Computing Bayes Nash Equilibrium Strategies in Auction Games via Simultaneous Online Dual Averaging

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Auctions are modeled as Bayesian games with continuous type and action spaces. Determining equilibria in auction games is computationally hard in general and no exact solution theory is known. We introduce an algorithmic framework in which we discretize type and action space and then learn distributional strategies via online optimization algorithms. One advantage of distributional strategies is that we do not have to make any assumptions on the shape of the bid function. Besides, the expected utility of agents is linear in the strategies. It follows that if our optimization algorithms converge to a pure strategy, then they converge to an approximate equilibrium of the discretized game with high precision. Importantly, we show that the equilibrium of the discretized game approximates an equilibrium in the continuous game. In a wide variety of auction games, we provide empirical evidence that the approach approximates the analytical (pure) Bayes Nash equilibrium closely. This speed and precision is remarkable, because in many finite games learning dynamics do not converge or are even chaotic. In standard models where agents are symmetric, we find equilibrium in seconds. While we focus on dual averaging, we show that the overall approach converges independent of the regularizer and alternative online convex optimization methods achieve similar results, even though the discretized game neither satisfies monotonicity nor variational stability globally. The method allows for interdependent valuations and different types of utility functions and provides a foundation for broadly applicable equilibrium solvers that can push the boundaries of equilibrium analysis in auction markets and beyond.

Key words: auctions, Bayes-Nash equilibrium, online convex optimization

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1. Introduction

Auction games are arguably some of the most important applications of game theory and they can be analyzed as continuous-type, continuous-action Bayesian games. Bidders' valuations or types in such an auction game are drawn from some continuous distribution and they can choose from a

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continuous range of possible actions (or bids). Early on, Nobel Prize laureate Vickrey (1961) showed how to derive a Bayes-Nash equilibrium (BNE) strategy in a single-object first-price auction in the independent-private values (IPV) model with symmetric bidders and quasi-linear utility functions. The first-order conditions together with the assumption of symmetric bidding behavior lead to an ordinary differential equation, which has a closed-form solution for the BNE bidding strategy.

The BNE provides a principled way to think about strategic interaction in auctions and a prescriptive model how rational bidders should behave. Unfortunately, deviations from the benchmark model by Vickrey (1961) lead to challenges in the equilibrium analysis (McAfee and McMillan 1987). For example, when the valuations of potential bidders are interdependent, then the system of first-order partial differential equations that characterizes a BNE often becomes intractable (Campo et al. 2003). Computing Nash equilibria (NE) in complete-information finite games is already known to be PPAD-hard. However, computing of exact Bayesian Nash equilibria (BNE) can even be PP-hard, a complexity class that is clearly intractable (Cai and Papadimitriou 2014). Overall, the analytical derivation of BNE strategies has been elusive for all but very simple auction games. Even existence of BNE has only been shown for a limited set of auction models (Jackson and Swinkels 2005). As a result, Bayes-Nash equilibrium analysis remained in the realm of academic research and it is rarely used by bidders in real-world auctions.

There have been a few approaches to develop numerical techniques for specific environments. For example, Armantier et al. (2008) introduced a BNE-computation method that is based on expressing the Bayesian game as the limit of a sequence of complete-information games. Rabinovich et al. (2013) study best-response dynamics in auctions with finite action spaces, while Bosshard et al. (2020) contribute an iterated best-response algorithm for combinatorial auctions with an elaborate empirical verification method. Recently, Bichler et al. (2021) introduced a versatile technique to compute approximate Bayes-Nash equilibria (BNE) in a variety of auction models using neural networks and self-play. Their use of neural networks and evolutionary strategies leads to a relatively complex algorithm, which leverages massive parallelization on GPU hardware. In all prior approaches, numerical techniques are required to certify that the strategies found are indeed an approximate BNE and there are no guarantees that the process converges or that a BNE emerges if the algorithm converges. All these techniques are computationally expensive, even for simple symmetric auction models.

We introduce an algorithmic framework based on a discretization of the type and action space, in which we can use online convex optimization to learn distributional strategies (Milgrom and Weber 1985), which are a form of mixed strategies for Bayesian games. In contrast to learning algorithms for complete-information games, auction games require us to consider the prior type

distributions. The distributional strategies allow us to derive gradients and implement gradientbased optimization algorithms without relying on neural networks with self-play as in Bichler et al. (2021). In Simultaneous Online Dual Averaging (SODA) we focus on dual averaging as learning algorithm, which is one of the most effective online convex optimization algorithms. However, empirically we show that alternative algorithms such as mirror ascent or the Frank-Wolfe algorithm achieve very similar results in a wide variety of auction models and contests. SODA allows for interdependent types and different utility functions (e.g., risk aversion), which makes it a very fast and generic algorithm compared to existing approaches. It is straightforward to incorporate risk aversion or other behavioral motives in the utility function, which leads to complications in analytical derivations. Importantly, it does not make any assumptions on the parametric form of the bid function, allowing us to find non-smooth equilibria as well. An advantage of dual averaging is that the expected utility is linear in the distributional strategies as we show, which allows us to show that if the algorithm converges to a pure strategy, then it has to be an equilibrium of the discretized game. This is an advantage over prior numerical methods, which rely on numerical estimates of the utility loss to certify an approximate equilibrium. Importantly, we can show for single-object auctions that the distributional ε -BNE found in the discretized auction approximates a continuous equilibrium, if one exists. Note that there are examples where equilibria exist only in the discretized game and not in the continuous game (Jackson and Swinkels 2005).

Ex ante conditions that certify when gradient-based optimization algorithms converge to equilibrium even in finite, complete-information games turned out to be challenging. A number of recent results on matrix games showed that gradient-based algorithms either circle, diverge, or are even chaotic (Sanders et al. 2018). Independent learning dynamics do not generally obtain a Nash equilibrium (Benaim and Hirsch 1999). Actually, the study of gradient dynamics in games is akin to studying dynamical systems and characterizing environments where gradient dynamics converge to a Nash equilibrium (if one exists) can be arbitrarily complex (Andrade et al. 2021). The analysis of Bayesian games with continuous type and action spaces is difficult: for a convergence analysis we need to study the properties of an expected utility function that is based on the characteristics of an unknown equilibrium bid function. While this is not the case in the discretized version of the game, ex-ante guarantees are still very challenging as we discuss in Section 3.6. For example, we show that conditions such as monotonicity or variational stability do not hold globally in the discretized game. Yet, given that the algorithms are fast for standard auction and contest models, the ex-post verification we get with SODA is very useful.

We provide extensive experimental results where we approximate the analytical pure BNE closely in a wide variety of auction games and contests. We could actually compute close approximations of the BNE with only a few bidders in seconds even for complex core-selecting combinatorial auctions. If we restrict ourselves to independent private values, we can solve large instances with dozens of bidders within seconds. This allows for a quick exploration of auction models with different priors or different utility functions.

The wide range of environments where SODA converges is remarkable. We illustrate results of SODA for environments where an analytical solution is known, but also provide equilibrium strategies for models where no Bayes Nash equilibrium was available so far. Experimental results are reported for single-object auctions with interdependent valuations, combinatorial auctions with independent and interdependent values, combinatorial split-award auctions, all-pay auctions and Tullock contests.

Convergence of SODA to equilibrium is guaranteed, if the utility gradients are monotone or they satisfy relaxed notions such as variational stability (Geiger and Kanzow 2013, Mertikopoulos and Zhou 2019, Grossmann et al. 2007). With monotone utility gradients, the expected utility function is concave. Without knowing the parametric form of the bid function it is difficult to understand a priori whether concavity of the expected utility is satisfied in a specific Bayesian auction game. Numerical analysis with parametric assumptions on the bid function and the distribution function suggest that the expected utility function of several well-known auction games is concave or pseudoconcave for large ranges of the bid space, which explains the surprisingly positive results compared to several recent studies showing that gradient-based optimization algorithms and the resulting dynamics often do not converge in finite normal-form games.

Overall, the paper shows that important applications of equilibrium computation problems in auctions and contests are tractable and we can find approximate equilibria quickly. This provides a foundation for numerical tools that allow us to push the boundaries of equilibrium analysis. Tools of this sort will prove useful for market designers to understand specific market rules, but also for bidders to study strategic interaction in high-stakes auctions.

Section 2 provides a brief overview of related literature. Then, Section 3 introduces the notation and the algorithm, and discusses convergence and scalability. Section 4 reports results for various single-object and combinatorial auction models, before Section 6 provides conclusions.

2. Related Literature

Our research primarily relates to the extensive economic literature on equilibrium in auctions and contests and to the literature on equilibrium learning.

2.1. Equilibrium in Auction Games

Our paper primarily deals with Bayesian auction games where type- and action-spaces are continuous. A first question is whether BNE always exist in such games. Auctions and contests are prime applications, central to economic theory. For finite, complete-information games, we know

that a mixed Nash equilibrium exists (Nash et al. 1950) and that the computation is generally PPAD-hard (Daskalakis et al. 2009). Glicksberg (1952) extended the existence result to games with continuous and compact action sets. For Bayesian games with continuous action space, Jackson and Swinkels (2005) provide assumptions for the existence of equilibria in distributional strategies. For example, first-price and second-price single-unit auctions, all-pay auctions, double auctions, and multi-unit discriminatory or uniform price auctions were shown to have an equilibrium in distributional strategies. It is interesting to note that there are auction models where there is no Bayesian Nash equilibrium of the continuous game, but there are equilibria in the discretized game (Jackson et al. 2002). Overall, we neither know of the existence of Bayes-Nash equilibria in general continuous-type and -action auction games, nor do we know how hard they are to find if they exist. Cai and Papadimitriou (2014) showed that finding an exact BNE in specific simultaneous auctions for individual items is at least hard for PP, a complexity class higher than the polynomial hierarchy and close to PSPACE, and we know little about the complexity of finding BNE in other multi-item auctions.

2.2. Equilibrium Learning

Our research is best situated in the literature on equilibrium learning. The theory of learning in games examines what kind of equilibrium arises as a consequence of a process of learning and adaptation, in which agents are trying to maximize their payoff while learning about the actions of other agents (Fudenberg and Levine 2009). Fictitious play is a natural method by which agents iteratively search for a pure Nash equilibrium and play a best response to the empirical frequency of play of other players (Brown 1951). Several algorithms have been proposed based on best or better response dynamics. Besides, gradient-based online optimization algorithms have been proposed for normal-form games (Singh et al. 2000, Zinkevich 2003).

While such online gradient ascent algorithms lead to zero regret for the participating agents, their strategies do not generally converge. Even in simple matching pennies games, the gradient dynamics circle (Bowling 2005). Hence, no-regret learning algorithms do not find a BNE in general games. However, due to their simplicity, learning algorithms have been used to solve games for a long time. While there is no comprehensive characterization of games that are "learnable," and one cannot expect that uncoupled dynamics lead to Nash equilibrium in all games (Hart and Mas-Colell 2003), there are some important results regarding no-regret learners. First, one can distinguish between internal (or conditional) regret and a weaker version called external (or unconditional) regret. External regret compares the performance of an algorithm to the best single action in retrospect, while internal regret allows one to modify the online action sequence by changing every occurrence of a given action with an alternative one. For learning rules that satisfy the stronger no-internal regret condition, the empirical frequency of play converges to the game's set of correlated

equilibria (Foster and Vohra 1997, Hart and Mas-Colell 2000, Stoltz and Lugosi 2007). The set of correlated equilibria (CE) is a nonempty convex polytope that contains the convex hull of the game's Nash equilibria. The coordination in CE can be implicit via the history of play (Foster and Vohra 1997, Stoltz and Lugosi 2007). On the other hand, algorithms that are no-external-regret learners converge by definition to the set of coarse correlated equilibria (CCE). This set, in turn, contains the set of CE such that we get $NE \subset CE \subset CCE$. In contrast to correlated equilibria, coarse correlated equilibria may contain strictly dominated (pure) strategy profiles with positive probability (Viossat and Zapechelnyuk 2013), which makes them a relatively weak solution concept.

Recent work shows that gradient dynamics often do not converge (Daskalakis et al. 2010, Vlatakis-Gkaragkounis et al. 2020). Standard learning algorithms can cycle, diverge, or even be chaotic in zero-sum games (Mertikopoulos et al. 2018, Bailey and Piliouras 2018, Cheung and Piliouras 2020). Actually, Sanders et al. (2018) suggest that chaos is, in fact, typical behavior for more general matrix games. Simple examples where reasonable gradient-based methods cannot converge leave little hope for general gradient-based methods in the broader class of differential games (Letcher et al. 2019). Notably, the dynamics of general matrix games can be arbitrarily complex and hard to characterize a priori (Andrade et al. 2021). On the positive side, there is a long literature on monotonicity conditions that guarantee convergence to a Nash equilibrium (Kinderlehrer and Stampacchia 2000, Grossmann et al. 2007, Geiger and Kanzow 2013, Mertikopoulos and Zhou 2019). Apart from monotonicity, Even-Dar et al. (2009) introduce the notion of socially concave games. These are games where a convex combination of all agents' utilities exists that is concave, and each agent's utility is convex in the other agents' strategies. In contrast to monotonicity and variational stability, which suffices for convergence of the last iterate, social concavity only implies convergence of the mean of iterates to a BNE. However, it is also a strong assumption on the utility functions.

A large part of the literature on equilibrium learning has focused on complete-information games (Foster and Vohra 1997, Hart and Mas-Colell 2000, Jafari et al. 2001, Stoltz and Lugosi 2007, Hartline et al. 2015, Syrgkanis et al. 2015, Foster et al. 2016). Uncertainty about other players has also received attention. For example, there is work on Stackelberg games with uncertainty about the follower (Balcan et al. 2015), and there is a stream of literature on imperfect-information games as in Poker (see for example Sandholm (2015), Brown and Sandholm (2019)). The literature is too large to provide a comprehensive survey here. Bayesian games with continuous type and action spaces as they are used to model auctions or contests are less well studied. Solving such problems is challenging because it requires learning a bid function over infinitely many types. Such problems can be formulated as systems of differential equations. We lack a solution theory for such problems in general. Given how hard it is to find Bayes-Nash equilibria even in simultaneous multi-object

auctions in the worst case (Cai and Papadimitriou 2014), it is far from obvious that gradient-based algorithms can find a BNE in continuous-type and -action Bayesian games. It is not even clear how gradient dynamics would be implemented in games with continuous type space.

Neural Pseudogradient Ascent (NPGA) by Bichler et al. (2021) was recently published to address equilibrium computation in auction games: it is the first numerical method to compute BNE in a wide variety of auction games, including multi-object auctions with interdependent types. Therefore, it will serve as our benchmark when we report our experimental results. NPGA uses neural networks as a bid function to be learned via self-play. The authors employ evolutionary strategies as a smoothing technique to deal with the discontinuities of the ex-post utility function, which allows them to compute BNE in a finite-dimensional parameter space of neural networks. However, the use of neural networks and specific training methods makes it hard to derive theoretical guarantees. Moreover, it takes an expensive empirical validation procedure to verify if the strategies found by the algorithm are approximate BNE.

Our technique is quite different in that we discretize the type and action spaces and implement gradient dynamics in the discretized version of the game without using neural networks. We apply various well-known online learning algorithms to the discretized game. Our focus is on the dual averaging algorithm with entropy regularization, as it is often the method of choice in theoretical analyses (Mertikopoulos and Zhou 2019), and it enjoys particularly good regret bounds (Shalev-Shwartz and Singer 2007). While NPGA searches for pure Bayesian Nash equilibria, we compute distributional strategies in a discretized version of the game. Our technique is much faster for environments with a few players and items and can solve equilibria in symmetric auction games in seconds. Compared to NPGA, SODA is much easier to implement. Additionally, as a consequence of the no-regret property of our algorithm, if SODA converges, the limit point must necessarily be a Bayes-Nash equilibrium. Importantly, we can bound the approximation error to the original auction game with continuous type and action space. These theoretical guarantees are a significant advantage over NPGA because we do not require an expensive experimental verification of the solution.

3. Model and Algorithm

We will first introduce the necessary notation before we discuss a small illustrative example, and then describe the algorithm more generally.

3.1. Notation

An incomplete-information or Bayesian game is given by a sextuplet $G = (\mathcal{I}, \mathcal{V}, \mathcal{O}, \mathcal{A}, f, u)$. Here $\mathcal{I} = \{1, \dots, n\}$ denotes the set of agents participating in the game. Agent i's private observation is then given as a realization $o_i \in \mathcal{O}_i$, with $\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_n$ being the set of possible observation

profiles. Similarly, \mathcal{V} denotes the set of "true" but possibly unobserved valuations. Crucially, we make this distinction to model interdependencies in settings beyond purely private values or purely common values. Based on the observation o_i , the agent chooses an action, or bid, $b_i \in \mathcal{A}_i$, and the set of possible action profiles is given by $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. The joint probability density function $f: \mathcal{O} \times \mathcal{V} \to \mathbb{R}_{\geq 0}$ describes an atomless prior distribution over agents' types, given by tuples (o_i, v_i) of observations and valuations. We make no further restrictions on f, thus allowing for arbitrary correlations. f is assumed to be common knowledge and we will denote its marginals by f_{v_i} , f_{o_i} , etc.; its conditionals by $f_{v_i|o_i}$, etc.; and its associated probability measure by F.

For each possible action and valuation profile, the vector $u = (u_1, ..., u_n)$ of F-integrable, individual (ex-post) utility functions $u_i : \mathcal{A} \times \mathcal{V}_i \to \mathbb{R}$ assigns the game outcome to each player. Ex-ante, before the game, agents neither have observations nor valuations, only knowledge about f. In the interim stage, agents additionally observe o_i providing (possibly partial or noisy) information about their own valuations v_i . Full access to the outcomes u(v,b) is given only after taking actions (ex-post). In our formulation, we do not assume explicit ex-post access to any values (e.g., v_i, v_{-i}, b_{-i}) beyond the outcome u itself. An index -i denotes a partial profile of all agents but agent i.

Table 1	1 Types of interdependencie	s.
	Private v 's $(v = o)$	Common v 's (CV)
Independent o's (PDF of o is product of marginal PDFs)	Independent private values model (IPV)	Independent non-private values or common values
Correlated o's	Correlated or affiliated (APV) private values	Correlated non-private values

Taking an ex-ante view, players are tasked with finding strategies that link observations and bids. Instead of pure strategies, which are measurable functions $\beta_i : \mathcal{O}_i \to \mathcal{A}_i$ that map observations to bids, we are interested in distributional strategies that induce a probability measure on the space of observations and actions (Milgrom and Weber 1985).

DEFINITION 1. In the private values model, a distributional strategy for player i is a probability measure σ on $\mathcal{O}_i \times \mathcal{A}_i$ for which the marginal distribution on \mathcal{O}_i is f_{o_i} . Formally, the marginal condition can be written as $\sigma(O \times \mathcal{A}_i) = F_{o_i}(O)$ for all measurable sets $O \subset \mathcal{O}_i$. When players adopt distributional strategies $(\sigma_1, ..., \sigma_n)$ the expected utility is given by

$$\tilde{u}_i(\sigma_1, ..., \sigma_n) = \int u_i(b, o_i) \sigma_1(db_1|o_1) ... \sigma_n(db_n|o_n) F(do)$$
(1)

The strategy profile $(\sigma_1, ..., \sigma_n)$ is a ε -Bayes-Nash equilibrium $(\varepsilon$ -BNE) if no bidder i can increase its utility by more than $\varepsilon \geq 0$ by unilaterally deviating from its distributional strategy σ_i , i.e.,

$$\tilde{u}_i(\sigma'_i, \sigma_{-i}) - \tilde{u}_i(\sigma_i, \sigma_{-i}) \le \varepsilon \quad \forall \sigma'_i \text{ and } \forall i \in \mathcal{I},$$
 (2)

where σ_{-i} denotes the partial strategy profile for all bidders but bidder i. If $\varepsilon = 0$, the strategy profile corresponds to a Bayes-Nash equilibrium (BNE).

The primary Bayesian games we'll consider are sealed-bid auctions on I indivisible items. In general combinatorial auctions we thus have a set K of possible bundles of items and the valuationand action-spaces are therefore of dimension $|K| = 2^I$. In the private values setting, we always have $o_i = v_i$; in the common values setting, there is some unobserved constant $v_c = v_1 = \cdots = v_n$ and the o_i can be considered noisy measurements of v_c . Mixed settings are likewise possible. In any case, based on bid profile b, an auction mechanism will determine two things: An allocation $x = x(b) = (x_1, \dots x_n)$ which constitutes a partition of the m items, where bidder i is allocated the bundle x_i ; and a price vector $p(b) \in \mathbb{R}^n$, where p_i is the monetary amount bidder i has to pay in order to receive x_i . Formally, one may consider the individual allocations to be one-hot-encoded vectors $x_i \in \{0,1\}^{|K|}$. In the standard risk-neutral model the utilities u_i are then described by quasilinear payoff functions $u_i^{QL}(b, v_i) = (x_i(b) \cdot v_i - p_i(b))$, i.e., by how much players value their allocated bundle minus the price they have to pay.

An extension to this basic setting includes risk-aversion. Here, we model risk-aversion via utilities $u^{RA} = (u^{QL})^{\rho}$ where $\rho \in (0,1]$ is the risk attitude; $\rho = 1$ describes risk-neutrality, smaller values lead to strictly concave, risk-averse transformations of u^{QL} . Risk aversion is an established way to explain why in field studies of single-object first-price sealed-bid (FPSB) auctions, bidders bid higher than their risk-neutral counterparts in analytical BNE (Bichler et al. 2015). However, different types of utility functions are possible.

3.2. An Illustrative Example

Before we introduce the model and our algorithm in general, let us discuss a simplified setting: a single-object first-price sealed-bid auction with two symmetric bidders. We focus on the IPV model, where both bidders $i \in \{1,2\}$ observe their true valuation $o_i = v_i$ for an item, which is drawn independently according to some prior marginal distribution F_{o_i} from a real interval $\mathcal{O}_i \subset \mathbb{R}$. Therefore, the common prior is the product of the two marginal distributions. After both bidders submit their bids (b_1, b_2) the (ex-post) utility of player 1 (analogously for player 2) is given by

$$u_1(b_1, b_2, o_1) = \begin{cases} o_1 - b_1 & \text{if } b_1 > b_2\\ \frac{1}{2}(o_1 - b_1) & \text{if } b_1 = b_2 \\ 0 & \text{else} \end{cases}$$
 (3)

The bidders are risk neutral and want to maximize their expected profits/utilities. To analyze such auction formats, we are interested in equilibrium strategies, where no bidder has the incentive to deviate from the current strategy. Instead of pure strategies $\beta: \mathcal{O}_i \to \mathcal{A}_i$, we focus on distributional strategies σ , which are probability measures over $\mathcal{O}_i \times \mathcal{A}_i$. This means rather than first observing

o and then choosing an action b according to β , distributional strategies assign probabilities to observation-action pairs.

This idea becomes more tangible when we apply it in a discrete setting. Let us consider the auction in discretized versions of the observation and action space i.e., $\mathcal{O}_i^d = \{o_1,...,o_K\} \subset \mathcal{O}_i$ and $\mathcal{A}_i^d = \{b_1,...,b_L\} \subset \mathcal{A}_i$. The discrete distributional strategy s_i can be seen as a form of mixed strategy for Bayesian games over the discretized observation and action space, i.e., $s_i \in \Delta(\mathcal{O}_i^d \times \mathcal{A}_i^d)$. For each observation o_k , the strategy s_i induces a mixed strategy $s_i(\cdot|o_k) \in \Delta(\mathcal{A}^d)$ over the action space. This is similar to imperfect-information extensive-form games, where behavioral strategies induce mixed strategies at each information set (Shoham and Leyton-Brown 2008). In distributional strategies, these different mixed strategies are now combined by weighting each mixed strategy with the probability $(f_{o_i}^d)_k$ of the respective observation o_k , induced by the prior distribution. That is, we obtain a matrix $s_i \in \Delta(\mathcal{O}_i^d \times \mathcal{A}_i^d) \subset \mathbb{R}^{K \times L}$ in which each entry $(s_i)_{kl} = (f_{o_i}^d)_k \cdot s_i(b_l|o_k)$ indicates the probability that o_k is observed and b_l is played. By construction, this matrix satisfies the marginal condition as described in Definition 1. Note that the set of such distributional strategies, denoted by \mathcal{S}_i^d , is convex. Given a strategy profile $(s_1, s_2) \in \mathcal{S}_1^d \times \mathcal{S}_2^d$, the expected utility \tilde{u}_1 for player 1 is the sum of all outcomes weighted by their respective probability induced by the strategies:

$$\tilde{u}_1(s_1, s_2) = \sum_{k_1, l_1=1}^{K, L} \sum_{k_2, l_2=1}^{K, L} u_1(b_{l_1}, b_{l_2}, o_{k_1})(s_1)_{l_1 k_1}(s_2)_{l_2 k_2}$$

$$\tag{4}$$

$$= \sum_{k_1, l_1=1}^{K, L} (s_1)_{k_1 l_1} \left(\sum_{k_2, l_2=1}^{K, L} u_1(b_{l_1}, b_{l_2}, o_{k_1})(s_2)_{k_2 l_2} \right) =: \langle s_1, c_1 \rangle.$$
 (5)

This linear structure of the utility function \tilde{u}_i allows for two things. First, the function is obviously differentiable with $\nabla_{s_1}\tilde{u}_1(s_1;s_2)=c_1$. And secondly, the best response $s_1^{br}=\arg\max\{\tilde{u}_1(s,s_2):s_1\in\mathcal{S}^d\}$, given the opponents strategy s_2 , is the solution of the following linear program

$$\max_{s \in \mathbb{R}^{K \times L}} \langle s, c_1 \rangle \text{ s.t. } \sum_{l=1}^{L} s_{kl} = (f_{o_1}^d)_k \quad \forall k \in \{1, \dots, K\}$$

$$s_{kl} \ge 0 \quad \forall k \in \{1, \dots, K\}, \ l \in \{1, \dots, L\}.$$

$$(6)$$

This allows us to compute the utility loss, i.e., the utility gap ε of a ε -NE.

Overall, we can define a complete-information game $\Gamma = (\mathcal{I}, \mathcal{S}_i^d, \tilde{u}_i)$ based on the discretized incomplete-information game. The distributional strategies correspond to a compact, convex action set, and the expected utility functions \tilde{u}_i to differentiable utility functions. This enables us to draw on standard equilibrium learning methods from online convex optimization. In this example we will focus on dual averaging (DA) (Nesterov 2009). DA is based on two steps. Both players simultaneously use their gradients to update a variable in the dual space and then mirror this updated dual variable back to the feasible set \mathcal{S}_i^d to get an updated strategy s_i . In this setting, this is

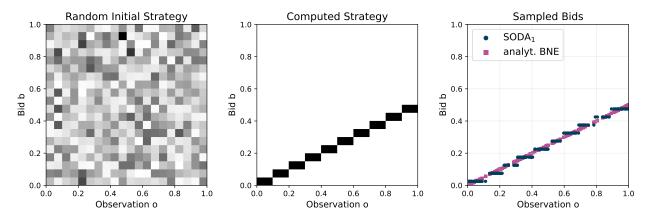


Figure 1 SODA applied to discretized FPSB auction with two symmetric bidders.

Note. In the first plot we can see a random strategy $s_i \in \mathcal{S}_i^d$. The color of each square represents the probability of the respective observation-action pair. The second plot shows the computed strategy using SODA. We can observe that the probabilities concentrate as expected near $\frac{1}{2}o$. In the last plot we compare the computed strategy with the analytical BNE from the continuous setting. This is done by sampling 150 observations o according to the uniform prior. Bids from the computed strategy are then obtained by identifying the nearest discrete observation o_k and sample a bid from the induced mixed strategy $s_i(\cdot|o_k) \in \Delta(\mathcal{A}_i^d)$ (blue dots). For the analytical BNE we simply plug in the sampled observations to the equilibrium function $\beta_i(o) = \frac{1}{2}o$ (purple squares). This way, we can evaluate the approximation and compute the metrics as explained in Section 4.1.

equivalent to Follow-the-Regularized-Leader (FTRL) (Shalev-Shwartz 2012). We repeat these steps until the strategy profile is close enough to an equilibrium i.e., until the utility loss with respect to the best response in the current strategy profile is sufficiently small. In Figure 1 we can see an application of this simultaneous online dual averaging (SODA) algorithm to our discretized first-price sealed-bid auction with two bidders as introduced above. We consider uniformly distributed observations over $\mathcal{O}_1 = \mathcal{O}_2 = [0, 1]$ and allow for bids within the same interval, i.e., $\mathcal{A}_1 = \mathcal{A}_2 = [0, 1]$. Both spaces are discretized using K = L = 20 equidistant points.

After this illustration, let us now introduce the model and the algorithm formally.

3.3. Discretization

Our algorithms are based on a discrete version of the game and distributional strategies. As illustrated by the example in the previous section, these discrete distributional strategies are constructed by restricting ourselves to finite subsets of the observation, valuation, and action sets and considering finitely atomic measures as a counterpart to the distributional strategies in the continuous setting. This constitutes a specific discretized game formalization that the algorithms operate on, which we also refer to as approximation game.

Formally speaking, we construct a discrete version $G^d = (\mathcal{I}, \mathcal{V}^d, \mathcal{O}^d, \mathcal{A}^d, f^d, u)$ of the incomplete-information game G. This is done by defining a set of discrete observations $\mathcal{O}^d = \mathcal{O}_1^d \times ... \times \mathcal{O}_n^d$ where

 $\mathcal{O}_i^d := \{o_1^i, ..., o_K^i\} \subset \mathcal{O}_i$. Similarly we define $\mathcal{A}_i^d := \{b_1^i, ..., b_L^i\} \subset \mathcal{A}_i$ and $\mathcal{V}_i^d := \{v_1^i, ..., v_M^i\} \subset \mathcal{V}_i$. We further approximate the joint probability density function f by a discrete version f^d over $\mathcal{V}^d \times \mathcal{O}^d$. The marginal distribution of f^d over \mathcal{O}_i^d can be written as $f_{o_i}^d \in \Delta(\mathcal{O}_i^d) \subset \mathbb{R}^K$. For simplicity we assume that the spaces are discretized with the same number of points for all agents. But this does not have to be the case.

The discrete version s_i of a distributional strategy σ_i for bidder i is now measure over $O_i^d \times A_i^d$ and can be identified with a matrix $s_i \in \Delta(O_i^d \times A_i^d) \subset \mathbb{R}^{K \times L}$. The marginal condition for distributional strategies translates to $\sum_l (s_i)_{kl} = (f_{o_i}^d)_k$ for all k = 1, ..., K. Therefore the set of all possible discrete distributional strategies for bidder i can be identified by matrices of the form:

$$S_i^d := \left\{ s_i \in \mathbb{R}^{K \times L} : (s_i)_{kl} \ge 0 \ \forall k, l, \text{ and } \sum_{l} s_{kl} = (f_{o_i}^d)_k \ \forall k \right\}$$
 (7)

For a given strategy profile $(s_1,...,s_n) \in \mathcal{S}_1^d \times ... \times \mathcal{S}_n^d$ we can compute the expected utility. This corresponds to the discretized version of equation (1).

$$\tilde{u}_i(s_1, ..., s_n) = \sum_{k,l,m} u_i(b_l, v_{m_i}) \prod_{j=1}^n (s_j)_{k_j l_j} \frac{(f^d)_{m,k}}{(f^d_{o_1})_{k_1} \cdots (f^d_{o_n})_{k_n}}$$
(8)

$$= \sum_{k_i, l_i} (s_i)_{k_i l_i} \sum_{m, k_{-i}, l_{-i}} u_i(b_l, v_{m_i}) \prod_{j \neq i} (s_j)_{k_j l_j} \frac{(f^d)_{m, k}}{\prod_{j'} (f^d_{o_{j'}})_{k_{j'}}}$$
(9)

For all k_i, l_i we denote the second sum, which only depends on s_{-i} , as $(c_i)_{k_i, l_i}$ and write

$$\tilde{u}_i(s_1, ..., s_n) = \sum_{k_i, l_i} (s_i)_{k_i l_i} (c_i)_{k_i, l_i} = \langle s_i, c_i \rangle$$
(10)

Note that in these equations, $l = (l_1, ..., l_n)$ is a multi-index and $b_l = (b_{l_1}^1, ..., b_{l_n}^n)$ the action profile of all bidders (same for v and o respectively). Since the second sum (c_i) does not depend on s_i , the expected utility function for bidder i is linear in the bidder's own strategy. Instead of considering the discretized incomplete-information game G^d , we can use the expected utility \tilde{u}_i and the sets of discrete distributional strategies S_i^d to define a complete-information game.

DEFINITION 2. Given the Bayesian game $G = (\mathcal{I}, \mathcal{V}, \mathcal{O}, \mathcal{A}, f, u)$, we construct a discrete version $G^d = (\mathcal{I}, \mathcal{V}^d, \mathcal{O}^d, \mathcal{A}^d, f^d, u)$ of the game by discretizing the respective spaces and probability distributions. The resulting sets of discrete distributional strategies \mathcal{S}_i^d and the expected utility \tilde{u}_i define a complete-information game $\Gamma = (\mathcal{I}, \mathcal{S}^d, \tilde{u})$, which we call the approximation game of G.

Observe that the Nash equilibria $s \in \mathcal{S}^d$ of the approximation game Γ , characterized by

$$\tilde{u}_i(s_i, s_{-i}) \ge \tilde{u}_i(s_i', s_{-i}) \quad \forall s' \in \mathcal{S}_i^d, \, \forall i \in \mathcal{I},$$

$$\tag{11}$$

correspond to Bayes Nash equilibria in the discretized Bayesian game G^d .

3.4. Algorithm

The approximation game $\Gamma = (\mathcal{I}, \mathcal{S}^d, \tilde{u})$ is a well-behaved complete-information game with linear (in s_i) utility functions \tilde{u}_i and compact, convex action sets $\mathcal{S}_i^d \subset \mathbb{R}^{K \times L}$. This structure allows us to use algorithms from online convex optimization, where all agents simultaneously compute the gradient given the current strategy profile and update their strategies according to some chosen method (Algorithm 1). In particular, we focus on Dual Averaging (DA) (Nesterov 2009) since Mertikopoulos and Zhou (2019) provide an ex-post certificate for the computed strategies if we converge (see Corollary 1). While DA is our baseline algorithm, we also analyze alternative gradient-based algorithms. This will help us understand whether convergence in these games is restricted to a specific type of algorithm or regularizer. Specifically, we provide results for gradient-based methods such as Mirror Descent (MD) (Nemirovskij and Yudin 1983) and the Frank-Wolfe Algorithm (Frank and Wolfe 1956).

Let us briefly summarize the gradient-based methods we are considering. Mirror Descent can be interpreted as a generalized projected gradient descent, where the projection is with respect to the Bregman divergence induced by a distance-generating mirror map g (Beck and Teboulle 2003). Commonly used mirror maps are strongly convex functions such as the negative entropy $g_1(x) = \sum_i x_i \log x_i$ with $g_1(0) = 0$ and $g(x) = \infty$ for all $x \notin \mathbb{R}^n_{\geq 0}$, and the Euclidean distance squared $g_2(x) = ||x||_2^2$. While g_2 leads to the standard projected gradient algorithm, the update step generated by g_1 is known as the entropic descent algorithm (Beck and Teboulle 2003).

In Dual Averaging one distinguishes between dual and primal iterates. It is considered to be a lazy version of Mirror Descent since the gradient update is only done in the dual space. To get the next iterate in the primal space, the updated dual variable is projected onto the feasible set in the primal space. The projection is done with respect to some regularizer h which again is induced by a strongly convex function. For our examples the mirror maps g_a with $a \in \{1,2\}$ induce regularization functions h_a by $h_a(x) = g_a(x) + I_{\mathcal{S}_i}(x)$, where $I_{\mathcal{S}_i}(x) = 0$ if $x \in \mathcal{S}_i$ and $+\infty$ else. For h_1 we get the same update step as in MD, namely the entropic descent algorithm. But for the Euclidean regularizer, DA leads to a lazy version of the projected gradient descent which is equal to the (linearized) FTRL with Euclidean regularizer. A pseudo-code can be found in Algorithm 1.

MD and DA are widely used and no-regret learners (Shalev-Shwartz 2012). Juditsky et al. (2022) provides a detailed analysis of Mirror Descent and Dual Averaging, unifying both approaches and explaining the differences between mirror maps (MD) and regularizers (DA). They also provide intuition for the cases where both methods coincide, as we can observe for the negative entropy.

Another method we consider is the Frank-Wolfe (FW) algorithm, also known as conditional gradient. This method uses gradient feedback to solve the linear program induced by the first-order approximation of the objective function. The next iterate is a convex combination of this optimal

Algorithm 1: Simultaneous Online Dual Averaging (SODA)

 $\begin{array}{c|c} \mathbf{4} & y_{i,t+1} \leftarrow y_{i,t} + \eta_t \cdot c_{i,t} \\ \mathbf{5} & s_{i,t+1} \leftarrow \nabla h^*(y_{i,t+1}) \end{array}$

solution and the previous iterate. Since the feasible set is convex, one avoids the potentially expensive projection which has to be computed in the other methods. Hazan and Kale (2012) introduced an online version of the Frank-Wolfe, where the solution of the linear program is computed with respect to the aggregated objective functions of all previous iterates. In contrast to the standard version, the online version also has the no-regret property. But due to better performance in our experiments, we stick with the standard Frank-Wolfe algorithm.

An overview of the different update rules we used in our experiments is provided in Table 2. Interestingly, we find that all algorithms in Table 2 converge to equilibrium quickly and the results

	Method	Update Rule
$\begin{array}{ c c c c c }\hline SODA_1 \\ (SOMA_1) \\ \hline \end{array}$	Dual Averaging + entropic regularizer	$(s_{i,t+1})_{kl} = (f_{o_i}^d)_k \frac{(s_{i,t})_{kl} \exp(\eta_t(c_{i,t})_{kl})}{\sum_{l'} (s_{i,t})_{kl'} \exp(\eta_t(c_{i,t})_{kl'})} \forall k, l$
$SODA_2$	Dual Averaging + Euclid. regularizer	$y_{i,t+1} = y_{i,t} + \eta_t c_{i,t}$ $s_{i,t+1} = \arg \max\{\ s - y_{i,t+1}\ _2^2 \text{ s.t. } s \in \mathcal{S}_i\}$
$SOMA_2$	Mirror Ascent + Euclid. mirror map	$s_{i,t+1} = \arg\max\{\ s - (s_{i,t} + \eta_t c_{i,t})\ _2^2 \text{ s.t. } s \in \mathcal{S}_i\}$
SOFW	Frank-Wolfe	$\begin{aligned} \operatorname{br}_{i,t+1} &= \operatorname{argmax} \{ \langle c_{i,t}, s \rangle \text{ s.t. } s \in \mathcal{S}_i \} \\ s_{i,t+1} &= (1 - \eta_t) s_{i,t} + \eta_t \operatorname{br}_{i,t+1} \end{aligned}$

Table 2 Overview of simultaneous online (SO) gradient updates.

Note that agents want to maximize their utilities in our examples, which is why we change all update rules to ascent instead of descent methods. The gradient for each agent is denoted by $c_i := \nabla_{s_i} u_i(s_i, s_{-i})$. We use a non-increasing sequence of step sizes $\{\eta_t\}$ of the form $\eta_t = \eta_0 t^{-\beta}$ for some $\beta \in (0,1]$ for DA and MD and the commonly used step size $\eta_t = \frac{2}{1+t}$ for Frank-Wolfe.

are similar. We also report results for Fictitious Play (FP), as an algorithm that is not gradient-based. FP is the oldest and best-known equilibrium learning technique (Brown 1951). At each round, each player best responds to the empirical frequency of play of their opponent. Also FP converges in the analyzed model, but it is not as efficient as the gradient-based algorithms.

3.5. Approximation via Discretization

Next, we show that approximate BNEs of the discrete game G^d naturally induce approximate BNEs of the continuous game G, where the quality of the approximation depends on the coarseness of the discretization. Thus, if our algorithm finds a good solution to the discretized setting, this also induces a good solution for the continuous setting, where the quality depends on the coarseness of the discretization. We only consider some specific single-object auctions here. Apart from that, we do not postulate any strong assumptions, such as symmetry or independence. The precise formal statement of the following theorem, together with its proof, can be found in Appendix B. The assumptions for the proof include single-object auctions such as the first-price and second-price sealed bid auctions as well as first-price and second-price all pay auctions (e.g., war of attrition).

THEOREM 1. Let $s \in \mathcal{S}^d$ be an ε -BNE of the discrete game G^d of a single-object auction. Let $\sigma \in \mathcal{S}$ be the strategy profile, where each σ_i is the strategy induced by s_i . Then σ is an $\varepsilon + \mathcal{O}(\delta_\alpha + \delta_\tau)$ -BNE of the continuous game G.

Here δ_{τ} and δ_{α} denote the coarseness of the discretization of the valuation and the action space. The central message of the proposition is that if we find an approximate BNE for the discrete game, we also find an approximate BNE for the continuous game with an additional error term decreasing linearly with the coarseness of the discretization.

The idea of the proof is as follows. Given an arbitrary strategy profile $s \in \mathcal{S}^d$ of the discrete game, we show that s naturally induces a feasible strategy profile $\sigma \in \mathcal{S}$ of the continuous game and that the difference of utilities of these two solutions is small. Conversely, we can construct a feasible discrete strategy profile s from a given continuous strategy profile s. Our central argument is that if we start with a continuous strategy profile $s \in \mathcal{S}$, and consider the induced discrete strategy profile $s \in \mathcal{S}^d$, which in turn induces a continuous strategy $\tilde{\sigma}_i \in \mathcal{S}_i$ for each agent s, the loss of utility is in $\mathcal{O}(\delta_\tau + \delta_\alpha)$. Now suppose we find an s-BNE s of the discrete game and consider the induced continuous strategy profile s. Let s be a best response to s is an s-BNE. But by the result mentioned above, the utility of the continuous strategy $\tilde{\sigma}_i$ neither differs by much from s, nor from s. Thus, the gain of utility from switching to s is in s in s in s the s-BNE.

In all our experiments not only the utility loss converged with finer discretization, but also the strategies converged. However, this is not necessarily the case. There are auction models where there is an approximate equilibrium in the discretized auction, but not in the continuous case (Jackson and Swinkels 2005). However, there are also cases where we know that pure, symmetric equilibria exist, but there may be no corresponding equilibrium in the discretized case. Rasooly and Gavidia-Calderon show that for different tie-breaking rules there are only asymmetric pure

equilibria for some simple first-price sealed bid auction settings and only mixed equilibria for some all-pay auction settings. Although such situations can happen, we can observe that SODA approximates the continuous equilibria well due to the richer class of symmetric distributional strategies that are being learned.

3.6. Ex-Post Certificates

An advantage of SODA over earlier methods for equilibrium computation in auctions (Bosshard et al. 2020, Bichler et al. 2021) is that SODA does not need an empirical verifier, which is computationally expensive. This insight relies on Mertikopoulos and Zhou (2019), who prove in their Theorem 4.1 that if a sequence of pure strategy profiles resulting from dual averaging converges to a strategy profile for all players, then this profile is a Nash equilibrium. A consequence of the distributional strategies that we learn is that the expected utility $\tilde{u}(s_1, \dots, s_n)$ is linear in the bidder's own strategy, satisfying the assumption of the theorem, that the utility functions are (pseudoconcave in the bidders' own strategies. Consequently, if SODA converges to a pure strategy, it also converges to a Nash equilibrium.

COROLLARY 1 (to Mertikopoulos and Zhou (2019), Theorem 4.1). Suppose that SODA is run with a step-size sequence that is square summable but not summable and produces the sequence $(s^t)_{t\in T}$ of action profiles. If the sequence of strategy profiles $(s_i^t)_{t\in T}$ converges to $s_i^* \in \mathcal{S}_i^d$ for all $i \in \mathcal{I}$, then s^* is a Nash equilibrium.

Of course, checking empirically whether an infinite sequence of iterates converges by inspecting finitely many of them is not possible. However, we believe that the rapidly decreasing distance between consecutive iterates we observe in our experiments strongly indicates that we indeed approximate exact BNEs with high precision.

4. Experimental Evaluation

We illustrate the versatility of our method by analyzing a number of very different auctions and contests. We report results on single-object auctions with interdependent valuations, combinatorial auctions with single-minded bidders and multi-minded bidders, single-object auctions with risk-averse bidders, and Tullock contests with a randomized contest success function. In some of these models the analytical BNE is given, which provides an unambigous baseline to compare against. However, we also explore models where no BNE was known so far, which includes all-pay auctions with risk-averse bidders and the Tullock contest. With only a few bidders we can compute BNE within a few minutes or seconds. We compare our results to those in Bichler et al. (2021) on NPGA to illustrate the performance increase we get for these environments.

4.1. Parameter and Evaluation Criteria

We start by constructing the approximation game by discretizing each dimension of the respective spaces with K = L = M = 64 equidistant points, if not stated otherwise. The discrete prior distribution is computed by evaluating the density function at these discrete points and normalizing the resulting probability vector. Since ties happen with a positive probability in the discretized game, we also have to define a tie-breaking rule. Due to better performance in our experiments, we deviate from the standard random tie-breaking and implement a rule where no agent wins if the maximal bid is not unique.

Given the constructed approximation game, we apply the learning algorithms as defined in Section 3.4. The algorithms stop either after a fixed number of iterations $(T=1\,000)$ or whenever the stopping criterion is satisfied, i.e., $\ell_i < \varepsilon_{\text{tol}} = 10^{-4}$, for each agent i. We use the relative utility loss ℓ_i as the stopping criterion, which denotes the relative improvement of the expected utility \tilde{u} an agent can achieve in the approximation game when fixing the opponents strategies s_{-i} and playing the best response s_i^{br} instead of s_i . The best response is the solution of a simple LP (see Equation 6). A low relative utility loss means that we are in some approximate NE in the approximation game. If bidders are symmetric, we learn a single strategy for all of them. After computing an approximate discrete distributional equilibrium strategy, we want to evaluate the computed solution within the initial continuous setting of the auction game. To do this, we sample observations according to the prior distribution and determine the corresponding bids from our strategies. Note that, unlike to pure strategy functions β_i , we cannot simply plug in the sampled observations o_i and get a bid $b_i = \beta_i(o_i)$. Instead, we have to identify the closest discrete observation o_{k_i} and sample a discrete bid $b_i \sim s_i(\cdot | o_{k_i})$ according to the induced mixed strategy by s_i . To compare our results with NPGA from Bichler et al. (2021), we choose the same approach and focus on two metrics. First, given the opponents' strategies β_{-i} we estimate the ex-ante utility using the sample-mean of the expost utilities $\hat{u}_i(\cdot, \beta_{-i}) := \frac{1}{n_o} \sum_o u_i(\cdot, \beta_{-i}((o_{-i})))$. We then compare the outcome of a player bidding according to the computed strategy s_i versus bidding according to the known equilibrium strategy β_i , while all opponents j play the equilibrium strategy β_j . This leads to the relative ex-ante utility loss $\mathcal{L}(s_i;\beta) = 1 - \frac{\hat{u}_i(s_i,\beta_{-i})}{\hat{u}_i(\beta_i,\beta_{-i})}$. Secondly, we report the probability-weighted root mean squared error of the sampled bids from s_i and the bids from the equilibrium strategy β_i , which approximates the L_2 distance of two functions or in our case between the function and the sampled bids b_i , i.e., $L_2(s_i, \beta_i) = \left(\frac{1}{n_o} \sum_{o_i} (s_i(\cdot | o_i) - \beta(o_i))^2\right)^{\frac{1}{2}}$. This metric ensures that we not only achieve a low utility loss but also approximate the equilibrium strategies. Similar to Bichler et al. (2021) we sample $n_o = 2^{22}$ observation (or valuation) profiles for both metrics. We report the mean and standard deviation of all metrics over ten runs with random initial strategies.

All experiments are run on a computer with an Intel Core i7-8565U CPU @ 1.80 GHz and 16GB of RAM. The implementation of the algorithm uses Python 3.8.5.

4.2. Single-Object Auctions

We start with single-item auctions with interdependencies. The most well-known examples of interdependencies are the common value model (with independent observations o) and the affiliated value model for single-item auctions (Krishna 2009). We explore the second-price auction in an en-

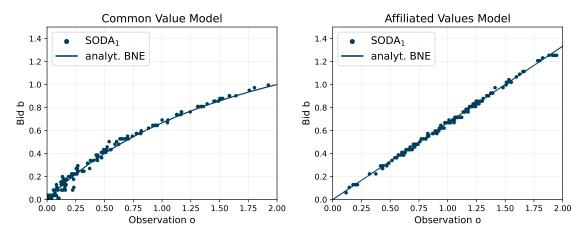


Figure 2 Computed strategies for single-item auctions with interdependencies.

Note. We draw 150 observations according to the prior distribution and sample the corresponding bids from the computed discrete distributional strategies (blue dots). The colored lines indicate the analytical equilibrium strategies in these settings.

vironment where there is one pure common value that is the same among all bidders. Three bidders $i \in \{1,2,3\}$ share a common $\mathcal{U}(0,1)$ -distributed value for the item of interest. Conditioned on this value, the observation o_i of bidder i is uniformly—and independently from the other observations—distributed on the interval from zero to two times the common value. Formally, we can define the joint prior probability density function f with a four-dimensional uniformly distributed random variable $\Omega = [0,1]^4$. For a draw $\omega \sim \mathcal{U}(\Omega)$ we set each player's type to $v_i(\omega) = \omega_4$ and each observation to be $o_i(\omega) = 2 \cdot \omega_i \cdot \omega_4$. Notice, all agents have the same value (or type), but they learn their value only if they win the auction. In this model, the symmetric BNE strategy profile can be stated in closed form as $\beta_i^*(o_i) = \frac{2o_i}{2+o_i}$. For our algorithms we restrict the spaces to the intervals $\mathcal{O}_i = [0,2]$, $\mathcal{V}_i = [0,1]$, and $\mathcal{A}_i = [0,1.5]$ and discretize them. Since all bidders are symmetric we learn a single strategy for all of them. We observe (see Table 3) that the standard projected gradient ascent (SOMA₂) converges within seconds, while all other methods run for several minutes. The strategies computed with SOFW deviate significantly from the equilibrium strategy for low valuations, which explains the high L_2 norm. But since this only happens for low valuations with low bids, there is little effect on the utility loss \mathcal{L} .

In the affiliated values model the individual observations are correlated. In a model with two bidders (see also Krishna (2009, Example 6.2)), we can set $\Omega = [0,1]^3$ and bidder $i \in \{1,2\}$ then

Table 3 Results for the Common Value Model.

Algorithm	step size	runtime	\mathcal{L}	L_2
$SODA_1$	$\beta = 0.50, \eta_0 = 100$	10-13 min	0.007 (0.001)	0.034 (0.000)
$SODA_2$	$\beta = 0.05, \eta_0 = 1$	$14-16 \min$	0.003(0.000)	0.019(0.000)
$SOMA_2$	$\beta = 0.50, \eta_0 = 50$	7-9 s	$0.003\ (0.000)$	0.018 (0.000)
SOFW	-	$2-14 \min$	$0.000 \ (0.001)$	$0.196 \ (0.002)$
FP	-	$9-14 \min$	$0.000 \ (0.001)$	0.439 (0.014)
NPGA	-	$15 \min$	0.000 (0.000)	$0.009 \ (0.002)$

The mean (and standard deviation) of the approximated utility loss \mathcal{L} and L_2 distance, as well as the step size and runtime is reported.

makes the observation $o_i(\omega) = \omega_i + \omega_3$ and both have a common value of $v(\omega) = \frac{1}{2}(\omega_1 + \omega_2) + \omega_3$. The symmetric BNE strategy for both agents under a second-price payment rule is to bid truthfully and for a first-price payment rule to bid according to $\beta_i^*(o_i) = \frac{2}{3}o_i$. In contrast to the common value model, we do not need an additional valuation space and only discretize the spaces $\mathcal{O}_i = [0, 2]$ and $\mathcal{A}_i = [0, 1.5]$. Together with fewer bidders (i.e., two symmetric bidders), this leads to significantly faster computations of the equilibrium strategies as we can see in Table 4.

Table 4 Results for the Affiliated Values Model.

Algorithm	step size	runtime	$\mathcal L$	L_2
$SODA_1$	$\beta = 0.5, \eta_0 = 100$	$15\text{-}16 \mathrm{\ s}$	0.002 (0.000)	0.014 (0.000)
$SODA_2$	$\beta = 0.5, \eta_0 = 1$	11-12 s	0.002 (0.000)	0.012(0.000)
$SOMA_2$	$\beta = 0.5, \eta_0 = 1$	11 s	$0.002 \ (0.000)$	$0.014 \ (0.000)$
SOFW	-	11 s	$0.004 \ (0.001)$	$0.020 \ (0.002)$
FP	-	12 - 13 s	0.005 (0.000)	0.025 (0.001)
NPGA	-	$15 \min$	$0.002\ (0.001)$	$0.018 \; (0.009)$

The mean (and standard deviation) of the approximated utility loss \mathcal{L} and L_2 distance, as well as the step size and runtime is reported.

4.3. Combinatorial Auctions in the Local-Local-Global Model

Bayesian Nash equilibria are rarely available for multi-object auctions. Combinatorial auctions have received significant attention due to their use in spectrum sales and other applications (Bichler and Goeree 2017). The local-local-global (LLG) model has received significant attention in the analysis of core-selecting combinatorial auctions (Goeree and Lien 2016). The *core* of an auction game describes the set of outcomes such that no *coalition* of bidders (and possibly the auctioneer) can profitably deviate given the bids. This LLG model is simple enough to allow for the derivation of analytical results (Ausubel and Baranov 2019). At the same time, core-selecting auction mechanisms are challenging and among the most complex auction formats used today, which provides an interesting benchmark for equilibrium computation.

The LLG model consists of two objects $\{1,2\}$, two local bidders $i \in \{1,2\}$ and one global bidder i=3, each being interested only in one specific bundle (of the single object i (locals) or both objects (global)), and we denote the valuation of each bidder's single bundle by $v_i \in \mathbb{R}$. We consider a private values (but not independent private values) setting with $o_i = v_i$ which allows for correlation. It was shown that with independent private values and risk-neutral bidders, core-selecting payment rules lead to significant inefficiencies in equilibrium (Goeree and Lien 2016) in combinatorial auctions. Essentially, the two local bidders attempt to free-ride on each other. Depending on the prior value distributions, it can happen that both local bidders bid too low in total and they fail to outbid the global bidder, even if their combined valuations are higher than the global bidder's. This results in an inefficient outcome and it has been used as an argument against core-selecting combinatorial auctions (Bichler and Goeree 2017). Now, it is interesting to understand equilibria with different assumptions. For example, it is reasonable to believe that bidder valuations in spectrum auctions are correlated because telecoms face the same downstream market.

Ausubel and Baranov (2019) investigate two models of correlation among local bidders' private values and derive analytical BNE. We will focus on the *Bernoulli weights model* and use it as a baseline in our experiments in addition to the results of NPGA. Let's define the joint prior f to be the five-dimensional uniform distribution of a latent random variable $\omega \sim \mathcal{U}[0,1]^5$. Then let $v_3 = 2\omega_3$ be the valuation of the global bidder and

$$v_1(\omega) = w\omega_4 + (1 - w)\omega_1, \quad v_2(\omega) = w\omega_4 + (1 - w)\omega_2$$
 (12)

be the valuations of the local bidders where the weight w is a random variable depending on ω_5 only. The valuations of the local bidders can be thought of as a linear combination of an individual component ω_i and a common component ω_4 . Now given an exogenous correlation parameter $\gamma \in [0,1]$, Ausubel and Baranov (2019) choose w such that $\operatorname{corr}(v_1,v_2)=\gamma$ via the Bernoulli weights model: $w(\omega)=1$ if $\omega_5<\gamma$ and $w(\omega)=0$ else. The authors analytically derive the unique symmetric BNE strategies for multiple bidder-optimal core-selecting payment rules including the nearest-zero (NZ), nearest-VCG (NVCG), and nearest-bid (NB) rule in the Bernoulli weights model. These rules all choose the efficient allocation x (according to the submitted bids) but select different price vectors p from the set of core-stable outcomes. For example, the nearest-VCG rule picks the point in the core that minimizes the Euclidean distance to the (unique) Vickrey-Clarke-Groves payments. Similarly, the nearest-zero point takes the origin of the coordinate system as a reference point, while the nearest-bid rule minimizes the distance to the vector of submitted bids b. We report the results for these core-selecting payment rules with different Bernoulli weights $\gamma \in \{0.1, 0.5, 0.9\}$ in Table 5-7. Since truthful bidding is a dominant strategy for the global bidder, which is easier to

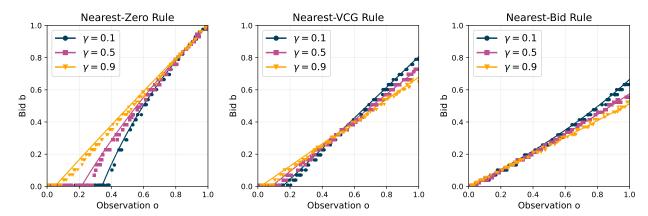


Figure 3 Computed strategies for the local bidders in the LLG model.

Note. We draw 150 observations according to the prior distribution and sample the corresponding bids from the computed discrete distributional strategies using SODA₂ (colored shapes). The colored lines indicate the analytical equilibrium strategies for these settings. We consider the three core-selecting payment-rules and different correlations according to the Bernoulli weights model with parameter $\gamma \in \{0.1, 0.5, 0.9\}$.

approximate and leads to more accurate results in all instances, we only report the results for the local bidders. For $\gamma = 0.5$ we compare our results to NPGA.

We construct the approximation game by discretizing the spaces $\mathcal{O}_L = [0,1]$ and $\mathcal{O}_G = [0,2]$, according to the prior distribution, and the action spaces $\mathcal{A}_i = \mathcal{O}_i$, $i \in \{L,G\}$. Since the local bidders are symmetric we learn a single strategy for both. For each update method, we use a single step rule for all different settings, i.e., SODA₁ ($\beta = 0.05$, $\eta_0 = 100$), SODA₂ ($\beta = 0.05$, $\eta_0 = 50$), and SOMA₂ ($\beta = 0.05$, $\eta_0 = 50$).

Overall, we can observe that SODA shows an comparable low utility loss to NPGA. However, NPGA was again run for 15 minutes while SODA₁ converged in less than 0.5 minutes and often even within a few seconds. Across all experiments all methods except for fictitious play converge, i.e.,

Table 5 Results for the local bluders in the ELG Model with Nearest-Zero Rule.							
Algorithm	$\gamma = 0.1$		$\gamma =$	$\gamma = 0.5$		$\gamma = 0.9$	
	\mathcal{L}	L_2	$\mathcal L$	L_2	${\cal L}$	L_2	
$SODA_1$	0.002 (0.000)	0.022 (0.001)	0.001 (0.000)	0.022 (0.001)	0.000 (0.000)	0.025 (0.000)	
$SODA_2$	0.002(0.000)	$0.021\ (0.001)$	$0.001\ (0.000)$	$0.024 \ (0.002)$	0.000(0.000)	0.025 (0.001)	
$SOMA_2$	$0.002 \ (0.000)$	$0.018 \; (0.002)$	$0.001\ (0.000)$	$0.019\ (0.001)$	0.000 (0.000)	$0.021\ (0.000)$	
SOFW	$0.002 \ (0.000)$	$0.018 \; (0.000)$	$0.001\ (0.000)$	$0.023\ (0.000)$	0.000 (0.000)	$0.034\ (0.000)$	
FP	$0.002 \ (0.000)$	$0.021\ (0.000)$	$0.001\ (0.000)$	$0.023\ (0.000)$	$0.000 \ (0.000)$	$0.028 \ (0.000)$	
NPGA	-	-	$0.000 \ (0.000)$	$0.011\ (0.005)$	-	-	

Table 5 Results for the local bidders in the LLG Model with Nearest-Zero Rule.

We report the mean (and standard deviation) over ten runs for the utility loss \mathcal{L} and L_2 distance. SODA₁ takes 10-34 seconds, SODA₂ 1-6 seconds, and FP 31-39 seconds per run. All other methods run for less than 1 second.

achieve a relative utility loss of less than 10^{-4} in the discretized game within the 1000 iterations.

Especially Frank-Wolfe and the standard projected gradient ascent (SOMA₂) only need a few iterations until the stopping criterion is satisfied. Nevertheless, all computed strategies perform well when compared to the analytical BNE in the continuous setting.

Table 6 Results for the local bidders in the LLG Model with Nearest-VCG Rule.

Algorithm	$\gamma = 0.1$		$\gamma = 0.5$		$\gamma = 0.9$	
	\mathcal{L}	L_2	${\cal L}$	L_2	${\cal L}$	L_2
$SODA_1$	0.001 (0.000)	0.017 (0.001)	0.001 (0.000)	0.017 (0.001)	0.001 (0.000)	0.021 (0.001)
$SODA_2$	$0.001\ (0.000)$	0.017(0.000)	$0.001\ (0.000)$	$0.016\ (0.000)$	0.000(0.000)	$0.016\ (0.000)$
$SOMA_2$	0.001 (0.000)	0.015(0.001)	0.000(0.000)	0.014(0.001)	0.000(0.000)	0.016(0.001)
SOFW	$0.001\ (0.000)$	0.015(0.000)	0.000(0.000)	0.015(0.000)	0.000(0.000)	$0.016\ (0.000)$
FP	$0.001\ (0.000)$	$0.019\ (0.000)$	$0.001\ (0.000)$	$0.018\ (0.000)$	$0.001\ (0.000)$	0.019(0.000)
NPGA	-	-	0.000 (0.000)	0.016 (0.016)	-	-

We report the mean (and standard deviation) over ten runs for the utility loss \mathcal{L} and L_2 distance. SODA₁ takes for 8-16 seconds and FP up to 47 seconds to compute one strategy, while all other methods run for less than 2 seconds.

Table 7 Results for the local bidders in the LLG Model with Nearest-Bid Rule.

Algorithm	$\gamma = 0.1$		$\gamma = 0.5$		$\gamma = 0.9$	
	${\cal L}$	L_2	${\cal L}$	L_2	${\cal L}$	L_2
$SODA_1$	0.001 (0.000)	0.014 (0.001)	0.001 (0.000)	0.015 (0.002)	0.001 (0.001)	0.017 (0.001)
$SODA_2$	0.001 (0.000)	0.013(0.000)	0.000(0.000)	0.008(0.000)	0.000(0.000)	0.009(0.001)
$SOMA_2$	0.000(0.000)	0.012(0.000)	0.000(0.000)	0.008(0.001)	0.000(0.000)	0.009(0.000)
SOFW	0.000(0.000)	0.013(0.001)	0.000(0.000)	0.009(0.001)	0.000(0.000)	0.012(0.000)
FP	0.001 (0.000)	0.017(0.000)	0.001(0.000)	0.015(0.000)	0.001(0.000)	0.016(0.000)
NPGA	-	<u> </u>	0.001 (0.000)	0.021 (0.021)	-	

We report the mean (and standard deviation) over ten runs for the utility loss \mathcal{L} and L_2 distance. SODA₁ takes 8-23 seconds and FP 31-39 seconds per run, while all other methods run for less than 2 seconds.

A setting where no analytical equilibria are known is the LLG model with a first-price payment rule. This auction format is important as a number of countries used first-price combinatorial auctions in high-stakes spectrum auctions (Bichler and Goeree 2017). Using the Frank-Wolfe algorithm, we converge within 30 seconds in the discretized game. The corresponding equilibrium strategies are visualized in Figure with different levels of correlation 4. In contrast to the other settings, the global bidder has no simple dominant strategy. The resulting equilibrium strategy is not as smooth as in other models, but the relative ex-ante utility loss is very small as in other models ($\ell < 10^{-4}$).

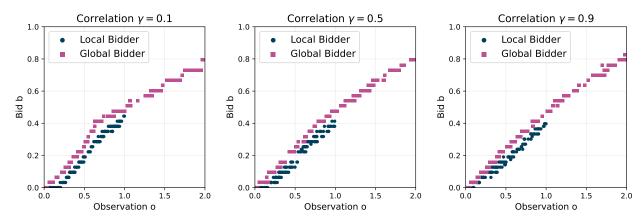


Figure 4 Computed strategies in the LLG model with a first-price payment rule.

Note. We draw 150 observations according to the prior distribution and sample the corresponding bids from the computed discrete distributional strategies using SOFW (colored shapes) for different correlation parameters γ .

4.4. Combinatorial Split-Award Auction

Another combinatorial auction environment for which the BNE strategies are known is that of combinatorial split-award procurement auctions (Kokott et al. 2019). In contrast to the LLG model, bidders are not single-minded but they are interested in either one share of a contract or the entire contract. Importantly, there are two pure BNE for the two symmetric bidders in the FPSB combinatorial procurement market, which makes the analysis interesting. A specific version with two suppliers and two lots has been analyzed by Anton and Yao (1992). Here, suppliers $i \in \{1, 2\}$ can bid on a 100% and a 50% share. With dis-economies of scale, we have the economically inefficient "winner-takes-all" (WTA) equilibrium where one bidder wins both lots (the 100% share) and a continuum of efficient "pooling equilibria" where both suppliers coordinate and each bidder wins one good (a 50% share) at a high pooling price. The equilibrium with the highest bids on one lot out of all the efficient pooling equilibria is the payoff-dominant strategy for each bidder.

We applied SODA to this setting with uniform and Gaussian (truncated with l=1.2 and $\sigma=0.1$) distributed observations. We consider dis-economies of scale and choose marginal costs for the split source of C=0.3. The parameters are consistent with experiments from Kokott et al. (2019). To compare our results to the analytical BNE (Table 8) we consider a truncated version of the Gaussian prior since the equilibrium analysis requires bounded observations. We can observe that for both priors, Gaussian (Figure 5) and uniform, SODA always finds the efficient equilibrium. This is remarkable, because coordination is strategically more challenging than in the WTA equilibrium in which bidders just compete on the 100% share similar to a single-object auction. In the pooling equilibrium bidders bid high on the 50% share, but they also need to find a bid on the bundle of both lots (the 100% share) such that it is not profitable for the opponent to deviate from the pooling equilibrium.

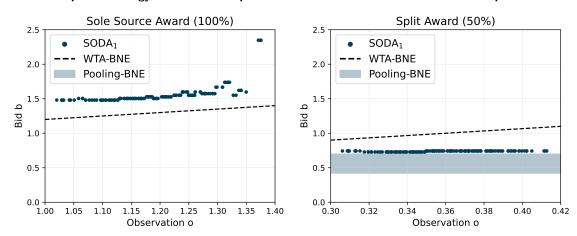


Figure 5 Computed strategy for the FPSB Split-Award Auction with a truncated Gaussian prior.

Note. We draw 150 observations from a truncated Gaussian prior distribution and sample the corresponding bids from the computed discrete distributional strategies (blue dots). The dashed lines indicate the winner-takes-all (WTA) equilibrium, while the shaded area denotes all possible pooling equilibria.

Table 8 Results for the FPSB split-award auction with a truncated Gaussian prior.

Algorithm	step size	time	$\mathcal L$	L_2
$SODA_1$	$\beta = 0.05, \eta_0 = 20$	3-5 min	-0.064 (0.001)	$0.050 \ (0.010)$
$SODA_2$	$\beta = 0.05, \eta_0 = 0.05$	$3-4 \min$	-0.077 (0.001)	0.067 (0.005)
$SOMA_2$	$\beta = 0.50, \eta_0 = 0.05$	$7-6 \min$	-0.086 (0.001)	$0.100 \ (0.009)$
SOFW	-	7-11 min	$0.031 \ (0.075)$	$0.029 \ (0.008)$
FP	-	7-12 min	$0.194\ (0.024)$	$0.078 \ (0.006)$

The mean (and standard deviation) of the approximated utility loss \mathcal{L} and L_2 distance (only for the 50% share), as well as the step size and runtime are reported.

The algorithms take several minutes since we have a two-dimensional action space $\mathcal{A}_i = [1.0, 2.5] \times [0.3, 1.2]$ where each interval is discretized using L = 64 equidistant points and the observation space $\mathcal{O}_i = [1.0, 1.4]$ which is discretized using K = 32 points. We choose a lower discretization for the observation space because otherwise we would run into memory issues for the computation of the gradients. In Figure 5 we can observe that in the case of the Gaussian prior, the agents bid slightly above the analytical BNE for the winning bid. This leads to a higher utility compared to the utility in BNE and thereby to a negative utility loss. This also explains the rather large L_2 distance in Table 8. We get more accurate results for the uniform prior, as we can see in Table 9. Note that we do not consider the L_2 distance for the bid on the 100% share, since the strategy is spread within the continuum of pooling BNE.

We observe that SODA₁ and SODA₂ converge within 5 minutes in all instances and achieve results similar to NPGA, which takes around 15 min to get $\mathcal{L} = 0.019$ and a $L_2 = 0.025$ for the uniform prior (Bichler et al. 2023). SOMA₂ and the Frank-Wolfe algorithm perform worse, espe-

Table 9 Results for the FPSB split-award auction with a uniform prior.

Algorithm	step size	runtime	\mathcal{L}	L_2
$SODA_1$	$\beta = 0.05, \eta_0 = 20$	$2 \min$	0.009 (0.000)	$0.024\ (0.028)$
$SODA_2$	$\beta = 0.05, \eta_0 = 0.05$	2-3 min	0.009 (0.000)	0.015 (0.000)
$SOMA_2$	$\beta = 0.50, \eta_0 = 0.01$	$7-9 \min$	$0.029 \ (0.002)$	0.097 (0.016)
SOFW	-	7-8 min	$0.191\ (0.032)$	0.075 (0.010)
FP	-	7-8 min	$0.177 \ (0.031)$	$0.039 \ (0.008)$

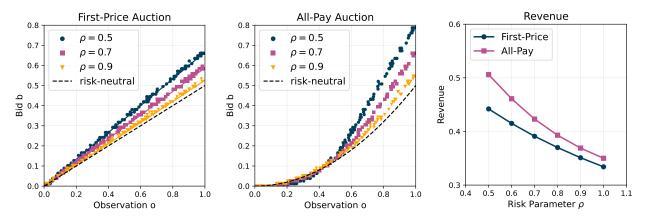
The mean (and standard deviation) of the approximated utility loss \mathcal{L} and L_2 distance (only for the 50% share), as well as the step size and runtime are reported.

cially for the uniform prior, and Fictitous Play doesn't achieve a sufficient accuracy in any setting. Nevertheless, all methods approximate the payoff-dominant equilibrium.

4.5. Single-Object Auctions with Risk-Averse Bidders

In addition to single-object auctions with the standard quasi-linear utility functions, we can also consider extensions such as risk-aversion. As described in Section 3.1, risk aversion can be modeled using a risk attitude $\rho \in (0,1]$ by transforming the standard quasi-linear utility u_i^{QL} into (strictly) concave payoff functions of the form $u_i^{RA} = (u_i^{QL})^{\rho}$. This model is also known as constant relative risk aversion (CRRA).

Figure 6 Computed strategies and revenue for the first-price and all-pay auction with risk-averse bidders.



Note. The first two plot shows the equilibrium strategies for the first-price and all-pay auction under risk-aversion compared to the risk-neutral equilibrium strategy (black line). The computed strategies are illustrated by drawing 150 observations according to the prior distribution and sampling the corresponding bids. In the last plot we visualize the approximated expected revenue under different risk parameters.

We consider settings with two symmetric bidders who observe their uniformly distributed and private valuations independently. For the first-price sealed-bid auction it is well known that risk-averse bidders ($\rho \in (0,1)$) bid higher than risk-neutral bidders ($\rho = 1$), which leads to a higher revenue for the seller (Maskin and Riley 1984). For all-pay auctions on the other hand, results are

much more limited. Fibich et al. (2006) analyze the first-order conditions and show that in the independent private value setting, risk-averse bidders bid lower for low valuations and higher for high valuations compared to the risk-neutral equilibrium strategy. But they are not able to derive explicit equilibrium strategies or to make statements about how risk-aversion affects the expected revenue. Here, our methods can add to the existing literature. While we observe the effects in the equilibrium strategies predicted by Fibich et al. (2006), we can also observe that risk aversion, similar to first-price auctions, increases the expected revenue in the all-pay auction (Figure 6).

Table 10 Results for risk-avers bidders in the FPSB auction with different risk parameter ρ .

Algorithm	ho = 0.5		$\rho = 0.7$		$\rho = 0.9$	
	\mathcal{L}	L_2	$\mathcal L$	L_2	$\mathcal L$	L_2
$SODA_1$	0.001 (0.000)	0.007 (0.000)	0.001 (0.000)	0.007 (0.000)	0.001 (0.000)	0.008 (0.000)
$SODA_2$	$0.001\ (0.000)$	0.007(0.000)	$0.001\ (0.000)$	0.007(0.000)	$0.001\ (0.000)$	0.008(0.001)
$SOMA_2$	$0.001\ (0.000)$	0.007(0.000)	0.001(0.000)	0.007(0.000)	0.001(0.000)	0.008(0.000)
SOFW	0.001 (0.000)	0.008(0.000)	0.001(0.000)	0.008(0.000)	0.001 (0.000)	0.009(0.000)
FP	0.002 (0.000)	0.013 (0.000)	0.002 (0.000)	0.013 (0.000)	0.003 (0.000)	0.013 (0.001)

We report the mean (and standard deviation) over ten runs for the utility loss \mathcal{L} and L_2 distance. The runtime for all methods is less than 1 second per run.

For the numerical experiments we consider first-price and all-pay auctions with two symmetric bidders. They independently observe their uniformly distributed valuations from $\mathcal{O}_i = [0,1]$. We restrict the action space to $\mathcal{A}_i = [0,0.8]$. The strategies in Figure 6 are computed using SODA₁ with parameters $\beta = 0.05$, $\eta_0 = 25$ for the all-pay and $\beta = 0.05$, $\eta_0 = 20$ for the first-price auctions. The revenue is the mean over 2^{22} simulated auctions using the computed strategies over ten runs. For risk-averse bidders in the first-price auction we can use the analytical solution to evaluate the computed strategies. The results are reported in Table 10. The parameters for the learning algorithms, i.e., SODA₂ with $\beta = 0.05$, $\eta_0 = 0.1$ and SOMA₂ with $\beta = 0.5$, $\eta_0 = 0.5$, are constant over the different risk parameters. Note that for our learning algorithm we have to extend the definition of CRRA to negative numbers. This is done by $u_i^{RA} = \text{sign}(u_i^{QL}) \cdot |u_i^{QL}|^{\rho}$.

4.6. Tullock Contests

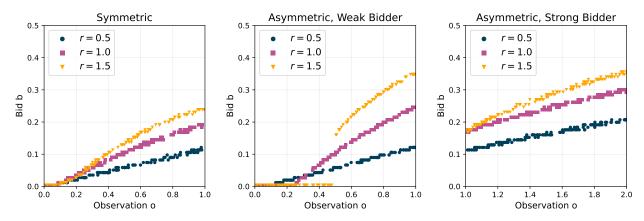
Finally, we consider Tullock contests. In contests, agents invest efforts toward winning one or more prizes, and these efforts are costly and irreversible. One distinguishes between perfectly discriminating contests, such as all-pay auctions, where the bidder with the highest effort wins the prize with certainty, and imperfectly discriminating contests, where the probability of winning is a monotonically increasing function of one's own effort (bid). Contests occur in various contexts such as rent-seeking, warfare conflicts, R&D competition, and the labor market (Vojnović 2016). The Tullock lottery (Tullock 1980) is the best known example of such a contest, where the probability

of winning the prize is proportional to agents effort. We will focus on the slightly more general r-Tullock contest with parameter r > 0, where the (ex-post) utility of player i is given by

$$u_{i}(b_{i}, b_{-i}, o_{i}) = \begin{cases} o_{i} \frac{b_{i}^{r}}{\sum_{j=1}^{n} b_{j}^{r}} - b_{i} & \text{if } \sum_{j=1}^{n} b_{j} > 0\\ o_{i} \frac{1}{n} & \text{else} \end{cases}$$
 (13)

If r=1, the contest corresponds to the aforementioned Tullock lottery. Due to the discontinuity at zero, the model is hard to analyze in the incomplete-information setting. Existence of pure BNE in the IPV model ($o_i = v_i$ independent for all i) is only known for the concave case, i.e., with $r \leq 1$, while we only get existence in behavioral strategies for r > 1 (Haimanko 2021). But even in the symmetric, concave case, no analytical equilibrium strategy is known. For $r \in \{0.5, 1\}$ numerical approximations were obtained by discretizing the integral in the first order condition and iteratively following the best response until convergence is reached (Fey 2008, Ryvkin 2010). In contrast, our method does not rely on the first-order condition and can be easily adapted to more general contests. Especially in settings with asymmetric bidders, where the first order condition becomes a system of non-linear ODEs, our approach becomes even more valuable.

Figure 7 Computed strategies for the generalized Tullock Contest with two symmetric and asymmetric bidders



Note. We draw 150 observations according to the prior distribution and sample the corresponding bids from the computed discrete distributional strategies using SODA₁ (colored shapes). The first plot shows the equilibrium strategies in the symmetric setting, while the other two plots depict the weak and strong bidder in the asymmetric case.

In Figure 7 we show the computed equilibrium strategies for 2 player r-Tullock contests with $r \in \{0.5, 1.0, 1.5\}$. Note that this also includes a non-concave setting (r = 1.5) where existence of pure BNE has not been shown yet. We consider a symmetric version where the valuations of both contestants are uniformly distributed on [0,1] and an asymmetric setting, where we have a weak bidder with $o_{\text{weak}} \sim U([0,1])$ and a strong bidder with $o_{\text{strong}} \sim U([1,2])$. We restrict the actions to $\mathcal{A}_i = [0,0.5]$ and discretize all spaces with K = L = 64 equidistant points. For dual averaging and

mirror descent we used the following parameter: SODA₁: $\eta = 100, \beta = 0.05$, SODA₂: $\eta = 10, \beta = 0.05$, SOMA₂: $\eta = 100, \beta = 0.5$. It takes all methods less than 0.1s in the symmetric and less than 2s in the asymmetric settings to converge, i.e., achieve a relative utility loss $\ell < 10^{-4}$ in the discretized game.

5. Discussion

SODA converges in a wide range of environments as illustrated in the previous section. In this section, we discuss what is known about convergence and scalability of the approach.

5.1. Convergence

Although we can certify equilibrium ex post, an intriguing question remains: why do gradient dynamics converge to an equilibrium in such a wide variety of auctions and contests, even though gradient dynamics don't converge in many finite games (Sanders et al. 2018)? This is a notoriously challenging question. Andrade et al. (2021) write that there is little hope for a general understanding of the behaviors arising from optimization-driven dynamics even in normal-form games. Whether learning algorithms converge or not depends on the properties of the game being played. Apparently, a wide variety of auctions and contests have properties that allow for SODA to converge to equilibrium.

There is a long literature on variational inequalities and how they are used to model equilibrium problems (Kinderlehrer and Stampacchia 2000, Grossmann et al. 2007, Geiger and Kanzow 2013). We know that projection algorithms converge if a complete-information game with continuous action spaces satisfies monotonicity or the weaker variational stability condition, but that they do not converge if there are only mixed equilibria (Mertikopoulos and Zhou 2019, Flokas et al. 2020). Variational stability coincides with the existence of (one or more) sharp equilibria in complete-information games (Mertikopoulos and Zhou 2019). Unfortunately, it is not easy to assess ex ante whether a specific game has a sharp or even only a pure Nash equilibrium.

The early theorems of Nash et al. (1950) and Debreu (1952) reveal that games possess a pure strategy Nash equilibrium if (1) the strategy spaces are nonempty, convex, and compact, and (2) players have continuous and quasi-concave payoff functions. These assumptions are necessary for the fixed-point theorems that the authors draw on. However, in many economic models, the payoffs are discontinuous. Bidders in an auction experience a discontinuous jump in their utility when their bid on some unit increases to the point where it is no longer a losing bid. This led to a literature on equilibrium existence in discontinuous games (see the survey by Reny (2020)). Athey (2001) introduced the single-crossing property: whenever each opponent uses a non-decreasing strategy in the sense that higher types choose higher actions, a player's best response strategy is also non-decreasing. When the property holds, a pure-strategy Nash equilibrium exists in every

finite-action game. Further, for games with discontinuous payoffs and a continuum of actions, there exists a sequence of pure-strategy Nash equilibria to finite-action games that converges to a PSNE of the continuum-action game. The condition was shown to hold for first-price, multi-unit, and all-pay auctions, as well as pricing games with incomplete-information about costs. Reny (2011) generalizes these results and also covers more general multi-unit auctions with risk-averse bidders. However, these ex-ante characteristics are not easy to verify and 70 years after Nash's original work understanding whether a game has a pure or even a strict equilibrium is still a challenge.

One could try to analyze the monotonicity of the continuous ex-ante game. But for this it is important to understand the individual utility functions and their gradients. However, the agents' utility functions are based on an unknown bid function. Without strong assumptions on the functional form of the bid function, it is hard to characterize the payoff gradient explicitly. Appendix C summarizes a number of plots, where we do make parametric assumptions on the prior distribution and the bid function. The plots suggest that under a variety of assumptions the resulting expected utility function is quasi-concave or at least unimodal. However, the parametric assumptions are hard to justify. SODA is based on the discretized approximation game, not the continuous ex-ante game. Unfortunately, we can show that the approximation game satisfies neither monotonicity nor variational stability globally. Yet, we find convergence in a wide variety of games. A longer discussion and definitions are provided in Appendix D.

5.2. Scalability

Although convergence is difficult to analyze, we want to provide some drivers for the computational complexity of SODA. The main factors are the number of players, the number of items or bundles (which drives the number of strategies), and the level of discretization. If the number of strategies is exponential in the number of items (as in a combinatorial auction with general valuations), then gradient-based optimization as in SODA explores all exponentially-many strategies. As a result, an algorithm learning even only approximate ε -BNE cannot be polynomial in the number of items. Cai and Papadimitriou (2014) showed with a similar argument that computing approximate ε -BNE in combinatorial auctions is NP-hard.

In most auction-theoretical models, the number of items or strategies per agent is small. Examples include single-minded bidders in combinatorial auctions or split-award auctions with two or three items only. Apart from this, a standard assumption in auction theory is that of symmetric priors and symmetric equilibrium strategies, which leads to the fact that we only need to explore the strategies of a single and not of multiple players. For example, if we further assume that the bidders are independent, the computational effort can be further reduced. In such a first-price sealed-bid auction, the expected utility can be written as

$$\tilde{u}_i(s_1, ..., s_n) = \sum_{k,l} (s_i)_{kl} (o_k - b_l) \mathbb{P}(b_l \text{ is highest bid}; s_{-i}).$$
 (14)

Compared to the very general formulation (10), where we sum over all combination of bids which grows exponentially in the number of bidders n, we compute the first order statistic. This way the complexity does not increase with the number of bidders, which allows us to analyze much larger settings (see Appendix A). So, while we know that the complexity of finding ε -BNE in general is NP-hard, computation is not necessarily a limiting factor in most of the models analyzed in auction theory, where we focus on small markets with a few players only.

6. Conclusions

Computing Bayesian Nash equilibria for continuous-type and -action auction games was considered intractable. Sixty years after Vickrey's seminal work on single-object auctions, we still only know equilibrium strategies for very restricted environments such as single-object auctions. These equilibrium problems can be modeled as systems of differential equations and for many model assumptions we don't have a complete mathematical solution theory.

SODA is a new numerical technique that relies on distributional strategies and a discretization of the type and action spaces that takes the prior distributions into account. The method is very fast for auction models with symmetric bidders. In first-price environments with independent private values, SODA computes approximate equilibrium also for large numbers of bidders in seconds, which makes SODA a convenient numerical tool for analysts. We analyzed very different types of auctions and contests and SODA converged in all of them. Ex-post verification upon convergence is very useful, because the algorithms are very fast for standard models and analysts are not required to perform costly numerical validation. While these formal convergence results have only been shown for SODA, we demonstrated empirically that other first-order methods are as effective in finding equilibrium strategies.

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Appendix A: Running Time with Different Discretizations and Symmetry Assumptions

In what follows, we report the impact of different levels of discretization and number of bidders on the running time, and we explore the performance gains from symmetric models.

First, we investigate the effect of finer discretizations in the approximation game on the accuracy of the approximation. We apply SODA₁ ($\eta_0 = 10, \beta = 0.05$) to a FPSB auction with two symmetric bidders and independent, uniformly distributed priors. The computations are repeated using different numbers of discrete points. More precisely, we discretize the action and observation spaces with $K = L \in \{16, 32, 64, 128\}$ equidistant points. We run SODA and stop the algorithm after 1 000 iterations. Afterwards we compare the computed strategies with the analytical BNEs in the continuous setting (i.e., approximate \mathcal{L} and L_2) as described in section 4.1. The results are reported in Table 11.

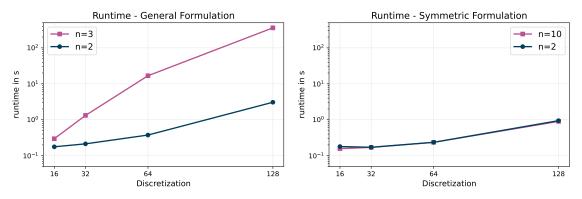
Table 11 Results for the FPSB with two bidders and different discretizations.

K, L	16	32	64	128	256
Utility Loss \mathcal{L} L_2 Distance	\ /	\ /	\ /	\ /	\ /

The mean (and standard deviation) of the approximated utility loss \mathcal{L} and L_2 distance over ten runs is reported.

As expected, increasing the number of discretization points leads to better approximations. But obviously this has a huge effect on the runtime of our algorithm. The computation of the gradient requires computing the weighted sum over $K \cdot L^n$ elements (all possible combinations of valuation and action profiles) in the general formulation as described in Section 3.3 for one-dimensional spaces. The number of possible outcomes increases exponentially in the number of bidders n (or the dimension of the spaces). Therefore, our method is limited to models with a small number of bidders or items.

Figure 8 Runtime for a FPSB using the general and symmetric formulation.



Note. We report the mean runtime for 1000 iterations over 10 runs for a single-item FPSB with uniform prior.

But there are settings, where our method can be used even for a very large number of bidders. As described in Section 5.2, one often considers independent symmetric agents in single-item first-price auctions. This

allows us to use an alternative way of calculating the gradient, where the computational effort does not depend on the number of bidders. In Figure 8 we can see that increasing the number of bidders from two to just three already increases the runtime for higher discretizations from a few seconds to minutes in the general formulation. Using the symmetric formulation on the other hand allows us to consider any number of bidders.

Appendix B: Proof of Theorem 1

Given the auction game $G = (\mathcal{I}, \mathcal{V}, \mathcal{A}, f, u)$, we make following assumptions.

Assumption 1. The type spaces V_i and action spaces A_i are compact intervals of \mathbb{R} .

Assumption 2. The associated probability measure of the common prior F is absolutely continuous with respect to its marginals F_i , with L_f -Lipschitz continuous Radon-Nikodym derivative f:

$$F(V) = \int_{V} f(v)dF_1(v_1) \dots dF_n(v_n), \quad V \subset \mathcal{V} \text{ measurable.}$$

We assume V_i to be the support of F_i . Since V is compact, there is also M > 0 such that $F(V_1 \times \cdots \times V_n) \leq MF_1(V_1) \cdot \cdots \cdot F_n(V_n)$.

ASSUMPTION 3. With each agent i there are associated two payment (or transfer) functions $t_i^l: \mathcal{A} \to \mathbb{R}$ and $t_i^w: \mathcal{A} \to \mathbb{R}$, determining the agent's payment when they lose (t_i^l) or win (t_i^w) the good. All t_i^l and t_i^w are L_t -Lipschitz continuous. Moreover, for fixed bids b_{-i} of the other agents, they are nondecreasing in b_i .

Assumption 4. Each agent has a nondecreasing von Neumann-Morgenstern utility function $U_i : \mathbb{R} \to \mathbb{R}$. Thus, the agent's utility for winning the good is $U_i(v_i - t_i^w(b))$, and for losing it is $U_i(-t_i^l(b))$. The U_i are L_U -Lipschitz-continuous.

Assumption 5. The allocation function $x_i : \mathcal{A} \to [0,1]$ denotes the probability of agent i winning the good, given the bids of all agents. Only maximal bids are winning, i.e., $x_i(b) > 0 \Rightarrow b_i \geq b_j \, \forall j$. We assume that x_i is nondecreasing in b_i for fixed b_{-i} , and $\sum_{i=1}^n x_i(b) \leq 1$ for all $b \in \mathcal{A}$.

The ex-post utility of agent i can thus be written as

$$u_i(b, v_i) = x_i(b)U_i(v_i - t_i^w(b)) + (1 - x_i(b))U_i(-t_i^l(b)).$$

It is easy to see that the Lipschitz-continuity of U_i results in the Lipschitz-continuity of u_i , e.g., for $v_i, v_i' \in \mathcal{V}_i$

$$|u_i(b, v_i) - u_i(b, v_i')| = |x_i(b)(U_i(v_i - t_i^w(b)) - x_i(b)U_i(v_i' - t_i^w(b)))|$$

$$\leq |U_i(v_i - t_i^w(b)) - U_i(v_i' - t_i^w(b))| \leq L_U|v_i - v_i'|.$$

Assumption 6. For each agent i, there is a function $p_i : A_i \to \mathbb{R}$, determining the marginal payment at ties: formally, if $b \in A$ is a bid vector such that b_i is a maximal bid and there is a $j \neq i$ with $b_i = b_j$, then $t_i^w(b) - t_i^l(b) = p_i(b_i)$. Hence, at ties marginal payments depend only on agent i's bid b_i . Note that the p_i are L_t -Lipschitz continuous.

These assumptions include single-object auction formats such as the first-price and the second-price sealed bid auctions, and first-price as well as second-price all-pay auctions (war of attrition) Jackson and Swinkels (2005).

To formally describe our discretized game $G^d(\mathcal{I}, \mathcal{V}^d, \mathcal{A}^d, F^d, u)$, we use the following definitions.

DEFINITION 3. The discrete type space of agent i is a finite subset $\mathcal{V}_i^d \subseteq \mathcal{V}_i$. There is a function $\tau_i : \mathcal{V}_i \to \mathcal{V}_i^d$, mapping each $v_i \in \mathcal{V}_i$ to its discrete representant $\tau_i(v_i)$ and mapping each $v_i^d \in \mathcal{V}_i^d$ to itself. Denote by $\delta_{\tau} = \max_i \sup_{v_i \in \mathcal{V}_i} |v_i - \tau_i(v_i)|$.

If we discretize the valuation space, for instance, using N equally sized sub-intervals and τ_i maps $v_i \in \mathcal{V}_i$ to the midpoint of the respective interval. Then we get $\delta_{\tau} = \frac{1}{2N} |\mathcal{V}_i|$.

DEFINITION 4. The discrete action space of agent i is a finite subset $\mathcal{A}_i^d \subseteq \mathcal{A}_i$. \mathcal{A}_i^d contains the minimal and maximal element of \mathcal{A}_i . Denote by $\alpha_i^+: \mathcal{A}_i \to \mathcal{A}_i^d$ the function mapping each $b_i \in \mathcal{A}_i$ to the minimal element in \mathcal{A}_i^d not smaller than b_i . Similarly, denote by $\alpha_i^-: \mathcal{A}_i \to \mathcal{A}_i^d$ the function mapping $b_i \in \mathcal{A}_i$ to the maximal element in \mathcal{A}_i^d not greater than b_i . Denote by $\delta_{\alpha} = \max_{s \in \{+,-\}} \max_i \sup_{b_i \in \mathcal{A}_i} |b_i - \alpha_i^s(b_i)|$.

DEFINITION 5. The valuations in $\mathcal{V}^d = \mathcal{V}_1^d \times \cdots \times \mathcal{V}_n^d$ are distributed according to probability measure F^d on \mathcal{V}^d , given by

$$F^d(\{v_1^d\}\times \dots \times \{v_n^d\}) = F(\tau_1^{-1}(v_1^d)\times \dots \times \tau_n^{-1}(v_n^d)) \text{ for all } v_i^d \in \mathcal{V}_i^d.$$

Consequently, F^d has marginals $F_i^d(\{v_i^d\}) = F_i(\tau_i^{-1}(v_i^d))$ and density $f^d(v_1^d, \dots, v_n^d) = F^d(\{v_1^d\} \times \dots \times \{v_n^d\})/\prod_i F_i^d(\{v_i^d\})$ with respect to the marginals F_i^d . Note that F^d can also be interpreted as a probability measure on $\mathcal V$ via $F^d(V) = F^d(V \cap \mathcal V^d)$ for $V \subseteq \mathcal V$ measurable.

We denote distributional strategies in G^d for agent i by s_i , and distributional strategies in G by σ_i . Given a discrete strategy s_i for agent i, possibly computed by our algorithm, it is straightforward to construct a corresponding distributional strategy σ_i which is feasible for the game G: Given an arbitrary type $v_i \in \mathcal{V}_i$, compute its discrete representant $\tau_i(v_i)$. Then choose strategy $b_i^d \in \mathcal{A}_i^d \subseteq \mathcal{A}_i$ with the same probability as b_i^d is chosen in the discrete game when agent i has type $\tau_i(v_i)$. Formally, we set

$$\sigma_i(V_i \times \{b_i^d\}) = \sum_{v_i^d \in \mathcal{V}_i^d} F_i(V_i \cap \tau_i^{-1}(v_i^d)) \frac{s_i(\{v_i^d\} \times \{b_i^d\})}{F_i^d(\{v_i^d\})}$$

for $V_i \subseteq \mathcal{V}_i$ measurable. We call this σ_i the strategy induced by s_i . Since s_i has \mathcal{V}_i^d -marginal F_i^d , $s_i(\{v_i^d\} \times \mathcal{A}_i^d) = F_i^d(\{v_i^d\})$, and $\sigma_i(V_i \times \mathcal{A}_i) = \sum_{v_i^d \in \mathcal{V}_i^d} F_i(V_i \cap \tau_i^{-1}(v_i^d)) = F_i(V_i)$, so σ_i is indeed feasible for the game Γ .

LEMMA 1. Let $s = (s_1, ..., s_n)$ be a strategy profile of the discretized game G^d and $\sigma = (\sigma_1, ..., \sigma_n)$ the strategy profile of the continuous game G, where the σ_i are induced by s_i . Then the difference in the expected utilities is $|\tilde{u}_i(\sigma) - \tilde{u}_i(s)| \leq L_U \delta_\tau$.

Proof: Consider fixed $b_i^d \in \mathcal{A}_i^d$ and $v_i^d \in \mathcal{V}_i^d$ for all agents i. Set $V_i = \tau_i^{-1}(v_i^d)$ and define $V = V_1 \times \cdots \times V_n$ and $A = \{b_1^d\} \times \cdots \times \{b_n^d\}$. Using the definitions of σ_i and f^d we get

$$\int_{V \times A} f(v) d\sigma_1(v_1, b_1) \dots d\sigma_n(v_n, b_n)$$

$$= \prod_i \frac{s_i(\{v_i^d\} \times \{b_i^d\})}{\prod_i F_i^d(\{v_i^d\})} \int_{V \times A} f(v) dF_1(v_1) \dots dF_n(v_n)$$

$$\begin{split} &= \Pi_i s_i(\{v_i^d\} \times \{b_i^d\}) \frac{F(V)}{\Pi_i F_i^d(\{v_i^d\})} = \Pi_i s_i(\{v_i^d\} \times \{b_i^d\}) f^d(v_i^d) \\ &= \int_{V \times A} f^d(v) ds_1(v_1, b_1) \dots ds_n(v_n, b_n). \end{split}$$

It follows that

$$\int_{V\times A} u_i(b,v_i) f^d(v) ds(v,b) = \int_{V\times A} u_i(b,v_i^d) f(v) d\sigma(v,b),$$

where $b^d = (b_1^d, \dots, b_n^d)$. Now

$$\left| \int_{V \times A} u_i(b, v_i) f(v) d\sigma(v, b) - \int_{V \times A} u_i(b, v_i^d) f(v) d\sigma(v, b) \right|$$

$$\leq L_U \delta_\tau \int_{V \times A} f(v) d\sigma(v, b).$$

In the last step we used that u_i is Lipschitz continuous and non-decreasing in v_i , i.e., $u_i(b, v_i) - u_i(b, v_i^d) \le L_U \delta_\tau$. Hence, summing over all such sets V and A, we get

$$|\tilde{u}_i(\sigma) - \tilde{u}_i(s)| \le L_U \delta_{\tau}.$$

In the next step, we want to compare the utility of a continuous strategy σ compared to the strategy $\tilde{\sigma}_i$ induced by the discrete strategy s_i , which was in return induced by σ_i . To do so, we have to define the discrete strategy s_i which is induced by σ_i .

Define a function $\psi: \mathcal{V}_i \times \mathcal{A}_i \to \mathcal{V}_i^d \times \mathcal{A}_i^d$ by

$$\psi(v_i, b_i) = \begin{cases} (\tau_i(v_i), \alpha_i^+(b_i)) \text{ if } v_i - p_i(b_i) \ge 0\\ (\tau_i(v_i), \alpha_i^-(b_i)) \text{ else.} \end{cases}$$

Thus, we define the discrete strategy s_i by $s_i(\{v_i^d\} \times \{b_i^d\}) = \sigma_i(\psi^{-1}(v_i^d, b_i^d))$.

LEMMA 2. Let σ be a strategy profile in the continuous game G and i an arbitrary agent. Then there is a strategy $\tilde{\sigma}_i$ that is induced by a strategy s_i of the discrete game G^d such that $\tilde{u}_i(\tilde{\sigma}_i, \sigma_{-i}) \geq \tilde{u}_i(\sigma) - L_U(4L_t\delta_{\alpha} + \delta_{\tau})$.

Proof: The proof is similar to the proof of Lemma 7 in Jackson and Swinkels (2005). We denote $\tilde{\sigma}_i$ the continuous strategy induced by s_i . Let $V_i = \tau_i^{-1}(v_i^d)$ and $A_i = \{b_i^d\}$.

First, we are going to show that for $(v_i, b_i) \in \psi^{-1}(v_i^d, b_i^d)$ and for arbitrary $(v_{-i}, b_{-i}) \in \mathcal{V}_{-i} \times \mathcal{A}_{-i}$, we have that $|u_i(b, v_i) - u_i(b_i^d, b_{-i}, v_i^d)|$ is small:

By the Lipschitz continuity of the payment functions, we have $|t_i^s(b_i, b_{-i}) - t_i^s(b_i^d, b_{-i})| \le L_t \delta_\alpha$ for $s \in \{w, l\}$, so

$$\begin{aligned} |U_i(v_i - t_i^w(b_i, b_{-i})) - U_i(v_i^d - t_i^w(b_i^d, b_{-i}))| &\leq L_U(\delta_\tau + L_t \delta_\alpha) \\ |U_i(-t_i^l(b_i, b_{-i})) - U_i(-t_i^l(b_i^d, b_{-i}))| &\leq L_U L_t \delta_\alpha. \end{aligned}$$

We distinguish two cases: either the allocation for agent i changes when the bid changes from b_i to b_i^d , or it does not change. If it does not change, i.e., $x_i(b_i, b_{-i}) = x_i(b_i^d, b_{-i})$, then

$$\begin{split} u_i(b_i^d, b_{-i}, v_i) &= x_i(b_i^d, b_{-i}) U_i(v_i - t_i^w(b_i^d, b_{-i})) + (1 - x_i(b_i^d, b_{-i})) U_i(-t_i^l(b_i^d, b_{-i})) \\ &\leq x_i(b_i, b_{-i}) U_i(v_i - t_i^w(b_i, b_{-i})) + (1 - x_i(b_i, b_{-i})) U_i(-t_i^l(b_i, b_{-i})) \\ &\quad + x_i(b_i, b_{-i}) L_U(\delta_\tau + L_t \delta_\alpha) + (1 - x_i(b_i, b_{-i})) L_U L_t \delta_\alpha \\ &\leq u_i(b, v_i) + L_U(\delta_\tau + L_t \delta_\alpha). \end{split}$$

Now consider the case where allocations differ, i.e., $x_i(b_i,b_{-i}) \neq x_i(b_i^d,b_{-i})$. Let us consider the case $b_i^d > b_i$, i.e., $b_i^d = \alpha_i^+(b_i)$ - the case $b_i^d < b_i$ can be treated similarly. Then there exists some bid $\tilde{b}_i \in [b_i,b_i^d]$ such that there is a tie between bidder i and some other bidder. Consequently, we have $t_i^w(\tilde{b}_i,b_{-i}) - t_i^l(\tilde{b}_i,b_{-i}) = p_i(\tilde{b}_i)$, so

$$\begin{split} &|(t_i^w(b_i,b_{-i})-t_i^l(b_i,b_{-i}))-p_i(b_i)|\\ &=|t_i^w(b_i,b_{-i})-t_i^w(\tilde{b}_i,b_{-i})-t_i^l(b_i,b_{-i})-t_i^l(\tilde{b}_i,b_{-i})-p_i(b_i)-p_i(\tilde{b}_i)|\\ &\leq 3L_t|b_i-\tilde{b}_i|. \end{split}$$

Since $v_i - p_i(b_i) \ge 0$, this implies

$$v_i - t_i^w(b_i, b_{-i}) \ge -t_i^l(b_i, b_{-i}) + 3L_t|b_i - \tilde{b}_i|,$$

and therefore

$$\begin{split} U_i(v_i - t_i^w(b_i, b_{-i})) &\geq U_i(-t_i^l(b_i, b_{-i}) + 3L_t|b_i - \tilde{b}_i|) \\ &\geq U_i(-t_i^l(b_i, b_{-i})) - 3L_UL_t|b_i - \tilde{b}_i| \end{split}$$

and thus, using that $x_i(b_i^d, b_{-i}) \ge x_i(b)$,

$$\begin{split} u_i(b,v_i) &= x_i(b)U_i(v_i - t_i^w(b_i,b_{-i})) + (1-x_i(b))U_i(-t_i^l(b_i,b_{-i})) \\ &= x_i(b_i^d,b_{-i})U_i(v_i - t_i^w(b_i,b_{-i})) + (1-x_i(b_i^d,b_{-i}))U_i(-t_i^l(b_i,b_{-i})) \\ &\quad + (x_i(b) - x_i(b_i^d,b_{-i}))(U_i(v_i - t_i^w(b_i,b_{-i})) - U_i(-t_i^l(b_i,b_{-i}))) \\ &\leq x_i(b_i^d,b_{-i})U_i(v_i - t_i^w(b_i,b_{-i})) + (1-x_i(b_i^d,b_{-i}))U_i(-t_i^l(b_i,b_{-i})) + 3L_UL_t\delta_\alpha \\ &\leq x_i(b_i^d,b_{-i})U_i(v_i - t_i^w(b_i^d,b_{-i})) + (1-x_i(b_i^d,b_{-i}))(U_i(-t_i^l(b_i^d,b_{-i})) + 4L_UL_t\delta_\alpha) \\ &= u_i(b_i^d,b_{-i},v_i) + 4L_UL_t\delta_\alpha \\ &\leq u_i(b_i^d,b_{-i},v_i^d) + 4L_UL_t\delta_\alpha + L_U\delta_\tau. \end{split}$$

By using an analogous argument, we arrive at the same bound for the case $b_i^d = \alpha_i^-(b_i)$. Let us now evaluate the expected utilities with respect to σ_i and $\tilde{\sigma}_i$. For $(v_i^d, b_i^d) \in \mathcal{V}_i^d \times \mathcal{A}_i^d$ and fixed v_{-i}, b_{-i} , we have that

$$\int_{\psi^{-1}(v_i^d, b_i^d)} u_i(b, v_i) f(v) d\sigma_i \leq \int_{\psi^{-1}(v_i^d, b_i^d)} (u_i(b_i^d, b_{-i}, v_i^d) + 4L_U L_t \delta_\alpha + L_U \delta_\tau) f(v) d\sigma_i
= \int_{\psi^{-1}(v_i^d, b_i^d)} (u_i(b_i^d, b_{-i}, v_i^d) + 4L_U L_t \delta_\alpha + L_U \delta_\tau) f(v) d\tilde{\sigma}_i.$$

By summing the integral over all sets $\psi^{-1}(v_i^d, b_i^d)$ and integrating with respect to σ_{-i} , we see that $\tilde{u}_i(\sigma) \leq \tilde{u}_i(\tilde{\sigma}_i, \sigma_{-i}) + L_U(4L_t\delta_\alpha + \delta_\tau)$.

Theorem 1 Let $s \in S^d$ be an ε -BNE of the discrete game G^d of a single-object auction that satisfies the assumptions 1-6. Let $\sigma \in S$ be the strategy profile, where each σ_i is the strategy induced by s_i . Then σ is an $\varepsilon + \mathcal{O}(\delta_{\alpha} + \delta_{\tau})$ -BNE of the continuous game G.

Proof: Let σ_i^* be a best response to σ_{-i} . Then σ_i^* induces a strategy \tilde{s}_i in the discrete game, which in turn induces a continuous strategy $\tilde{\sigma}_i$. By Lemma 2 we get the following bound of the expected utility of the best response with respect to $\tilde{\sigma}_i$.

$$\tilde{u}_i(\sigma_i^*, \sigma_{-i}) \leq \tilde{u}_i(\tilde{\sigma}_i, \sigma_{-i}) + L_U(4L_t\delta_\alpha + \delta_\tau)$$

Since $(\tilde{\sigma}_i, \sigma_{-i})$ is a strategy profile induced by the discrete profile (\tilde{s}_i, s_{-i}) , we can use Lemma 1 and further obtain

$$\leq \tilde{u}_i(\tilde{s}_i, s_{-i}) + L_U(4L_t\delta_\alpha + \delta_\tau) + L_U\delta_\tau.$$

Since (s_i, s_{-i}) is an ε -equilibrium the previous term is bounded by

$$\leq \tilde{u}_i(s_i, s_{-i}) + \varepsilon + L_U(4L_t\delta_\alpha + \delta_\tau) + L_U\delta_\tau$$

and applying again Lemma 1 to get a bound with respect to the induced profile σ , we obtain

$$\leq \tilde{u}_i(\sigma_i, \sigma_{-i}) + \varepsilon + L_U(4L_t\delta_\alpha + \delta_\tau) + 2L_U\delta_\tau.$$

Thus we get our claim that

$$\tilde{u}_i(\sigma_i^*, \sigma_{-i}) < \tilde{u}_i(\sigma_i, \sigma_{-i}) + \varepsilon + \mathcal{O}(\delta_\alpha + \delta_\tau).$$

Appendix C: Expected utility functions based on different parametric assumptions.

In this appendix we provide plots of the expected utility function for different parametric assumptions of the bid function and different distributional assumptions. Figures 9 to 12 illustrate ex-interim utilities for specific value draws ($o_i = 0.7$) of a bidder. Plots for lower or higher valuations have similar properties.

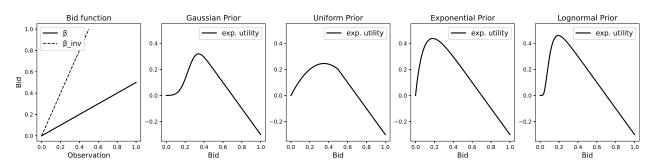


Figure 9 Expected utility with a linear bid function in a first-price sealed-bid auction.

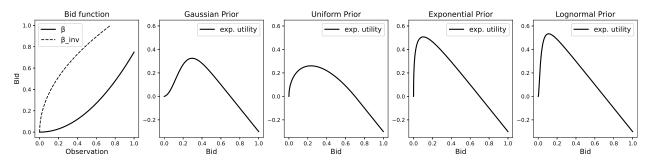


Figure 10 Expected utility with a convex bid function in a first-price sealed-bid auction.

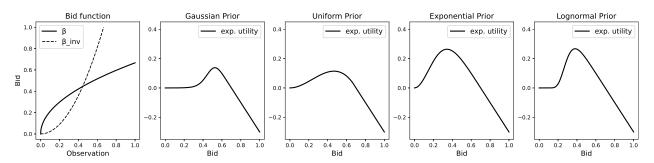


Figure 11 Expected utility with a concave bid function in a first-price sealed-bid auction.

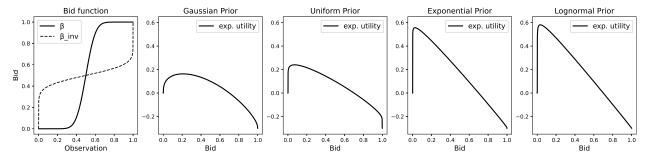


Figure 12 Expected utility with a sigmoid bid function in a first-price sealed-bid auction.

Appendix D: Monotonicity and Variational Stability in the Approximation Game

In this section we want to give evidence that variational stability and thereby monotonicity are not satisfied globally in our settings as discussed in Section 3.6. For this we consider a simple example of a FPSB with two symmetric bidders.

Consider our approximation game $\Gamma = (\mathcal{I}, \mathcal{S}^d, \tilde{u})$ as defined in Definition 2. Since the sets of discrete distributional strategies \mathcal{S}_i^d are compact, convex subsets of $\mathbb{R}^{K \times L}$ and the utility functions \tilde{u}_i are linear and therefore concave in s_i , we have a continuous, concave game as defined in (Mertikopoulos and Zhou 2019). Rosen (1965) refers to such games as n-person concave games. In this setting, Nash equilibria $s^* \in \mathcal{S}^d$ are precisely the solutions of the corresponding variational inequality $VI(F, \mathcal{S}^d)$

$$\langle F(s^*), s - s^* \rangle \le 0, \quad \forall s \in \mathcal{S}^d,$$
 (VI)

with $F = (F_i)_{i \in \mathcal{I}}$ and $F_i(s) := \nabla_i \tilde{u}_i(s_i, s_{-i})$. In the following, we want to give a short overview of the relevant definitions.

The game is said to satisfy the payoff monotonicity condition (MC), if

$$\langle F(s) - F(s'), s - s' \rangle = \sum_{i} \langle \nabla_i \tilde{u}_i(s_i, s_{-i}) - \nabla_i \tilde{u}_i(s'_i, s'_{-i}), s_i - s'_i \rangle \le 0, \quad \forall s, s' \in \mathcal{S}^d$$
 (MC)

with equality if and only if s = s'. Rosen (1965) uses this property, which he calls diagonally strict concavity, and shows that if the game satisfies (MC), it admits a unique Nash equilibrium. In the literature on variational inequalities, this is also known as strict monotonicity (Facchinei and Pang 2003). Furthermore, we say that a strategy profile $s^* \in \mathcal{S}^d$ is variationally stable (VS), if there exists a neighborhood $S \subseteq \mathcal{S}^d$ such that

$$\langle F(s), s - s^* \rangle = \sum_{i} \langle \nabla_i \tilde{u}_i(s_i, s_{-i}), s_i - s_i^* \rangle \le 0, \quad \forall s \in S$$
 (VS)

with equality if and only if $s = s^*$. In terms of variational inequalities, global variational stable points are in the set of weak solutions of the corresponding variational inequality $VI(F, \mathcal{S}^d)$. Mertikopoulos and Zhou (2019) extend results of monotone games and show that the weaker concept of (VS) suffices to get convergence for the no-regret algorithm dual averaging.

We will now show that, even in the simplest setting, variational stability is not satisfied, and therefore convergence does not follow from these results. Let us consider a first-price sealed bid with two symmetric bidders and i.i.d. observations (valuations) $o \sim U([0,1]]$. The observation and action space are discretized equally with K = L points, i.e., $\mathcal{O}_i^d = \mathcal{A}_i^d = \{0, \frac{1}{K-1}, \dots, \frac{K-2}{K-1}, 1\}$. Similar to our numerical experiments, we assume a tie-breaking rule where bidders win only if their bid is strictly greater than the opponents' bids. In that case, the discretized game has two symmetric, pure equilibria which basically correspond to $\beta_1(o) = \lceil \frac{o}{2} \rceil$ and $\beta_2(o) = \lfloor \frac{o}{2} \rfloor$ (Rasooly and Gavidia-Calderon 2021).

First, we observe that there are two equilibria, which immediately proves that the game cannot satisfy (MC). Second, none of the BNE is globally variationally stable. If both bidders stick to the collusive strategy s_i^c (i.e., bid approximately $\frac{1}{4}o_i$) each agent ends up with a higher utility, than in a situation where the agent deviates to a BNE s_i^* . This means

$$\tilde{u}_i(s_i^*, s_{-i}^c) < \tilde{u}_i(s_i^c, s_{-i}^c)$$

By symmetry, this stays true if we sum over i. Since the utility \tilde{u}_i is linear in s_i the gradient does not depend on s_i and we can write

$$\sum_i \langle \nabla_i \tilde{u}_i(s_i^*, s_{-i}^c), s_i^* \rangle = \sum_i \langle \nabla_i \tilde{u}_i(s_i^c, s_{-i}^c), s_i^* \rangle < \sum_i \langle \nabla_i \tilde{u}_i(s_i^c, s_{-i}^c), s_i^c \rangle.$$

Rearranging terms, we get an inequality which contradicts (VS)

$$\sum_{i} \langle \nabla_i \tilde{u}_i(s_i^*, s_{-i}^c), s_i^c - s_i^* \rangle > 0.$$

Therefore, the equilibrium s^* cannot be globally variationally stable. Note that this line of argument does not rely on the non-uniqueness of the equilibrium, or the specific tie-breaking rule. We only use individual linearity of the bidder's utility function, which is a consequence of the discretization, and the fact that we can construct collusive strategies, where deviating to the BNE reduces the expected utility.

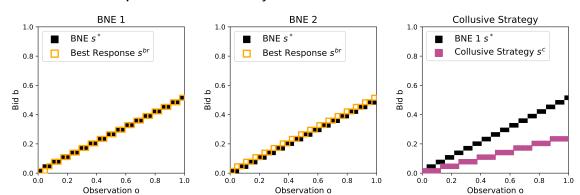


Figure 13 Counterexample for variational stability.

Note. In the first two plots, we see the two discrete distributional BNE for the 2-player FPSB, which correspond to β_1 and β_2 indicated by the black squares. While the equilibrium strategies are obviously best responses to themselves, the yellow squares indicate alternative best responses, which makes the BNE non-strict. On the third plot we illustrate a BNE (black) and a collusive strategy (purple). For these strategies we observe that $\tilde{u}_i(s_i^c, s_{-i}^c) > u_i(s_i^*, s_{-i}^c)$, i.e., unilaterally deviating from the collusive strategy profile to the BNE descreases the utility the bidder. This makes the collusive strategy profile a point, where VS w.r.t. the BNE is not satisfied.

Specifically, in this example, we can further show that both equilibria are not even locally variationally stable. In Figure 13 we illustrate that the equilibrium strategies are not strict since the best responses are not unique. By linearity of the utility functions, the inequality (VS) is equal to zero for a BNE s^* and any convex combination $s = \lambda s^* + (1 - \lambda)s^{br}$ of s^* and its best response $s^{br} \neq s^*$. This means that either the BNE is not locally variationally stable (for every neighborhood we can choose λ small enough), or the BNE and its best response are elements of some larger variationally stable set. But in the latter case, we know that this set has to be a convex set of equilibria (Mertikopoulos and Zhou 2019, Prop. 2.7). And since we can verify numerically that for instance, a convex combination of BNE2 and its best response BNE1 is not an equilibrium, this cannot be the case.

In conclusion, we have a setting which is not monotonic and not even locally variationally stable, but in which our methods still approximates the BNE.