# Hitting Topological Minor Models in Planar Graphs is Fixed Parameter Tractable<sup>1</sup>

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#### Abstract

For a finite collection of graphs  $\mathcal{F}$ , the  $\mathcal{F}$ -TM-DELETION problem has as input an n-vertex graph G and an integer k and asks whether there exists a set  $S \subseteq V(G)$  with  $|S| \leq k$  such that  $G \setminus S$  does not contain any of the graphs in  $\mathcal{F}$  as a topological minor. We prove that for every such  $\mathcal{F}$ ,  $\mathcal{F}$ -TM-DELETION is fixed parameter tractable on planar graphs. Our algorithm runs in a  $2^{\mathcal{O}(k^2)} \cdot n^2$  time or, alternatively in  $2^{\mathcal{O}(k)} \cdot n^4$  time. Our techniques can easily be extended to graphs that are embeddable on any fixed surface.

**Keywords:** Topological minors, irrelevant vertex technique, treewidth, vertex deletion problems

# 1 Introduction

## 1.1 The $\mathcal{P}$ -deletion problem and its variants

In general, a  $\mathcal{P}$ -DELETION problem is determined by some graph class  $\mathcal{P}$  and asks, given an n-vertex graph G and an integer k, whether G can be transformed to a graph in  $\mathcal{P}$  after the deletion of k vertices. In other words, the class  $\mathcal{P}$  represents some desired property that we want to impose on the input graph after deleting k vertices. This is a general graph modification problem with great expressive power as it encompasses many problems, depending on the choice of the property  $\mathcal{P}$ . Unfortunately for most instances of  $\mathcal{P}$ , this problem is not expected to admit a polynomial time algorithm. Lewis and Yannakakis showed in [42] that for any non-trivial and hereditary graph class  $\mathcal{P}$ , the  $\mathcal{P}$ -vertex deletion problem is NP-complete. Given this hardness result, an attractive alternative is to consider the standard parameterized version of the problem, called p- $\mathcal{P}$ -DELETION where the parameter is the number k of vertex deletions. In this case the challenge is to investigate for which instances of  $\mathcal{P}$ , p- $\mathcal{P}$ -DELETION is fixed parameter tractable (or, in short, is FPT), i.e., it can be solved by an  $f(k) \cdot n^{\mathcal{O}(1)}$ -time algorithm (also called an FPT-algorithm), for some function  $f: \mathbb{N} \to \mathbb{N}$ . There is a long line of research on this general question. In many cases, this concerns particular properties and possible optimizations of the p-arametric

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dependence f(k) (see e.g. [10]). However, it is interesting to notice that FPT-algorithms exist for general families of properties. In this direction the more general (and compact) results concern properties  $\mathcal{P}$  that can be characterized by the exclusion of some finite set  $\mathcal{F}$  of graphs of at most h vertices or edges with respect to some partial ordering relation  $\leq$ . We define

$$\mathcal{P}_{\mathcal{F},<} = \{G \mid \forall H \in \mathcal{F} : H \not\leq G\}$$

and ask whether  $p-\mathcal{P}_{\mathcal{F},\leq}$ -DELETION is FPT. Let us now consider the general status of this problem for the main known instances of the partial ordering relation  $\leq$ .

- (1)  $\leq$  is the contraction<sup>1</sup> relation: then there are graphs H such that  $\mathcal{P}_{\{H\},\leq}$ -DELETION is NP-complete even for the case where k=0. For instance one may take H to be the path on 4 vertices, as indicated in [12]. Using the terminology of fixed parameter complexity, this implies that there are choices of  $\mathcal{F}$  such that p- $\mathcal{P}_{\mathcal{F},<}$ -DELETION is para-NP-complete.
- (2)  $\leq$  is the *induced minor*<sup>2</sup> relation: as in the previous case, there are choices of  $\mathcal{F}$  such that  $p-\mathcal{P}_{\mathcal{F},\leq}$ -DELETION is para-NP-complete. For instance, one may consider  $\mathcal{F}$  to contain the graph in [23, Theorem 4.3].
- (3)  $\leq$  is the subgraph or the induced subgraph relation: because of the result of Cai in [15],  $p-\mathcal{P}_{\mathcal{F},\leq}$ -DELETION is FPT, for every  $\mathcal{F}$ . In particular, the result in [15] implies an  $\mathcal{O}(h^k n^{h+1})$ -time algorithm for both these problems. However, if instead we parameterize  $\mathcal{P}_{\mathcal{F},\leq}$ -DELETION by h, then there are instances of  $\mathcal{F}$  for which the problem is W[1]-hard even for k=0: just take  $\mathcal{F}=\{K_h\}$  in order to generate the p-CLIQUE problem.
- (4)  $\leq$  is the minor<sup>3</sup> relation: again p- $\mathcal{P}_{\mathcal{F},\leq}$ -DELETION is FPT, for every  $\mathcal{F}$ . To see this, observe that, for every k, the set of yes-instances of this problem is closed under taking of minors. On the other hand, Robertson and Seymour [47] proved that graphs are well-quasi-ordered with respect to the minor relation. These two facts together imply that there is a finite set  $\mathcal{B}_k$  (whose size depends on k and k) such that k0 is a yes-instance if and only if k2 contains no graph in k3 a minor. As minor-checking for a graph on k2 vertices can be done in k3 time [46], we derive the (non-constructive) existence of an k4 made constructive in [2]. Recently, a k5 time algorithm for k6 notation). This result was made constructive in [2]. Recently, a k6 time algorithm for k7 time algorithm for k8 notation.

### 1.2 Our contribution.

A graph H is a topological minor of a graph G if G contains as a subgraph some subdivision of H and we denote this by  $H \leq G$ . We consider the problem  $p \cdot \mathcal{P}_{\mathcal{F}, \leq}$ -DELETION that in the rest of this paper we call  $\mathcal{F}$ -TM-DELETION. Notice that this problem is more general than its counterpart for the minor relation (case (4) above) as, for every graph H, there exists a finite set of graphs  $\mathcal{H}$  such that a graph G contains H as a minor if and only if G contains some graph in  $\mathcal{H}$  as a topological minor. However as graphs are not well-quasi-ordered with respect to the topological minor relation, the parameterized complexity of  $\mathcal{F}$ -TM-DELETION remained open for a while.

<sup>&</sup>lt;sup>1</sup>A graph G is a contraction of a graph G' if G can be obtained from G by applying edge contractions.

<sup>&</sup>lt;sup>2</sup>A graph G is an *induced minor* of a graph G' if G can be obtained from some contraction of G' after removing vertices.

<sup>&</sup>lt;sup>3</sup>A graph G is an *minor* of a graph G' is G is the contraction of some subgraph of G'.

<sup>&</sup>lt;sup>4</sup>A graph G is a *subdivision* of a graph G' if G can be obtained from G' if we replace its edges by paths with the same endpoints.

In this paper we prove that  $\mathcal{F}$ -TM-DELETION is FPT for inputs restricted to planar graphs. Moreover, we develop results and techniques that may serve as the base for further FPT-algorithms for p- $\mathcal{P}_{\mathcal{F},\leq}$ -DELETION on planar graphs, when  $\leq$  is the induced minor or the contraction relation (see Subsection 6.4 for a discussion). We stress that, until very recently, the parameterized complexity of this problem was unknown. For an update on the current status of the general problem see Subsection 6.3.

Let  $\mathcal{F}$  be a finite set of graphs. We use  $h(\mathcal{F})$  for the maximum number of vertices or edges of a graph in  $\mathcal{F}$ , i.e.,  $h(\mathcal{F}) = \max\{|V(H)|, |E(H)| \mid H \in \mathcal{F}\}$ . We also write  $\mathcal{F} \npreceq G$  to denote the fact that none of the graphs in  $\mathcal{F}$  is a topological minor of G. We define the parameter  $\mathbf{tm}_{\mathcal{F}}$  so that, for every graph G,

$$\mathbf{tm}_{\mathcal{F}}(G) = \min\{k \mid \exists S \subseteq V(G) : |S| \le k \land \mathcal{F} \npreceq G \setminus S\}.$$

The main result of this paper is the following:

**Theorem 1.1.** There exists an algorithm that given a finite set of graphs  $\mathcal{F}$ , a  $k \in \mathbb{N}$ , and an n-vertex planar graph G, outputs whether  $\mathbf{tm}_{\mathcal{F}}(G) \leq k$  in  $2^{\mathcal{O}_h(k^2)} \cdot n^2$  time, or, alternatively,  $\mathcal{O}(k \cdot n^4) + \mathcal{O}_h(n^4) + 2^{\mathcal{O}_h(k)} \cdot n^2$  time, where  $h = h(\mathcal{F})$ .

We stress that the algorithm of Theorem 1.1 can be straightforwardly modified so as to output a set S of size at most k that intersects all models of the graphs in  $\mathcal{F}$ . A version of Theorem 1.1 without the explicit parametric dependences on the running times appeared in [30].

#### 1.3 High level description of our algorithm

Our main approach towards proving Theorem 1.1 is the application of the so-called *irrelevant* vertex technique. This technique was introduced for the first time by Roberston and Seymour in [46] for the design of an FPT-algorithm for the DISJOINT PATHS problem, parameterized by the number of terminals. Subsequently, it was applied, in diverse ways, for the design of FPT-algorithms for several graph-theoretical problems and is now considered as a powerful technique of parameterized algorithm design [3, 18, 29, 32, 33, 35, 37–40, 43, 44]. We also refer to [17, Chapter 7] for a high-level overview of the irrelevant vertex technique. The general algorithmic paradigm of the irrelevant vertex technique takes advantage of some structural characteristic of the input graph in order to detect, in FPT-time, some vertex, called *irrelevant*, whose removal from G generates an equivalent instance of the problem. By recursing on the produced equivalent instance we end up with a graph where the structural parameter is bounded (by some function of k). This fact permits the resolution of the problem with other techniques - typically by dynamic programming. Most of the times, this structural parameter is treewidth (see Section 2 for the formal definition) and this is the one that we use in this paper. Towards proving Theorem 1.1, the application of the irrelevant vertex technique is based on Theorem 1.2 that we present below.

Let G be a graph, R be a subset of V(G), and k be a non-negative integer. We say that (G, R, k) is a  $\mathbf{tm}_{\mathcal{F}}$ -triple if there exists an  $S \subseteq R$  such that  $|S| \leq k$  and  $\mathcal{F} \npreceq G \setminus S$ . Intuitively, the set R can be seen as the set of vertices that are possible candidates for a solution S. Aiming to remove (irrelevant) vertices from the given graph, we also make progress by reducing R. This is formulated in the next result.

**Theorem 1.2.** There exists a function  $f_1 : \mathbb{N} \to \mathbb{N}$ , and an algorithm with the following specifications:

 $Find\_Irrelevant\_Vertex(k, h, G, R)$ 

Input:  $k, h \in \mathbb{N}$ , an n-vertex planar graph G, and a set  $R \subseteq V(G)$ . Output:

- 1. an (irrelevant) vertex  $v \in V(G)$  and a set  $R' \subseteq R$  such that, for every graph class  $\mathcal{F}$  where  $h(\mathcal{F}) \leq h$ , (G, R, k) is a  $\mathbf{tm}_{\mathcal{F}}$ -triple if and only if  $(G \setminus v, R', k)$  is a  $\mathbf{tm}_{\mathcal{F}}$ -triple or
- 2. a tree decomposition of G of width at most  $f_1(h) \cdot k$ .

Moreover, this algorithm runs in  $2^{\mathcal{O}_h(k^2)} \cdot n$  time, or, alternatively,  $\mathcal{O}(k \cdot n^3) + \mathcal{O}_h(n^3) + 2^{\mathcal{O}_h(k)} \cdot n$  time.

After applying the algorithm of Theorem 1.2 at most n times, the problem is reduced to instances of bounded treewidth. As topological minor containment can be defined by a formula in Monadic Second Order Logic, i.e., an MSOL formula, [41, Appendix D] and vertex deletion to some MSOL definable property is also MSOL definable, it follows from Courcelle's Theorem [16] (see also [4, 11, 49]) that the problem for reduced instances can be solved in  $\mathcal{O}_{k,h}(n)$  time. To solve the version of the problem where a certificate of the solution is asked for, one can use the version of Courcelle's Theorem [16] that returns such a certificate, if it exists. However, to achieve the parametric dependencies in the running times of Theorem 1.1, we have to avoid the use of Courcelle's Theorem when solving the problem on instances of bounded treewidth. We devote Section 3 to describe how to develop a dynamic programming algorithm (Lemma 3.4) that can solve the problem on instances of treewidth at most w in  $2^{\mathcal{O}_h(w \log w)} \cdot n$  time, or, alternatively, in  $\mathcal{O}(n^3) + 2^{\mathcal{O}_h(w)} \cdot n$  time. Theorem 1.1 follows. We stress that each one of these running times has some advantage against the other. In the first case, we have a linear, in n, algorithm whose parametric dependence on k is super-exponential. In the second, we drop the parametric dependency to a single exponential one to the cost of a worst polynomial dependency on n.

In the rest of this section we give an outline on how Theorem 1.2 is proved. All combinatorial concepts used in this description are presented in an intuitive way; formal definitions can be found in Section 2. Given a tuple of variables  $\mathbf{x} = (x_1, \dots, x_q)$  by the term  $\mathbf{x}$ -big/small we refer to a quantity that is lower/upper bounded by some (unbounded) function of  $\mathbf{x}$ . Alternatively, we use the term  $\mathbf{x}$ -many/few that is defined analogously. We work on some embedding of G in the plane.

Walls and annuli. An important combinatorial object is the one of a r-wall, as the one in Figure 1, that can be seen as the union of r horizontal paths intersected by r vertical paths. The layers of a wall W are defined as indicated in Figure 1.

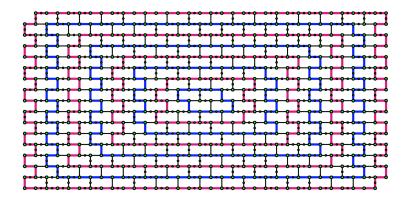


Figure 1: A 17-wall and its 8 layers.

We call the outermost layer *perimeter* of the wall W. Combining the results of [1, 8, 29, 34] we know that if the treewidth of a planar graph is (k, h)-big, then we can find a (k, h)-big wall

in G such that the subgraph of G, called the compass of W, inside the closed disk "cropped" by the perimeter of W has (k, h)-small treewidth (see Proposition 2.1). This additional property will permit us to compute (possible) partial solutions on subgraphs of the compass of W.

The next step is to detect some more structure in the wall W that is intuitively depicted in the left side of Figure 2. We first distinguish the collection C of the (k, h)-many outermost layers, drawn in yellow, and then we consider in the rest of W a packing of (k, h)-many (h)-big walls, drawn in green. This is done in Lemma 2.1.

We now work on the "annulus" of the (k,h)-many outer layers of W. For this, it is convenient to see those cycles as "crossed" by a collection  $\mathcal{P}$  of disjoint paths (that are monotone subpaths of the horizontal/vertical paths of W) called *rails*. We call this system of cycles and rails *railed annulus*, denoted by  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ . See the right side of Figure 2 for an example of a railed annulus with 5 cycles and 8 rails.

Combing topological minor models. Notice that if H is a topological minor of a graph G, then this is materialized by a pair (M,T) where M is a subgraph of G and T is a set of vertices of M, called branches, such that all vertices of  $V(M) \setminus T$  have degree 2. We say that (M,T) is a topological minor model of H in G if a graph isomorphic to H is created after dissolving in M all vertices in  $V(M) \setminus T$  (which means deleting every such vertex and making its two neighbors adjacent). For simplicity, assume that  $\mathcal{F} = \{H\}$  and recall that  $\mathbf{tm}_{\mathcal{F}}(G) \leq k$  if there is a set  $S \subseteq V(G), |S| \leq k$ , called from now on solution set, that intersects all topological minor models of H in G.

Our next aim is to analyze how topological minor models of H may cross the cycles and the rails of a railed annulus  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ . For this reason, using [31, Corollary 1] (see also [30] for a conference version), we prove that if the branches of (M,T) are situated *outside* the annulus and the annulus is (h)-big then it is possible to find an alternative "rail-combed" model (M',T') of G, whose intersection with the "middle cycle" of  $\mathcal{A}$  consists only of (h)-few rail vertices. We refer to this theorem as the "model combing theorem" (Theorem 2.1).

Representations of topological minor models. Using the model combing theorem, we can pick an (h)-small collection  $\mathcal{P}'$  of the rails of  $\mathcal{A}$  for which the following holds: for every topological minor model (M,T) of H that crosses  $\mathcal{A}$ , there is a disk  $\Delta$  bounded by some cycle C of  $\mathcal{A}$  and a "combed" (through  $\mathcal{P}'$ ) version (M',T') of (M,T) that represents (M,T) in the sense that a set of vertices that are "not so close" to C, intersects  $M \cap \Delta$  if and only if the same set intersects  $M' \cap \Delta$ . From now on we refer to the instances of  $M' \cap \Delta$  as the inner combed models of  $\mathcal{A}$  and we can see them as models representing the "inner part" of all annulus-crossing models.

Reducing the solution space. The next step is to compute, for every cycle C of A, a set  $S_C$  of at most (k, h)-many vertices intersecting each possible inner combed model of A (it is possible that  $S_C$  is an empty set). This computation can be done by the dynamic programming algorithm of Lemma 3.4 that can find (partial) solutions of the problem on subgraphs of the compass of W that have (k, h)-small treewidth. Let  $\Delta_{\text{in}}$  be the disk bounded by the innermost cycle of C (cycle  $C_5$  in Figure 2). We then compute  $S_{\text{in}} = \Delta_{\text{in}} \cap (\bigcup_{C \in C} S_C)$  and observe that  $S_{\text{in}}$  has (k, h)-small size. Based on the fact that the inner combed models represent the inner part of all models crossing A and the fact that all these models are intersected by subsets of at most k vertices whose restriction in  $\Delta_{\text{in}}$  is in  $S_{\text{in}}$ , we prove that if  $G \setminus S$  does not contain any topological minor model of H, then we can replace  $S \cap \Delta_{\text{in}}$  by vertices of  $S_{\text{in}}$  to obtain a new solution that is not larger than S (Lemma 4.1). This is an important restriction of the solution space of the problem in what concerns its intersection with  $\Delta_{\text{in}}$ . As there are (k, h)-many (h)-big subwalls

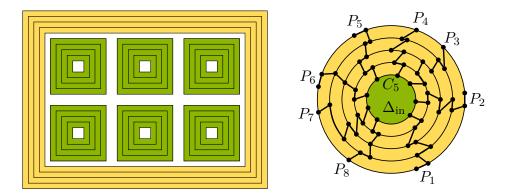


Figure 2: Left: The partition of a wall into a yellow annulus and several green subwalls. Right: An example of a (5,8)-railed annulus depicted in yellow; its innermost cycle is  $C_5$  and the disk  $\Delta_{\text{in}}$  bounded by  $C_5$  is depicted in green.

packed inside  $\Delta_{\text{in}}$ , there is a subwall whose compass can be avoided by all possible solution sets. The compass of such a wall is called *solution-free*. In the above, H might be any graph on h vertices, however its is more convenient to think about some specific (planar) graph H in  $\mathcal{F}$ .

Finding an irrelevant vertex. We now fix our attention to the solution-free compass of some (h)-big subwall of W. Once again, we see this wall as a railed annulus  $\mathcal{A}'$  and use the model combing theorem in order to represent all ways topological minor models of H can "invade" the compass of W by combed topological models going through the rails of  $\mathcal{A}'$ . This, in turn, permits us to detect a vertex v of the solution-free compass of W such that if a solution set S intersects a topological minor model that contains v, then it should also intersect some representation of it that avoids v, therefore v is irrelevant (Lemma 4.2).

### 1.4 Organization of the paper

In Section 2, we give some definitions and preliminaries. In Section 3 we present a way to design a dynamic programming algorithm that solves the problem in bounded treewidth graphs. In Section 4 we present the two main subroutines of the algorithm of Theorem 1.2 and in Section 5 we prove Theorem 1.2. We conclude in Section 6 by discussing the running time dependancy on h of our algorithm, the extension of our results to bounded Euler genus graphs, some recent advances on the study of the problem on general graphs, and some open problems.

# 2 Definitions and preliminaries

We denote by  $\mathbb{N}$  the set of all non-negative integers. Given an  $n \in \mathbb{N}$ , we denote by  $\mathbb{N}_{\geq n}$  the set containing all integers equal or greater than n. Given two integers x and y we define by  $[x,y]=\{x,x+1,\ldots,y-1,y\}$ . Given an  $n\in\mathbb{N}_{\geq 1}$ , we also define  $[n]=\{1,\ldots,n\}$ . Let U be a set,  $r\in\mathbb{N}_{\geq 1}$ , and  $\mathcal{A}=[A_1,\ldots,A_r]\subseteq (2^U)^r$ ,  $\mathcal{B}=[B_1,\ldots,B_r]\subseteq (2^U)^r$ . We say that  $\mathcal{A}\subseteq\mathcal{B}$  if for all  $i\in[r]$ ,  $A_i\subseteq B_i$ . Also, if  $S\subseteq U$  we denote  $\mathcal{A}\cap S=[A_1\cap S,\ldots,A_r\cap S]$ .

Let  $(x_1, \ldots, x_l) \in \mathbb{N}^l$  and  $\chi, \psi : \mathbb{N} \to \mathbb{N}$ . We use the notation  $\chi(n) = \mathcal{O}_{x_1, \ldots, x_l}(\psi(n))$  to denote that there exists a computable function  $f : \mathbb{N}^l \to \mathbb{N}$  such that  $\chi(n) = \mathcal{O}(f(x_1, \ldots, x_l) \cdot \psi(n))$ .

#### 2.1 Basic concepts on graphs

All graphs in this paper are undirected, finite, and they do not have loops or multiple edges. Unless stated otherwise, we denote by n the number of vertices of the graph under consideration. If  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  are graphs, then we denote  $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$  and  $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ . Also, given a graph G and a set  $S \subseteq V(G)$ , we denote by  $G \setminus S$  the graph obtained if we remove from G the vertices in S, along with their incident edges. Given a vertex  $v \in V(G)$ , we denote by  $N_G(v)$  the set of vertices of G that are adjacent to G in G. Also, given a set G is a set G in G that have a neighbor in G in G is a graph G, we say that the pair G is a separation of G if G in a graph G is a connected subgraph with all vertices of degree at most (exactly) 2. A path is trivial if it has only one vertex and is empty if it is the empty graph (i.e., the graph with empty vertex set).

Partially disk-embedded graphs. A closed disk (resp. open disk)  $\Delta$  is a set homeomorphic to the set  $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$  (resp.  $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ ). We use  $\mathbf{bor}(\Delta)$  to denote the boundary of  $\Delta$  and  $\mathsf{int}(\Delta)$  to denote the open disk  $\Delta \setminus \mathsf{bor}(\Delta)$ . When we embed a graph G in the plane or in a disk, we treat G as a set of points. This permits us to make set operations operations between graphs and sets of points. We say that a graph G is partially disk-embedded in some closed disk  $\Delta$ , if there is some subgraph K of G that is embedded in  $\Delta$  such that  $\mathsf{bor}(\Delta)$  is a cycle of K and  $(V(G) \cap \Delta, V(G) \setminus \mathsf{int}(\Delta))$  is a separation of G. From now on, we use the term partially  $\Delta$ -embedded graph G to denote that a graph G is partially G-embedded graph G and we always assume that we accompany a partially G-embedded graph G with the embedding of its compass in G, that is the set  $G \cap G$ .

Grids and walls. Let  $k, r \in \mathbb{N}$ . The  $(k \times r)$ -grid is the Cartesian product of two paths on k and r vertices, respectively. An elementary r-wall, for some odd  $r \geq 3$ , is the graph obtained from a  $(2r \times r)$ -grid with vertices  $(x,y), x \in [2r] \times [r]$ , after the removal of the "vertical" edges  $\{(x,y),(x,y+1)\}$  for odd x+y, and then the removal of all vertices of degree one. Notice that, as  $r \geq 3$ , an elementary r-wall is a planar graph that has a unique (up to topological isomorphism) embedding in the plane such that all its finite faces are incident to exactly six edges. The perimeter of an elementary r-wall is the cycle bounding its infinite face. Given an elementary wall  $\overline{W}$ , a vertical path of  $\overline{W}$  is one whose vertices, in order of appearance, are  $(i,1), (i,2), (i+1,2), (i+1,3), (i,3), (i,4), (i+1,4), (i+1,5), (i,5), \ldots, (i,r-2), (i,r-1), (i+1,r-1), (i+1,r),$  for some  $i \in \{1,3,\ldots,2r-1\}$ . Also, a horizontal path of  $\overline{W}$  is one whose vertices, in order of appearance, are  $(1,j), (2,j), \ldots, (2r,j)$ , for some  $j \in [2,r-1]$ , or  $(1,1), (2,1), \ldots, (2r-1,1)$  or  $(2,r), (3,r), \ldots, (2r,r)$ .

An r-wall is any graph W obtained from an elementary r-wall  $\overline{W}$  after subdividing edges (see Figure 1). We call the vertices that were added after the subdivision operations subdivision vertices. The perimeter of W, denoted by perim(W), is the cycle of W whose non-subdivision vertices are the vertices of the perimeter of  $\overline{W}$ . Also, a vertical (resp. horizontal) path of W is a subdivided vertical (resp. horizontal) path of  $\overline{W}$ . An r'-subwall W' of a wall W is any r'-wall that is a subgraph of W and whose horizontal/vertical paths are subpaths of the horizontal/vertical paths of W.

A subgraph W of a graph G is called a wall of G if W is an r-wall for some odd  $r \geq 3$  and we refer to r as the height of the wall W. Let W be a wall of a graph G and K' be the connected component of  $G \setminus V(\mathsf{perim}(W))$  that contains  $W \setminus V(\mathsf{perim}(W))$ . The compass of W,

denoted by  $\mathsf{compass}(W)$ , is the graph  $G[V(K') \cup V(\mathsf{perim}(W))]$ . Observe that W is a subgraph of  $\mathsf{compass}(W)$  and  $\mathsf{compass}(W)$  is connected.

The *layers* of an r-wall W are recursively defined as follows. The first layer of W is its perimeter. For  $i=2,\ldots,(r-1)/2$ , the i-th layer of W is the (i-1)-th layer of the subwall W' obtained from W after removing from W its perimeter and all occurring vertices of degree one. Notice that each (2r+1)-wall has r layers (see Figure 1).

**Treewidth.** A tree decomposition of a graph G is a pair  $(T,\chi)$  where T is a tree and  $\chi:V(T)\to 2^{V(G)}$  such that

- 1.  $\bigcup_{t \in V(T)} \chi(t) = V(G);$
- 2. for every edge e of G there is a  $t \in V(T)$  such that  $\chi(t)$  contains both endpoints of e and
- 3. for every  $v \in V(G)$ , the subgraph of T induced by  $\{t \in V(T) \mid v \in \chi(t)\}$  is connected.

The width of  $(T, \chi)$  is defined as  $\mathbf{w}(T, \chi) := \max \{ |\chi(t)| - 1 | t \in V(T) \}$ . The treewidth of G is defined as  $\mathbf{tw}(G) := \min \{ \mathbf{w}(T, \chi) | (T, \chi) \text{ is a tree decomposition of } G \}$ .

The following result follows combining the results of [1, 8, 29, 34]. It intuitively states that given a  $q \in \mathbb{N}$  and a planar graph G with "big" enough treewidth, we can find a q-wall of G whose compass has "small" enough treewidth.

**Proposition 2.1.** There exists a constant  $c_1$  and an algorithm with the following specifications: Find Wall(G, q)

Input: a planar graph G and a  $q \in \mathbb{N}_{\geq 3}$ . Output:

- 1. Either a q-wall W of G whose compass has treewidth at most  $c_1 \cdot q$  or
- 2. a tree decomposition of G of width at most  $c_1 \cdot q$ .

Moreover, this algorithm runs in  $2^{\mathcal{O}(q^2)} \cdot n$  time, or, alternatively, in  $2^{\mathcal{O}(q)} \cdot n^2$  time.

The above algorithm uses first the single exponential FPT-approximation of treewidth by [8] and as long as the treewidth is not small enough then it finds a q-wall W by either using the algorithm of [1], that runs in  $2^{\mathcal{O}(q^2)} \cdot n$  time, or the algorithm of [34] that runs in  $\mathcal{O}(n^2)$  time. The treewidth of the compass of W is bounded by applying the main idea of [29, Lemma 4.2]. We present a proof for completeness.

Proof of Proposition 2.1. The following algorithm is a slight modification of the algorithm Compass in [29, Subsection 4.2]. The version presented here uses [8, Theorem VI] and the algorithms of [1,34] to obtain the claimed running times.

We set  $c_1 := 94$ . We start by applying the single-exponential 5-approximation algorithm of Bodlaender et al. for treewidth [8, Theorem VI], which outputs either a report that the treewidth of G is larger than 18q + 1 or a tree decomposition of G of width at most  $5 \cdot (18q + 1) + 4$ . Observe that  $5 \cdot (18q + 1) + 4 \le 94q$ , for  $q \ge 3$ . In the latter case, we return the obtained tree decomposition of G. In the former case, i.e., where the treewidth of G is larger than 18q + 1, we know [29, Lemma 2.1] that G contains a 2q-wall as a minor. Such a q-wall W can be found using either the minor-checking algorithm of [1] that runs in  $2^{\mathcal{O}(q^2)} \cdot n$  time, or the algorithm of [34] that runs in  $\mathcal{O}(n^2)$  time. Next, among the four vertex-disjoint q-subwalls of W, we obtain the one, say W', whose compass has the minimum number of vertices. We recursively apply the algorithm of [8, Theorem VI] with input the compass of W' and the integer 18q + 1. For more details, we refer the reader to [29, Subsection 4.2].

#### 2.2 Railed annuli

In this subsection we present the notion of railed annulus, introduced in [36], a "wall-like" graph as in the right side of Figure 2, that is the union of a collection of cycles and a collection of paths "crossing" these cycles. In order to define railed annuli, we first give the definitions of nested sequences of cycles and annuli.

Nested cycles and annuli. Let G be a partially  $\Delta$ -embedded graph and let  $\mathcal{C} = [C_1, \ldots, C_r]$ ,  $r \geq 2$ , be a collection of vertex-disjoint cycles of the compass of G. We say that the sequence  $\mathcal{C}$  is a  $\Delta$ -nested sequence of cycles of G if every  $C_i$  is the boundary of an open disk  $D_i$  such that  $\Delta \supseteq D_1 \supseteq \cdots \supseteq D_r$ . From now on, each  $\Delta$ -nested sequence  $\mathcal{C}$  will be accompanied with the sequence  $[D_1, \ldots, D_r]$  of the corresponding open disks as well as the sequence  $[\overline{D}_1, \ldots, \overline{D}_r]$  of their closures. Given  $x, y \in [r]$  where  $x \leq y$ , we call the set  $\overline{D}_x \setminus D_y$  the (x, y)-annulus of  $\mathcal{C}$  and we denote it by  $\operatorname{ann}(\mathcal{C}, x, y)$ . Finally, we say that  $\operatorname{ann}(\mathcal{C}, 1, r)$  is the annulus of  $\mathcal{C}$  and we denote it by  $\operatorname{ann}(\mathcal{C})$ .

**Railed annuli.** Let  $r \in \mathbb{N}_{\geq 3}$  and  $q \in \mathbb{N}_{\geq 3}$ . Assume also that r is an odd number. An (r,q)-railed annulus of a partially  $\Delta$ -embedded graph G is a pair  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$  where  $\mathcal{C} = [C_1, \ldots, C_r]$  is a  $\Delta$ -nested collection of cycles of G and  $\mathcal{P} = [P_1, \ldots, P_q]$  is a collection of pairwise vertex-disjoint paths in G such that

- For every  $j \in [q], P_j \subseteq \operatorname{ann}(\mathcal{C})$ .
- For every  $(i,j) \in [r] \times [q]$ ,  $C_i \cap P_j$  is a non-empty path, that we denote  $P_{i,j}$ .

We refer to the paths of  $\mathcal{P}$  as the rails of  $\mathcal{A}$  and to the cycles of  $\mathcal{C}$  as the cycles of  $\mathcal{A}$ .

Let  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$  be an (r, q)-railed annulus of a partially  $\Delta$ -embedded graph G. We call  $\overline{D}_r$  (resp.  $\overline{D}_1$ ) the *inner (resp. outer) disk* of  $\mathcal{A}$ . We also extend the notion of an annulus and we say that the *annulus* of  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$  is the annulus of  $\mathcal{C}$ .

We now prove the following lemma which intuitively states that there is an algorithm that given a "big enough" wall, outputs a collection of railed annuli whose number and size will be useful in the proof of Theorem 1.2.

**Lemma 2.1.** There exists a function  $f_2 : \mathbb{N}^3 \to \mathbb{N}$  and an algorithm with the following specifications:

## Find\_Collection\_of\_Annuli $(x, y, z, \Delta, G, W)$

Input: two odd integers  $x, y \in \mathbb{N}_{\geq 3}$ , an integer  $z \in \mathbb{N}$ , a partially  $\Delta$ -embedded graph G and a q-wall W of the compass of G whose perimeter is the boundary of  $\Delta$  and such that  $q \geq f_2(x, y, z)$ . Output: a closed disk  $\Delta' \subseteq \Delta$  and a collection  $\mathfrak{A} = \{A_0, A_1, \ldots, A_z\}$  of railed annuli of the compass of G such that

- $A_0$  is an (x,x)-railed annulus whose outer disk is  $\Delta$  and whose inner disk is  $\Delta'$ ,
- for  $i \in [z]$ ,  $A_i$  is a (y,y)-railed annulus of  $G \cap \text{int}(\Delta')$ , and
- for every  $i, j \in [z]$  where  $i \neq j$ , the outer disk of  $A_i$  and the outer disk of  $A_j$  are disjoint.

Moreover, this algorithm runs in  $\mathcal{O}(n)$  time and  $f_2(x,y,z) = \mathcal{O}(x+y\sqrt{z})$ .

*Proof.* Let  $y' := 2y + \lceil y/4 \rceil$  and assume that y' is an odd integer (otherwise, make it odd by adding 1) and let  $f_2(x, y, z) = 2x + \max\{\lceil x/4 \rceil, \lceil \sqrt{z}/2 \rceil \cdot y'\} + 1$ . We argue that the following holds:

Claim: Let  $p \in \mathbb{Z}_{\geq 3}$  be an odd integer. If H is an h-wall of G, where h is an odd integer such that  $h \geq 2p + \lceil p/4 \rceil$ , then H contains a (p,p)-railed annulus  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ , where  $\mathcal{C} = [C_1, \ldots, C_p]$  and for every  $i \in [p]$ ,  $C_i$  is the i-th layer of H.

Proof of Claim: Let H be an h-wall of G, where  $h \geq 2p + \lceil p/4 \rceil$ . We define the  $\Delta$ -nested collection  $\mathcal{C} = [C_1, \ldots, C_p]$  of cycles of G, where, for every  $i \in [p]$ ,  $C_i$  is the i-th layer of H. Let  $\hat{\mathcal{P}}$  be the collection of the vertical and horizontal paths of H that contain branch vertices of W that are not in  $\bigcup_{i \in [p]} V(C_i)$ . Observe that, for every  $i \in [p]$ , every path in  $\hat{\mathcal{P}}$  also intersects  $C_i$  and that  $\hat{\mathcal{P}} \cap \operatorname{ann}(\mathcal{C})$  is a collection of pairwise-vertex disjoint paths of G. Also, notice that since  $h - 2p \geq \lceil p/4 \rceil$ ,  $\hat{\mathcal{P}} \cap \operatorname{ann}(\mathcal{C})$  contains at least p paths. Let  $\mathcal{P} := [P_1, \ldots, P_p]$  be a subset of  $\hat{\mathcal{P}} \cap \operatorname{ann}(\mathcal{C})$ . Then,  $\mathcal{P}$  is a collection of pairwise vertex-disjoint paths of G and it holds that for every  $j \in [p]$ ,  $P_j \subseteq \operatorname{ann}(\mathcal{C})$  and for every  $(i,j) \in [p] \times [p]$ ,  $C_i \cap P_j$  is a non-empty path. Therefore, H contains a (p,p)-railed annulus  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$  of G and the claim follows.

Following the claim above, for H:=W, h:=q, and p:=x, since  $q\geq 2x+\lceil x/4\rceil$ , we deduce the existence of an (x,x)-railed annulus  $A_0$  whose inner disk is  $\overline{D}_x$  and whose outer disk is  $\overline{D}_1$  - that is  $\Delta$ . Observe that since  $q-2x\geq \lceil \sqrt{z}/2\rceil\cdot y'+1$ , there exists an r-wall  $\hat{W}$  of G for some odd  $r\in \mathbb{Z}_{>3}$  such that  $r\geq \lceil \sqrt{z}/2\rceil\cdot y'$  and  $\hat{W}\subseteq G\cap D_x$ .

Now, notice that  $\hat{W}$  contains a collection  $\mathcal{W} = \{W'_1, \dots, W'_z\}$  of z y'-subwalls of W such that, for every  $i, j \in [z], i \neq j$ ,  $\mathsf{compass}(W'_i) \cap \mathsf{compass}(W'_j) = \emptyset$ . Therefore, for every  $i \in [z]$ , applying again the claim above for  $H := W'_i, \ h := y'$  and p := y, we deduce the existence of a (y, y)-railed annulus  $\mathcal{A}_i$  of  $W'_i$ . Furthermore, for every  $i, j \in [z], i \neq j$ , the fact that  $\mathsf{compass}(W'_i) \cap \mathsf{compass}(W'_j) = \emptyset$  implies that the outer disk of  $\mathcal{A}_i$  and the outer disk of  $\mathcal{A}_j$  are disjoint. The proof concludes by setting  $\Delta' = \overline{D}_x$  and  $\mathfrak{A} = \{\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_z\}$ .

## 2.3 Rerouting linkages inside railed annuli

In the rest of this section we show how to reroute topological minor models inside railed annuli. For this reason, in Subsection 2.3, we define the notion of a linkage, which we study as a subgraph of a partially disk-embedded graph. It has been proved [31, Corollary 1] that if a linkage L of a partially disk-embedded graph invades a sufficiently large railed annulus inside the disk, then there is an equivalent linkage that is "combed" through the rails of the annulus. In Subsection 2.4, we extend this result (Proposition 2.2) to topological minor models, by treating the paths of the model as paths of the linkage, and we conclude with the "model combing theorem" (Theorem 2.1) that allows us to reroute topological minor models in order to "comb" them through the rails of a sufficiently large railed annulus.

Before stating Proposition 2.2 we need some definitions.

**Linkages.** A linkage in a graph G is a subgraph L of G whose connected components are all non-trivial paths. The paths of a linkage are its connected components and we denote them by  $\mathcal{P}(L)$ . The size of L is the number of paths and is denoted by |L|. The terminals of a linkage L, denoted by T(L), are the endpoints of the paths in  $\mathcal{P}(L)$ , and the pattern of L is the set  $\{\{s,t\} \mid \mathcal{P}(L) \text{ contains some } (s,t)\text{-path}\}$ . Two linkages  $L_1, L_2$  of G are equivalent if they have the same pattern and we denote this fact by  $L_1 \equiv L_2$ .

**Linkages in railed annuli.** Let G be a partially  $\Delta$ -embedded graph, let  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$  be a (r,q)-railed annulus of G and L be a linkage of G. Given a set  $D \subseteq \Delta$ , then we say that L is D-avoiding if  $T(L) \cap D = \emptyset$ . We also say that L is A-avoiding if it is  $\operatorname{ann}(\mathcal{C})$ -avoiding (see Figure 3).

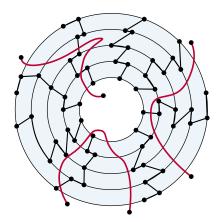


Figure 3: An example of a railed annulus  $\mathcal{A}$  and a linkage L (depicted in red) that is  $\mathcal{A}$ -avoiding.

Let r = 2t + 1. Let also  $s \in [r]$  where s = 2t' + 1. Given some  $I \subseteq [q]$ , we say that a linkage L is (s, I)-confined in A if

$$L\cap \mathsf{ann}(\mathcal{C},t+1-t',t+1+t')\subseteq \bigcup_{i\in I}P_i.$$

We are now ready to state the following result from [30], whose proof can be found in [31].

**Proposition 2.2** ([30,31]). There exist two functions  $f_3, f_4 : \mathbb{N} \to \mathbb{N}$ , where the images of  $f_4$  are even, such that for every odd  $s \in \mathbb{N}_{\geq 1}$  and every  $\ell \in \mathbb{N}$ , if G is a partially  $\Delta$ -embedded graph,  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$  is a (r, q)-railed annulus of G, where  $r = f_4(\ell) + s$  and  $q \geq 5/2 \cdot f_3(\ell)$ , L is an A-avoiding linkage of size at most  $\ell$ , and  $I \subseteq [q]$ , where  $|I| > f_3(\ell)$ , then G contains a linkage  $\tilde{L}$  where  $\tilde{L} \equiv L$ ,  $\tilde{L}$  is A-avoiding,  $\tilde{L} \setminus \mathsf{ann}(\mathcal{C}) \subseteq L \setminus \mathsf{ann}(\mathcal{C})$ , and  $\tilde{L}$  is (s, I)-confined in A. Moreover,  $f_4(\ell) = \mathcal{O}((f_3(\ell))^2)$ .

It follows from the result in [3], that  $f_3(\ell) = 2^{\mathcal{O}(\ell)}$ , when G is a planar graph. Furthermore, if G is a graph of Euler genus at most  $\gamma$ , then  $f_3(\ell) = 2^{\mathcal{O}_{\gamma}(\ell)}$ , because of the result of Mazoit in [45]. We stress that [31] contains the proof of a more general version of Proposition 2.2 where linkages are t-scattered, i.e., their paths are within distance at least t.

### 2.4 Rerouting topological minors

We say that (M,T) is a  $\operatorname{tm-pair}$  if M is a graph,  $T\subseteq V(M)$ , and all vertices in  $V(M)\setminus T$  have degree two. We denote by  $\operatorname{diss}(M,T)$  the graph obtained from M by dissolving all vertices in  $V(M)\setminus T$ . A  $\operatorname{tm-pair}$  of a graph G is a  $\operatorname{tm-pair}(M,T)$  where M is a subgraph of G. Given two graphs H and G, we say that a  $\operatorname{tm-pair}(M,T)$  of G, is a topological minor model of H in G if H is isomorphic to  $\operatorname{diss}(M,T)$ . We call the vertices in T branch vertices of (M,T).

**Topological minor models in railed annuli.** Let G be a partially  $\Delta$ -embedded graph, let H be a graph,  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$  be a (r, q)-railed annulus of G. Let r = 2t + 1. Let also  $s \in [r]$  where s = 2t' + 1. Given some  $I \subseteq [q]$ , we say that a topological minor model (M, T) of H in G is (s, I)-confined in  $\mathcal{A}$  if

$$M \cap \operatorname{ann}(\mathcal{C}, t+1-t', t+1+t') \subseteq \bigcup_{i \in I} P_i.$$

Intuitively, the above definition demands that M traverses the "middle" (s, q)-annulus by intersecting it only at the rails of A.

Our algorithms are strongly based on the following combinatorial result.

**Theorem 2.1** (Model Combing). There exist two functions  $f_3, f_4 : \mathbb{N} \to \mathbb{N}$ , where the images of  $f_4$  are even, such that if

- s is a positive odd integer,
- H is a graph on at most g edges,
- G is a partially  $\Delta$ -embedded graph,
- $\mathcal{A} = (\mathcal{C}, \mathcal{P})$  is an (r,q)-railed annulus of G, where  $r = f_4(g) + 2 + s$  and  $q \geq 5/2 \cdot f_3(g)$ ,
- (M,T) is a topological minor model of H in G such that  $T \cap \operatorname{ann}(A) = \emptyset$ , and
- $I \subseteq [q]$  where  $|I| > f_3(g)$ ,

then G contains a topological minor model  $(\tilde{M}, \tilde{T})$  of H in G such that

- 1.  $\tilde{T}=T$ ,
- 2.  $\tilde{M}$  is (s, I)-confined in A and
- 3.  $\tilde{M} \setminus \operatorname{ann}(\mathcal{A}) \subseteq M \setminus \operatorname{ann}(\mathcal{A})$ .

Moreover,  $f_4(g) = \mathcal{O}((f_3(g))^2)$ .

*Proof.* Let s be a positive odd integer, H be a graph on g edges, G be a partially  $\Delta$ -embedded graph,  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$  be a (r, q)-railed annulus of G, where  $r = f_4(g) + 2 + s$  and  $q \geq 5/2 \cdot f_3(g)$ , (M, T) be a topological minor model of H in G such that  $T \cap \mathsf{ann}(\mathcal{A}) = \emptyset$ .

Notice that all the connected components of  $M \setminus T$  are paths of G. Let L be the linkage of  $G \setminus T$  created by taking the union of all non-trivial connected components of  $M \setminus T$  (see Figure 4). Observe that  $\mathcal{P}(L)$  is the set of all paths of G connecting neighbors of branch vertices of M and consisting only of subdividing vertices of M and that there is an one-to-one correspondence of  $\mathcal{P}(L)$  with E(H). Thus  $|L| \leq g$ .

Let  $\mathcal{A}' = ([C_2, \ldots, C_{r-1}], \mathcal{P} \cap \mathsf{ann}(\mathcal{C}, 2, r-1))$  and keep in mind that  $\mathcal{A}'$  is a (r', q)-railed annulus of G, where  $r' = f_4(g) + s$  and  $q \geq 5/2 \cdot f_3(g)$ . The fact that  $T \cap \mathsf{ann}(\mathcal{A}) = \emptyset$  implies that  $T(L) \cap \mathsf{ann}(\mathcal{A}') = \emptyset$  and thus L is  $\mathcal{A}'$ -avoiding (see Figure 4).

Let  $I \subseteq [q]$ , where  $|I| > f_3(h)$ . By applying Proposition 2.2 for  $s, g, G, \mathcal{A}', L$ , and I we obtain a linkage  $\tilde{L}$  of G such that  $\tilde{L} \equiv L$ ,  $\tilde{L}$  is  $\mathcal{A}'$ -avoiding,  $\tilde{L} \setminus \mathsf{ann}(\mathcal{A}') \subseteq L \setminus \mathsf{ann}(\mathcal{A}')$ , and  $\tilde{L}(s, I)$ -confined in  $\mathcal{A}'$ . We define

$$\tilde{M} = (M \setminus L) \cup \tilde{L}.$$

By definition,  $(\tilde{M}, T)$  is a topological minor model of H in G. Also, since  $L, \tilde{L} \subseteq \mathsf{ann}(\mathcal{A})$ , then  $\tilde{M} \setminus \mathsf{ann}(\mathcal{A}) \subseteq M \setminus \mathsf{ann}(\mathcal{A})$ . Finally, as  $\tilde{L}$  is (s, I)-confined in  $\mathcal{A}'$  then  $\tilde{M}$  is (s, I)-confined in  $\mathcal{A}$  as well.

# 3 Optimizing the Dynamic programming

According to the classic meta-algorithmic results of [4,11] (see also [5,16]) computing  $\mathbf{tm}_{\mathcal{F}}(G)$  can be done in  $\mathcal{O}_{h,\mathbf{tw}}(n)$  time. As we want to optimize the contribution of k in our algorithm, we present in this section a way to design a dynamic programming algorithm for computing  $\mathbf{tm}_{\mathcal{F}}(G)$  in  $2^{\mathcal{O}_h(\mathbf{tw}\log\mathbf{tw})}n$  time. Actually, we give a more general statement of this result, Lemma 3.4, that will be useful in intermediate steps of our algorithm, presented in Subsection 4.1 and Subsection 4.2. In particular, Lemma 3.4 will allow us to find (partial or complete) solutions to the (partial or complete) instances of the problem in bounded treewidth graphs. In order

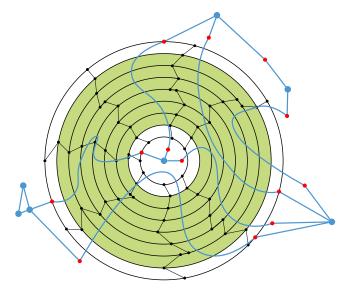


Figure 4: An example of a topological minor model (M,T) of H in G. Vertices of T are depicted in blue while the neighbors of vertices of T that are also subdividing vertices are depicted in red. Also,  $\operatorname{ann}(\mathcal{A}')$  is depicted in green.

to prove Lemma 3.4, we adapt the main ideas of [6] in our context. Thus, in Subsection 3.1, we define boundaried graphs and an equivalence relation among them with respect to the existence of certain topological minor models as subgraphs, which gives rise to a minimum-sized representative of each equivalence class. In Subsection 3.2, we use known results from the protrusion machinery and bidimensionality theory [6,7,26] to deduce that the size of each representative is bounded by a function of its boundary size. Finally, in Subsection 3.3, we define a notion of annotated boundaried graphs, the enhanced boundaried graphs, we extend the notions of equivalence of boundaried graphs to meta-equivalence of enhanced boundaried graphs. We also prove that the number of different meta-representatives is bounded by a function of the boundary size (Lemma 3.3) and, using the dynamic programming tools of [7], we conclude with an algorithm (Lemma 3.4) that computes the minimal size modulator of an enhanced boundaried graph to a given meta-representative.

#### 3.1 Boundaried graphs and representatives

We begin with some definitions, which originate in the seminal work of Bodlaender et al. [9].

Boundaried graphs. Let  $t \in \mathbb{N}$ . A t-boundaried graph is a triple  $\mathbf{G} = (G, B, \rho)$  where G is a graph,  $B \subseteq V(G)$ , |B| = t, and  $\rho : B \to [t]$  is an bijective function. We call B the boundary of  $\mathbf{G}$  and we call the vertices of B the boundary vertices of  $\mathbf{G}$ . We also call G the underlying graph of  $\mathbf{G}$ . We say that the t-boundaried  $\mathbf{G}' = (G', B', \rho')$  is a subgraph of  $\mathbf{G}$  if G' is a subgraph of G, B' = B, and  $\rho' = \rho$ . For  $S \subseteq V(G) \setminus B$ , we define  $\mathbf{G} \setminus S$  to be the t-boundaried graph  $(G', B, \rho)$  where  $G' = G \setminus S$ . Also, for  $B' \subseteq B$ , we define the bijection  $\rho[B'] : B' \to [|B'|]$  such that for every  $v \in B'$ ,  $\rho[B'](v) = |\{u \in B' \mid \rho(u) \leq \rho(v)\}|$ . Two t-boundaried graphs  $\mathbf{G}_1 = (G_1, B_1, \rho_1)$  and  $\mathbf{G}_2 = (G_2, B_2, \rho_2)$  are isomorphic if  $G_1$  is isomorphic to  $G_2$  via a bijection  $\phi : V(G_1) \to V(G_2)$  such that  $\rho_1 = \rho_2 \circ \phi|_{B_1}$ , i.e., the vertices of  $B_1$  are mapped via  $\phi$  to equally indexed vertices of  $B_2$ . A boundaried graph is any t-boundaried graph for some  $t \in \mathbb{N}$ . As in [46] (see also [6]), we define the detail of a boundaried graph  $\mathbf{G} = (G, B, \rho)$  as detail  $\mathbf{G} = \max\{|E(G)|, |V(G) \setminus B|\}$ .

Let  $h, t \in \mathbb{N}$ . We denote by  $\mathcal{B}^{(t)}$  the set of all (pairwise non-isomorphic) t-boundaried graphs and by  $\mathcal{B}_h^{(t)}$  the set of all (pairwise non-isomorphic) t-boundaried graphs with detail at most h. We set  $\mathcal{B} = \bigcup_{t \in \mathbb{N}} \mathcal{B}^{(t)}$ . We say that a boundaried graph  $\mathbf{G} = (G, B, \rho)$  is planar if G is planar.

We also define the *treewidth* of a boundaried graph  $\mathbf{G} = (G, B, \rho)$ , denoted by  $\mathbf{tw}(\mathbf{G})$ , as the minimum width of a tree decomposition  $(T, \chi)$  of G for which there is some  $u \in V(T)$  such that  $B \subseteq \chi(u)$ . Notice that the treewidth of a t-boundaried graph is always lower bounded by t-1.

Equivalent boundaried graphs and representatives. We say that two boundaried graphs  $\mathbf{G}_1 = (G_1, B_1, \rho_1)$  and  $\mathbf{G}_2 = (G_2, B_2, \rho_2)$  are compatible if  $\rho_2^{-1} \circ \rho_1$  is an isomorphism from  $G_1[B_1]$  to  $G_2[B_2]$ . Given two compatible boundaried graphs  $\mathbf{G}_1 = (G_1, B_1, \rho_1)$  and  $\mathbf{G}_2 = (G_2, B_2, \rho_2)$ , we define  $\mathbf{G}_1 \oplus \mathbf{G}_2$  as the graph obtained if we take the disjoint union of  $G_1$  and  $G_2$  and, for every  $i \in [|B_1|]$  we identify vertices  $\rho_1^{-1}(i)$  and  $\rho_2^{-1}(i)$ .

Given an  $h \in \mathbb{N}$ , we say that two boundaried graphs  $\mathbf{G}_1$  and  $\mathbf{G}_2$  are h-equivalent, denoted by  $\mathbf{G}_1 \equiv_h \mathbf{G}_2$ , if they are compatible and, for every graph H on at most h vertices and h edges (or, in other words, every 0-boundaried graph H with detail at most h) and every boundaried graph  $\mathbf{F}$  that is compatible with  $\mathbf{G}_1$  (hence, with  $\mathbf{G}_2$  as well), it holds that

$$H \preceq \mathbf{F} \oplus \mathbf{G}_1 \Longleftrightarrow H \preceq \mathbf{F} \oplus \mathbf{G}_2.$$

Note that  $\equiv_h$  is an equivalence relation on  $\mathcal{B}$ . In the rest of this section we insist that  $\mathcal{B}$  contains only planar graphs, therefore  $\equiv_h$  is seen as an equivalence relation on boundaried planar graphs.

A minimum-sized (first in terms of edges and then in terms of vertices) member of an equivalence class of  $\equiv_h$  is called *representative* of  $\equiv_h$ . For every  $t \in \mathbb{N}$ , we denote by  $\mathcal{R}_h^{(t)}$  the set of all t-boundaried graphs that are representatives of equivalence classes of  $\equiv_h$ . We also define the function  $\operatorname{rep} : \mathcal{B} \to \bigcup_{t \in \mathbb{N}} \mathcal{R}_h^{(t)}$  that maps each boundaried graph to the representative of the equivalence class of  $\equiv_h$  it belongs to.

### 3.2 Bounding the size of a representative

In this subsection we briefly present how the main idea of [6] is applied in our context so as to bound the size of the representatives in  $\mathcal{R}_h^{(t)}$ . We define a graph parameter that measures the minimum amount of vertices needed to affect every wall of the graph under consideration. Following [6, Corollary 25], this parameter on representatives in  $\mathcal{R}_h^{(t)}$  is linear in terms of t (Proposition 3.1). Moreover, we argue that, under the light of bidimensionality theory, this parameter is contraction-bidimensional and linear-separable. We combine all above facts and employ a result of Fomin et al. [26, Theorem 3.11] and a slight extension of a result of Baste et al. [7, Lemma 7.2] to obtain the desired linear bound on the size of every representative (Lemma 3.2). To achieve this, we have to deal with protrusion decompositions, a notion which is also defined in this subsection and was introduced in [9].

**Affecting walls in planar graphs.** Let  $r \in \mathbb{N}_{\geq 3}$ , G be a planar graph,  $S \subseteq V(G)$ , and W be an r-wall of G. We say that S affects W if the following condition holds: for every embedding  $\Gamma$  of G on the plane and every closed disk  $\Delta$ , if the connected component  $\Delta_W$  of  $\mathbb{R}^2 \setminus \mathsf{perim}(W)$  that intersects W is a subset of  $\Delta$ , then  $S \cap \Delta_W \neq \emptyset$ . Given an  $r \in \mathbb{N}_{\geq 3}$ , we define the following graph parameter on planar graphs

$$\mathbf{p}_r(G) = \min\{k \mid \exists S \subseteq V(G) : |S| \le k \land S \text{ affects every } r\text{-wall of } G\}.$$

From now on, functions  $f_3$ ,  $f_4$  will always denote the functions of Theorem 2.1. The following result is [6, Corollary 25] in the special case of planar graphs.

**Proposition 3.1.** Let  $h, t \in \mathbb{N}$ . There exists a function  $f_5 : \mathbb{N} \to \mathbb{N}$  such that for every  $\mathbf{G} \in \mathcal{R}_h^{(t)}$  it holds that  $\mathbf{p}_{f_5(h)}(G) \leq t$ . Moreover,  $f_5(h) = \mathcal{O}((f_3(h))^3)$ .

We comment that the function  $f_5$  is obtained by [6, Theorem 23 and Corollary 25] taking into account that in our case we deal with walls whose compass is planar.

Protrusion decompositions of boundaried graphs. Given a graph G, a set  $X \subseteq V(G)$  is a  $\beta$ -protrusion of G if  $|\partial(X)| \leq \beta$  and  $\mathbf{tw}(G[X]) \leq \beta - 1$ . Given a boundaried graph  $\mathbf{G} = (G, B, \rho)$ , a boundaried graph  $\mathbf{G}' = (G', B', \rho')$  is a  $\beta$ -protrusion of  $\mathbf{G}$  if

- V(G') is a  $\beta$ -protrusion of G,
- $\mathbf{tw}(\mathbf{G}') \leq \beta 1$ ,
- $\partial(V(G')) \subseteq B'$ ,
- $B \cap V(G') \subseteq B'$ , and
- $\rho' = \rho[B']$ .

Given  $\alpha, \ell \in \mathbb{N}$ , an  $(\alpha, \beta)$ -protrusion decomposition of **G** is a sequence  $\mathcal{P} = \langle R_0, \dots, R_\ell \rangle$  of pairwise disjoint subsets of V(G) such that

- $\bigcup_{i \in [0,\ell]} R_i = V(G),$
- $\max\{\ell, |R_0|\} \leq \alpha$ ,
- $B \subseteq R_0$ ,
- for  $i \in [\ell]$ , the triple  $\mathbf{G}'_i = (G'_i, B'_i, \rho'_i)$ , where  $G'_i = G[N_G[R_i]]$ ,  $B'_i = \partial(N_G[R_i])$ , and  $\rho'_i = \rho[B'_i] \to [|B'_i|]$ , is a  $\beta$ -protrusion of  $\mathbf{G}$ , and
- for  $i \in [\ell]$ ,  $N_G(R_i) \subseteq R_0$ .

Linear protrusion decompositions of representatives. Before we proceed, we give the definition of a contraction-bidimensional parameter, as defined in [26]. A graph parameter is a function  $\pi$  mapping graphs to non-negative integers. We say that a graph parameter  $\pi$  is contraction-bidimensional if for every graph G and every  $e \in E(G)$  it holds that  $\pi(G/e) \leq \pi(G)$  and  $\pi(\Gamma_k) = \Omega(k^2)$ , where  $\Gamma_k$  is the  $(k \times k)$ -triangulated grid<sup>5</sup>. Notice that  $\mathbf{p}_r$  is contraction-bidimensional.

Moreover, it easy to observe that  $\mathbf{p}_r$  is *linear-separable*, i.e., for every graph G, every set  $S \subseteq V(G)$  of size  $\mathbf{p}_r(G)$  that affects every r-wall of G, and every separation (L,R) of G it holds that  $||S \cap L| - \mathbf{p}_r(G[L])| = \mathcal{O}(|L \cap R|)$ .

Since  $\mathbf{p}_r$  is a contraction-bidimensional and linear-separable parameter on planar graphs, [26, Theorem 3.11] together with Proposition 3.1 imply the following result.

**Lemma 3.1.** Let  $h, t \in \mathbb{N}$ . There is a constant  $c_h$  such that every  $\mathbf{G} \in \mathcal{R}_h^{(t)}$  admits a  $(c_h \cdot t, c_h)$ -protrusion decomposition. Moreover,  $c_h = (f_3(h))^{\mathcal{O}(1)}$ .

<sup>&</sup>lt;sup>5</sup>the graph obtained from the  $(k \times k)$ -grid by adding, for all  $1 \le x, y \le k-1$ , the edge with endpoints (x+1,y) and (x,y+1) and additionally making vertex (k,k) adjacent to all the other vertices (x,y) with  $x \in \{1,k\}$  or  $y \in \{1,k\}$ , i.e., to the whole perimetric border of the grid.

Using Lemma 3.1, the definition of  $(\alpha, \beta)$ -protrusion decompositions, and the arguments of the proof of [7, Lemma 7.2], it follows that the size of every  $\mathbf{G} \in \mathcal{R}_h^{(t)}$  is  $\mathcal{O}_h(t)$ .

**Lemma 3.2.** There is a function  $f_6: \mathbb{N} \to \mathbb{N}$  such that for every  $t, h \in \mathbb{N}$ , if **G** is a boundaried graph in  $\mathcal{R}_h^{(t)}$ , then the underlying graph of **G** has at most  $f_6(h) \cdot t$  vertices. Moreover,  $f_6(h) = 2^{2^{2(f_3(h))^{\mathcal{O}(1)} \log f_3(h)}}$ .

*Proof.* Let  $s := f(c_h, h)$ , where f is the function of [7, Lemma 7.2] and  $c_h$  is the constant from Lemma 3.1. From the proof of [7, Lemma 7.2] we can derive that  $f(c_h, h) = 2^{2^{2^{\mathcal{O}(c_h \log c_h)}}}$ . Also, we set  $f_6(h) = (s+1) \cdot c_h$ .

Let  $\mathbf{G} = (G, B, \rho)$  be a boundaried graph in  $\mathcal{R}_h^{(t)}$ . We will prove that G has at most  $f_6(h) \cdot t$  vertices. By Lemma 3.1,  $\mathbf{G}$  admits a  $(c_h \cdot t, c_h)$ -protrusion decomposition. Therefore, by definition, there is an  $\ell \in [0, c_h \cdot t]$  and a sequence  $\mathcal{P} = \langle R_0, \dots, R_\ell \rangle$  of pairwise disjoint subsets of V(G) such that

- $\bigcup_{i \in [0,\ell]} R_i = V(G)$ ,
- $\max\{\ell, |R_0|\} \le c_h \cdot t$ ,
- $B \subseteq R_0$ ,
- for  $i \in [\ell]$ , the triple  $\mathbf{G}'_i = (G'_i, B'_i, \rho'_i)$ , where  $G'_i = G[N_G[R_i]]$ ,  $B'_i = \partial(N_G[R_i])$ , and  $\rho'_i = \rho[B'_i] \to [|B'_i|]$ , is a  $c_h$ -protrusion of  $\mathbf{G}$ , and
- for  $i \in [\ell]$ ,  $N_G(R_i) \subseteq R_0$ .

For every  $i \in [\ell]$ , we will show that  $G'_i$  has at most s vertices. Suppose that, towards a contradiction,  $|V(G_i)| > s$ . We sketch the proof of [7, Lemma 7.2] and we comment how to adjust it to our setting in order to obtain a contradiction to the fact that  $\mathbf{G} \in \mathcal{R}_h^{(t)}$ . First of all, in the statement of [7, Lemma 7.2] it is required that the family  $\mathcal{F}$  contains a planar graph, an assumption that is not true in our case. However, inside the proof this is only used in order to bound the treewidth of  $G'_i$  (in fact, to bound its branchwidth, which we know that is upper-bounded by its treewidth). Here, the fact that the treewidth of  $G'_i$  is at most  $c_h$  is implied by the fact that  $G'_i$  is a  $c_h$ -protrusion of G. After this step, the proof of our lemma follows the same arguments as the one of [7, Lemma 7.2]. Intuitively, having a bound on the branchwidth of  $G'_i$  we can consider a branch decomposition of  $G'_i$  of bounded width. Since we assume that  $|V(G_i')| > s$  and we have that  $\mathbf{G} \in \mathcal{R}_h^{(t)}$ , the number of edges of  $G_i'$  is "large enough" and therefore the branch decomposition contains a "long enough" path from the root of the decomposition to a leaf. Along this "long enough" path, we can find two boundaried graphs  $\mathbf{G}_1''$  and  $\mathbf{G}_2'$  such that the underlying graph of  $\mathbf{G}_2''$  is a subgraph of the underlying graph of  $\mathbf{G}_1''$  that has less edges and  $\mathbf{G}_1'' \equiv_h \mathbf{G}_2'$ . Therefore, by replacing  $\mathbf{G}_1''$  with  $\mathbf{G}_2'$ , we obtain a boundaried graph that is h-equivalent to  $\mathbf{G}$  and whose underlying graph has less edges than G, a contradiction to the fact that  $\mathbf{G} \in \mathcal{R}_h^{(t)}$ . For more details, we refer the reader to the proof of [7, Lemma 7.2]. 

#### 3.3 Enhanced boundaried graphs

In this subsection we define a notion of annotated boundaried graphs, the *enhanced boundaried graphs*. Mirroring the equivalence relation defined in Subsection 3.1 for boundaried graphs, in this subsection we also define *meta-equivalences* and *meta-representatives* of enhanced boundaried graphs and we prove that the size of a meta-representative is also linear bounded in terms

of its boundary size. This directly implies that an almost single-exponential (with a logarithmic factor error) bound on the number of different meta-representatives (Corollary 3.1). Finally, using the ideas of [7] concerning dynamic programming, we arrive to the main result of this section, Lemma 3.4.

Enhanced boundaried graphs. Let  $t \in \mathbb{N}$  and  $q \in \mathbb{N}_{\geq 1}$ . A q-enhanced t-boundaried graph is a triple  $(\mathbf{G}, \mathcal{Z}, \mathcal{V})$ , where  $\mathbf{G} = (G, B, \rho)$  is a t-boundaried graph,  $\mathcal{Z} = \{Z_1, \ldots, Z_q\}$  is a collection of non-empty subsets of B, and  $\mathcal{V} = \{V_1, \ldots, V_q\}$  is a collection of non-empty subsets of V(G), such that for every  $i \in [q]$ ,  $(G[V_i], Z_i, \rho[Z_i])$  is a boundaried graph. A q-enhanced boundaried graph is a q-enhanced t-boundaried graph for some  $t \in \mathbb{N}$ . The treewidth of a q-enhanced boundaried graph  $(\mathbf{G}, \mathcal{Z}, \mathcal{V})$  is the treewidth of  $\mathbf{G}$ . As we did in the previous subsection, we consider only triples  $(\mathbf{G}, \mathcal{Z}, \mathcal{V})$  where G is planar.

Meta-representatives of enhanced boundaried graphs. We say that two q-enhanced boundaried graphs  $(\mathbf{G}, \mathcal{Z}, \mathcal{V})$  and  $(\mathbf{G}', \mathcal{Z}', \mathcal{V}')$  are h-meta-equivalent, denoted by  $(\mathbf{G}, \mathcal{Z}, \mathcal{V}) \equiv_h^{(q)} (\mathbf{G}', \mathcal{Z}', \mathcal{V}')$ , if the following hold:

- **G** and **G**' are compatible (via the isomorphism  $\rho'^{-1} \circ \rho$  from G[B] to G'[B']) and
- for every  $i \in [q]$ , it holds that  $(G[V_i], Z_i, \rho_i) \equiv_h (G'[V_i'], Z_i', \rho_i')$ .

Notice that  $\equiv_h^{(q)}$  defines an equivalence relation on q-enhanced boundaried graphs. The minimum-sized (first in terms of edges and then in terms of vertices) member of each equivalence class of  $\equiv_h^{(q)}$  is called a meta-representative of  $\equiv_h^{(q)}$ . We denote by  $\mathcal{R}_h^{(q,t)}$  the set of all q-enhanced t-boundaried graphs that are meta-representatives of  $\equiv_h^{(q)}$ . We call |V(G)| the size of a meta-representative  $(\mathbf{G}, \mathcal{Z}, \mathcal{V})$  of  $\equiv_h^{(q)}$ .

**Lemma 3.3.** There is a function  $f_6: \mathbb{N} \to \mathbb{N}$  such that for every  $t, h \in \mathbb{N}$  and  $q \in \mathbb{N}_{\geq 1}$ , every  $\bar{\mathbf{G}} \in \mathcal{R}_h^{(q,t)}$  has size at most  $f_6(h) \cdot q \cdot t$ .

Proof. Let  $t, h \in \mathbb{N}$ ,  $q \in \mathbb{N}_{\geq 1}$  and let  $(\mathbf{G}, \mathcal{V}, \mathcal{Z})$  be a q-enhanced t-boundaried graph, where  $\mathbf{G} = (G, B, \rho)$ . For every  $i \in [q]$ , we denote by  $\mathbf{H}_i$  the graph  $\operatorname{rep}((G[V_i], Z_i, \rho_i))$  and notice that, due to Lemma 3.2, the underlying graph of  $\mathbf{H}_i$  has at most  $f_6(h) \cdot t$  vertices. By setting  $\mathbf{G}' = (\bigcup_{i \in [q]} H_i, B, \rho)$  and  $\mathcal{V}' = \{V(H_1), \dots, V(H_q)\}$ , we observe that the triple  $(\mathbf{G}', \mathcal{V}', \mathcal{Z})$  is a q-enhanced t-boundaried graph that is h-meta-equivalent with  $(\mathbf{G}, \mathcal{V}, \mathcal{Z})$  and its size is at most  $f_6(h) \cdot q \cdot t$ . Thus, the meta-representative of the equivalence class of  $\equiv_h^{(q)}$  that contains  $(\mathbf{G}, \mathcal{V}, \mathcal{Z})$  has size at most  $f_6(h) \cdot q \cdot t$ .

The following result is a direct consequence of Lemma 3.3, using the fact that the underlying graph G of every meta-representative of  $\equiv_h^{(q)}$  is planar, hence it has  $\mathcal{O}(|V(G)|)$  edges.

**Corollary 3.1.** There is a function  $f_7: \mathbb{N}^3 \to \mathbb{N}$  such that for every  $t, h \in \mathbb{N}$  and  $q \in \mathbb{N}_{\geq 1}$ ,  $|\mathcal{R}_h^{(q,t)}| \leq f_7(h,q,t)$ . Moreover, it holds that  $f_7(h,q,t) = 2^{\mathcal{O}(f_6(h) \cdot q \cdot t \cdot \log(q \cdot t))}$ .

Let  $q \in \mathbb{N}_{\geq 1}$  and  $(\mathbf{G}, \mathcal{V}, \mathcal{Z})$  be a q-enhanced boundaried graph, where  $\mathbf{G} = (G, B, \rho)$ . We say that a set  $S \subseteq V(G)$  is boundary-avoiding if  $S \cap B = \emptyset$ . The dynamic programming machinery of [7, Theorem 1 and Theorem 5] together with Corollary 3.1 and the single-exponential 5-approximation algorithm of Bodlaender et al. for treewidth [8, Theorem VI] yield the following result.

**Lemma 3.4.** There is an algorithm with the following specifications:

Compute  $rep(h, q, t, w, k, \bar{\mathbf{G}}, R, \bar{\mathbf{J}})$ 

Input: five integers  $t, w, k \in \mathbb{N}$  and  $h, q \in \mathbb{N}_{\geq 1}$ , where  $h \geq t, q$ , a q-enhanced (planar) t-boundaried graph  $\bar{\mathbf{G}}$  of treewidth at most w, a boundary-avoiding set  $R \subseteq V(G)$ , and a meta-representative  $\bar{\mathbf{J}} \in \mathcal{R}_h^{(q,t)}$ .

Output: if exists, the minimum-size set  $S_{\bar{\mathbf{J}}} \subseteq R$  of size at most k such that  $\bar{\mathbf{G}} \setminus S_{\bar{\mathbf{J}}} \equiv_h^{(q)} \bar{\mathbf{J}}$ . This algorithm runs in  $2^{\mathcal{O}(f_6(h) \cdot w \log w)} \cdot n$  time or, alternatively, in  $\mathcal{O}(n^3) + 2^{\mathcal{O}(f_6(h) \cdot w)} \cdot n$  time.

Proof. The algorithm first applies the single-exponential 5-approximation algorithm of Bodlaender et al. for treewidth [8, Theorem VI] to compute a tree decomposition of the underlying graph of  $\bar{\mathbf{G}}$  of width at most 5w. Then, using the dynamic programming algorithm of [7, Theorem 1], it checks, for every  $\bar{\mathbf{I}} \in \mathcal{R}_h^{(q,t)}$ , whether there is a set  $S_{\bar{\mathbf{I}}} \subseteq V(G)$  of size at most k such that  $\bar{\mathbf{G}} \setminus S_{\bar{\mathbf{I}}} \equiv_h^{(q)} \bar{\mathbf{I}}$ . We can easily modify this dynamic programming algorithm so as it checks whether such a set  $S_{\bar{\mathbf{I}}}$  is a subset of R and to also output a minimum-size  $S_{\bar{\mathbf{I}}}$  satisfying all above properties, if such exists. As, due to Corollary 3.1,  $|\mathcal{R}_h^{(q,t)}| \leq f_7(h,q,t)$ , this algorithm runs in time  $2^{\mathcal{O}(f_6(h) \cdot w \log w)} \cdot n$ . Moreover, we can replace the algorithm of [7, Theorem 1] with the one of [7, Theorem 5], which runs in a special branch decomposition of the underlying graph of  $\bar{\mathbf{G}}$  (called sphere-cut decomposition) and performs in time  $\mathcal{O}(n^3) + 2^{\mathcal{O}(f_6(h) \cdot w)} \cdot n$ .

We stress that, in the case q=1, we simply refer to q-enhanced t-boundaried graphs as t-boundaried graphs, and to the equivalence relation  $\equiv_h^{(q)}$  as  $\equiv_h$ . Following this, meta-representatives of  $\equiv_h^{(1)}$  are just called representatives. Lemma 3.4 is applied three times in this paper. First we use it in the proof of Lemma 4.1 in Subsection 4.1 towards reducing the solution size, second we use it with q=1 and k=0 in the proof of Lemma 4.2 in Subsection 4.2, and we also use it with t=0, t=0, and t=0, in order to deduce Theorem 1.1 from Theorem 1.2.

# 4 The two main subroutines of the algorithm

In this section, we provide two main subroutines that will be useful in the proof of Theorem 1.2. In subsection Subsection 4.1, we provide an algorithm (Lemma 4.1), that allows us to "safely" reduce the set of possible candidates to a solution, while in Subsection 4.2, we provide an algorithm (Lemma 4.2) that outputs a "big enough" wall such that the vertices in its compass are irrelevant with respect to the existence of a solution to the problem.

Before proceeding to the algorithmic results, we provide some definitions that will facilitate the presentation of the proofs.

Boundaried graphs in railed annuli. Let  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$  be a (r, q)-railed annulus of a partially  $\Delta$ -embedded graph G. We can see each path  $P_j$  in  $\mathcal{P}$  as being oriented towards the "inner" part of  $\Delta$ , i.e., starting from an endpoint of  $P_{1,j}$  and finishing to an endpoint of  $P_{r,j}$ . For every  $(i,j) \in [r] \times [q]$ , we define  $r_{i,j}$  as the first vertex of  $P_j$  that appears in  $P_{i,j}$  while traversing  $P_j$  according to this orientation. Given an  $i \in [r]$  and a  $t \in [q]$ , we define the t-boundaried graph  $G_{i,t} = (G_i, B_{i,t}, \rho_{i,t})$  where  $G_i = G \cap \overline{D}_i$ ,  $B_{i,t} = \{r_{i,1}, \ldots, r_{i,t}\}$  and, for  $j \in [t]$ ,  $\rho_{i,t}(r_{i,j}) = j$ .

#### 4.1 Reducing the solution space

We now prove the following lemma that intuitively states that there is an algorithm that given a graph G and a "big enough" railed annulus  $\mathcal{A}$  of G, it "reduces" the set of vertices that are candidates to the set S that certifies that  $\mathbf{tm}_{\mathcal{F}}(G) \leq k$ .

**Lemma 4.1.** There are three functions  $f_8, f_9 : \mathbb{N}^2 \to \mathbb{N}$ ,  $f_{10} : \mathbb{N} \to \mathbb{N}$  and an algorithm with the following specifications:

Reduce\_Solution\_Space $(k, h, w, \mathcal{F}, \Delta, G, R, \mathcal{C}, \mathcal{P})$ 

Input: three integers  $k, h, w \in \mathbb{N}$ , a finite set  $\mathcal{F}$  of graphs such that  $h \leq h(\mathcal{F})$ , a partially  $\Delta$ embedded graph G whose compass has treewidth at most w, a set  $R \subseteq V(G)$ , and an (r,q)-railed
annulus  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$  of G, where  $r = f_9(h, k)$  and  $q \geq f_{10}(h)$ .
Output: a set  $R' \subseteq R$  such that

- $|R' \cap D_r| \leq f_8(h,k)$  and
- if (G, R, k) is a  $\mathbf{tm}_{\mathcal{F}}$ -triple then (G, R', k) is a  $\mathbf{tm}_{\mathcal{F}}$ -triple.

Moreover,  $f_9(h, k) = \mathcal{O}_h(k)$ ,  $f_8(h, k) = \mathcal{O}_h(k^2)$ , and the algorithm runs in  $2^{\mathcal{O}_h(w \log w)} \cdot k \cdot n$  time, or, alternatively, in  $\mathcal{O}(k \cdot n^3) + 2^{\mathcal{O}_h(w)} \cdot k \cdot n$  time.

*Proof.* Let 
$$g := \binom{h}{2}$$
,  $\lambda := f_3(g) + 1$ ,  $\mu := f_4(g) + 3$ ,

$$f_9(h,k) := (k+1)(h+1)\mu,$$
  
 $f_8(h,k) := f_7(h+\lambda,h+1,(h+1)\cdot\lambda)\cdot k(k+1),$  and  
 $f_{10}(h) := 5/2\cdot f_3(g).$ 

Given an  $i \in [k+1]$ , we define  $A_i = \operatorname{ann}(\mathcal{C}, (i-1)(h+1)\mu+1, i(h+1)\mu)$  and for every  $j \in [h+1]$  we define  $Q_{i,j} = \operatorname{ann}(\mathcal{C}, (i-1)(h+1)\mu+(j-1)\mu+1, (i-1)(h+1)\mu+j\mu)$ . Intuitively, we partition  $\mathcal{C}$  into k+1 sets of consecutive cycles (i.e., the cycles of  $A_i$ ,  $i \in [k+1]$ ) and then, for every  $i \in [k+1]$  we further partition the set of cycles of  $A_i$  into h+1 sets of consecutive cycles (i.e., the cycles of  $Q_{i,j}$ ,  $j \in [h+1]$ ). Notice that for every  $i, j \in [k+1] \times [h+1]$ ,  $|Q_{i,j} \cap \mathcal{C}| = \mu$  (see Figure 5).

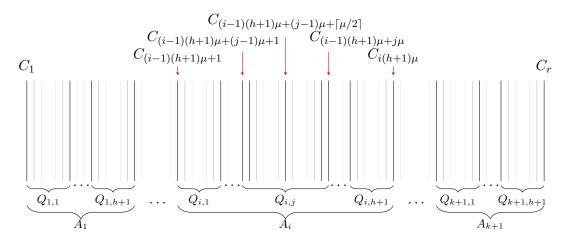


Figure 5: Visualization of the partition of the cycles of  $\mathcal{A}$  into sets  $A_i, i \in [k+1]$  and of the partition in sets  $Q_{i,j}, i, j \in [k+1] \times [h+1]$ .

Also, we define for every  $(i, j) \in [k+1] \times [h+1]$  the  $\lambda$ -boundaried graph

$$(G_{i,j}, B_{i,j}, \rho_{i,j}) := \mathbf{G}_{(i-1)(h+1)\mu + (j-1)\mu + \lceil \mu/2 \rceil, \lambda}.$$

To get some intuition, notice that the vertices of  $B_{i,j}$  lie on the "middle" cycle of  $Q_{i,j}$  – see Figure 5.

Now, for every  $i \in [k+1]$ , we aim to define a (h+1)-enhanced  $((h+1) \cdot \lambda)$ -boundaried graph obtained by the union of the  $\lambda$ -boundaried graphs  $(G_{i,j}, B_{i,j}, \rho_{i,j})$ , for  $j \in [h+1]$ . Let  $i \in [k+1]$ . We set  $\bar{\rho}_i : \bigcup_{j \in [h+1]} B_{i,j} \to [(h+1) \cdot \lambda]$  to be the function that, for every  $j \in [h+1]$ ,

$$\bar{\rho}_i(v) = (j-1) \cdot \lambda + \rho_{i,j}(v), \text{ if } v \in B_{i,j}.$$

Notice that  $\bar{\rho}_i$  is a bijection and that, for every  $j \in [h+1]$ ,  $\bar{\rho}_i[B_{i,j}] = \rho_{i,j}$ . Thus, if we set

$$\bar{\mathbf{G}}_{i} := (\bigcup_{j \in [h+1]} G_{i,j}, \bigcup_{j \in [h+1]} B_{i,j}, \bar{\rho}_{i}), 
\mathcal{Z}_{i} := \{B_{i,1}, \dots, B_{i,h+1}\}, \text{ and } 
\mathcal{V}_{i} := \{V(G_{i,1}), \dots, V(G_{i,h+1})\},$$

we have that  $(\bar{\mathbf{G}}_i, \mathcal{Z}_i, \mathcal{V}_i)$  is an (h+1)-enhanced  $((h+1) \cdot \lambda)$ -boundaried graph.

For every  $i \in [k+1]$  and every meta-representative  $\bar{\mathbf{J}} \in \mathcal{R}_{h+\lambda}^{(h+1,(h+1)\cdot\lambda)}$ , let  $S_{i,\bar{\mathbf{J}}}$  be the minimum-size subset of  $R \cap D_{i(h+1)\mu}$  of at most k vertices such that  $(\bar{\mathbf{G}}_i, \mathcal{Z}_i, \mathcal{V}_i) \setminus S_{i,\bar{\mathbf{J}}} \equiv_{h+\lambda}^{(h+1)} \bar{\mathbf{J}}$ . If such a set does not exist, then we set  $S_{i,\bar{\mathbf{J}}} = \emptyset$ . We define

$$R^* = (\bigcup_{\substack{i \in [k+1]\\ \bar{\mathbf{J}} \in \mathcal{R}_{h+\lambda}^{(h+1,(h+1)\cdot\lambda)}}} S_{i,\bar{\mathbf{J}}}) \cap D_r$$

and  $R' = (R \setminus D_r) \cup R^*$ . Observe that  $|R^*| \leq (k+1) \cdot |\mathcal{R}_{h+\lambda}^{(h+1,(h+1)\cdot\lambda)}| \cdot k = f_8(h,k)$ , and therefore  $|R' \cap D_r| \leq f_8(h,k)$ . Also, notice that as the underlying graph of  $\bar{\mathbf{G}}_i$  is a subgraph of the compass of G, it has treewidth at most w. Moreover, since  $S_{i,\bar{\mathbf{J}}} \subseteq R \cap D_{i(h+1)\mu}$ , it holds that  $R \cap \bigcup_{j \in [h+1]} B_{i,j} = \emptyset$ . Therefore, for every  $i \in [k+1]$  and every  $\bar{\mathbf{J}} \in \mathcal{R}_{h+\lambda}^{(h+1,(h+1)\cdot\lambda)}$ , we compute  $S_{i,\bar{\mathbf{J}}}$  by using the algorithm of Lemma 3.4 for  $h := h + \lambda$ , q := h + 1,  $t := (h+1) \cdot \lambda$ ,  $\bar{\mathbf{G}} := \bar{\mathbf{G}}_i$ , and  $R := R \cap D_{i(h+1)\mu}$ . This algorithm runs in  $2^{\mathcal{O}_h(w \log w)} \cdot n$  time, or, alternatively, in  $\mathcal{O}(n^3) + 2^{\mathcal{O}_h(w)} \cdot n$  time, since the underlying graph of each  $\bar{\mathbf{G}}_i$  is planar. Therefore, we can compute R' as well, in  $2^{\mathcal{O}_h(w \log w)} \cdot (k+1) \cdot n$  time, or, alternatively, in  $(k+1) \cdot \mathcal{O}(n^3) + (k+1) \cdot 2^{\mathcal{O}_h(w)} \cdot n$  time.

We now prove that if (G, R, k) is a  $\mathbf{tm}_{\mathcal{F}}$ -triple then (G, R', k) is also a  $\mathbf{tm}_{\mathcal{F}}$ -triple. In particular, we prove that for every graph H on at most h vertices and every  $S \subseteq R$ , if  $|S| \leq k$  and  $H \npreceq G \setminus S$ , then there is some  $S' \subseteq R'$  such that  $|S'| \leq k$  and  $H \npreceq G \setminus S'$ .

Let H be graph on at most h vertices (and, therefore, of at most g edges) and let  $S \subseteq R$  such that  $|S| \leq k$  and  $H \npreceq G \setminus S$ . As  $r = (k+1)(h+1)\mu$  and  $|S| \leq k$ , then by the pigeonhole principle there is some  $\ell \in [k+1]$  such that  $S \cap A_{\ell} = \emptyset$ . (In case there are many such  $\ell$ 's, we take the minimum one.) Let  $S_{\rm in} = S \cap D_{\ell(h+1)\mu}$  and  $S_{\rm out} = S \setminus \overline{D}_{(\ell-1)(h+1)\mu+1}$ . Let also  $k_{\rm in} := |S_{\rm in}|$  and  $k_{\rm out} := |S_{\rm out}|$  and keep in mind that  $k_{\rm in} + k_{\rm out} = |S| \leq k$ .

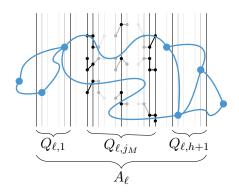
Let  $\bar{\mathbf{J}}_S$  be the meta-representative of  $(\bar{\mathbf{G}}_\ell, \mathcal{Z}_\ell, \mathcal{V}_\ell) \setminus S_{\text{in}}$ . Let also  $S_{\bar{\mathbf{J}}_S}$  be the minimum-size subset of  $R \cap D_{\ell(h+1)\mu}$  such that  $(\bar{\mathbf{G}}_\ell, \mathcal{Z}_\ell, \mathcal{V}_\ell) \setminus S_{\bar{\mathbf{J}}_S} \equiv_{h+\lambda}^{(h+1)} \bar{\mathbf{J}}_S$ . Clearly,  $|S_{\bar{\mathbf{J}}_S}| \leq |S_{\text{in}}| = k_{\text{in}}$  and therefore  $S_{\bar{\mathbf{J}}_S} = S_{\ell,\bar{\mathbf{J}}_S}$ . We now set  $S' = S_{\bar{\mathbf{J}}_S} \cup S_{\text{out}}$  and observe that, since  $S_{\bar{\mathbf{J}}_S} = S_{\ell,\bar{\mathbf{J}}_S}$ ,  $S_{\bar{\mathbf{J}}_S} \cap D_r \subseteq R^*$  and therefore  $S' \subseteq R'$ . Since  $|S'| \leq k$ , it remains to prove that  $H \not\preceq G \setminus S'$ .

Let  $\mathcal{H}$  be the set of all topological minor models of H in G and notice that for every  $(M,T) \in \mathcal{H}$  it holds that  $S \cap V(M) \neq \emptyset$ , i.e., S intersects at least one vertex of each graph in  $\mathcal{H}$ . Let  $\mathcal{H}_{\ell}$  be the members of  $\mathcal{H}$  that are intersected *only* by vertices in  $S_{\text{in}}$ .

The next claim shows that there is a collection of cycles of  $\mathcal{A}$ , the "middle" cycles of  $Q_{\ell,j}$ 's, such that for every tm-pair  $(M,T) \in \mathcal{H}_{\ell}$  there is a cycle C of this collection and a tm-pair

 $(\tilde{M}, \tilde{T}) \in \mathcal{H}_{\ell}$  that is equivalent to (M, T) and is "combed in C" in the sense that  $\tilde{M} \cap C$  is a subgraph of the rails of  $\mathcal{A}$ .

Claim: For every  $(M,T) \in \mathcal{H}_{\ell}$ , there exists a  $j_M \in [h+1]$  and a tm-pair  $(M,T) \in \mathcal{H}_{\ell}$ , such that  $\tilde{M} \setminus A_{\ell} \subseteq M \setminus A_{\ell}$  and the graph  $\tilde{M} \cap C_{y_M}$  is the union of the paths  $\{P_{y_M,c_1^M},\ldots,P_{y_M,c_{z_M}^M}\}$ , where  $y_M = (\ell-1)(h+1)\mu + (j_M-1)\mu + \lceil \mu/2 \rceil$  and  $\{c_1^M,\ldots,c_{z_M}^M\} \subseteq [\lambda]$  (see Figure 6).



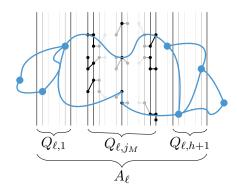


Figure 6: Visualization of the statement of the Claim. (M,T) is depicted in the left figure, while  $(\tilde{M},\tilde{T})$  is depicted in the right figure.

Proof of Claim: Let  $(M,T) \in \mathcal{H}_{\ell}$  and notice that  $S_{\text{in}} \cap V(M) \neq \emptyset$ . As  $|T| \leq h$ , there is some  $j_M \in [h+1]$  such that  $T \cap Q_{\ell,j_M} = \emptyset$  (if many such  $j_M$ 's exist, take the minimum one). We use notation  $\mathcal{A}^{(M)} = (\mathcal{C}^{(M)}, \mathcal{P}^{(M)})$  instead of  $\mathcal{A} \cap Q_{\ell,j_M}$ . Since  $|\mathcal{C}^{(M)}| = \mu = f_4(g) + 3$  and  $|\mathcal{P}^{(M)}| = q \geq f_{10}(h) = 5/2 \cdot f_3(g)$ , we can now apply Theorem 2.1 for s = 1,  $\mathcal{A} := \mathcal{A}^{(M)}$ , and  $I = [\lambda]$  and obtain a topological minor model  $(\tilde{M}, \tilde{T})$  of H in G such that  $\tilde{T} = T$ ,  $\tilde{M}$  is (1, I)-confined in  $\mathcal{A}^{(M)}$  and  $\tilde{M} \setminus Q_{\ell,j_M} \subseteq M \setminus Q_{\ell,j_M}$ , which implies that  $\tilde{M} \setminus A_{\ell} \subseteq M \setminus A_{\ell}$ . Let  $y_M = (\ell - 1)(h + 1)\mu + (i_M - 1)\mu + \lceil \mu/2 \rceil$ . Notice that  $(\tilde{M}, \tilde{T})$  is a topological minor model in  $\mathcal{H}_{\ell}$  whose intersection with  $C_{y_M}$  is the union of some of the paths in  $\{P_{y_M,1}, \dots, P_{y_M,\lambda}\}$ , namely  $\{P_{y_M,c_{2_M}^{I}}, \dots, P_{y_M,c_{2_M}^{M}}\}$ , where  $\{c_1^M, \dots, c_{2_M}^M\} \subseteq [\lambda]$ . The claim follows.

Suppose, towards a contradiction, that the graph  $G\setminus S'$  contains some topological minor model (M,T) of H as a subgraph. Since  $H\not\preceq G\setminus S$ , it holds that (M,T) is intersected only by vertices in  $S_{\text{in}}$  - thus  $(M,T)\in\mathcal{H}_\ell$ . According to the Claim above, there is an  $j_M\in[h+1]$  and a topological minor model  $(\tilde{M},\tilde{T})\in\mathcal{H}_\ell$  such that  $\tilde{M}\setminus A_\ell\subseteq M\setminus A_\ell$  and the graph  $\tilde{M}\cap C_{y_M}$  is the union of the paths  $\{P_{y_M,c_1^M},\ldots,P_{y_M,c_{z_M}^M}\}$  where  $y_M=(\ell-1)(h+1)\mu+(j_M-1)\mu+\lceil\mu/2\rceil+1$  and  $\{c_1^M,\ldots,c_{z_M}^M\}\subseteq [\lambda]$ . Note that  $B_{\ell,j_M}\subseteq V(C_{j_M})$ . Moreover, since (M,T) is a topological minor model of H in  $G\setminus S'$  and  $\tilde{M}\setminus A_\ell\subseteq M\setminus A_\ell$ , we have that  $(\tilde{M},\tilde{T})$  is a topological minor model of H in  $G\setminus S'$ .

We consider the  $\lambda$ -boundaried graph  $\tilde{\mathbf{M}}_{\text{in}} = (\tilde{M}_{\text{in}}, B_{\ell, j_M}, \rho_{\ell, j_M})$  where  $\tilde{M}_{\text{in}} = (\tilde{M} \cap \overline{D}_{y_M}) \cup (B_{\ell, j_M}, \emptyset)$ , i.e., the graph  $\tilde{M} \cap \overline{D}_{y_M}$  together with the isolated vertices  $B_{\ell, j_M}$ . We also define

$$\tilde{M}_{\mathrm{out}} = (\tilde{M} \setminus (\overline{D}_{y_M} \setminus B_{\ell,j_M})) \cup (B_{\ell,j_M}, \emptyset) \text{ and } \tilde{\mathbf{M}}_{\mathrm{out}} = (\tilde{M}_{\mathrm{out}}, B_{\ell,j_M}, \rho_{\ell,j_M}).$$

Intuitively,  $\tilde{M}_{\text{out}}$  is the graph obtained from  $\tilde{M}$  by removing all vertices in  $\overline{D}_{y_M}$  except the vertices in  $B_{\ell,j_M}$  and adding the isolated vertices  $B_{\ell,j_M} \setminus V(\tilde{M})$ .

We now observe that, since (M,T) is a topological minor model of H in  $G \setminus S'$  and  $S_{\bar{\mathbf{J}}}$  is a subset of S',  $\tilde{\mathbf{M}}_{\text{in}}$  is a subgraph of  $(G_{\ell,j_M}, B_{\ell,j_M}, \rho_{\ell,j_M}) \setminus S_{\bar{\mathbf{J}}}$ . The latter, together with the fact that  $\tilde{\mathbf{M}}_{\text{out}}$  is compatible with  $(G_{\ell,j_M}, B_{\ell,j_M}, \rho_{\ell,j_M}) \setminus S_{\bar{\mathbf{J}}}$ , implies that

$$H \leq \tilde{\mathbf{M}}_{\text{out}} \oplus (G_{\ell,j_M}, B_{\ell,j_M}, \rho_{\ell,j_M}) \setminus S_{\bar{\mathbf{J}}}.$$
 (1)

Also, the fact that  $(\bar{\mathbf{G}}_{\ell}, \mathcal{Z}_{\ell}, \mathcal{V}_{\ell}) \setminus S_{\bar{\mathbf{J}}_{S}} \equiv_{h+\lambda}^{(h+1)} (\bar{\mathbf{G}}_{\ell}, \mathcal{Z}_{\ell}, \mathcal{V}_{\ell}) \setminus S_{\text{in}}$  implies that

$$(G_{\ell,j_M}, B_{\ell,j_M}, \rho_{\ell,j_M}) \setminus S_{\text{in}} \equiv_{h+\lambda} (G_{\ell,j_M}, B_{\ell,j_M}, \rho_{\ell,j_M}) \setminus S_{\bar{\mathbf{J}}}.$$

$$(2)$$

By (1) and (2), we obtain that  $H \leq \tilde{\mathbf{M}}_{\text{out}} \oplus (G_{\ell,j_M}, B_{\ell,j_M}, \rho_{\ell,j_M}) \setminus S_{\text{in}}$ , which, in turn, implies that  $H \leq G \setminus S$ , a contradiction.

## 4.2 Finding an irrelevant area

The next lemma intuitively states that there exists an algorithm that given a partially  $\Delta$ -embedded graph G and a "big enough" railed annulus of G, outputs a "big enough" wall W of G whose compass is a subset of  $\Delta$  and such that for every hitting set S outside  $\Delta$ , the vertex set of the compass of W in G is an irrelevant part of the instance.

**Lemma 4.2.** There exist two functions  $f_{11}, f_{12} : \mathbb{N}^2 \to \mathbb{N}$ , and an algorithm with the following specifications:

Find\_irrelevant\_area $(b, h, w, \mathcal{F}, \Delta, G, R, \mathcal{C}, \mathcal{P})$ 

Input: three integers  $b \in \mathbb{N}_{\geq 3}$  and  $h, w \in \mathbb{N}$ , a finite set of graphs  $\mathcal{F}$  where  $h \leq h(\mathcal{F})$ , a partially  $\Delta$ -embedded graph G whose compass has treewidth at most w, a set  $R \subseteq V(G) \setminus \Delta$ , and an  $(f_{11}(h,b), f_{12}(h,b))$ -railed annulus  $\mathcal{A} = (\mathcal{C}, \mathcal{P})$  of G.

Output: a b-wall W of G such that

- $V(\mathsf{compass}(W)) \subseteq \Delta \ and$
- if  $(G \setminus V(\mathsf{compass}(W)), R, k)$  is a  $\mathsf{tm}_{\mathcal{F}}$ -triple then (G, R, k) is a  $\mathsf{tm}_{\mathcal{F}}$ -triple.

Moreover,  $f_{11}(h, b) = \mathcal{O}_h(b)$ ,  $f_{12}(h, b) = \mathcal{O}_h(b)$ , and this algorithm runs in  $2^{\mathcal{O}_h(w \log w)} \cdot b \cdot n$  time, or, alternatively, in  $\mathcal{O}_h(b \cdot n^3) + 2^{\mathcal{O}_h(w)} \cdot b \cdot n$  time.

Proof. Let 
$$g := \binom{h}{2}$$
,  $\lambda := f_3(g) + 1$ , and  $\mu := f_4(g) + 3$ . We set 
$$\ell := (h+2)\mu + b + 1,$$
 
$$r := f_7(h+\lambda,1,\lambda) \cdot \ell,$$
 
$$q := \max\{5/2 \cdot f_3(g), \lambda + 2b\},$$
 
$$f_{11}(h,b) := r, \text{ and }$$
 
$$f_{12}(h,b) := q.$$

For every  $i \in [r]$ , we consider the  $\lambda$ -boundaried graph  $\mathbf{G}_{i,\lambda} = (G_i, B_{i,\lambda}, \rho_{i,\lambda})$  where  $G_i = G \cap \overline{D}_i$ ,  $B_{i,\lambda} = \{r_{i,1} \dots, r_{i,\lambda}\}$  and, for  $j \in [\lambda]$ ,  $\rho_{i,\lambda}(r_{i,\lambda}) = j$ . Also, for every  $i \in [r]$ , let  $\mathbf{J}_i$  be the representative of  $\mathbf{G}_{i,\lambda}$ . To compute  $\mathbf{J}_i$ , we first observe that as the underlying graph of each  $\mathbf{G}_{i,\lambda}$  is a subgraph of the compass of G, we have that  $\mathbf{tw}(\mathbf{G}_{i,\lambda}) \leq w + \lambda = \mathcal{O}_h(w)$ . Therefore, for each representative  $\mathbf{J} \in \mathcal{R}_{h+\lambda}^{(\lambda)}$ , we call the algorithm  $\mathbf{Compute\_rep}(h+\lambda,1,\lambda,w+\lambda,0,\mathbf{G}_{i,\lambda},\emptyset,\mathbf{J})$  of Lemma 3.4 to check whether  $\mathbf{G}_{i,\lambda} \equiv_{h+\lambda} \mathbf{J}$ . The overall running time needed to compute the representative of each  $\mathbf{G}_{i,\lambda}$  is  $r \cdot |\mathcal{R}_{h+\lambda}^{(\lambda)}| \cdot 2^{\mathcal{O}_h(w \log w)} \cdot n = 2^{\mathcal{O}_h(w \log w)} \cdot b \cdot n$ , or, alternatively,  $r \cdot \mathcal{O}(n^3) + r \cdot |\mathcal{R}_{h+\lambda}^{(\lambda)}| \cdot 2^{\mathcal{O}_h(w)} \cdot n = \mathcal{O}_h(b \cdot n^3) + 2^{\mathcal{O}_h(w)} \cdot b \cdot n$ .

Since  $|\mathcal{R}_{h+\lambda}^{(\lambda)}| \leq f_7(h+\lambda,1,\lambda)$ , then there is a set  $I \subseteq [r]$  of size  $\ell$  such that for every  $p,q \in I$ ,  $\mathbf{J}_p = \mathbf{J}_q$ . Let i' be the maximum element of I and note that  $i' \geq (h+2)\mu + b + 1$ . We define  $\Delta'$  to be the arc-wise connected component of  $\mathsf{ann}(\mathcal{C}, i' - b, i' - 1) \setminus (P_{\lambda+1} \cup P_{\lambda+2b})$  that does not intersect  $P_1$  (see Figure 7).

Notice that the graph obtained by the union of  $\bigcup_{j\in[b]} C_{i'-j}$  and  $\bigcup_{j\in[2b]} P_{\lambda+j}$  contains a b-wall W as a subgraph such that  $V(\mathsf{compass}(W))$  is a subset of the closure of  $\Delta'$ . We set

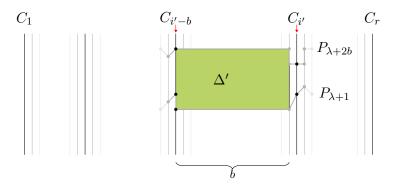


Figure 7: An example showing the disk  $\Delta'$ , whose closure contains the vertices of the compass of the obtained b-wall W.

 $K := \mathsf{compass}(W)$  and keep in mind that V(K) (and, therefore, also  $\Delta'$ ) is a subset of  $\Delta$  that does not intersect the cycle  $C_{i'}$ .

We now aim to prove that if  $(G \setminus V(K), R, k)$  is a  $\mathbf{tm}_{\mathcal{F}}$ -triple then (G, R, k) is a  $\mathbf{tm}_{\mathcal{F}}$ -triple. In order to prove this, we argue that if H is a graph on at most h vertices and S is a subset of R such that  $H \not\preceq (G \setminus V(K)) \setminus S$ , then it holds that  $H \not\preceq G \setminus S$ .

Let H be a graph on at most h vertices (and, therefore, of at most g edges) and  $S \subseteq R \subseteq V(G) \setminus \Delta$  such that  $H \npreceq (G \setminus V(K)) \setminus S$ . Suppose towards a contradiction that the graph  $G \setminus S$  contains some topological minor model (M,T) of H as a subgraph. In what follows, we argue how to obtain a subgraph of  $(G \setminus V(K)) \setminus S$  that is a subdivision of H, thus arriving at a contradiction.

As  $|T| \le h$  and  $\ell = (h+2)\mu + b + 1$ , there is some  $y \in I \setminus \{i'\}$  such that

$$T \cap \operatorname{ann}(\mathcal{C}, y - |\mu/2|, y + |\mu/2|) = \emptyset.$$

We consider the  $(\mu, q)$ -railed annulus  $\mathcal{A}' = (\mathcal{C}', \mathcal{P}')$  of G where

- $\mathcal{C}'=[C_1',\dots,C_\mu']:=[C_{y-\lfloor \mu/2\rfloor},\dots,C_{y+\lfloor \mu/2\rfloor}]$  and
- $\mathcal{P}' = [P'_1, \dots, P'_q] := [P_1 \cap \mathsf{ann}(\mathcal{C}'), \dots, P_q \cap \mathsf{ann}(\mathcal{C}')]$ . (See Figure 8.)

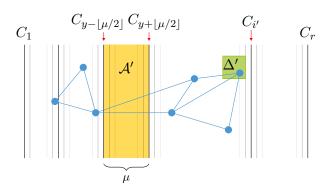


Figure 8: An example showing the  $(\mu, q)$ -railed annulus  $\mathcal{A}'$ .

Observe that, since  $S \subseteq V(G) \setminus \Delta$ ,  $\mathcal{A}'$  remains intact after removing the vertices of S from G, i.e.,  $\mathcal{A}'$  is also an  $(\mu, q)$ -railed annulus of  $G \setminus S$ . Since  $\mu = f_4(g) + 3$  and  $q \geq 5/2 \cdot f_3(g)$ , we are in position to apply Theorem 2.1 for s := 1, H,  $G := G \setminus S$ ,  $\mathcal{A} := \mathcal{A}'$ ,  $r := \mu$ , M, and

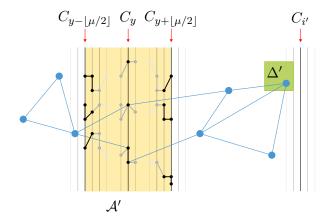


Figure 9: An example of  $(\tilde{M}, \tilde{T})$ , the result of applying Theorem 2.1 in the railed annulus  $\mathcal{A}'$ .

 $I = [\lambda]$ . We deduce the existence of a topological minor model  $(\tilde{M}, \tilde{T})$  of H in  $G \setminus S$  such that  $\tilde{T} = T$ ,  $\tilde{M}$  is (1, I)-confined in A', and  $\tilde{M} \setminus \mathsf{ann}(A') \subseteq M \setminus \mathsf{ann}(A')$  (see Figure 9).

Notice now that  $\tilde{M} \cap C_y$  is the union of some of the paths in  $\{P_{y,1}, \ldots, P_{y,\lambda}\}$ . Suppose that these paths are  $\{P_{y,c_1}, \ldots, P_{y,c_z}\}$  where  $\{c_1, \ldots, c_z\} \subseteq [\lambda]$ . We consider the  $\lambda$ -boundaried graph  $\mathbf{M}_y = (M_y, B_{y,\lambda}, \rho_{y,\lambda})$  where  $M_y = (\tilde{M} \cap \overline{D}_y) \cup (B_{y,\lambda}, \emptyset)$  (i.e. the graph  $\tilde{M} \cap \overline{D}_y$  together with the isolated vertices  $B_{y,\lambda}$ ). We also define

$$\tilde{M}_{\mathrm{out}} = (\tilde{M} \setminus (\overline{D}_y \setminus B_{y,\lambda})) \cup (B_{y,\lambda}, \emptyset) \text{ and } \tilde{\mathbf{M}}_{\mathrm{out}} = (\tilde{M}_{\mathrm{out}}, B_{y,\lambda}, \rho_{y,\lambda}).$$

Keep in mind that  $\tilde{M}_{\text{out}}$  is a subgraph of  $G \setminus S$  that does not contain vertices of K (see Figure 10).

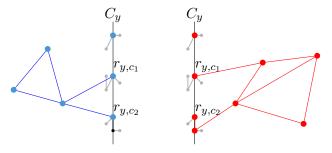


Figure 10: The graphs  $\tilde{M}_{\text{out}}$  (depicted in blue) and  $M_y$  (depicted in red).

Notice that  $\mathbf{M}_y$  is a subgraph of  $\mathbf{G}_{y,\lambda}$  and therefore the graph  $\tilde{\mathbf{M}}_{\text{out}} \oplus \mathbf{M}_y$  (that is the graph  $\tilde{M} \cup (B_{y,\lambda},\emptyset)$ ) is a subgraph of  $\tilde{\mathbf{M}}_{\text{out}} \oplus \mathbf{G}_{y,\lambda}$ . Thus,  $H \preceq \tilde{\mathbf{M}}_{\text{out}} \oplus \mathbf{G}_{y,\lambda}$ . Since  $\mathbf{J}_y = \mathbf{J}_{i'}$ , we have that  $H \preceq \tilde{\mathbf{M}}_{\text{out}} \oplus \mathbf{G}_{i',\lambda}$ . Therefore,  $\mathbf{G}_{i',\lambda}$  contains as a subgraph a  $\lambda$ -boundaried graph  $\mathbf{M}_{i'} = (M_{i'}, B_{i',\lambda}, \rho_{i',\lambda})$ , such that  $\tilde{\mathbf{M}}_{\text{out}} \oplus \mathbf{M}_{i'}$  contains H as a topological minor. Notice that  $M_{i'}$  is a subgraph of  $G \setminus S$  that does not intersect  $V(\mathsf{compass}(W))$ .

For every  $j \in [z]$ , we define  $P_j^*$  to be the path in  $P_j$  starting from  $r_{y,j}$  and finishing to  $r_{i',y}$ , i.e.,  $P_j^* = (P_j \cap \overline{D}_y) \setminus (\overline{D}_{i'} \setminus r_{i',j})$  and  $\mathcal{P}^* = \{P_j^* \mid j \in [z]\}$ . Observe that none of the paths in  $\mathcal{P}^*$  intersects V(K). Let  $\hat{M}_{i'}$  (resp.  $\hat{M}_{\text{out}}$ ) be the graph obtained from  $M_{i'}$  (resp.  $\tilde{M}_{\text{out}}$ ) after removing, for every  $j \in [\lambda] \setminus \{c_1, \ldots, c_z\}$ , the vertices  $r_{i',j}$  (resp.  $r_{y,j}$ ). Consider now the graph  $M_0 := \hat{M}_{\text{out}} \cup \hat{M}_{i'} \cup \bigcup \mathcal{P}^*$  and observe that  $M_0$  is a subgraph of  $(G \setminus V(K)) \setminus S$  that is a subdivision of H. Therefore  $H \preceq (G \setminus V(K)) \setminus S$ , a contradiction (see Figure 11).

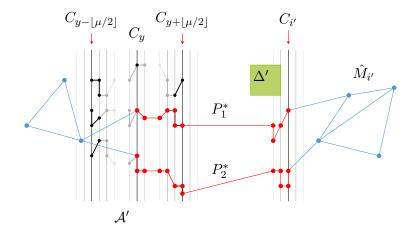


Figure 11: Visualization of the last part of the proof.

# 5 Proof of Theorem 1.2

In this section, having all necessary results, we are in position to present the proof of Theorem 1.2.

Proof of Theorem 1.2. Let

$$x := \max\{f_9(h, k), f_{10}(h)\} = \mathcal{O}_h(k),$$

$$y := \max\{f_{11}(h, 3), f_{12}(h, 3)\},$$

$$z := f_8(h, k) + 1 = \mathcal{O}_h(k^2), \text{ and}$$

$$q := f_2(x, y, z) = \mathcal{O}(x + y\sqrt{z}) = \mathcal{O}_h(k).$$

We first call the algorithm  $\mathbf{Find}_{-}\mathbf{Wall}(G,q)$  of Proposition 2.1 which outputs either a q-wall W of G whose compass has treewidth at most  $c_1 \cdot q$  or a tree decomposition of G of width at most  $c_1 \cdot q$ . This algorithm runs in  $2^{\mathcal{O}(q^2)} \cdot n = 2^{\mathcal{O}_h(k^2)} \cdot n$  time, or, alternatively, in  $\mathcal{O}(n^2)$  time. We consider the first case.

Let  $\Delta$  be the closed disk whose boundary is the perimeter of W and contains  $\mathsf{compass}(W)$ . We call the algorithm  $\mathsf{Find\_Collection\_of\_Annuli}(x,y,z,\Delta,G,W)$  of Lemma 2.1 which, in  $\mathcal{O}(n)$  time, outputs a closed disk  $\Delta' \subseteq \Delta$  and a collection  $\mathfrak{A} = \{\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_z\}$  of railed annuli of the compass of G such that

- $A_0$  is an (x,x)-railed annulus whose outer disk is  $\Delta$  and whose inner disk is  $\Delta'$ ,
- for  $i \in [z]$ ,  $A_i$  is a (y,y)-railed annulus of  $G \cap \operatorname{int}(\Delta')$ , and
- for every  $i, j \in [z]$  where  $i \neq j$ , the outer disk of  $A_i$  and the outer disk of  $A_j$  are disjoint.

Then, we call the algorithm **Reduce\_Solution\_Space** $(k, h, w, \Delta, G, R, C, P)$  of Lemma 4.1 for  $(C, P) := A_0$  and  $w := c_1 \cdot q$  which outputs a set  $R' \subseteq R$  such that

- $|R' \cap \text{int}(\Delta')| < f_8(h, k) = z 1$  and
- if (G, R, k) is a  $\mathbf{tm}_{\mathcal{F}}$ -triple then (G, R', k) is a  $\mathbf{tm}_{\mathcal{F}}$ -triple.

This algorithm runs in  $2^{\mathcal{O}_h(q\log q)} \cdot k \cdot n = 2^{\mathcal{O}_h(k\log k)} \cdot n$  time, or, alternatively, in  $\mathcal{O}(k \cdot n^3) + 2^{\mathcal{O}_h(q)} \cdot k \cdot n = \mathcal{O}(k \cdot n^3) + 2^{\mathcal{O}_h(k)} \cdot n$  time. Since |R'| < z then there exists a  $j \in [z]$  such that  $R' \cap \mathsf{ann}(\mathcal{A}_j) = \emptyset$ . Let  $(\mathcal{C}^{(j)}, \mathcal{P}^{(j)}) := \mathcal{A}_j$  and  $\Delta_j$  be the closure of the outer disk of  $\mathcal{A}_j$ . Now,

for b := 3, the algorithm **Find\_irrelevant\_area** $(h, b, w, \Delta_j, G, R', \mathcal{C}^{(j)}, \mathcal{P}^{(j)})$  of Lemma 4.2 outputs a b-wall W of G such that

- $V(\mathsf{compass}(W)) \subseteq \Delta$  and if and only if
- if  $(G \setminus V(\mathsf{compass}(W)), R, k)$  is a  $\mathsf{tm}_{\mathcal{F}}$ -triple then (G, R, k) is a  $\mathsf{tm}_{\mathcal{F}}$ -triple.

This algorithm runs in  $2^{\mathcal{O}_h(q \log q)} \cdot n = 2^{\mathcal{O}_h(k \log k)} \cdot n$  time, or, alternatively, in  $\mathcal{O}_h(n^3) + 2^{\mathcal{O}_h(q)} \cdot n = \mathcal{O}_h(n^3) + 2^{\mathcal{O}_h(k)} \cdot n$  time.

Therefore, if we pick a vertex  $v \in V(G) \cap \Delta''$  then it holds that (G, R, k) is a  $\mathbf{tm}_{\mathcal{F}}$ -triple if and only if  $(G \setminus v, R', k)$  is a  $\mathbf{tm}_{\mathcal{F}}$ -triple. The overall running time of the whole procedure is  $2^{\mathcal{O}_h(k^2)} \cdot n$ , or, alternatively,  $\mathcal{O}(k \cdot n^3) + \mathcal{O}_h(n^3) + 2^{\mathcal{O}_h(k)} \cdot n$ .

## 6 Discussion

In this paper we prove that  $\mathcal{F}$ -TM-Deletion is Fixed Parameter Tractable on planar graphs.

## 6.1 Running time dependency on h

The parametric dependency of our FPT-algorithm is  $2^{\mathcal{O}_h(k^2)}$  and it can be dropped to  $2^{\mathcal{O}_h(k)}$  if we admit a cubic polynomial dependency on n. However both these parametric dependencies hide huge dependency on h. To estimate this, one may observe that the complexity of the dynamic programming algorithm of Lemma 3.4 dominates the overall running time of the algorithm of Theorem 1.2, in terms of the contribution of h. This permits us to estimate that the algorithm of Theorem 1.1 runs in  $2^{k^2 \cdot 2^{2^{2^{\mathcal{O}(h)}}}} \cdot n^2$  time, or, alternatively, in  $\mathcal{O}(k \cdot n^4) + 2^{2^{2^{2^{\mathcal{O}(h)}}}} \cdot n^4 + 2^{2^{2^{\mathcal{O}(h)}}}$ 

#### 6.2 Extensions to bounded genus graphs

In this subsection, we show how to extend our results to graphs of Euler genus at most  $\gamma$ . In particular, we obtain an algorithm for  $\mathcal{F}$ -TM-DELETION on graphs of Euler genus at most  $\gamma$  that runs in  $2^{\mathcal{O}_{h,\gamma}(k^2)} \cdot n^2$  time, or, alternatively, in  $\mathcal{O}_{\gamma}(k \cdot n^4) + \mathcal{O}_{h,\gamma}(n^4) + 2^{\mathcal{O}_{h,\gamma}(k)} \cdot n^{\mathcal{O}(1)}$  time.

**Theorem 6.1.** There exists an algorithm that given a finite set of graphs  $\mathcal{F}$ , a  $k \in \mathbb{N}$ , and an n-vertex graph G of Euler genus at most  $\gamma$ , outputs whether  $\mathbf{tm}_{\mathcal{F}}(G) \leq k$  in  $2^{\mathcal{O}_{h,\gamma}(k^2)} \cdot n^2$  time, or, alternatively,  $\mathcal{O}_{\gamma}(k \cdot n^4) + \mathcal{O}_{h,\gamma}(n^4) + 2^{\mathcal{O}_{h,\gamma}(k)} \cdot n^{\mathcal{O}(1)}$  time, where  $h = h(\mathcal{F})$ .

To prove Theorem 6.1, we can follow the same approach as in the proof of Theorem 1.1, i.e., reduce the problem to instances of bounded treewidth by removing vertices and reducing the set R. This is done using the following result, which is an analogue of Theorem 1.2 for graphs of bounded Euler genus.

**Lemma 6.1.** There exists a function  $f_{13}: \mathbb{N} \to \mathbb{N}$ , and an algorithm that given two integers  $k, h \in \mathbb{N}$ , an n-vertex graph G of Euler genus at most  $\gamma$ , and a set  $R \subseteq V(G)$ , outputs either a vertex  $v \in V(G)$  and a set  $R' \subseteq R$  such that, for every graph class  $\mathcal{F}$  where  $h(\mathcal{F}) \leq h$ , (G, R, k) is a  $\mathbf{tm}_{\mathcal{F}}$ -triple if and only if  $(G \setminus v, R', k)$  is a  $\mathbf{tm}_{\mathcal{F}}$ -triple or a tree decomposition of G of width at most  $f_{13}(h) \cdot k$ . Moreover, this algorithm runs in  $2^{\mathcal{O}_{h,\gamma}(k^2)} \cdot n$  time, or, alternatively,  $\mathcal{O}_{\gamma}(k \cdot n^3) + \mathcal{O}_{h,\gamma}(n^3) + 2^{\mathcal{O}_{h,\gamma}(k)} \cdot n$  time.

The proof of Lemma 6.1 is analogous to the proof of Theorem 1.2. We can use both subroutines in Section 4 (i.e., the algorithms of Lemma 4.1 and Lemma 4.2) since they are designed to work when the input graph is partially  $\Delta$ -embedded (and not necessarily planar). The only missing ingredient for the proof of Lemma 6.1 is an extension of Proposition 2.1 on graphs of bounded Euler genus.

**Lemma 6.2.** There exists a constant  $c_2$  and an algorithm that given an n-vertex graph G of Euler genus at most  $\gamma$  and an integer  $q \in \mathbb{N}_{\geq 3}$ , outputs either a disk-embedded q-wall W of G whose compass has treewidth at most  $c_2 \cdot q$  or a tree decomposition of G of width at most  $c_2 \cdot q$ . Moreover, this algorithm runs in  $2^{\mathcal{O}_{\gamma}(q^2)} \cdot n$  time, or, alternatively, in  $2^{\mathcal{O}_{\gamma}(q)} \cdot n^2$  time.

Proof. As a first step we use the single exponential 5-approximation algorithm of [8] in order to check whether  $\mathbf{tw}(G) = \mathcal{O}(q)$ . If not, we aim to find a disk-embedded q-wall, whose existence is guaranteed by the grid exclusion theorem on bounded genus graphs (see e.g., [19–21,24]). To find the q-wall, we may again use the algorithm of [1] to first detect a wall of G and then a subwall of it that is disk-embedded (see [28]). This way, we derive an algorithm running in  $2^{\mathcal{O}_{\gamma}(q^2)} \cdot n$  time. If we want to avoid the exponential dependence on  $q^2$ , we may find the q-wall by the following alternative approach: as a first step, we may find a set S of  $\mathcal{O}_{\gamma}(q)$  vertices whose removal from G will give either a planar graph of treewidth  $\Omega(q)$  or a non-planar embedded graph of face-width  $\Omega(q)$ . This can be done by successively finding minimum-size non-contractible cycles on  $\mathcal{O}(q)$  vertices by using a polynomial time algorithm (see e.g., [13, 14, 22, 27, 50]). If the outcome is that S is a set of vertices whose removal from G gives a planar graph, then, as in the planar case, we use the polynomial algorithm of [34] to find the q-wall. Otherwise, the q-wall can be found by using the polynomial algorithmic procedure described in the proof of [19, Lemma 3.3]. The overall running time of the above procedure is  $2^{\mathcal{O}_{\gamma}(q)} \cdot n^{\mathcal{O}(1)}$ .

To complete the proof of Theorem 6.1 and achieve the claimed parametric dependencies in its running times, we may adapt the dynamic programming algorithm of Lemma 3.4 which runs in  $2^{\mathcal{O}_{h,\gamma}(w\log w)} \cdot n$  time. Again, if we want to avoid the  $\log w$  contribution in the exponent, we may alternatively use dynamic programming in [7, Theorem 10.1] (based on an extension of sphere-cut decompositions called *surface-cut decompositions*) and derive a dynamic programming algorithm that runs in  $\mathcal{O}_{\gamma}(n^3) + 2^{\mathcal{O}_{h,\gamma}(w)} \cdot n$  time.

We stress that, in the above analysis, we insisted on a single-exponential dependence on k on the running time of the algorithms. The reason is that this implies that, for every finite set of graphs  $\mathcal{F}$  with detail h and every  $\gamma \in \mathbb{N}$ , the following problem is polynomially solvable.

 $(\mathcal{F}, \gamma)$ -Log-TM-Deletion

**Input**: an *n*-vertex graph G of Euler genus  $\gamma$ .

**Question:** Does G contain a set S of  $\log n$  vertices such that  $G \setminus S$  excludes every graph in  $\mathcal{F}$  as a topological minor?

#### 6.3 Recent advances on the general problem

The remaining question is whether the same result can be derived for all graphs. Recently an  $\mathcal{O}_{h,k}(n^4)$  algorithm for the general  $\mathcal{F}$ -TM-DELETION problem was proposed by Fomin et al. [25]. Using the words of [25], the parametric dependency of this algorithm, on k and h, is humongous. However, in the same paper, it was proven that better parametric dependencies can be achieved when restricting the problem to graphs of bounded Euler genus. According to the results of [25],  $\mathcal{F}$ -TM-DELETION on graphs of Euler genus at most  $\gamma$  can be solved by an algorithm running in  $2^{2^{\mathcal{O}_{h,\gamma}(k)}} \cdot n^2$  time. The algorithms claimed in Subsection 6.2 can be seen as improvements of this result.

#### 6.4 Open problems

We believe that the techniques developed in this paper can be applied to other instances of the  $\mathcal{P}$ -DELETION problem. In particular, we conjecture the following.

Conjecture 6.1. If  $\mathcal{F}$  is a finite set of graphs and  $\leq$  is the contraction relation, then the problem  $\mathcal{P}_{\mathcal{F},\leq}$ -DELETION, with inputs restricted on graphs of Euler genus at most  $\gamma$ , can be solved by an algorithm that runs in  $\mathcal{O}_{h,k}(n^c)$  time, for some constant c.

Conjecture 6.2. If  $\mathcal{F}$  is a finite set of graphs and  $\leq$  is the induced minor relation, then the problem  $\mathcal{P}_{\mathcal{F},\leq}$ -DELETION, with inputs restricted on graphs of Euler genus at most  $\gamma$ , can be solved by an algorithm that runs in  $\mathcal{O}_{h,k}(n^c)$  time, for some constant c.

A possible pathway for proving Conjecture 6.1 is to use the fact that, given an graph embedding  $\Gamma$ , edge contractions on G correspond to topological minors on the dual embedding  $\Gamma^*$ . This "translation" of contractions to topological minors was proposed in [36] in order to devise an algorithm for the problem of checking whether a graph of Euler genus at most  $\gamma$  contains a graph H as a contraction (this result is the case k=0 of Conjecture 6.1). Under this setting, the only significant change is that instead of looking for a set of vertices to remove, we must find a set of faces to "shrink". Therefore, the main missing ingredient for a proof of Conjecture 6.1 is a dynamic programming framework for this shrinking variant of the problem on surfaces.

For Conjecture 6.2 one may directly attempt to build counterparts of all the algorithms of this paper for the induced minor relation. The only missing combinatorial ingredient for this is an "induced" version of the model combing theorem (Theorem 2.1), that is proved in [31].

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