)$ (for p with nonzero $\varphi_\psi(\cot^{-1}(p))$ to get $\frac{\lambda_X L(p)}{p^2} = \frac{1}{L(p)}\varphi_\psi(\cot^{-1}(p))\sin(\cot^{-1}(p))$), integrate from p_0 to ∞, and apply the second item of Lemma 4.10.
A similar argument (integrating from 0 to p_0) gives the expression on $\lambda_Y Y_0$.
(2) If $Y_0 > 0$, then the right side of the equation in the previous part is nonzero, so λ_Y must be nonzero. The case of λ_X is identical.
(3) Follows from points 1 and 2 of Lemma 4.11.
(4) The right sides of the equations in the first statement add up to the objective.
(5) Follows from point 3 of Lemma 4.11 □

C.3 Omitted Proof of Corollary 4.6

Restatement (Corollary 4.6). *Any two beliefs ψ_1, ψ_2 give the same optimal liquidity allocations if there exists a constant $\alpha > 0$ such that for every θ,*

$$\int_r \psi_1(r\cos(\theta), r\sin(\theta))dr = \alpha \int_r \psi_2(r\cos(\theta), r\sin(\theta))dr$$

Proof. Follows by substitution. α rescales the derivative of the objective with respect to every variable by the same constant, and thus does not affect whether an allocation is optimal. □

C.4 Omitted Proof of Corollary 4.7

Restatement (Corollary 4.7). *Define $\varphi_\psi(\theta) = \int_r \psi(r\cos(\theta), r\sin(\theta))dr$. Then*

$$\iint_{p_X, p_Y} \frac{\psi(p_X, p_Y)}{p_Y L(p_X/p_Y)} dp_X\ dp_Y = \int_p \frac{\varphi_\psi(\cot^{-1}(p))\sin(\cot^{-1}(p))}{L(p)} dp$$

PROOF.

$$\int_{p_X,p_Y} \frac{\psi(p_X,p_Y)}{p_Y L(p_X/p_Y)} dp_X\, dp_Y = \int_\theta \frac{\varphi_\psi(\theta)}{L(\cot(\theta))\sin(\theta)} d\theta$$
$$= \int_p \frac{\varphi_\psi(\theta)\sin^2(\theta)}{L(\cot(\theta))\sin(\theta)} dp$$
$$= \int_p \frac{\varphi_\psi(\cot^{-1}(p))\sin(\cot^{-1}(p))}{L(p)} dp$$

The first line follows by Lemma 4.5 (recall that $dp_X\, dp_Y = r\, dr\, d\theta$), the second by substitution of $p = \cot(\theta)$ and $d\theta = -\sin^2(\theta)dp$ (and changing the direction of integration — recall $\theta = 0$ when $p = \infty$), and the third by substitution. □

C.5 Omitted Proof of Lemma 4.8

RESTATEMENT (LEMMA 4.8). *The optimization problem of Theorem 4.4 always has a solution with finite objective value.*

PROOF. Set $L(p) = 1$ for $p \le 1$ and $L(p) = \varphi_\psi(\cot^{-1}(p))/p^2$ otherwise. Then

$$\int_p \frac{\varphi_\psi(\cot^{-1}(p))\sin(\cot^{-1}(p))}{L(p)} dp$$
$$\le \int_p \frac{\varphi_\psi(\cot^{-1}(p))}{L(p)} dp$$
$$\le \int_0^1 \varphi_\psi(\cot^{-1}(p))dp + \int_1^\infty \frac{dp}{p^2}$$

The first term of the last line is finite, as per our assumption on trader beliefs.

Set $Y_0 = \int_0^{p_0} \frac{L(p)dp}{p}$ and $X_0 = \int_{p_0}^\infty \frac{L(p)dp}{p^2}$. Clearly both X_0 and Y_0 are finite. Finally, rescale each $L(p)$, X_0, and Y_0 by a factor of $\frac{B}{P_X X_0 + P_Y Y_0}$ to get a new allocation $L'(p)$, X_0', and Y_0' satisfing the constraints and that still gives a finite objective value. □

C.6 Omitted Proof of Lemma 4.10

RESTATEMENT. *The following hold at any optimal solution.*

(1) $\int_0^{p_0} \frac{L(p)}{p} dp = Y_0$
(2) $\int_{p_0}^\infty \frac{L(p)}{p^2} dp = X_0$
(3) $X_0 P_X + Y_0 P_Y = B$

PROOF. The third equation holds since the objective function is strictly decreasing in at least one $L(p)$ (where the belief puts a nonzero probability on the exchange rate p), so any unallocated capital could be allocated to increase this $L(\cdot)$ on a neighborhood of $L(p)$ and reduce the objective.

The first equation holds because any unallocated units of X could be allocated to $L(p')$ for a set of p' in a neighborhood of some $p \le p_0$ and thereby reduce the objective. If there is no $p \le p_0$ where the belief puts a nonzero probability, then all of the capital allocated by the third constraint to X_0 could be reallocated into increasing Y_0 and thereby decreasing the objective.

The second equation follows by symmetry with the argument for the first. □

C.7 Omitted Proof of Corollary 4.12

Restatement (Corollary 4.12). *The integral $\mathcal{Y}(\tilde{p}) = \int_0^{\tilde{p}} \frac{L(p)dp}{p}$ is well defined for every $\tilde{p}$ and $\mathcal{Y}(\cdot)$ is monotone nondecreasing and continuous.*

Proof. The last item of Lemma 4.10 shows that $L(p) \neq 0$ if and only if $\varphi_\psi(\cot^{-1}(p))\sin(\cot^{-1}(p)) \neq 0$. When $L(p)$ is nonzero, it is either $\sqrt{\frac{p^2}{\lambda_X}\varphi_\psi(\cot^{-1}(p))\sin(\cot^{-1}(p))}$ or $\sqrt{\frac{p}{\lambda_Y}\varphi_\psi(\cot^{-1}(p))\sin(\cot^{-1}(p))}$ (depending on the value of p).

By our assumption on trader beliefs, $\varphi_\psi(\cot^{-1}(p))\sin(\cot^{-1}(p))$ is a well-defined function of p and is integrable. Thus, both $\sqrt{\frac{p^2}{\lambda_X}\varphi_\psi(\cot^{-1}(p))\sin(\cot^{-1}(p))}$ and $\sqrt{\frac{p}{\lambda_Y}\varphi_\psi(\cot^{-1}(p))\sin(\cot^{-1}(p))}$ are integrable. Monotonicity follows from $L(p) \geq 0$ and continuity from basic facts about integrals. □

C.8 Omitted Proof of Corollary 4.15

Restatement (Corollary 4.15). *A liquidity allocation $L(\cdot)$ and an initial spot exchange rate p_0 are sufficient to uniquely specify an equivalence class of beliefs (as defined in Corollary 4.6) for which $L(\cdot)$ is optimal.*

Proof. It suffices to uniquely identify $\varphi_\psi(\cot^{-1}(p))$ for each p, up to some scalar. Lemma 4.13 shows that $\varphi_\psi(\cot^{-1}(p))$ is a function of an optimal $L(p)$ and Lagrange multipliers λ_X or λ_Y, and because $\lambda_X \frac{P_X}{P_Y} = \lambda_Y$, we must have that φ_ψ is specified by $L(p)$ and p_0 up to some scalar λ_X. □

C.9 Omitted Proof of Corollary 4.16

Restatement (Corollary 4.16). *Let P_X and P_Y be initial reference valuations, and let $L(\cdot)$ denote a liquidity allocation. Define the belief $\psi(p_X, p_Y)$ to be $\frac{(L(p_X/p_Y))^2}{p_X/p_Y}$ when $p_X \in (0, P_X]$ and $p_Y \in (0, P_Y]$, and to be* 0 *otherwise. Then $L(\cdot)$ is the optimal allocation for $\psi(\cdot,\cdot)$.*

Proof. Recall the definition of $\varphi_\psi(\cdot)$ in 4.7. For the given belief function ψ, standard trigonometric arguments show that when $p \geq p_0$, we have $\varphi_\psi(\cot^{-1}(p)) = \frac{P_X L(p)^2/p}{\cos(\cot^{-1}(p))}$ and that when $p \leq p_0$, we have $\varphi_\psi(\cot^{-1}(p)) = \frac{P_Y L(p)^2/p}{\sin(\cot^{-1}(p))}$.

Let $\hat{L}(p)$ be the allocation that results from solving the optimization problem for minimising the expected CFMM inefficiency for belief ψ. Lemma 4.13 part 3, gives the complementary slackness condition of $\hat{L}(p)$ and its corresponding Lagrange multiplier. With this, Lemma 4.11 gives the following: when $p \geq p_0$, $\frac{\lambda_B P_X}{p^2} = \frac{1}{\hat{L}(p)^2}(P_X L(p)^2/p)/p$, and when $p \leq p_0$, $\frac{\lambda_B P_Y}{p} = \frac{1}{\hat{L}(p)^2}(P_Y L(p)^2/p)$.

In other words, for all p, $\lambda_B = \frac{L(p)^2}{\hat{L}(p)^2}$, so $L(\cdot)$ and $\hat{L}(\cdot)$ differ by at most a constant multiplicative factor. But both allocations use the same budget, so it must be that $\lambda_B = 1$ and $\hat{L}(\cdot) = L(\cdot)$. □

C.10 Omitted Proof of Corollary 4.17

Restatement (Corollary 4.17). *Let ψ_1, ψ_2 be any two belief functions (that give φ_{ψ_1} and φ_{ψ_2}) with optimal allocations $L_1(\cdot)$ and $L_2(\cdot)$, and let $L(\cdot)$ be the optimal allocation for $\psi_1 + \psi_2$. Then $L^2(\cdot)$ is a linear combination of $L_1^2(\cdot)$ and $L_2^2(\cdot)$.*

Further, when φ_{ψ_1} and φ_{ψ_2} have disjoint support, $L(\cdot)$ is a linear combination of $L_1(\cdot)$ and $L_2(\cdot)$.

Proof. Note that $\int_r \psi(r\cos(\theta), r\sin(\theta)dr$ is a linear function of each $\psi(p_X, p_Y)$, and thus $\varphi_{\psi_1+\psi_2}(\cdot) = \varphi_{\psi_1}(\cdot)+\varphi_{\psi_2}(\cdot)$ For any p with $p \geq p_0$ and nonzero $\varphi_{\psi_1}(\cot^{-1}(p))$, $L_1(p)^2 = \frac{p^2}{\lambda_{1,X}}\varphi_{\psi_1}(\cot^{-1}(p))\sin(\cot^{-1}(p))$.

Similarly, for nonzero $\varphi_{\psi_2}(\cot^{-1}(p))$, $L_2(p)^2 = \frac{p^2}{\lambda_{2,X}}\varphi_{\psi_2}(\cot^{-1}(p))\sin(\cot^{-1}(p))$.

If either φ_{ψ_1} or φ_{ψ_2} is nonzero at $\cot^{-1}(p)$, then

$$L(p)^2 = \frac{p^2}{\lambda_X}(\varphi_{\psi_1}(\cot^{-1}(p))\sin(\cot^{-1}(p)) + \varphi_{\psi_2}(\cot^{-1}(p))\sin(\cot^{-1}(p)))$$

Therefore,

$$L(p)^2 = \frac{\lambda_{1,X}}{\lambda_X}L_1(p)^2 + \frac{\lambda_{2,X}}{\lambda_X}L_2(p)^2$$

An analogous argument holds for $p \leq p_0$.

The second statement follows from the fact that that when only one of $L_1(p)$ or $L_2(p)$ is nonzero, we must have that either $L(p) = \sqrt{\frac{\lambda_{1,X}}{\lambda_X}}L_1(p)$ or $L(p) = \sqrt{\frac{\lambda_{2,X}}{\lambda_X}}L_2(p)$. □

C.11 Omitted Proof of Proposition 5.3

Restatement (Proposition 5.3). *Let $p_{min} < p_{max}$ be two arbitrary exchange rates, and let $\psi(p_X, p_Y) = 1$ if and only if $0 \leq p_X \leq P_X$, $0 \leq p_Y \leq P_Y$, and $p_{min} \leq p_X/p_Y \leq p_{max}$, and 0 otherwise. The allocation $L(\cdot)$ that maximizes the fraction of successful trades is the allocation implied by a concentrated liquidity position with price range defined by p_{min} and p_{max}.*

Proof. A concentrated liquidity position trades exactly as a constant product market maker within its price bounds p_{min} and p_{max}, and makes no trades outside of that range.

By Lemma 4.10, the optimal $L(p)$ is 0 for p outside of the range $[p_{min}, p_{max}]$. Inside that range, by Proposition 4.14, $L(p)$ differs from the optimal liquidity allocation for the constant product market maker by a constant, multiplicative factor (the same factor for every p). Thus, the resulting liquidity allocation has the same behavior as a concentrated liquidity position. □

C.12 Omitted Proof of Proposition 5.5

Restatement (Proposition 5.5). *The belief function $\psi(p_X, p_Y) = \left(\frac{p_X}{p_Y}\right)^{\frac{\alpha-1}{\alpha+1}}$ when $(p_X, p_Y) \in (0, P_X] \times (0, P_Y]$ and 0 otherwise corresponds to the weighted product market maker $f(x, y) = x^\alpha y$.*

Proof. This trading function gives the relation $p = \frac{y\alpha}{x}$ and thus $\mathcal{Y}(p) = p^{\frac{\alpha}{\alpha+1}}(\frac{K}{\alpha^\alpha})^{\frac{1}{\alpha}}$, for $K = f(\hat{X}, \hat{Y})$ and $\hat{X}, \hat{Y}$ is some initial state of the CFMM reserves.

Thus, as defined by the trading function, $L(p) = p^{\frac{\alpha}{\alpha+1}}\frac{\alpha}{\alpha+1}\left(\frac{K}{\alpha^\alpha}\right)^{\frac{1}{\alpha+1}}$.

Corollary 4.16 shows that a belief that leads to this liquidity allocation is

$$\frac{L(p)^2}{p} = p^{-1}p^{\frac{2\alpha}{\alpha+1}}\left(\frac{\alpha}{\alpha+1}\right)^2\left(\frac{K}{\alpha^\alpha}\right)^{\frac{2}{\alpha+1}} = p^{\frac{\alpha-1}{\cdot}\alpha+1}C$$

on the rectangle $(0, P_X] \times (0, P_Y]$ and 0 elsewhere, for some constant C. The result follows by rescaling the belief function (Corollary 4.6). □

C.13 Omitted Proof of Proposition 5.6

Restatement (Proposition 5.6). *The optimal trading function to minimize the expected CFMM inefficiency for the belief $\psi(p_X, p_Y) = \frac{p_X p_Y}{(p_X+p_Y)^2}$ when $(p_X, p_Y) \in (0, P_X] \times (0, P_Y]$ and 0 otherwise, is $f(x, y) = 2 - e^{-x} - e^{-y}$.*

Proof. This trading function implies the relationship $p = e^{y-x}$.

Combining this with the equation $e^{-y} + e^{-x} = K$ (for some constant K) gives $(1 + p)e^{-y} = K$ and thus $\mathcal{Y}(p) = \ln(\frac{1+p}{K})$. From the definition of liquidity, we obtain $L(p) = \frac{p}{1+p}$.

Corollary 4.16 shows that a belief function that leads to this liquidity allocation is (with $p = p_X/p_Y$)

$$\frac{L(p)^2}{p} = \frac{p}{(1+p)^2} = \frac{p_X p_Y}{(p_X + p_Y)^2}$$

for $(p_X, p_Y) \in (0, P_X] \times (0, P_Y]$ and 0 elsewhere. The result follows by rescaling the belief function (Corollary 4.6). □

C.14 Omitted Proof of Proposition 6.4

Restatement (Proposition 6.4). *The expected future value of the CFMM's reserves, as per belief $\psi(p_X, p_Y)$, is*

$$v(\psi) = \frac{1}{N_\psi} \iint\limits_{p_X,p_Y} \psi(p_X,p_Y)\,(p_X \mathcal{X}(p_X/p_Y) + p_Y \mathcal{Y}(p_X/p_Y))\,dp_X\,dp_Y$$

$$= \frac{1}{N_\psi} \int_0^\infty \left(\frac{L(p)}{p^2} \iint\limits_{p_X,p_Y} p_X \psi(p_X,p_Y) \mathbb{1}_{\{\frac{p_X}{p_Y} \le p\}} dp_X\,dp_Y + \frac{L(p)}{p} \iint\limits_{p_X,p_Y} p_Y \psi(p_X,p_Y) \mathbb{1}_{\{\frac{p_X}{p_Y} \ge p\}} dp_X\,dp_Y \right) dp$$

where $\mathcal{X}(p)$ and $\mathcal{Y}(p)$ denote the amounts of X and Y held in the reserves at spot exchange rate p, and $\mathbb{1}_E$ is the characteristic function of the event E.

This expression for $v(\psi)$ is a linear function of each $L(p)$.

Proof.

$$\frac{1}{N_\psi} \int\limits_{p_X,p_Y} \psi(p_X,p_Y)\,(p_X \mathcal{X}(p_X/p_Y)) + p_Y \mathcal{Y}(p_X/p_Y))\,dp_X\,dp_Y$$

$$= \frac{1}{N_\psi} \int\limits_{p_X,p_Y} \psi(p_X,p_Y) \left(p_X \int_{p_X/p_Y}^{\infty} \frac{L(p)}{p^2} dp + p_Y \int_0^{p_X/p_Y} \frac{L(p)}{p} dp \right) dp_X\,dp_Y$$

$$= \frac{1}{N_\psi} \int_0^\infty \left(\frac{L(p)}{p^2} \int\limits_{p_X,p_Y} p_X \psi(p_X,p_Y) \mathbb{1}_{\{\frac{p_X}{p_Y} \le p\}} dp_X\,dp_Y + \frac{L(p)}{p} \int\limits_{p_X,p_Y} p_Y \psi(p_X,p_Y) \mathbb{1}_{\{\frac{p_X}{p_Y} \ge p\}} dp_X\,dp_Y \right) dp$$

The first equation follows by substitution of the equations in Observation 4.

Note that for any p_X, p_Y, the term $\frac{L(p)}{p^2}$ for any p appears in the integral $\int_{p_X/p_Y}^{\infty} \frac{L(p)dp}{p^2}$ if and only if $p \ge p_X/p_Y$. The result follows from rearranging the integral to group terms by $L(p)$. □

C.15 Omitted Proof of Theorem 6.6

Restatement (Theorem 6.6). *Let $L_1(p)$ be the optimal liquidity allocation that maximizes fee revenue — the solution to the optimization problem for the objective of minimizing the following:*

$$-\frac{\delta}{N_\psi} \iint_{p_X,p_Y} rate_\delta(p_X,p_Y)\psi(p_X,p_Y) \left(1 - \frac{s}{p_Y L(p_X/p_Y)}\right) dp_X\,dp_Y$$

Let $L_2(p)$ be the optimal liquidity allocation that maximizes fee revenue while accounting for divergence loss — the solution to the optimization problem for the objective of minimizing the following:

$$-v(\psi) - \frac{\delta}{N_\psi} \iint_{p_X,p_Y} rate_\delta(p_X,p_Y)\psi(p_X,p_Y) \left(1 - \frac{s}{p_Y L(p_X/p_Y)}\right) dp_X\,dp_Y$$

Let $X_1 = \int_{p_0}^{\infty} \frac{L_1(p)}{p^2}dp$ and $X_2 = \int_{p_0}^{\infty} \frac{L_2(p)}{p^2}dp$ be the optimal initial quantities of X for the above two problems respectively.

Then there exists some $p_1 > p_0$ such that for $p_0 \le p \le p_1$, $\frac{L_1(p)}{X_1} \ge \frac{L_2(p)}{X_2}$ and for $p \ge p_1$, $\frac{L_1(p)}{X_1} \le \frac{L_2(p)}{X_2}$. An analogous statement holds for the allocations of Y.

Proof. Define $\varphi'(\theta) = \delta \int_r rate_\delta(r\cos(\theta), r\sin(\theta)\psi(r\cos(\theta), r\sin(\theta))dr$.

The KKT conditions for the first problem give the following (nearly identically to those in Lemma 4.11, just using $L_1(\cdot)$ in place of $L(\cdot)$):

(1) For all p with $p \ge p_0$, $\frac{\lambda_X}{p^2} = \frac{1}{L_1(p)^2}\varphi'(\cot^{-1}(p))\sin(\cot^{-1}(p)) + \lambda_{L_1(p)}$.
(2) For all p with $p \le p_0$, $\frac{\lambda_Y}{p} = \frac{1}{L_1(p)^2}\varphi'(\cot^{-1}(p))\sin(\cot^{-1}(p)) + \lambda_{L_1(p)}$.
(3) $\lambda_X = P_X\lambda_B$ and $\lambda_Y = P_Y\lambda_B$.

Observe that the derivative, with respect to $L(p)$, of the divergence loss, is

$$\kappa(p) = \frac{1}{N_\psi}\left(\frac{1}{p^2}\iint_{p_X,p_Y} p_X\psi(p_X,p_Y)\mathbb{1}_{\{\frac{p_X}{p_Y}\le p\}}dp_X\ dp_Y + \frac{1}{p}\iint_{p_X,p_Y} p_Y\psi(p_X,p_Y)\mathbb{1}_{\{\frac{p_X}{p_Y}\ge p\}}dp_X\ dp_Y\right).$$

Computing the KKT conditions for the second problem gives the following:

(1) For all p with $p \ge p_0$, $\frac{\lambda_X}{p^2} = \frac{1}{L_2(p)^2}\varphi'(\cot^{-1}(p))\sin(\cot^{-1}(p)) + \kappa(p) + \lambda_{L_2(p)}$.
(2) For all p with $p \le p_0$, $\frac{\lambda_Y}{p} = \frac{1}{L_2(p)^2}\varphi'(\cot^{-1}(p))\sin(\cot^{-1}(p)) + \kappa(p) + \lambda_{L_2(p)}$.

As defined in the theorem statement, $X_1 = \int_{p_0}^{\infty} \frac{L_1(p)}{p^2}dp$ and $X_2 = \int_{p_0}^{\infty} \frac{L_2(p)}{p^2}dp$ are the optimal initial quantities of X. Normalizing L_1 and L_2 by X_1 and X_2 respectively gives the equation

$$\int_{p_0}^{\infty} \frac{L_1(p)}{X_1p^2}dp = \int_{p_0}^{\infty} \frac{L_2(p)}{X_2p^2}dp \tag{3}$$

This implies that, when normalized by X_1 and X_2, $p \ge p_0$, and $\varphi'(\cot^{-1}(p)) \ne 0$, we have that

$$\begin{aligned} L_2(p) &= \sqrt{\frac{\varphi'(\cot^{-1}(p))\sin(\cot^{-1}(p))}{\frac{\lambda_X}{p^2} - \kappa(p)}} \\ &= \sqrt{\frac{\varphi'(\cot^{-1}(p))\sin(\cot^{-1}(p))}{\frac{\lambda_X}{p^2}} * \frac{\frac{\lambda_X}{p^2}}{\frac{\lambda_X}{p_2} - \kappa(p)}} \\ &= L_1(p)\frac{X_2}{X_1}\sqrt{\frac{\lambda_X}{\lambda_X - p^2\kappa(p)}} \end{aligned}$$

A similar argument shows that when $p \le p_0$,

$$L_2(p) = L_1(p)\frac{Y_2}{Y_1}\sqrt{\frac{\lambda_Y}{\lambda_Y - p\kappa(p)}}$$

Arithmetic calculation gives that

$$\begin{aligned} \kappa(p)p^2 = \frac{1}{N_\psi}\int_r\int_{\theta'=\theta}^{\pi/2} r^2\cos(\theta')\psi(r\cos(\theta'), r\sin(\theta'))dr\ d\theta' \\ + \cot(\theta)\int_r\int_{\theta'=0}^{\theta} r^2\sin(\theta')\psi(r\cos(\theta'), r\sin(\theta'))dr\ d\theta' \end{aligned}$$

and thus that

$$\frac{d(\kappa(\cot(\theta))\cot(\theta)^2)}{d\theta} = -\frac{\csc^2(\theta)}{N_\psi}\int_r\int_{\theta'=0}^{\theta} r^2\sin(\theta')\psi(r\cos(\theta'), r\sin(\theta'))dr\ d\theta' \leq 0$$

$\kappa(\cot(\theta))\cot(\theta)^2$ is therefore decreasing in θ, so $\kappa(p)p^2$ is increasing in p (since $p = \cot(\theta)$). Therefore, $\sqrt{\frac{\lambda_X}{\lambda_X - p^2\kappa(p)}}$ increases as p goes to ∞.

By an analogous argument,

$$\kappa(p)p = \frac{\tan(\theta)}{N_\psi}\int_r\int_{\theta'=\theta}^{\pi/2} r^2\cos(\theta')\psi(r\cos(\theta'), r\sin(\theta'))dr\ \ d\theta' + \int_r\int_{\theta'=0}^{\theta} r^2\sin(\theta')\psi(r\cos(\theta'), r\sin(\theta'))dr\ d\theta'$$

and thus

$$\frac{d(\kappa(\cot(\theta))\cot(\theta))}{d\theta} = \frac{\sec^2(\theta)}{N_\psi}\int_r\int_{\theta'=\theta}^{\pi/2} r^2\sin(\theta')\psi(r\cos(\theta'), r\sin(\theta'))dr\ d\theta' \geq 0$$

$\kappa(\cot(\theta))\cot(\theta)$ is therefore increasing in θ, so $\kappa(p)p$ increases as p decreases.

Therefore, $\sqrt{\frac{\lambda_Y}{\lambda_Y - p\kappa(p)}}$ increases as p goes to 0.

Equation 3 implies that the quantities $\frac{L_1(p)}{X_1p^2}$ and $\frac{L_2(p)}{X_2p^2}$ integrate to the same value, but $L_2(\cdot)$ increases strictly more quickly than $L_1(\cdot)$, so there must be a point $p_1 > p_0$ beyond which $\frac{L_1(p)}{X_1} \leq \frac{L_2(p)}{X_2}$.

An analogous argument holds for $p < p_0$. □