
3-OBJECTIVE PARETO OPTIMIZATION FOR PROBLEMS WITH CHANCE CONSTRAINTS

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ABSTRACT

Evolutionary multi-objective algorithms have successfully been used in the context of Pareto optimization where a given constraint is relaxed into an additional objective. In this paper, we explore the use of 3-objective formulations for problems with chance constraints. Our formulation trades off the expected cost and variance of the stochastic component as well as the given deterministic constraint. We point out benefits that this 3-objective formulation has compared to a bi-objective one recently investigated for chance constraints with Normally distributed stochastic components. Our analysis shows that the 3-objective formulation allows to compute all required trade-offs using 1-bit flips only, when dealing with a deterministic cardinality constraint. Furthermore, we carry out experimental investigations for the chance constrained dominating set problem and show the benefit for this classical NP-hard problem.

Keywords Chance constraints, evolutionary multi-objective optimization, theory, runtime analysis

1 Introduction

Evolutionary algorithms have been shown to be successful for a wide range of optimization problems and the development and application of evolutionary multi-objective algorithms [1, 2] is one of the great success stories in the area of evolutionary computation. This includes both solving classical problems with multiple objectives (see e.g. [3, 4, 5]) as well as using multi-objective models to solve single-objective problems by relaxing constraints into additional objectives [6, 7, 8] or adding helper objectives [9, 10, 11]. In the context of submodular optimization using multi-objective formulations that relax a given constraint into an additional objective has been shown to achieve best possible performance guarantees for a wide range of submodular problem while outperforming classical approaches based on greedy algorithms in practice [12, 13, 14].

Tackling stochastic problems in terms of uncertainties can use objectives such as the expected cost (or value) and uncertainties such as variances or quantiles [15, 16]. Recently, chance constrained problems [17] have gained increasing attention in the area of evolutionary computation [18, 19, 20, 21]. These problems involve stochastic components and constraints that should be met with a given probability α . Furthermore, formulations using chance constraints can be used to guarantee high quality objective function values with a high probability of α if the stochastic components influence the objective function and not (only) the constraint. It has been shown in [22] that a bi-objective formulation taking into account the expected cost and variance of a solution is highly effective for minimizing the stochastic cost of a solution for every possible confidence level α when considering uniform or spanning tree constraints.

We investigate the use of 3-objective formulations instead of bi-objective formulations for the set up studied in [22] by adding the constraint as an additional objective. Adding the constraint as an additional objective usually implies that the number of trade-offs according to the objectives functions grows significantly. On the other hand, the additional

objective can enable a different way of searching for high quality solutions. We investigate in detail how 3-objective formulations can be used for chance constrained problems.

Our first contribution is a theoretical runtime analysis which generalizes the results obtained in [22] to the 3-objective formulation. Here we show that the 3-objective formulation allows to compute all possible trade-offs for independent Normally distributed weight for the whole set of possible uniform constraints. Investigating the problem furthermore, we show that in order to compute the whole set of trade-offs, only 1-bit flips are required in the 3-objective formulation which significantly improves upper the results given in [22]. Afterwards, we investigate the 3-objective formulation and compare it to the bi-objective one through experimental investigations. We consider the chance constrained dominating set problem in the same set up as done in [22]. Our results show that the 3-objective formulation provides a clear advantage for graphs with up to 500 nodes.

The outline of the paper is as follows. In Section 2, we introduce the two chance constrained problem setups that we are investigating in this paper. In Section 3, we introduce the 3-objective formulation that is subject to our investigations. Sections 4 and 5 provide a rigorous runtime analysis of our approach which shows that it efficiently computes a set up solutions that includes optimal solutions for a wide range of constrained settings of the two considered problems. We present our experimental results for the chance constrained dominating set problem in Section 6 and finish with some conclusions.

2 Preliminaries

Pareto optimization approaches are usually used to tackle constrained single-objective optimization problems by taking the constraint as an additional objective. Chance constrained problems involve constraints that are impacted by the expected (cost) value as well as its variance. In [22], a chance constrained problem has been considered which involves such stochastic components and has an additional deterministic constraint. We motivate our multi-objective settings by these recent investigations.

We consider the chance constrained problem investigated in [22]. Given a set of n items $I = \{1, \dots, n\}$ with stochastic weights w_i , $1 \leq i \leq n$, we want to solve

$$\min W \quad \text{subject to} \quad (Pr(w(x) \leq W) \geq \alpha) \wedge (|x|_1 \geq k) \quad (1)$$

where $w(x) = \sum_{i=1}^n w_i x_i$, $x \in \{0, 1\}^n$, and $\alpha \in [1/2, 1[$. The weights are independent and each w_i is distributed according to a Normal distribution $N(\mu_i, \sigma_i^2)$, $1 \leq i \leq n$, where $\mu_i \geq 1$ and $\sigma_i \geq 1$, $1 \leq i \leq n$. We denote by $\mu(x) = \sum_{i=1}^n \mu_i x_i$ the expected weight and by $v(x) = \sum_{i=1}^n \sigma_i^2 x_i$ the variance of the weight of solution x .

As stated in [22], the problem given in Equation 1 is equivalent to minimizing

$$\hat{w}(x) = \mu(x) + K_\alpha \sqrt{v(x)}, \quad (2)$$

under the constraint that $|x|_1 \geq k$ holds. Here, K_α denotes the α -fractional point of the standard Normal distribution.

The uniform constraint $|x|_1 \geq k$ requires that each feasible solution has to contain at least k elements. As expected weights and variances are strictly positive, an optimal solution has exactly k elements. Depending on the choice of α , the difficulties lies in finding the right trade-off between the expected weight and variance among all solutions with exactly k elements.

It has been shown that this problem given in Equation 1 can be solved by the following bi-objective formulation [22]. The objective function is given as $f_{2D}(x) = (\hat{\mu}(x), \hat{v}(x))$ where

$$\hat{\mu}(x) = \begin{cases} \sum_{i=1}^n \mu_i x_i & |x|_1 \geq k \\ (k - |x|_1) \cdot (1 + \sum_{i=1}^n \mu_i) & |x|_1 < k \end{cases}$$

$$\hat{v}(x) = \begin{cases} \sum_{i=1}^n \sigma_i^2 x_i & |x|_1 \geq k \\ (k - |x|_1) \cdot (1 + \sum_{i=1}^n \sigma_i^2) & |x|_1 < k \end{cases}$$

We say that a solution x dominates a solution y ($x \succeq y$) iff $\hat{\mu}(x) \leq \hat{\mu}(y) \wedge \hat{v}(x) \leq \hat{v}(y)$. Furthermore, a solution x strongly dominates a solution y ($x \succ y$) iff $x \succeq y$ and $f_{2D}(x) \neq f_{2D}(y)$. The setup can be generalized by using $c(x) \geq k$ for a constraint function $c(x)$ instead of $|x|_1 = k$. In the experimental investigations carried out in [22], $c(x)$ is counting the number of dominated nodes in the dominating set problem in graphs with n nodes, and $c(x) = n$ is required for a solution to be feasible.

The key idea of the result given in [22] is to show that the algorithm computes the extremal points of the Pareto front of the given problem. Note that as the expected costs and variances are strictly positive, each Pareto optimal solution contains exactly k elements when considering this bi-objective formulation.

We also consider the problem of maximizing a given deterministic objective $c(x)$ under a given chance constraint, i.e

$$\max c(x) \quad \text{subject to} \quad Pr(w(x) \leq B) \geq \alpha. \quad (3)$$

with $w(x) = \sum_{i=1}^n w_i x_i$ where each w_i is chosen independently of the other according to a Normal distribution $N(\mu_i, \sigma_i^2)$, and B and $\alpha \in [1/2, 1]$ are a given weight bound and reliability probability.

Such a problem formulation includes for example the maximum coverage problem in graphs with so-called chance constraints [23, 14], where $c(x)$ denotes the nodes of covered by a given solution x and the costs are stochastic. Furthermore, the chance constrained knapsack problem as investigated in [20, 24] fits into this problem formulation.

3 3-Objective Pareto Optimization

The now introduce the 3-objective formulation of the problems given in Equation 1 and 3 and the algorithms that we study in this paper.

3.1 3-Objective Formulation

We investigate the 3-objective formulation given as

$$f_{3D}(x) = (\mu(x), v(x), c(x))$$

where $c(x)$ is the constraint value of a given solution that should be maximized. In our theoretical study, we focus on the case $c(x) = |x|_1$, which turns the constraint $|x|_1 \geq k$ into the additional objective of maximizing the number of bits in the given bitstring.

Similar to the bi-objective model we minimize the expected weight $\mu(x) = \sum_{i=1}^n \mu_i x_i$ and the variance $v(x) = \sum_{i=1}^n \sigma_i^2 x_i$ of the weight of solution x . Note that here we do not consider penalty terms for violating the constraint $|x|_1 \geq k$ as done in the bi-objective formulation. We say that a solution x dominates a solution y ($x \succeq y$) iff $c(x) \geq c(y) \wedge \mu(x) \leq \mu(y) \wedge v(x) \leq v(y)$. Furthermore, a solution x strongly dominates y ($x \succ y$) iff $x \succeq y$ and $f_{3D}(x) \neq f_{3D}(y)$.

Generalizing the results given in [22], we show that our problem formulation solves the problem given in Equation 1 for every possible value of k and α . Furthermore, we use the 3-objective problem to compute, for any possible pair of B and α values, a solution with the highest possible $c(x)$ -value according to Equation 3.

Using the expected cost and variance as objectives for the problem given in Equation 3, allows here to explore the trade-offs with respect to the expected cost and variance for the different values of B and α that lead to a maximum possible value of $c(x)$. We will show that the 3-objective formulation is obtaining for any possible B and $\alpha \in [1/2, 1]$ a feasible solution with the maximal value for $c(x) = |x|_1$ in expected pseudo-polynomial time. We first show this by adapting the proof given in [22] to the 3-objective setting. The proof makes use of specific 2-bit flips that allow to compute all convex points of the Pareto front when constraining the number of elements to one particular constraint value k and thereby solving the problem given in Equation 1 as well.

Afterwards, we improve our upper bound by showing that the 3-objective formulation enables an additional search direction for evolutionary multi-objective algorithms which only relies on the use of 1-bit flips. As specific 1-bits occur more frequently than specific 2-bit flips, we obtain an improved upper bound.

3.2 Algorithms

For our investigations, we consider variants of the well-known Global Simple Evolutionary Multi-Objective Optimizer (GSEMO) [25, 26] given in Algorithm 1. The algorithm starts with one initial solution chosen uniformly at random and produces in each iteration a single offspring by standard bit mutations. GSEMO maintains at each point in the time for each non-dominated objective vector found so far one single solution. The variant of GSEMO called SEMO originally introduced in [25] differs from GSEMO by flipping in each mutation step exactly one randomly chosen bit.

We investigate the algorithms GSEMO2D and GSEMO3D which are using our bi-objective and 3-objective problem formulation together with standard bit-mutations as outlined in Algorithm 1. As the proofs for the bi-objective formulation carried out in [22] rely on 1- and 2-bit flips, we consider the algorithm SEMO2D which with probability

Algorithm 1: GSEMO

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Choose  $x \in \{0, 1\}^n$  uniformly at random;
 $P \leftarrow \{x\}$ ;
repeat
  Choose  $x \in P$  uniformly at random;
  Create  $y$  by flipping each bit  $x_i$  of  $x$  with probability  $\frac{1}{n}$ ;
  if  $\nexists w \in P : w \prec y$  then
     $P \leftarrow (P \setminus \{z \in P \mid y \preceq z\}) \cup \{y\}$ ;
until stop;

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1/2 carries out a 1-bit flip and otherwise carries out a 2-bit flip in the mutation step. Similarly, as our investigations for the 3-objective model show that it performs well with 1-bit operations only, we consider the algorithm SEMO3D which flips in each mutation step one single bit. Note that SEMO3D is exactly the algorithm variant introduced in [25] although it has only been applied to bi-objective problems in that paper.

For our theoretical investigations, we measure time in terms of the number of fitness evaluations to achieve a desired goal. The expected number of fitness evaluations is also called the *expected time* to achieve the given goal.

4 Analysis Based on 2-Bit Flips

We investigate the problem given in Equation 3 for the case where $c(x) = |x|_1$, and each w_i is chosen according to the Normal distribution $N(\mu_i, \sigma_i^2)$ independently of the others. As done in [22], we assume $\mu_i \geq 1$ and $\sigma_i^2 \geq 1$, $1 \leq i \leq n$, in the following. We claim that GSEMO computes an optimal solution for any combination of B and $\alpha \in [1/2, 1[$ in expected pseudo-polynomial time for the Problem given in Equation 3.

Inspired by the analysis of chance-constrained minimum spanning trees [27], we consider sets of Pareto optimal search points having exactly k , $0 \leq k \leq n$, elements that are minimal with respect to

$$\hat{w}(x) = \mu(x) + K_\alpha \sqrt{v(x)}$$

for each fixed k and α .

To do this, we follow the ideas given in [22]. Our goal is to minimize $f_\lambda(x) = \lambda\mu(x) + (1 - \lambda)v(x)$. This can be done by choosing iteratively k minimal elements with respect to $f_\lambda(e_i) = \lambda\mu_i + (1 - \lambda)\sigma_i^2$, $0 < \lambda < 1$. For $\lambda = 0$ and $\lambda = 1$, f_λ is minimized by minimizing $f_0(x) = (v(x), \mu(x))$ and $f_1(x) = (\mu(x), v(x))$ with respect to the lexicographic order. Note that for each $\lambda \in [0, 1]$, an optimal solution for f_λ can be obtained by selecting the first k items in increasing order with respect to f_λ . In terms of notation, we use f_λ for the evaluation of a search point x as well as the evaluation of an element e_i in the following.

We denote by $X_{k,\lambda}^* \subseteq \{0, 1\}^n$ the set of minimal elements with respect to f_λ having exactly k elements. Note that all points in the sets $X_{k,\lambda}^*$, $0 \leq \lambda \leq 1$, are not strongly dominated in $\{0, 1\}^n$ as the expected cost and variance strictly increase when adding any additional element. Therefore, the sets $X_{k,\lambda}^*$, $0 \leq \lambda \leq 1$, $0 \leq k \leq n$, constitute Pareto optimal points. Note there may be other Pareto optimal points not included in these sets.

Definition 1 (Extreme point of set X). *For a given set $X \subseteq \{0, 1\}^n$, we call $f(x) = (\mu(x), v(x))$ an extreme point of X if there is a $\lambda \in [0, 1]$ such that $x \in X_{k,\lambda}^*$ and $v(x) = \max_{y \in X_{k,\lambda}^*} v(y)$.*

We denote by P_{\max} the maximum population size that GSEMO encounters during the run of the algorithm, i. e., before reaching its goal of optimization.

Let $v_{\max} = \max_{1 \leq i \leq n} \sigma_i^2$ and $\mu_{\max} = \max_{1 \leq i \leq n} \mu_i$. we assume that $v_{\max} \leq \mu_{\max}$ holds. Otherwise, the bounds in Theorem 1 and 2 and can be tightened by replacing v_{\max} by μ_{\max} .

Let $X^k = \{x \in \{0, 1\}^n \mid |x|_1 = k\}$ be the set of all solutions having exactly k elements. The following theorem shows that GSEMO computes for each k and α an optimal solution for the problem given in Equation 1, which has been investigated in [22] in the context of the 2-objective formulation given in Section 2.

Theorem 1. *GSEMO computes a population P which contains for each $\alpha \in [1/2, 1[$ and $k \in \{0, \dots, n\}$ a solution*

$$x_\alpha^k = \arg \min_{x \in X^k} \left\{ \mu(x) + K_\alpha \sqrt{v(x)} \right\} \quad (4)$$

in expected time $O(P_{\max} n^4 (\log n + \log v_{\max}))$.

Proof. For $k \in \{0, n\}$, the search points 0^n and 1^n are the unique corresponding optimal solutions. As GSEMO minimizes $\mu(x)$, the search point 0^n is obtained in expected time $O(P_{\max} n \log n)$ using the bound on the population size together with standard fitness level argument from the analysis of the (1+1) EA for OneMax [28]. Similarly, as GSEMO maximizes $c(x)$, the search point 1^n is obtained in expected time $O(P_{\max} n \log n)$. These runtime bounds can be obtained by considering 1-bit flips only.

We now consider the time to compute all Pareto optimal solutions x_α^k with exactly k elements for $\alpha = 1/2$ and $0 \leq k \leq n$ (the case of general α will be studied below). These are solution with the smallest expected weight as $\alpha = 1/2$ implies $K_\alpha = 0$. Having computed all Pareto optimal solutions for $\alpha = 1/2$ with at most j , $0 \leq j < n - 1$ elements, we pick the Pareto optimal solution x_α^j for $\alpha = 1/2$ and j elements. Inserting the smallest element i currently not x_α^j according to (μ_i, σ_i^2) in lexicographic order leads to a Pareto optimal solution x_α^{j+1} which has $j + 1$ elements and is optimal for $\alpha = 1/2$. The expected time for GSEMO to produce this solution is $O(P_{\max} n)$ as one can obtain it from x_α^j by flipping the single specific bit for element i . There are $n + 1$ different values of k . Hence the expected time until an optimal solution x_α^k for $\alpha = 1/2$ and all k , $0 \leq k \leq n$, has been obtained is $O(P_{\max} n^2)$.

We now analyze the time to get optimal solutions x_α^k , $1 \leq k \leq n - 1$, for any value of $\alpha > 1/2$ after having obtained all optimal solutions for $\alpha = 1/2$. Thereby, we make use of the arguments given in [22] where it is shown that the set of Pareto optimal solutions when minimizing the bi-objective problem given as $g(x) = (\mu(x), v(x))$ under the constraint $|x|_1 \geq k$, contains all extreme points of the problem. As we have $\mu_i > 0$, $\sigma_i^2 > 0$, $1 \leq i \leq n$, each Pareto optimal solution for minimizing $g(x)$ under the constraint $|x|_1 \geq k$ has exactly k elements. Let $X^k = \{x \in \{0, 1\}^n \mid |x|_1 \geq k\}$ be the set of all solutions containing at least k elements. Furthermore let $Y^k = \{0, 1\}^n \setminus X^k$ be the set of all search points having strictly less than k elements. As the first objective is to maximize $p(x)$ no search point in X^k is dominated by any search point in Y^k . Therefore creating a new search point in Y^k does not lead to the removal of any element from X^k if its currently contained in the population of GSEMO.

As done in [22], we define $\lambda_{i,j} = \frac{\sigma_j^2 - \sigma_i^2}{(\mu_i - \mu_j) + (\sigma_j^2 - \sigma_i^2)}$ for the pair of items i and j where $\sigma_i^2 < \sigma_j^2$ and $\mu_i > \mu_j$ holds, $1 \leq i < j \leq n$. The set $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_\ell, \lambda_{\ell+1}\}$ where $\lambda_1, \dots, \lambda_\ell$ are the values $\lambda_{i,j}$ in increasing order and $\lambda_0 = 0$ and $\lambda_{\ell+1} = 1$. The set of Pareto optimal solutions with exactly k elements contains all optimal solutions f_λ and every $\lambda \in \Lambda$.

We will now analyze the time until GSEMO until GSEMO has computed a population which includes a solution x_α^k according to (4) for any choice of $\alpha \in [1/2, 1[$ and $k \in \{1, \dots, n - 1\}$. To this end, we build until the analysis leading to Theorem 4 in [22] but consider the progress of the algorithm for the $n - 1$ different values for $|x|_1$ (excluding the trivial values 0 and n) in parallel. Recall that the bi-objective formulation in [22] considers only a fixed value (to be precise, a lower bound) for $|x|_1$. The analysis in that paper proceeds by conducting a sequence of multiplicative drift analyses for the fixed k ; more precisely the expected time is bounded until a search point of minimal variance is obtained for $\lambda_0, \lambda_1, \dots, \lambda_\ell + 1$, using a multiplicative drift argument for each individual λ . Since there probability of choosing an individual that can be improved by a two-bit flip is at least $1/(P_{\max})$ and the probability of a two-bit flip is $\Omega(1/n^2)$, this results in a bound of $O(P_{\max} n^2 \ell (\log n + \log v_{\max}))$ in [22].

In our three-objective formulation, the probability of picking an individual whose first objective value equals k is at least $1/P_{\max}$. To progress of GSEMO towards its overall goal of including a solution x_α^k in the population for each $\alpha \in [1/2, 1[$ and each number of one-bits $k \in \{1, \dots, n - 1\}$ is now estimated with $n - 1$ parallel sequences of multiplicative drift processes in the following way. Considering a fixed $k \in \{1, \dots, n - 1\}$, sequence k corresponds to finding the optimal solutions for the uniform constraint of having at least k one-bits. As explained above, this is guaranteed after completion of ℓ consecutive multiplicative drift processes indexed $1, \dots, \ell_k$. Each each point of time, each sequence k is “performing” its p_k th multiplicative drift process, where p_k ranges from 1 to ℓ_k and increases by 1 when a process from the sequence has reached its target.

At each point of time, exactly one k is chosen and process p_k from sequence k decreases its state by a multiplicative factor of no larger than $1 - \delta = 1 - 1/(en^2)$ since there is always at least one two-bit flip available that decreases the state. For each multiplicative drift process within each of the k sequences, it holds that it reaches the target after time $T^* = O(n^2 (\log n + \log w_{\max}))$ with probability at least $1 - 2/n^3$, where we used the tail bounds for multiplicative drift. We also know from Lemma 2 in [22] that $\ell_k \leq n^2$ for every k .

For each $k \in \{1, \dots, n - 1\}$, a step of sequence k happens with probability at least $1/P_{\max}$. Within $2P_{\max} n^2 T^*$ steps, each sequence is chosen at least $n^2 T^*$ times with probability $1 - e^{-\Omega(P_{\max} n^2 T^*)} = 1 - e^{-\Omega(n^2)}$ according to Chernoff bounds. Assuming this to happen, by a union bound over $n - 1$ sequences and at most n^2 processes per sequence, each process reaches its target within time T^* with probability at least $1 - (n - 1)n^2/(2n^3) \geq 1/2$, implying that the every process from every sequence has reached its target within time $n^2 T^*$. The total failure probability is at most $1/2 + e^{-\Omega(n^2)} = 1/2 + o(1)$.

In case of a failure, we repeat the argumentation. The expected number of repetitions is at most $2+o(1)$, so that the total expected time until all processes have reached the target is $O((2+o(1))2P_{\max}n^2T^*) = O(P_{\max}n^4(\log n + \log w_{\max}))$. As explained above, this also bounds the time to include a solution x_{α}^k in the population of GSEMO for all $\alpha \in [1/2, 1[$ and all $k \in \{1, \dots, n-1\}$. \square

The previous theorem bounds the time to include solutions of the type described in (4), which may look abstract. The following theorem shows that these solutions allow us to find solutions for the chance-constrained problem in (3) efficiently from the population of GSEMO for different settings of confidence level and constraint.

Theorem 2. *The expected time until GSEMO has computed a population which includes an optimal solution for the problem given in Equation 3 with $c(x) = |x|_1$ for any possible choice of B and $\alpha \in [1/2, 1[$ is $O(P_{\max}n^4(\log n + \log v_{\max}))$,*

Proof. We show that the population $P \supseteq \{x_{\alpha}^k \mid 0 \leq k \leq n, \alpha \in [1/2, 1[$ given in Theorem 1 contains the optimal solutions for any choice of $\alpha \in [1/2, 1[$. Let x_{α}^* be an optimal solution for a given value of $\alpha \in [1/2, 1[$.

For a given α , the solution x_{α}^j with the maximum number of elements for which

$$\hat{w}(x) = \mu(x) + K_{\alpha}\sqrt{v(x)} \leq B$$

holds satisfies $|x_{\alpha}^j| = |x_{\alpha}^*|$ as otherwise x_{α}^j would not be a solution with the maximal number of elements for which the constraint holds or x_{α}^* would not be optimal for α . This implies that x_{α}^j is an optimal solution for α . \square

5 Improved Upper Bound Based on 1-Bit Flips Only

The analysis from the previous section relied on specific 2-bit flips that allow to produce the solutions for each value of α by swapping elements to produce new Pareto optimal solutions for a given number of k elements.

We now show that 2-bit flips are not necessary in the 3-objective formulation and also improve the upper bound by considering only 1-bit flips. We note that the upper bound is by an asymptotic factor $\Omega(n^2)$ lower compared to Theorem 1 and includes the same P_{\max} .

Theorem 3. *The expected time until SEMO3D and GSEMO have computed a population which includes an optimal solution for the problems given in Equation 1 (for any choice of k and α) and Equation 3 (with $c(x) = |x|_1$ for any choice of B and α) is $O(P_{\max}n^2)$ and it is at most $2eP_{\max}n^2$ with probability $1 - e^{-\Omega(n)}$.*

Proof. To prove the theorem, we show that the same set of Pareto optimal objective vectors can be computed by GSEMO as in the proof of Theorem 1 when considering 1-bit flips only. Theorem 2 implies that then not only all optimal solutions with respect to Equation 1 but also with respect to Equation 3 have been computed.

By a simple fitness-level argument, the expected time until the Pareto optimal search point 0^n has been included in the population is $O(P_{\max}n \log n)$. This search point will never be removed from the population as it is the unique search point with minimum expected cost and variance.

As done in [22], we define $\lambda_{i,j} = \frac{\sigma_j^2 - \sigma_i^2}{(\mu_i - \mu_j) + (\sigma_j^2 - \sigma_i^2)}$ for the pair of items i and j where $\sigma_i^2 < \sigma_j^2$ and $\mu_i > \mu_j$ holds, $1 \leq i < j \leq n$. The set $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_{\ell}, \lambda_{\ell+1}\}$ where $\lambda_1, \dots, \lambda_{\ell}$ are the values $\lambda_{i,j}$ in increasing order and $\lambda_0 = 0$ and $\lambda_{\ell+1} = 1$.

Following [22], we define the function

$$f_{\lambda}(x) = \lambda\mu(x) + (1 - \lambda)v(x)$$

and also use it applied to elements e_i of the given input, i.e.

$$f_{\lambda}(e_i) = \lambda\mu_i + (1 - \lambda)\sigma_i^2.$$

Note that for a given λ the function f_{λ} can be optimized by a greedy approach which iteratively selects a set of k smallest elements according to $f_{\lambda}(e_i)$. For any $\lambda \in [0, 1[$ an optimal solution for f_{λ} with k elements is Pareto optimal as there is no other solution with at least k elements that improves the expected cost or variance without impairing the other. Hence, once obtained a solution with the resulting objective vector will remain in the population for the rest of the optimization process. Furthermore, the set of optimal solutions for different λ values only change at the λ values of the set Λ as these λ values constitute the weighting where the order of items according to f_{λ} can switch [27, 22].

We consider a $\lambda_i \in \Lambda$ with $0 \leq i \leq \ell$ and similar to [27] define $\lambda_i^* = (\lambda_i + \lambda_{i+1})/2$. The order of items according to the weighting of expected value and variance can only change at values $\lambda_i \in \Lambda$ and the resulting objective vectors are not necessarily unique for values $\lambda_i \in \Lambda$. Choosing the λ_i^* -values in the defined way gives optimal solutions for all $\lambda \in [\lambda_i, \lambda_{i+1}]$ which means that we consider all orders of the items that can lead to optimal solutions when inserting the items greedily according to any fixed weighting of expected weights and variances.

In the following, we analyze the time until an optimal solution with exactly k elements has been produced for

$$f_{\lambda_i^*}(x) = \lambda_i^* \mu(x) + (1 - \lambda_i^*) v(x)$$

for any k , $0 \leq k \leq n$. Note that these λ_i^* values allow to obtain all optimal solutions for the set of functions f_{λ} , $\lambda \in [0, 1]$.

For a given λ_i^* , let the items be ordered such that $f_{\lambda_i^*}(e_1) \leq \dots \leq f_{\lambda_i^*}(e_k) \leq \dots \leq f_{\lambda_i^*}(e_n)$ holds. An optimal solution for k elements and λ_i^* consists of k elements with the smallest $f_{\lambda_i^*}(e_i)$ value. If there is more than one element with the value $f_{\lambda_i^*}(e_k)$ then reordering these elements does not change the objective vector or $f_{\lambda_i^*}$ -value.

Assume that optimal solution with k elements for $f_{\lambda_i^*}$ has already been included in the population. Note that for $k = 0$ the search point 0^n is optimal for any $\lambda \in [0, 1]$ and we assume that this search point has already been included in the population. Picking an optimal solution with k elements for $f_{\lambda_i^*}$ and inserting an element with value $f_{\lambda_i^*}(e_{k+1})$ leads to an optimal solution for $f_{\lambda_i^*}$ with $k + 1$ elements. We call such a step, picking the solution that is optimal for $f_{\lambda_i^*}$ with k elements and inserting an element with value $f_{\lambda_i^*}(e_{k+1})$, a success. Let X_i be the indicator variable for a success in the i th step.

We have

$$\Pr(X_i = 1) \geq \frac{1}{P_{\max} e n}$$

as long as an optimal solution has not been obtained for all values of k , $1 \leq k \leq n$. We consider a phase of $T = 2eP_{\max}n^2$ steps. Let $X = \sum_{i=1}^T X_i$ be the number of successes. The expected number of successes within a phase of T steps where the success in each step is at least $1/(P_{\max}en)$ is at least $2n$. The probability to have less than n success is at most $e^{-n/2}$ using Chernoff bounds (see for example Theorem 1.10.5 in [29]). The number of different values of λ_i^* is upper bounded by the number of pairs of elements and therefore at most n^2 . The probability to have not obtained all optimal solutions for all $f_{\lambda_i^*}$ is therefore at most $n^2 e^{-n/2}$. Hence, all optimal solutions for all $f_{\lambda_i^*}$ are obtained with probability of at least $1 - n^2 e^{-n/2} = 1 - e^{-\Omega(n)}$ within T steps. As each phase of T steps only requires that the search point 0^n is already included in the population and each phase of T steps is successful with probability at least $1 - e^{-\Omega(n)}$, the expected number of phases of length T is at most 2 which completes the proof for the runtime. As mentioned above, as the same Pareto optimal objectives have been obtained as in the proof of Theorem 1, optimal solutions for the problems stated in Equation 1 and 3 have been obtained. \square

6 Experimental Investigations

We now carry out experimental investigations to see when the 3-objective formulation is preferable over the bi-objective formulation given in [22] in practice. As done in [22], we investigate a chance constrained version of the minimum dominating set problem where the cost of each node is chosen independently of the others according to a given Normal distribution. Our goal is to provide complementary insights to the theoretical analysis carried out in the previous sections and show when the 3-dimensional approach whose analysis is based on 1-bit flips is superior to the bi-objective one. Furthermore, we provide insights into the population sizes obtained during the runs of the algorithms. This is a crucial aspect as a larger population size occurred in the 3-objective approach can potentially make the approach less effective.

We consider the following problem. Given graph $G = (V, E)$ with $n = |V|$ nodes and weights on the nodes, the aim is to obtain a set of nodes $D \subseteq V$ of minimal weight such that each node of the graph is dominated by D , i.e. either contained in D or adjacent to a node in D . Here the weight w_i of each node v_i is chosen independently of the others according to a Normal distribution $N(\mu_i, \sigma_i^2)$. Let $c(x)$ be the number of nodes dominated by the given search point x . Note that a solution x is feasible iff $c(x) = n$ holds. For the bi-objective formulation, we work with the constraint $c(x) = n$, i.e. each feasible solution has to be a dominating set. For the 3-objective formulation $c(x)$ counts the number of nodes dominated by x and we pick the best feasible solution, i.e. a solution x with $c(x) = n$, when the algorithm terminates.

As done in [22], we use the medium size graphs cfat200-1, cfat200-2, ca-netscience from the network repository [30] for our experiments. cfat200-1, cfat200-2 have 200 nodes each whereas ca-netscience has 379 nodes. We also investi-

Graph	weight gype	SEMO2D		SEMO3D		GSEMO2D		GSEMO3D	
		Mean	Std	Mean	Std	Mean	Std	Mean	Std
cfat200-1	uniform	57	19	2921	964	56	16	2923	929
cfat200-2	uniform	29	11	348	128	23	10	361	128
ca-netscience	uniform	69	22	5531	997	40	11	4631	678
ca-GrQc	uniform	4	3	7519	488	3	1	3921	262
Erdos992	uniform	2	1	4476	338	1	1	2173	153
cfat200-1	uniform-fixed	1	0	66	12	1	0	67	11
cfat200-2	uniform-fixed	1	0	18	4	1	0	18	4
ca-netscience	uniform-fixed	1	0	401	43	1	0	403	40
ca-GrQc	uniform-fixed	1	0	3565	251	1	0	1942	133
Erdos992	uniform-fixed	1	0	2217	102	1	0	1313	61
cfat200-1	degree	2	1	335	36	2	1	340	35
cfat200-2	degree	1	1	42	6	1	0	42	6
ca-netscience	degree	22	8	3293	764	18	7	2981	585
ca-GrQc	degree	3	2	6112	371	3	2	3240	252
Erdos992	degree	2	1	3128	166	2	1	1725	84

Table 1: Maximum population size for stochastic minimum weight dominating set.

gate larger graphs, namely ca-GrQC and Erdoes992 which have 4158 and 6100 nodes, respectively, in order to show the limitations of the 3-objective approach.

We consider the following categories for choosing the weights as done in [22]. In the *uniform* setting each weight $\mu(u)$ is an integer chosen independently and uniformly at random in $\{n, \dots, 2n\}$. The variance $v(u)$ is an integer chosen independently and uniformly at random in $\{n^2, \dots, 2n^2\}$. In the *degree-based* setting, we have $\mu(u) = (n + \deg(u))^5/n^4$ where $\deg(u)$ is the degree of node u in the given graph. The variance $v(u)$ is an integer chosen independently and uniformly at random in $\{n^2, \dots, 2n^2\}$. Furthermore, we consider the *uniform-fixed* setting where the expected weights are chosen as in the *uniform* setting, but the variances are set to $2n^2$ for each given node. Our goal here is to study how fixed the variance for each node that therefore making it determined by the number of chosen nodes influences the results compared to the uniform setting. As done in [22], we consider for each combination of graph and weight setting values of $\alpha = 1 - \beta$ where $\beta \in \{0.2, 0.1, 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}, 10^{-10}, 10^{-12}, 10^{-14}, 10^{-16}\}$.

The 3-objective formulation is expected to produce much more trade-offs than the bi-objective formulation. We compare SEMO2D and SEMO3D, and GSEMO2D and GSEMO3D, respectively, in terms of the results that they obtain. Comparing SEMO2D and SEMO3D allows to judge whether the additional objective that increases the population size is helpful in practice even if the mutation operation is highly restrictive. Each algorithm is run for each setting 30 times whereas each run consists of 10M iterations. The results for the 30 runs carried out for each algorithm are obtained on a the same set of 30 instances that are generated in the way described above. Note that one run produces results for each considered value of α as we chose from the final population the feasible solution with the smallest weight according to Equation 2. Obviously, finding these solutions can be done in time linear in the size of the final population produced by the considered algorithm.

The results for the maximum population sizes of the considered settings are shown in Table 1. For each setting and algorithm, we show the average maximum population size within the 30 runs and their standard deviations. It can be observed that the population sizes encountered by the approaches using the 3-objective formulations are significantly higher than the maximum population sizes for the bi-objective formulations. Even for relatively small graphs such as cfat200-1, the maximum population sizes for SEMO3D and GSEMO3D are close to 3000, and reach up to 7500 for SEMO3D and ca-GrQc in the uniform setting. Comparing SEMO3D and GSEMO3D, it is interesting to see that the maximum population size encountered by GSEMO3D is in most cases smaller than for SEMO3D. A possible explanations is that the standard bit mutations used in GSEMO3D are often able to create objective vectors that dominate part of the current population, which reduces the population size.

The optimization results for the different graphs and chance constrained settings are shown in Table 2. For each setting, we show the average weight value according to Equation 2 and standard deviation for the algorithms. The best average value in the direct comparison of SEMO2D and SEMO3D, and GSEMO2D and GSEMO3D, respectively, are highlighted in bold. Furthermore, we highlight in grey the best result among all 4 algorithms. We also display the p -value obtained by the Mann-Whitney test for the comparison of SEMO2D and SEMO3D, and GSEMO2D and GSEMO3D, respectively. We call a result statistically significant if the p -value is at most 0.05.

Our results show that for the graphs cfat200-1, cfat200-2, and ca-netscience, there is usually a strong benefit of using the 3-objective formulation instead of the bi-objective one. For the uniform setting where the expected weights and

Graph	weight gype	β	SEMO2D		SEMO3D		p-value	GSEMO2D		GSEMO3D		p-value
			Mean	Std	Mean	Std		Mean	Std	Mean	Std	
cfat200-1	uniform	0.2	3618	76	3599	82	0.308	3615	91	3599	79	0.544
		0.1	3994	82	3970	82	0.268	3989	96	3972	80	0.544
		0.01	4877	101	4842	86	0.169	4866	109	4845	86	0.535
		1.0E-4	6030	123	5985	94	0.128	6015	126	5991	98	0.455
		1.0E-6	6870	139	6824	103	0.201	6855	138	6832	108	0.605
		1.0E-8	7562	152	7519	113	0.326	7546	147	7525	118	0.641
		1.0E-10	8163	163	8122	123	0.469	8145	154	8125	125	0.751
		1.0E-12	8700	172	8660	130	0.535	8680	159	8660	130	0.859
		1.0E-14	9190	180	9150	136	0.657	9169	164	9148	133	0.842
1.0E-16	9633	188	9593	142	0.636	9611	168	9589	137	0.865		
cfat200-2	uniform	0.2	1797	72	1788	53	0.923	1791	49	1767	32	0.049
		0.1	2049	78	2035	55	0.865	2040	54	2016	37	0.074
		0.01	2634	92	2617	69	0.739	2621	72	2593	51	0.162
		1.0E-4	3394	111	3369	85	0.535	3381	97	3336	65	0.070
		1.0E-6	3948	125	3918	95	0.511	3937	113	3880	71	0.044
		1.0E-8	4403	134	4372	106	0.496	4394	124	4329	77	0.032
		1.0E-10	4799	143	4768	117	0.549	4793	132	4720	82	0.028
		1.0E-12	5153	150	5123	125	0.559	5149	139	5071	85	0.024
		1.0E-14	5476	157	5447	133	0.589	5475	145	5391	88	0.020
1.0E-16	5769	164	5740	141	0.559	5769	150	5681	91	0.021		
ca-netscience	uniform	0.2	32922	1308	32608	904	0.506	33042	1289	33007	1023	0.712
		0.1	34456	1323	34115	907	0.544	34568	1302	34514	1028	0.745
		0.01	38097	1361	37694	919	0.408	38189	1334	38089	1040	0.848
		1.0E-4	42938	1414	42461	938	0.274	43012	1380	42846	1054	1.000
		1.0E-6	46527	1457	45995	951	0.255	46591	1415	46377	1065	0.824
		1.0E-8	49500	1493	48923	960	0.198	49557	1442	49303	1076	0.712
		1.0E-10	52091	1526	51478	970	0.165	52145	1465	51857	1087	0.615
		1.0E-12	54416	1554	53773	979	0.147	54467	1487	54150	1096	0.564
		1.0E-14	56542	1581	55873	987	0.132	56592	1507	56249	1105	0.487
1.0E-16	58469	1605	57776	996	0.117	58517	1526	58151	1114	0.478		
cfat200-1	uniform-fixed	0.2	3891	183	3851	129	0.530	3813	125	3721	59	0.006
		0.1	4353	195	4306	135	0.464	4269	133	4169	59	0.006
		0.01	5450	224	5385	149	0.333	5352	151	5235	59	0.006
		1.0E-4	6913	264	6823	169	0.258	6795	177	6655	59	0.006
		1.0E-6	7999	293	7890	183	0.234	7868	196	7710	59	0.006
		1.0E-8	8901	317	8776	195	0.223	8757	212	8586	59	0.006
		1.0E-10	9688	339	9549	206	0.217	9534	226	9350	59	0.006
		1.0E-12	10395	358	10243	216	0.206	10232	239	10036	59	0.006
		1.0E-14	11043	376	10878	225	0.191	10871	250	10665	59	0.006
1.0E-16	11630	392	11455	234	0.186	11450	261	11235	59	0.006		
cfat200-2	random-fixed	0.2	1989	116	1980	112	0.690	1937	104	1866	50	0.011
		0.1	2307	128	2297	123	0.679	2249	115	2171	54	0.011
		0.01	3064	157	3048	149	0.554	2990	141	2897	63	0.011
		1.0E-4	4073	196	4049	185	0.554	3978	176	3864	76	0.011
		1.0E-6	4822	225	4792	213	0.554	4712	203	4583	86	0.011
		1.0E-8	5444	249	5410	236	0.554	5321	225	5179	94	0.011
		1.0E-10	5986	269	5948	257	0.554	5853	244	5700	102	0.011
		1.0E-12	6474	288	6432	275	0.554	6330	261	6168	108	0.011
		1.0E-14	6920	305	6875	292	0.554	6768	277	6596	114	0.011
1.0E-16	7325	321	7276	308	0.554	7164	291	6984	120	0.011		
ca-netscience	uniform-fixed	0.2	35378	1891	32956	844	0.000	34936	1747	32926	816	0.000
		0.1	37239	1934	34718	844	0.000	36779	1785	34687	819	0.000
		0.01	41659	2038	38901	844	0.000	41156	1878	38869	825	0.000
		1.0E-4	47551	2178	44475	844	0.000	46991	2004	44439	831	0.000
		1.0E-6	51927	2282	48614	843	0.000	51325	2098	48576	835	0.000
		1.0E-8	55559	2369	52049	842	0.000	54922	2177	52009	838	0.000
		1.0E-10	58729	2445	55048	842	0.000	58061	2246	55006	841	0.000
		1.0E-12	61577	2513	57741	843	0.000	60881	2309	57698	844	0.000
		1.0E-14	64184	2576	60207	843	0.000	63463	2366	60162	847	0.000
1.0E-16	66548	2633	62443	844	0.000	65805	2418	62397	850	0.000		
cfat200-1	degree	0.2	4495	143	4392	10	0.002	4444	115	4387	6	0.001
		0.1	4835	148	4727	14	0.002	4781	119	4721	9	0.003
		0.01	5642	158	5523	25	0.001	5582	129	5512	16	0.004
		1.0E-4	6718	172	6584	39	0.001	6650	143	6566	26	0.003
		1.0E-6	7517	184	7372	50	0.001	7443	154	7349	34	0.003
		1.0E-8	8180	193	8025	59	0.001	8101	163	7999	40	0.003
		1.0E-10	8758	202	8596	67	0.001	8675	171	8567	45	0.003
		1.0E-12	9278	210	9108	74	0.001	9191	178	9076	50	0.003
		1.0E-14	9754	217	9578	81	0.001	9663	185	9542	55	0.003
1.0E-16	10185	223	10003	87	0.001	10091	191	9965	59	0.003		
cfat200-2	degree	0.2	3218	227	3029	154	0.033	3041	172	2963	4	0.027
		0.1	3448	235	3256	160	0.033	3267	178	3185	6	0.027
		0.01	3996	255	3795	173	0.033	3803	194	3713	11	0.027
		1.0E-4	4726	280	4514	193	0.033	4518	216	4416	17	0.027
		1.0E-6	5268	300	5048	209	0.033	5049	232	4938	22	0.027
		1.0E-8	5718	316	5491	223	0.033	5490	245	5371	26	0.027
		1.0E-10	6110	329	5878	235	0.033	5875	257	5749	30	0.027
		1.0E-12	6463	342	6225	247	0.033	6220	267	6089	33	0.027
		1.0E-14	6786	354	6543	257	0.033	6537	277	6400	36	0.027
1.0E-16	7079	364	6832	267	0.033	6823	286	6682	38	0.027		
ca-netscience	degree	0.2	28587	1535	26148	201	0.000	28164	1002	26169	196	0.000
		0.1	30122	1580	27636	207	0.000	29689	1029	27657	200	0.000
		0.01	33758	1686	31158	228	0.000	33300	1098	31183	216	0.000
		1.0E-4	38593	1828	35840	269	0.000	38103	1192	35874	251	0.000
		1.0E-6	42180	1936	39313	306	0.000	41665	1265	39355	285	0.000
		1.0E-8	45155	2026	42192	338	0.000	44620	1327	42243	317	0.000
		1.0E-10	47751	2104	44705	367	0.000	47198	1381	44763	347	0.000
		1.0E-12	50082	2175	46962	394	0.000	49514	1429	47026	374	0.000
		1.0E-14	52216	2239	49027	420	0.000	51633	1474	49098	400	0.000
1.0E-16	54151	2297	50900	444	0.000	53555	1515	50977	423	0.000		

Table 2: Results for stochastic minimum weight dominating set with different confidence levels of α where $\alpha = 1 - \beta$.

variances are chosen uniformly at random, we can see that either SEMO3D or GSEMO3D obtain the best average weight value for each setting. Interestingly SEMO3D obtains better results than GSEMO3D and SEMO2D better results than GSEMO2D for the ca-netscience graph which might be due to the larger set of trade-offs occurring in the uniform setting, enabling success by carrying out local mutations only. Considering the uniform-fixed setting, we see that GSEMO3D obtains the best results and significantly outperforms GSEMO2D in all settings. Comparing SEMO3D and SEMO2D for the uniform-fixed setting, we observe smaller average weight values for SEMO3D although the results are only statistically significant for the instances of the graph ca-netscience.

The degree-based setting shows clear advantages for the 3-objective setting with SEMO3D outperforming SEMO2D and GSEMO3D outperforming GSEMO2D. All results are statistically significant. Interestingly, GSEMO3D has the smallest average cost values for cfat200-1 and cfat200-2, and SEMO3D the smallest average cost values for ca-netscience. We have already seen before that SEMO3D has smaller average costs than GSEMO3D for ca-netscience in the uniform setting which suggest that using only 1-bit flips instead of standard bit mutations is preferable when working with the given evaluation budget of 10M for this graph of around 400 nodes and variable variances chosen uniformly at random in $\{n^2, \dots, 2n^2\}$.

For the graphs ca-GrQc and Erdos992 which consist of 4158 and 6100 nodes, respectively, the 3-objective models often do not obtain a feasible solution within 10M iterations when starting with a solution uniformly at random. A reason for this is the large number of trade-offs resulting in large population sizes before obtaining a feasible solution for the first time. In contrast to this the bi-objective formulation has a clear advantage here as it always works with a population of size 1 and can only increase its population size when producing feasible solutions. Therefore, we do not display results for these graphs.

7 Conclusions

We explored the use of 3-objective formulation for chance constrained optimization problems using evolutionary multi-objective algorithms. Our three objective formulation takes the expected weight and its variance each as one objective and adds an additional objective which can be the deterministic objective function or deterministic constraint. For the case of independent Normally distributed uncertainties, we have show that this approach computes optimal solution for the case of the objective function counting the number of chosen elements. Furthermore, we have pointed out that single 1-bit flips are the key element for the success of our 3-objective formulation. We showed this through an improved upper bound for the considered problem that showed that the crucial Pareto optimal objective vectors can be obtained through 1-bit flips. Our experimental investigations for the chance constrained dominating set problem show the clear advantage of our 3-objective setting for graphs of moderate size in various stochastic settings.

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