Chains, Koch Chains, and Point Sets with Many Triangulations

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Abstract

We introduce the abstract notion of a chain, which is a sequence of n points in the plane, ordered by x-coordinates, so that the edge between any two consecutive points is unavoidable as far as triangulations are concerned. A general theory of the structural properties of chains is developed, alongside a general understanding of their number of triangulations.

We also describe an intriguing new and concrete configuration, which we call the Koch chain due to its similarities to the Koch curve. A specific construction based on Koch chains is then shown to have $\Omega(9.08^n)$ triangulations. This is a significant improvement over the previous and long-standing lower bound of $\Omega(8.65^n)$ for the maximum number of triangulations of planar point sets.

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1 Introduction

Let P be a set of n points in the Euclidean plane. Throughout the paper, P is assumed to be in *general position*, which means for us that no two points have the same x-coordinate and that no three points are on a common line. A *geometric graph* on P is a graph with vertex set P combined with an embedding into the plane where edges are realized as straight-line segments between the corresponding endpoints. It is called *crossing-free* if the edges have no pairwise intersection, except possibly in a common endpoint.

Triangulations. Perhaps the most prominent and most studied family of crossing-free geometric graphs is the family of *triangulations*, which may be defined simply as edge-maximal crossing-free geometric graphs on P. It is easy to see that such a definition implies that the edges of any triangulation subdivide the convex hull of P into triangular regions.

Let $\mathrm{tr}(P)$ denote the number of triangulations on a given point set P. Trying to better understand this quantity is a fundamental question in combinatorial and computational geometry. For very specific families of point sets, exact formulas or at least asymptotic

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estimates can be derived. For example, it is well-known that if P is in *convex position*, then $\mathrm{tr}(P) = C_{n-2}$ where $C_k = \frac{1}{k+1}\binom{2k}{k} = \Theta(k^{-3/2}4^k)$ is the k-th Catalan number [1]. In general, however, this problem turns out to be much more elusive.

There is an elegant algorithm by Alvarez and Seidel [7] from 2013 that computes $\mathrm{tr}(P)$ in exponential time $O(2^n n^2)$. It was surpassed by Marx and Miltzow [16] in 2016, who showed how to compute $\mathrm{tr}(P)$ in subexponential time $n^{O(\sqrt{n})}$. Moreover, Avis and Fukuda [9] have shown already in 1996 how to enumerate the set of all triangulations on P (i.e., to compute an explicit representation of each element) by using a general technique called reverse search in time $\mathrm{tr}(P) \cdot p(n)$ for some polynomial p. A particularly efficient implementation of that technique with $p(n) = O(\log\log n)$ has been described by Bespamyatnikh [10].

Extensive research has also gone into extremal upper and lower bounds in terms of the number of points. That is, if we define

$$\mathrm{tr}_{\max}(n) = \max_{P\colon |P|=n} \mathrm{tr}(P), \qquad\qquad \mathrm{tr}_{\min}(n) = \min_{P\colon |P|=n} \mathrm{tr}(P)$$

to be the respectively largest and smallest numbers of triangulations attainable by a set P of n points in general position, then various authors have attempted to establish and improve upper and lower bounds on these quantities.

As far as the maximum is concerned, a seminal result by Ajtai, Chvátal, Newborn, and Szemerédi [6] from 1982 shows that the number of triangulations – and, more generally, the number of all crossing-free geometric graphs – is at most 10^{13n}. A long series of successive improvements [23, 11, 20, 19, 22] using a variety of different techniques has culminated in the currently best upper bound $\mathrm{tr}_{\max}(n) \le 30^n$ due to Sharir and Sheffer [21], which has remained uncontested for over a decade. Coming from the other side, attempts have been made to construct point sets with a particularly large number of triangulations. For some time, the *double chain* by García, Noy, and Tejel [17] with approximately $\Theta(8^n)$ triangulations was conjectured to have the largest possible number of triangulations. However, variants like the *double zig-zag chain* by Aichholzer et al. [5] with $\Theta(8.48^n)$ triangulations and a specific instance of the *generalized double zig-zag chain* by Dumitrescu, Schulz, Sheffer, and Tóth [12] with $\Omega(8.65^n)$ triangulations have since been discovered. But also on this front, no further progress on the lower bound $\mathrm{tr}_{\max}(n) = \Omega(8.65^n)$ has been made for a decade.

The situation for the minimum is different insofar that the *double circle* with $\Theta(3.47^n)$ triangulations, as analyzed by Hurtado and Noy [15] in 1997, is still conjectured by many to have the smallest number of triangulations. In other words, it is believed that the resulting upper bound $\mathrm{tr}_{\min}(n) = O(3.47^n)$ is best possible. On the other hand, Aichholzer et al. [4] have shown that every point set has at least $\Omega(2.63^n)$ triangulations, thereby establishing the lower bound $\mathrm{tr}_{\min}(n) = \Omega(2.63^n)$.

The focus of this paper lies on $\mathrm{tr}_{\max}(n)$ and, more specifically, on establishing an improved lower bound on that quantity. Ultimately, we show how to construct a new infinite family of point sets with $\Omega(9.08^n)$ triangulations, thereby proving $\mathrm{tr}_{\max}(n) = \Omega(9.08^n)$.

General chains. It has occurred to us that almost all families of point sets whose numbers of triangulations have been analyzed over the years have a very special structure, which we are trying to capture in the following definition.

▶ **Definition 1.** *A* chain *C is a sequence of points $p_0, \dots, p_n$ sorted by increasing x-coordinates, such that the edge $p_{i-1}p_i$ is* unavoidable *(i.e., contained in every triangulation of C) for each $i = 1, \dots, n$. These specific unavoidable edges are also referred to as* chain edges.

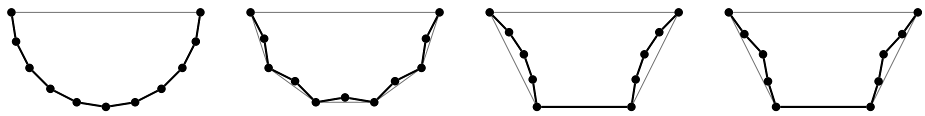


Figure 1 Some classic point sets realized as chains. For the double circle, we need to remove one of the inner points. Chain edges are displayed black and bold, other unavoidable hull edges in gray.

In contrast to previous convention, we use the parameter n to denote the number of chain edges and not the number of points in C, which is $n+1$. Also note that Definition 1 implies that the edge p_0p_n is an edge of the convex hull and, hence, also unavoidable. Indeed, since all chain edges are unavoidable, the edge p_0p_n cannot possibly cross any of them and, hence, is either above or below all the points in between. Therefore, a chain always admits a spanning cycle of unavoidable edges with at least one hull edge. We prove in Section 2 that this is also a characterization of chains in terms of *order types* (see [14] for a definition).

▶ **Theorem 2.** *For every point set that admits a spanning cycle of unavoidable edges including at least one convex hull edge, there exists a chain with the same order type.*

All of the mentioned families of point sets (convex position, double chain, and so on) are usually neither defined nor depicted in a way that makes it clear that they may be thought of as chains as in Definition 1. Still, the premise of Theorem 2 is easily verified for all of them except for the double circle, which may however be transformed into a chain by removing one of the inner points. Figure 1 shows realizations of some such point sets as chains.

Imagine walking along the chain edges and recording at each point the information whether we make a left turn or a right turn. It can be noted already now that such information – while crucial – is not enough to really capture all of the relevant combinatorial structure of a given chain. Instead, the right way of looking at it turns out to be recording for each edge p_ip_j whether it lies above or below all the chain edges in between.

The simple linear structure inherent to chains allows us to develop a combinatorial theory in Section 2, by which every chain admits a unique construction starting from the primitive chain with only one edge. Two types of sum operations, so-called convex and concave sums, are used to “concatenate” chains, while an inversion allows to “flip” a chain on its head. This yields for every chain a concise and unique description as an algebraic formula. Based on this, we will also see that the number of combinatorially different chains is equal to S_{n-1}, where $S_k = \sum_{i=0}^{k} \frac{1}{i+1}\binom{k}{i}\binom{k+i}{i} = \Theta(k^{-3/2}(3+\sqrt{8})^k)$ is the k-th large Schröder number [2].

Triangulations of chains. The unavoidable chain edges separate every triangulation cleanly into an *upper triangulation* of the region above the chain edges and into a *lower triangulation* of the region below. Therefore, both upper and lower triangulations may be analyzed separately. It also follows that there is no further complication due to inner vertices as one would typically encounter them in general point sets.

There is a simple cubic time dynamic programming algorithm for counting triangulations of simple polygons [13]. Such an algorithm can of course also be used to count both the upper and lower triangulations of a given chain. However, we show in Section 3 that the additional structure of chains allows us to devise an improved quadratic time algorithm, which plays a crucial role in the derivation of our main result.

▶ **Theorem 3.** *Given a chain C with n chain edges as input, it is possible to compute the number $\mathrm{tr}(C)$ by using only $O(n^2)$ integer additions and multiplications.*

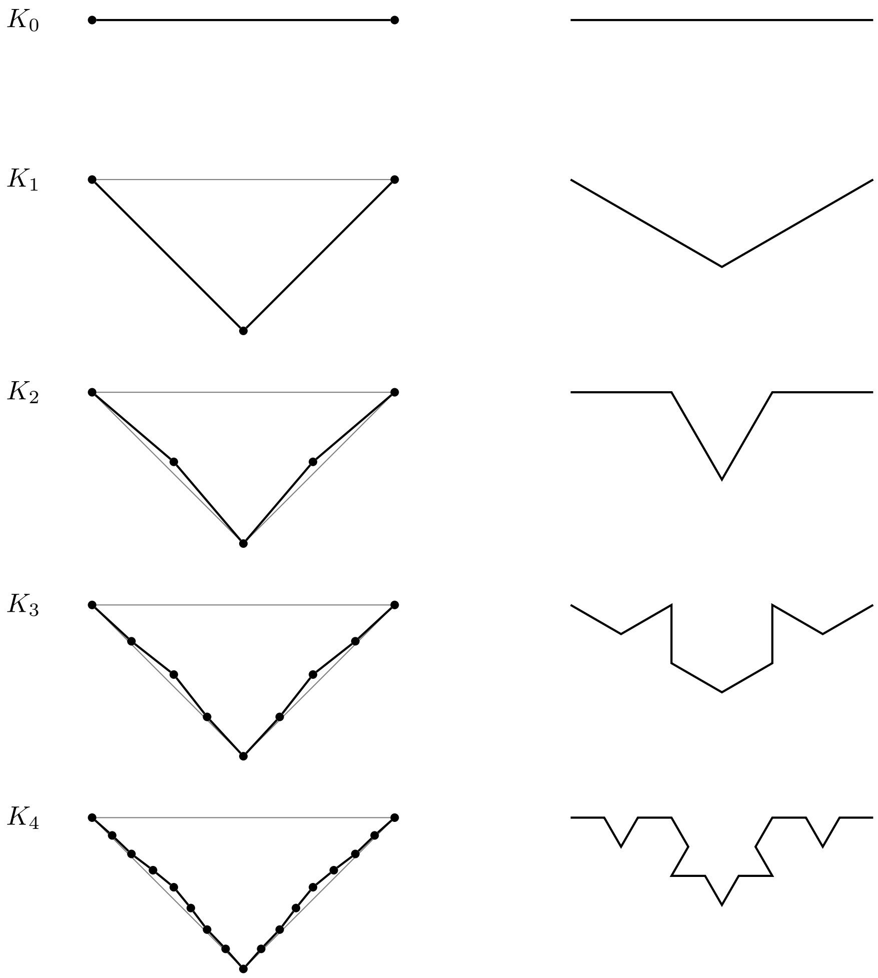


Figure 2 The Koch chains K_s for $s = 0, \ldots, 4$ and the corresponding Koch curves. Even though it is hard to recognize for larger values of s, the changes in direction along the Koch curve on the right are reflected one-to-one by the chain edges of the corresponding Koch chain on the left.

The Koch chain. There is a particular type of chain that has caught our interest and which, to the best of our knowledge, has not been described in the literature before. We call it the *Koch chain* due to its striking similarity in appearance and definition to the famous Koch curve. More precise definitions follow later in Definition 14; for now, suppose K_0 is a primitive chain with just one chain edge, and let the s-th iteration K_s of the Koch chain be defined by concatenating two flipped and sufficiently flattened copies of K_{s-1} in such a way that the chain edges at the point of concatenation form a left turn, see Figure 2.

Koch chains turn out to have a particularly large number of triangulations, much more so than any other known point sets. For values of s up to 21, we have computed the corresponding numbers of upper and lower triangulations, as well as complete triangulations, by using our algorithm from Theorem 3. The results are displayed in Table 1.

In consequence, concatenating copies of K_{21} side by side results in an infinite family of point sets with at least 9.082798^n triangulations. This alone already establishes the improved lower bound of $\mathrm{tr}_{\max}(n) = \Omega(9.082798^n)$.

Table 1 The computed numbers of triangulations of the Koch chain K_s for $s = 0, \ldots, 21$. As usual, n is the number of chain edges, whereas U, L, and T stand, respectively, for the numbers of upper, lower, and complete triangulations of the corresponding Koch chain.

s	n	$\sqrt[n]{U}$	$\sqrt[n]{L}$	$\sqrt[n]{T}$	s	n	$\sqrt[n]{U}$	$\sqrt[n]{L}$	$\sqrt[n]{T}$
0	1	1.0	1.0	1.0	11	2048	3.121029	2.858643	8.921910
1	2	1.0	1.0	1.0	12	4096	2.882177	3.121029	8.995359
2	4	1.189207	1.0	1.189207	13	8192	3.134955	2.882177	9.035496
3	8	1.791279	1.189207	2.130201	14	16384	2.889213	3.134955	9.057554
4	16	2.035453	1.791279	3.646065	15	32768	3.139056	2.889213	9.069406
5	32	2.558954	2.035453	5.208633	16	65536	2.891256	3.139056	9.075820
6	64	2.564646	2.558954	6.562814	17	131072	3.140236	2.891256	9.079229
7	128	2.935733	2.564646	7.529118	18	262144	2.891838	3.140236	9.081055
8	256	2.783587	2.935733	8.171870	19	524288	3.140569	2.891838	9.082019
9	512	3.075469	2.783587	8.560839	20	1048576	2.892001	3.140569	9.082530
10	1024	2.858643	3.075469	8.791671	21	2097152	3.140662	2.892001	9.082799

Poly chains and Twin chains. We were unable to nail down the exact asymptotic behavior of the number of triangulations of K_s as s approaches infinity. It is also unclear how much is lost due to undercounting by not considering any interactions between the different copies of K_{21} in our simple lower bound construction from just before.

To remedy the situation somewhat, in Section 4 we define and analyze more carefully the *poly-C chain* (a specific way of concatenating k copies of a given chain C) and the *twin-C chain* (a construction where two copies of a poly-C chain face each other, similar in spirit to the classic double chain). Based on these considerations, we get a slightly improved lower bound construction, and we are also able to conclude that the numbers in the last column of Table 1 will not grow significantly larger than what we already have.

▶ **Theorem 4.** *Let C_k be the twin-K_{21} chain that uses $2k$ copies of K_{21} in total. Then,*

$$\lim_{k\to\infty} \sqrt[n]{\operatorname{tr}(C_k)} = 9.083095\ldots, \qquad \operatorname{tr}_{\max}(n) = \Omega(9.083095^n).$$

▶ **Theorem 5.** *For the Koch chain K_s with $n = 2^s$ chain edges, we have*

$$9.082798 \le \lim_{s\to\infty} \sqrt[n]{\operatorname{tr}(K_s)} \le 9.083139.$$

2 Structural Properties of Chains

Recall Definition 1 from the introduction. Note that the unavoidable chain edges form an x-monotone curve $p_0p_1\ldots p_n$, to which we refer as the *chain curve*. An edge p_ip_j that is not a chain edge cannot cross the chain curve, and so it lies either above or below that curve.

▶ **Definition 6.** *To every chain C we associate a* visibility triangle $V(C)$ *with entries*

$$V(C)_{i,j} = \begin{cases} +1, & \text{if } p_ip_j \text{ lies above the chain curve;} \\ -1, & \text{if } p_ip_j \text{ lies below the chain curve;} \\ 0, & \text{if } p_ip_j \text{ is a chain edge (i.e., } i+1=j\text{);} \end{cases} \qquad (0 \le i < j \le n).$$

As an example, the visibility triangles of the chains that correspond to the classic point sets from the introduction can be seen in Figure 3.

$$\begin{array}{cccccccccc}
0 & + & + & + & + & + & + & + & + & + \\
 & 0 & + & + & + & + & + & + & + & + \\
 & & 0 & + & + & + & + & + & + & + \\
 & & & 0 & + & + & + & + & + & + \\
 & & & & 0 & + & + & + & + & + \\
 & & & & & 0 & + & + & + & + \\
 & & & & & & 0 & + & + & + \\
 & & & & & & & 0 & + & + \\
 & & & & & & & & 0 & + \\
 & & & & & & & & & 0
\end{array}$$

convex position

$$\begin{array}{cccccccccc}
0 & - & + & + & + & + & + & + & + & + \\
 & 0 & + & + & + & + & + & + & + & + \\
 & & 0 & - & + & + & + & + & + & + \\
 & & & 0 & + & + & + & + & + & + \\
 & & & & 0 & - & + & + & + & + \\
 & & & & & 0 & + & + & + & + \\
 & & & & & & 0 & - & + & + \\
 & & & & & & & 0 & + & + \\
 & & & & & & & & 0 & - \\
 & & & & & & & & & 0
\end{array}$$

"double circle"

$$\begin{array}{ccccccccc}
0 & - & - & - & + & + & + & + & + \\
 & 0 & - & - & + & + & + & + & + \\
 & & 0 & - & + & + & + & + & + \\
 & & & 0 & + & + & + & + & + \\
 & & & & 0 & + & + & + & + \\
 & & & & & 0 & - & - & - \\
 & & & & & & 0 & - & - \\
 & & & & & & & 0 & - \\
 & & & & & & & & 0
\end{array}$$

double chain

$$\begin{array}{ccccccccc}
0 & + & - & - & + & + & + & + & + \\
 & 0 & - & - & + & + & + & + & + \\
 & & 0 & + & + & + & + & + & + \\
 & & & 0 & + & + & + & + & + \\
 & & & & 0 & + & + & + & + \\
 & & & & & 0 & + & - & - \\
 & & & & & & 0 & - & - \\
 & & & & & & & 0 & + \\
 & & & & & & & & 0
\end{array}$$

double zig-zag chain

Figure 3 The visibility triangles corresponding to the chains depicted earlier in Figure 1. For improved clarity, we only display the signs of the respective entries.

For $i < j < k$, the triangle $p_ip_jp_k$ is oriented counter-clockwise if and only if p_j lies below the edge p_ip_k or, equivalently, if and only if $V(C)_{i,k} = +1$. It follows that two chains with the same visibility triangle have the same order type and, therefore, the same set of crossing-free geometric graphs and triangulations. For this reason, we consider from now on two chains to be equal if their visibility triangles are identical.

The edge p_0p_n plays a crucial role in determining the shape of a chain. For example, if $V(C)_{0,n} = +1$, then this edge is the only edge on the upper convex hull and, from a global perspective, the chain curve looks like it is curving upwards. Conversely, if $V(C)_{0,n} = -1$, then the chain curve looks like it is curving downwards. Correspondingly, we call a chain C with $V(C)_{0,n} \geq 0$ an *upward chain*, and a chain with $V(C)_{0,n} \leq 0$ a *downward chain*. This implies that every chain is either an upward or a downward chain, and the primitive chain with only $n = 1$ chain edge is the only chain that is both.

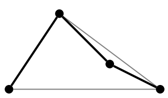
2.1 Flips

Chains may be flipped upside-down by reflection at the x-axis, thus turning an upward chain into a downward chain, and vice versa. See Figure 4 for an example.

▶ **Proposition 7.** *Let C be a chain with n chain edges. Then, there is another chain (which we denote by $\overline{C}$ and call the* flipped version *of C) with n chain edges and visibility triangle*

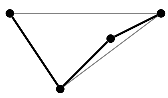
$$V(\overline{C})_{i,j} = -V(C)_{i,j} \qquad (0 \leq i < j \leq n).$$

C



$$\begin{array}{ccc}
0 & - & - \\
 & 0 & + \\
 & & 0
\end{array}$$

$\overline{C}$



$$\begin{array}{ccc}
0 & + & + \\
 & 0 & - \\
 & & 0
\end{array}$$

Figure 4 A chain C and its flipped version $\overline{C}$ with the corresponding visibility triangles.

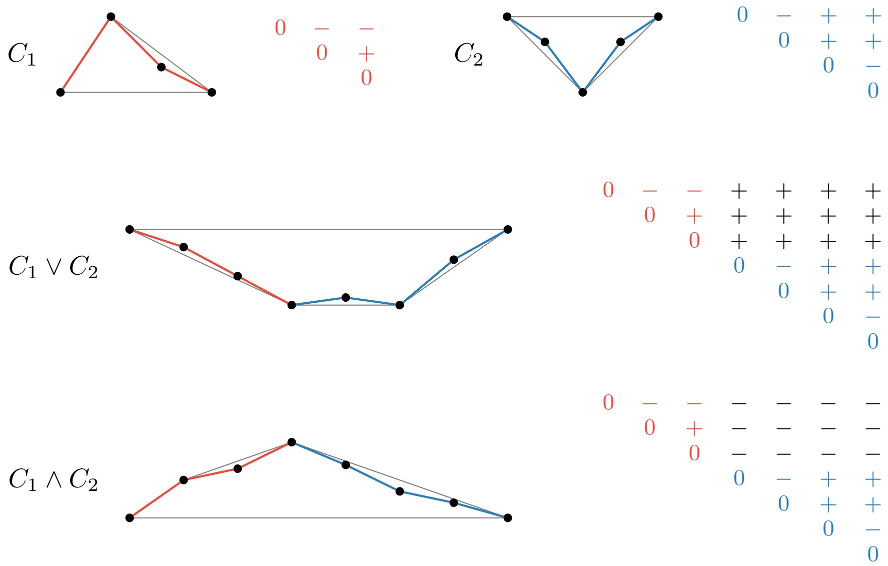


Figure 5 In the top row, two chains C_1 and C_2 with their visibility triangles. Below, the corresponding convex and concave sums $C_1 \vee C_2$ and $C_1 \wedge C_2$. Red and blue color is used to highlight the contained substructures and their origin.

2.2 Convex and Concave Sums

Given two chains C_1 and C_2, we would like to concatenate them so that we get a new chain containing C_1 and C_2 as substructures. As shown in Figure 5, there are two ways to do so.

▶ **Proposition 8.** *Let C_1 and C_2 be chains with n_1 and n_2 chain edges, respectively. Then, there is an upward chain (which we denote by $C_1 \vee C_2$ and call the* convex sum *of C_1* and *C_2) with $n_1 + n_2$ chain edges and visibility triangle*

$$V(C_1 \vee C_2)_{i,j} = \begin{cases} V(C_1)_{i,j}, & \text{if } i,j \in [0,n_1]; \\ V(C_2)_{i-n_1,j-n_1}, & \text{if } i,j \in [n_1, n_1+n_2]; \\ +1, & \text{if } i < n_1 < j; \end{cases} \qquad (0 \le i < j \le n_1 + n_2).$$

▶ **Proposition 9.** *Similarly, there is a downward chain (which we denote by $C_1 \wedge C_2$ and call the* concave sum *of C_1* and *C_2) with $n_1 + n_2$ chain edges and visibility triangle*

$$V(C_1 \wedge C_2)_{i,j} = \begin{cases} V(C_1)_{i,j}, & \text{if } i,j \in [0,n_1]; \\ V(C_2)_{i-n_1,j-n_1}, & \text{if } i,j \in [n_1, n_1+n_2]; \\ -1, & \text{if } i < n_1 < j; \end{cases} \qquad (0 \le i < j \le n_1 + n_2).$$

Proof of Proposition 8. We focus on the convex sum; the proof for the concave sum is analogous. We have to show that there is a point set that forms a chain with the specified visibility triangle. Intuitively speaking, this is achieved by first flattening the two given chains and then arranging them in a $\vee$-shape.

To be more precise, we employ vertical shearings, which are maps $(x, y) \mapsto (x, y + \lambda x)$ in $\mathbb{R}^2$ for some $\lambda \in \mathbb{R}$. Vertical shearings preserve signed areas and x-coordinates. Hence, if a point set realizes a specific chain, then so does its image under any vertical shearing.

With the help of an appropriate vertical shearing, we may realize C_1 as a point set in the rectangle $[-1,0] \times [-1,1]$ in such a way that the first point is at $(-1,0)$ and the last point is at $(0,0)$. Then, given any $\varepsilon \geq 0$, we may rescale vertically to get a point set $Q_1(\varepsilon)$ in the rectangle $[-1,0] \times [-\varepsilon,\varepsilon]$. Let now $R_1(\varepsilon)$ be the image of $Q_1(\varepsilon)$ under the vertical shearing with $\lambda = -1$. Then, the first point of $R_1(\varepsilon)$ lies at $(-1,1)$, while the last point remains at $(0,0)$. For $\varepsilon > 0$, since $Q_1(\varepsilon)$ is a realization of C_1, so is $R_1(\varepsilon)$. On the other hand, for $\varepsilon = 0$, the points of $R_1(\varepsilon)$ all lie on the segment between $(-1,1)$ and $(0,0)$.

With C_2 we proceed similarly to get a point set $Q_2(\varepsilon)$ in the rectangle $[0,1] \times [-\varepsilon,\varepsilon]$, but we now apply the vertical shearing with $\lambda = 1$ to get $R_2(\varepsilon)$ with the first point at $(0,0)$ and the last point at $(1,1)$.

Let $T(\varepsilon) = R_1(\varepsilon) \cup R_2(\varepsilon)$. We claim that for $\varepsilon > 0$ small enough, $T(\varepsilon)$ is a chain with visibility triangle $V(C_1 \vee C_2)$ as specified. Indeed, as $R_i(\varepsilon)$ is a realization of C_i, we only need to check that the edges between any point of $R_1(\varepsilon)$ and any point of $R_2(\varepsilon)$ (excluding the common point at the origin) lie above all the points in between. Since this is the case for $\varepsilon = 0$ and $T(\varepsilon)$ depends continuously on ε, the claim follows. ◀

2.3 Algebraic Properties

Using the formulas for the visibility triangles from the corresponding transformations in Propositions 7–9, it can be checked easily that the following algebraic laws hold.

▶ **Lemma 10.** *Let C_1, C_2, C_3 be arbitrary chains. Then, the following are all true.*

Involution: $\overline{\overline{C_1}} = C_1$

De Morgan: $\overline{C_1 \vee C_2} = \overline{C_1} \wedge \overline{C_2}$, $\quad \overline{C_1 \wedge C_2} = \overline{C_1} \vee \overline{C_2}$

Associativity: $(C_1 \vee C_2) \vee C_3 = C_1 \vee (C_2 \vee C_3)$, $\quad (C_1 \wedge C_2) \wedge C_3 = C_1 \wedge (C_2 \wedge C_3)$

However, note that for example $(C_1 \wedge C_2) \vee C_3$ is not the same chain as $C_1 \wedge (C_2 \vee C_3)$.

2.4 Examples

We denote by E the primitive chain with only $n = 1$ chain edge; that is, the visibility triangle has just the entry $V(E)_{0,1} = 0$. Using this as a building block, we may define two more fundamental chains, the *convex chain* $C_{\mathsf{cvx}}(n)$ and the *concave chain* $C_{\mathsf{ccv}}(n)$, by setting

$$C_{\mathsf{cvx}}(n) = \underbrace{E \vee \cdots \vee E}_{n \text{ copies}}, \qquad C_{\mathsf{ccv}}(n) = \underbrace{E \wedge \cdots \wedge E}_{n \text{ copies}}.$$

The convex chain is an upward chain, while the concave chain is a downward chain. Also, since $\overline{E} = E$, we get $\overline{C_{\mathsf{cvx}}(n)} = C_{\mathsf{ccv}}(n)$ by using De Morgan's law. Finally, note that $C_{\mathsf{cvx}}(n)$ and $C_{\mathsf{ccv}}(n)$ are distinct as chains, even though they both are in convex position.

As already mentioned in the introduction, many previously studied point sets are in fact chains, or can be seen as such. Using flips as well as convex and concave sums, we can now describe these configurations with very concise formulas.

▶ **Example 11.** The *double chain* with $n = 2k + 1$ chain edges is the chain

$$C_{\mathsf{dbl}}(n) = C_{\mathsf{ccv}}(k) \vee E \vee C_{\mathsf{ccv}}(k).$$

▶ **Example 12.** The *zig-zag chain* with $n = 2k$ chain edges (which, in essence, is a double circle with one of the inner points removed) is the chain

$$C_{\mathsf{zz}}(n) = \underbrace{C_{\mathsf{ccv}}(2) \vee \cdots \vee C_{\mathsf{ccv}}(2)}_{k \text{ copies}}.$$

▶ **Example 13.** The *double zig-zag chain* with $n = 4k + 1$ chain edges is the chain

$$C_{\mathsf{dzz}}(n) = \overline{C_{\mathsf{zz}}(2k)} \vee E \vee \overline{C_{\mathsf{zz}}(2k)}.$$

All these examples involve formulas of constant nesting depth only. But the tools developed up to this point allow us to also define more complicated chains via formulas of non-constant nesting depth, without having to worry about questions of realizability. One such chain with logarithmic nesting depth is indeed the Koch chain.

▶ **Definition 14.** *The* Koch chain K_s *is an upward chain with* $n = 2^s$ *chain edges, defined recursively via* $K_0 = E$ *and* $K_s = \overline{K_{s-1}} \vee \overline{K_{s-1}}$ *for all* $s \geq 1$*.*

Indeed, after expanding the recursive definition twice and using De Morgan's law on both sides, we see that the formula $K_s = (K_{s-2} \wedge K_{s-2}) \vee (K_{s-2} \wedge K_{s-2})$ has a complete binary parse tree with alternating convex and concave sums on any path from the root to a leaf.

2.5 Unique Construction

We want to prove the following result. In essence, it states that every chain can be constructed in a unique way by using only convex and concave sums.

▶ **Theorem 15.** *Every chain can be expressed as a formula involving convex sums, concave sums, parentheses, and copies of the primitive chain with only one chain edge. This formula is unique up to redundant parentheses (redundant due to associativity as in Lemma 10).*

In particular, the above theorem allows us to encode a chain with $O(n)$ bits (as opposed to the $O(n^2)$ bits required for the visibility triangle) and to easily enumerate all chains of a fixed size. We further see that the number of upward chains is given by the little Schröder numbers [3] and the number of all chains is given by the large Schröder numbers [2].

The theorem follows by induction from the following proposition (and from an analogous proposition that expresses downward chains as a unique concave sum of upward chains).

▶ **Proposition 16.** *Let* C *be an upward chain with* $n > 1$ *chain edges. Suppose that the lower convex hull of* C *is* $p_{i_0}p_{i_1}\dots p_{i_k}$ *with* $0 = i_0 < \cdots < i_k = n$*. For* $j = 1, \dots, k$*, let* C_j *be the chain with points* $p_{i_{j-1}}, \dots, p_{i_j}$*. Then, each* C_j *is a downward chain. Moreover,* $C = C_1 \vee \cdots \vee C_k$ *and any formula that evaluates to* C *has the same top-level structure.*

Proof. As $p_{i_{j-1}}p_{i_j}$ is an edge of the lower convex hull of C, it is below all the points in between. Hence, each C_j is indeed a downward chain.

To prove $C = C_1 \vee \cdots \vee C_k$, we have to show that both chains have the same visibility triangle. By definition of the C_j, the visibility triangles clearly agree on all entries that stem from an edge $p_a p_b$ where p_a and p_b are both part of the same C_j. On the other hand, if p_a and p_b are not part of the same C_j, then there is a j with $a < i_j < b$. As p_{i_j} is a vertex of the lower convex hull, it lies below the edge $p_a p_b$ and hence $V(C)_{a,b} = +1$. But this is precisely what we also get for the visibility triangle of the convex sum $C_1 \vee \cdots \vee C_k$.

For uniqueness, suppose we are given any formula for C. Since C is assumed to be an upward chain and since any concave sum is a downward chain, the formula must be of the form $C'_1 \vee \cdots \vee C'_{k'}$. We may further assume that each C'_j is a downward chain by omitting redundant parentheses. Observe now that in any such convex sum of downward chains, the resulting lower convex hull is determined by the points that are shared by any two consecutive chains C'_j. Since the given formula evaluates to C, we must have $k' = k$ and $C'_j = C_j$. ◀

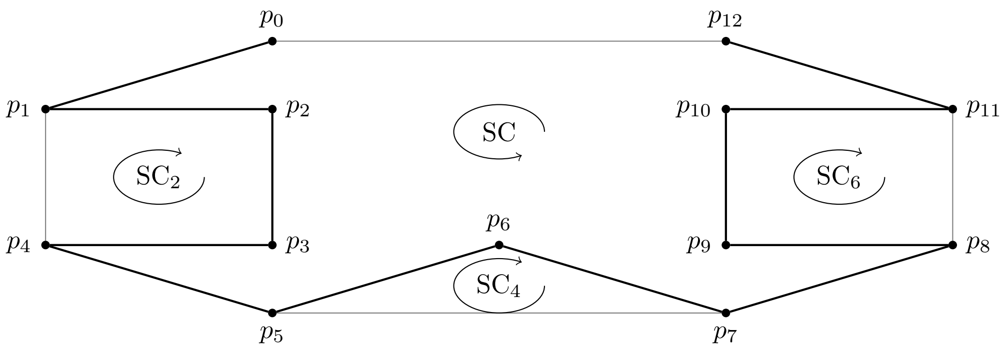


Figure 6 The situation in the proof of Theorem 2. Beware that this is just a sketch; in reality, the pockets would need to be much more narrow in order to make all edges of SC unavoidable.

2.6 Geometric Characterization

As already mentioned in the introduction, the chain edges together with the hull edge p_0p_n form a spanning cycle of unavoidable edges. We are now ready to prove that this property characterizes chains geometrically.

Proof of Theorem 2. Let $\mathrm{SC} = p_0p_1 \dots p_n$ be the spanning cycle in counter-clockwise order, with p_0p_n an edge of the convex hull, which we call the *base edge*. As SC consists of unavoidable edges only, it cannot be crossed by any edge that is not part of SC. Hence, we can associate a visibility triangle with the given point set, similar to the visibility triangle of a chain, by setting

$$V_{i,j} = \begin{cases} +1, & \text{if } p_ip_j \text{ is inside SC or the base edge;} \\ -1, & \text{if } p_ip_j \text{ is outside SC;} \\ 0, & \text{if } p_ip_j \text{ is part of SC (i.e., } i+1=j\text{);} \end{cases} \qquad (0 \le i < j \le n).$$

By using that p_0p_n is a hull edge and by some geometric considerations[2], one can then show that for $i < j < k$, the triangle $p_ip_jp_k$ is oriented counter-clockwise if and only if $V_{i,k} = +1$. Hence, it suffices to construct a chain whose visibility triangle agrees with V.

Let $p_{i_0}, p_{i_1}, \dots, p_{i_k}$ be the vertices of the convex hull with $0 = i_0 < \dots < i_k = n$. For $1 \le j \le k$, let $P_j = \{p_{i_{j-1}}, \dots, p_{i_j}\}$. We now see that either P_j consists of only two points or that it admits a spanning cycle of unavoidable edges, namely $\mathrm{SC}_j = p_{i_{j-1}}p_{i_{j-1}+1} \dots p_{i_j}$ with base edge $p_{i_{j-1}}p_{i_j}$. The situation is depicted in Figure 6. Note that the inside of SC_j is outside of SC. In fact, SC_j forms a so-called pocket of SC, which means that all edges of the cycle SC_j except for $p_{i_{j-1}}p_{i_j}$ are also edges of SC.

By induction, there is a chain C_j with the same order type as P_j, that is, with

$$V(C_j)_{a,b} = \begin{cases} +1, & \text{if } p_{i_{j-1}+a}p_{i_{j-1}+b} \text{ is inside } \mathrm{SC}_j \text{ or the base edge;} \\ -1, & \text{if } p_{i_{j-1}+a}p_{i_{j-1}+b} \text{ is outside } \mathrm{SC}_j\text{;} \\ 0, & \text{if } p_{i_{j-1}+a}p_{i_{j-1}+b} \text{ is part of } \mathrm{SC}_j \text{ (i.e., } a+1=b\text{);} \end{cases} \qquad (0 \le a < b \le i_j - i_{j-1}).$$

[2] This involves a lengthy case distinction that does not add much insight. We omit the details here.

Let us consider the flipped version $\overline{C_j}$. As noted before, the inside of SC_j is outside of SC. As SC_j moreover forms a pocket of SC, any edge outside of SC_j is inside SC. Hence,

$$V(\overline{C_j})_{a,b} = \begin{cases} +1, & \text{if } p_{i_{j-1}+a}p_{i_{j-1}+b} \text{ is inside SC;} \\ -1, & \text{if } p_{i_{j-1}+a}p_{i_{j-1}+b} \text{ is outside SC;} \\ 0, & \text{if } p_{i_{j-1}+a}p_{i_{j-1}+b} \text{ is part of SC (i.e., } a+1=b\text{);} \end{cases} \qquad (0 \le a < b \le i_j - i_{j-1}).$$

We claim that $C = \overline{C_1} \vee \cdots \vee \overline{C_k}$ has the desired visibility triangle V. We have just seen that the entries stemming from the individual $\overline{C_j}$ are correct. So, all that is left to observe is that edges between different pockets lie inside of SC, which is indeed the case. ◀

3 Triangulations of Chains

In the previous section, we have seen that any chain can be expressed as a formula involving only convex and concave sums. Our goal here is to understand how triangulations behave with respect to such convex and concave sums. In order for this to work out, we have to consider not just triangulations, but a more general notion of partial triangulations.

We start by decomposing triangulations of a chain C into an upper and a lower part. An edge p_ip_j is an *upper edge* if $V(C)_{i,j} = +1$, a *chain edge* if $V(C)_{i,j} = 0$, and a *lower edge* if $V(C)_{i,j} = -1$. That is, upper edges lie above the chain curve, while lower edges lie below.

▶ **Definition 17.** *An* upper (lower) triangulation *of a given chain C is a crossing-free geometric graph on C that is edge-maximal subject to only containing chain edges and upper (lower) edges. We denote the number of upper and lower triangulations by $U(C)$ and $L(C)$, respectively, and as always the number of (complete) triangulations by* $\mathrm{tr}(C)$.

Note that the chain edges are contained in every upper and lower triangulation. Moreover, every triangulation is the union of a unique upper and a unique lower triangulation, which implies $\mathrm{tr}(C) = U(C) \cdot L(C)$. A lower triangulation of a chain C is an upper triangulation of the flipped version $\overline{C}$, and therefore $L(C) = U(\overline{C})$. For this reason, we may restrict our attention to studying only upper triangulations.

Intuitively speaking, we can create a partial upper triangulation by combining all the chain edges with some upper edges, in such a way that all bounded faces are triangles. Note that then, only some of the used edges are visible from above.

▶ **Definition 18.** *Let C be any chain with n chain edges, and let $V = p_{i_0}p_{i_1}\dots p_{i_v}$ with $0 = i_0 < i_1 < \cdots < i_v = n$ be an (x-monotone) curve composed of chain edges and upper edges only. A* partial upper triangulation of C (with visible edges V) *consists of all chain edges, all edges in V, and a triangulation of the areas between the two.*

Figure 7 depicts some partial upper triangulations and their visible edges. We are interested in counting such triangulations parameterized by the number of triangles. It can be noted that a partial upper triangulation with k triangles has $n-k$ visible edges.

▶ **Definition 19.** *Let C be any chain with n chain edges. For $k = 0, \dots, n-1$, let $t_k(C)$ be the number of partial upper triangulations of C with k triangles (i.e., with $n-k$ visible edges). The* upper triangulation polynomial of C *is the corresponding generating function*

$$T_C(x) = \sum_{k=0}^{n-1} t_k(C)x^k.$$

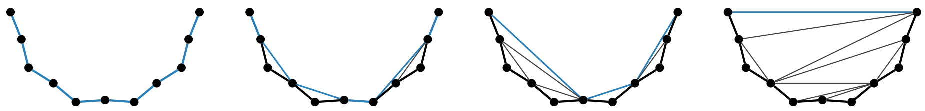


Figure 7 Four partial upper triangulations of the "double circle" with ten, six, three, and one visible edge, respectively. As usual, chain edges are in bold, while visible edges are in blue.

As an example, enumerating all partial upper triangulations of the convex chain $C_{\mathsf{cvx}}(4)$ shows that $T_{C_{\mathsf{cvx}}(4)}(x) = 1 + 3x + 5x^2 + 5x^3$. In general, note that for every chain C we have $t_0(C) = 1$ and that the leading coefficient of $T_C(x)$ is equal to $U(C)$. Moreover, we may again think of $T_{\overline{C}}(x)$ as the "lower triangulation polynomial" of C.

3.1 Convex and Concave Sums

Let us start with the easy case. For concave sums, we can establish the following relation.

▶ **Lemma 20.** *A partial upper triangulation of $C_1 \wedge C_2$ is the union of a unique partial upper triangulation of C_1 and a unique partial upper triangulation of C_2. Hence,*

$$T_{C_1 \wedge C_2}(x) = T_{C_1}(x) \cdot T_{C_2}(x), \qquad\qquad U(C_1 \wedge C_2) = U(C_1) \cdot U(C_2).$$

Convex sums are more tricky. The main insight is that every partial upper triangulation of $C_1 \vee C_2$ consists of a partial upper triangulation of C_1, a partial upper triangulation of C_2, and some edges between C_1 and C_2. More precisely:

▶ **Proposition 21.** *There is a triangle-preserving bijection between*

- *all triples (T_1, T_2, T_3) where T_1 is a partial upper triangulation of C_1 (with v_1 visible edges), T_2 is a partial upper triangulation of C_2 (with v_2 visible edges), and T_3 is a partial upper triangulation of the convex sum $C_{\mathsf{ccv}}(v_1) \vee C_{\mathsf{ccv}}(v_2)$, and*
- *all partial upper triangulations of $C_1 \vee C_2$.*

This bijection is defined by taking the union of all triangles, see Figure 8. The proposition then directly implies the following equation for the upper triangulation polynomial.

▶ **Lemma 22.** *Let C_1 and C_2 be chains with n_1 and n_2 chain edges, respectively. Then,*

$$T_{C_1 \vee C_2}(x) = \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} t_{k_1}(C_1) \cdot t_{k_2}(C_2) \cdot x^{k_1+k_2} \cdot T_{C_{\mathsf{ccv}}(n_1-k_1) \vee C_{\mathsf{ccv}}(n_2-k_2)}(x).$$

Let us consider the special case of a convex sum of two concave chains with n_1 and n_2 chain edges, respectively. Note that any partial upper triangulation of such a chain has at most one upper edge that is visible. Summing over all possibilities for that edge, we get

$$T_{C_{\mathsf{ccv}}(n_1) \vee C_{\mathsf{ccv}}(n_2)}(x) = 1 + \sum_{l=1}^{n_1} \sum_{r=1}^{n_2} \binom{l+r-2}{l-1} x^{l+r-1}.$$

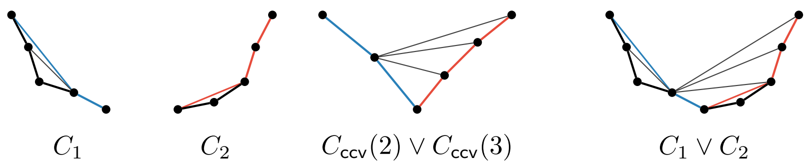


Figure 8 From left to right, the respective partial upper triangulations T_1 of C_1, T_2 of C_2, T_3 of $C_{\mathsf{ccv}}(2) \vee C_{\mathsf{ccv}}(3)$, and the resulting partial upper triangulation of $C_1 \vee C_2$ as in Proposition 21.

Combining the above equation with Lemma 22 allows us to compute $T_{C_1 \vee C_2}(x)$ from $T_{C_1}(x)$ and $T_{C_2}(x)$. Furthermore, by comparing the leading coefficients in the formulas from Lemmas 20 and 22, we get the following obvious but important fact.

▶ **Corollary 23.** *$C_1 \vee C_2$ has at least as many upper triangulations as $C_1 \wedge C_2$. That is,*

$$U(C_1 \vee C_2) \geq U(C_1 \wedge C_2).$$

Finally, note that the two chains $C_1 \vee C_2$ and $C_2 \vee C_1$ can be quite different from a geometric point of view. But in terms of the number of triangulations, they are the same.

▶ **Corollary 24.** *For any two chains C_1 and C_2, we have*

$$T_{C_1 \vee C_2}(x) = T_{C_2 \vee C_1}(x), \qquad\qquad T_{C_1 \wedge C_2}(x) = T_{C_2 \wedge C_1}(x).$$

3.2 Dynamic Programming

In this subsection, we show how to use dynamic programming in order to speed up the computations for a convex sum. To simplify the analysis, we assume a computational model where all additions and multiplications take only constant time.

▶ **Proposition 25.** *Let C_1 and C_2 be chains with n_1 and n_2 chain edges, respectively. Given the coefficients of $T_{C_1}(x)$ and $T_{C_2}(x)$, we can compute $T_{C_1 \vee C_2}(x)$ in $O(n_1 n_2)$ time.*

Recall that by Theorem 15, we can write any chain C as a formula involving only convex sums, concave sums, and primitive chains with only one chain edge. Therefore, using Proposition 25 for convex sums and Lemma 20 for concave sums, we are able to compute $T_C(x)$ in quadratic time. Clearly, this proves Theorem 3 from the introduction.

Proof of Proposition 25. Observe that every partial upper triangulation of $C_1 \vee C_2$ either corresponds to a partial upper triangulation of $C_1 \wedge C_2$, or it has a unique visible upper edge that connects a vertex of C_1 with a vertex of C_2. Let us call this edge the *bridge*. Let further $\mathrm{DP}[l][r]$ be the number of partial upper triangulations whose visible edges consist of l visible edges in C_1, followed by the bridge, followed by r visible edges in C_2. Then,

$$T_{C_1 \vee C_2}(x) = T_{C_1 \wedge C_2}(x) + \sum_{l=0}^{n_1-1} \sum_{r=0}^{n_2-1} \mathrm{DP}[l][r] \cdot x^{n_1+n_2-l-r-1}.$$

To compute the table DP, let us see what happens when we remove the bridge. We either end up with a partial upper triangulation of $C_1 \wedge C_2$ with $l+1$ and $r+1$ visible edges in C_1 and C_2, respectively, or we get a new bridge, which used to be an edge of the triangle below the old bridge. In the latter case, depending on which of the two possible edges this is, we end up with one more visible edge in either C_1 or C_2. Figure 9 depicts these three cases. To summarize, for all l and r ($0 \leq l < n_1, 0 \leq r < n_2$),

$$\mathrm{DP}[l][r] = t_{n_1-l-1}(C_1) \cdot t_{n_2-r-1}(C_2) + \mathrm{DP}[l+1][r] + \mathrm{DP}[l][r+1],$$

with the base case $\mathrm{DP}[n_1][r] = \mathrm{DP}[l][n_2] = 0$. Therefore, filling up the table DP takes $O(n_1 n_2)$ time, as desired. ◀

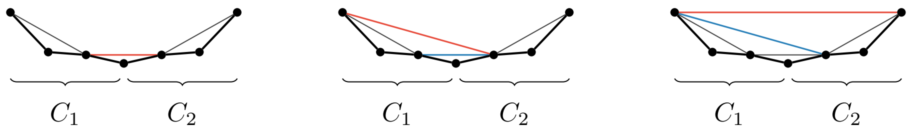


Figure 9 The three cases when removing the bridge from a partial upper triangulation of $C_1 \vee C_2$ in the proof of Proposition 25. On the left, both C_1 and C_2 gain a visible edge. In the middle, only C_1 gains a visible edge. On the right, only C_2 gains a visible edge. The current bridge is red, and the edge that becomes the new bridge is blue.

3.3 Koch Chains

Recall Definition 14 and that the formula for Koch chains expands to the nested expression

$$K_s = (K_{s-2} \wedge K_{s-2}) \vee (K_{s-2} \wedge K_{s-2})$$

with alternating convex and concave sums. This repeated mixing of the two types of sums appears to make an exact analysis of the number of triangulations of K_s very difficult.

Instead, we have implemented the quadratic time algorithm from the previous subsection and used it to compute $T_{K_s}(x)$ and $T_{\overline{K_s}}(x)$ for all $s \le 21$. To deal with the exponentially growing coefficients, we rely on a custom floating point type with a 64 bit mantissa and a 32 bit exponent from the boost multiprecision library. As only additions and multiplications are involved, we do not have to deal with numerical issues; in fact, the rounding errors grow at most linearly. In addition, we make use of multi-threading and take advantage of symmetries of K_s for a constant factor speed-up. This allows us to compute $T_{K_{21}}(x)$ in around a day on a regular workstation (Intel i7-6700HQ, 2.6GHz).

Table 1 from the introduction lists the resulting numbers. For example, K_{21} has approximately 9.082799^n triangulations, where $n = 2^{21}$. In the next section, we show how the computed coefficients of $T_{K_{21}}(x)$ can be used to give bounds on $\mathrm{tr}(K_s)$ as $s \to \infty$.

4 Poly Chains and Twin Chains

Let C_0 be a chain with m chain edges. We want to define two particular families of chains that can be built from many copies of C_0 via concave and convex sums.

▶ **Definition 26.** *For $N \ge 1$, the* poly-C_0 chains (of length $n = mN$) *are the chains*

$$C_{\mathsf{poly}}(C_0, N) = \underbrace{\overline{C_0} \vee \cdots \vee \overline{C_0}}_{N \text{ copies}}.$$

▶ **Definition 27.** *For $N \ge 1$, the* twin-C_0 chains (of length $n = 2mN + 1$) *are the chains*

$$C_{\mathsf{twin}}(C_0, N) = \overline{C_{\mathsf{poly}}(C_0, N)} \vee E \vee \overline{C_{\mathsf{poly}}(C_0, N)}.$$

Note that both resulting chains are upward chains, as long as $N > 1$. For example, the poly-E chains are the convex chains, the twin-E chains are the classic double chains, and the twin-$(E \vee E)$ chains are the double zig-zag chains.

We are interested in the asymptotic behavior of the number of triangulations of these constructions as $N \to \infty$. Lemma 20 gives us the number of lower triangulations.

$$L(C_{\mathsf{poly}}(C_0, N)) = U(C_0 \wedge \cdots \wedge C_0) = U(C_0)^N$$
$$L(C_{\mathsf{twin}}(C_0, N)) = U(C_{\mathsf{poly}}(C_0, N) \wedge E \wedge C_{\mathsf{poly}}(C_0, N)) = U(C_{\mathsf{poly}}(C_0, N))^2$$

For the upper triangulations, we make use of the following general result.

▶ **Theorem 28.** *The chains $C_{\mathsf{poly}}(C_0, N)$ have $\widetilde{\Theta}(\lambda^n)$ upper triangulations, while the chains $C_{\mathsf{twin}}(C_0, N)$ have $\widetilde{\Theta}(\tau^n)$ upper triangulations, where*

$$\lambda = \sqrt[m]{\sum_{k=1}^{m} 2^k (k+1) \cdot t_{m-k}(\overline{C_0})}, \qquad \tau = \sqrt[m]{\sum_{k=1}^{m} 2^k \cdot t_{m-k}(C_0)}.$$

It follows that the chains $C_{\mathsf{twin}}(C_0, N)$ have $\widetilde{\Theta}((\lambda\tau)^n)$ complete triangulations.

▶ **Example 29.** Let us analyze the poly-$C_{\mathsf{cvx}}(4)$ chains and twin-$C_{\mathsf{cvx}}(4)$ chains. We have

$$T_{\overline{C_{\mathsf{cvx}}(4)}}(x) = 1, \qquad T_{C_{\mathsf{cvx}}(4)}(x) = 1 + 3x + 5x^2 + 5x^3,$$

which yields $\lambda = \sqrt[4]{80}$ and $\tau = \sqrt[4]{70}$. Therefore, the twin-$C_{\mathsf{cvx}}(4)$ chains have $\widetilde{\Theta}(\sqrt[4]{5600}^{\,n})$ triangulations, where $\sqrt[4]{5600} \approx 8.6506154$. Note that these chains are the generalized double zig-zag chains from [12]. By comparison, the numerical bound there was $\widetilde{\Omega}(8.6504^n)$.

Using the coefficients of $T_{K_{21}}(x)$ and $T_{\overline{K_{21}}}(x)$ that we computed with our algorithm, we can also analyze the twin-K_{21} chains and, therefore, prove Theorem 4 from the introduction.

▶ **Corollary 30.** *The chains $C_{\mathsf{twin}}(K_{21}, N)$ have $\widetilde{\Theta}(\lambda^n)$ triangulations, for $\lambda \approx 9.083095$.*

The next lemma, combined with the first part of Theorem 28, can further be used to show asymptotic upper bounds for families of chains that are built from the same C_0.

▶ **Lemma 31.** *Let C be any chain that can be written as a formula involving convex sums, concave sums and exactly N copies of C_0. Then,*

$$U(C_0)^N \le U(C) \le U(C_{\mathsf{poly}}(\overline{C_0}, N)).$$

Proof. Use induction on N with Corollary 23. ◀

▶ **Corollary 32.** *In the same setting, we have*

$$\mathrm{tr}(C_0)^N \le \mathrm{tr}(C) \le U(C_{\mathsf{poly}}(C_0, N)) \cdot U(C_{\mathsf{poly}}(\overline{C_0}, N)).$$

Proof. Apply Lemma 31 twice. First to C with C_0, then to $\overline{C}$ with $\overline{C_0}$. ◀

The Koch chains K_s with $s \ge 21$ can be written as formulas involving copies of K_{21}, so Corollary 32 applies to them. We get $9.082798^n \le \mathrm{tr}(K_s) \le 9.083139^n$, as in Theorem 5.

4.1 Tools for the proof of Theorem 28

We only sketch the main steps here. We use similar ideas as Section 2 of [12] with three improvements that yield an exact $\widetilde{\Theta}$ instead of a numerical lower bound. The first improvement is that our chain framework allows us to analyze even more general "double circles".

▶ **Theorem 33.** *Let $c_1, \dots, c_m \ge 0$ be integers. Define*

$$V(c_1, \dots, c_m) = C_{\mathsf{poly}}(C_{\mathsf{cvx}}(1), c_1) \vee \dots \vee C_{\mathsf{poly}}(C_{\mathsf{cvx}}(m), c_m)$$

where we omit poly chains with $c_k = 0$. Then,

$$U(V(c_1, \dots, c_m)) \in \widetilde{\Omega}\Big(\prod_{k=1}^{m} \left(2^k (k+1)\right)^{c_k}\Big)$$

where the polynomial factors in the $\widetilde{\Omega}$ only depend on m (and not on the c_k).

Proof. By Corollary 23 and Corollary 24, we get

$$U(V(c_1,\dots,c_m)) \geq U(C_{\mathsf{poly}}(C_{\mathsf{cvx}}(1),c_1))\cdots U(C_{\mathsf{poly}}(C_{\mathsf{cvx}}(m),c_m)).$$

In [8] it is shown that $U(C_{\mathsf{poly}}(C_{\mathsf{cvx}}(k),N)) \in \widetilde{\Omega}((2^k(k+1))^N)$. ◀

The second improvement is to replace the numerical optimization in [12] by this lemma.

▶ **Lemma 34.** *Let $u_1,\dots,u_m \geq 0$ be given. Let $H(\alpha_1,\dots,\alpha_m) = -\sum_k \alpha_k \ln \alpha_k$ be the entropy function. Then,*

$$\max_{\substack{0\leq\alpha_1,\dots,\alpha_m\leq 1\\ \alpha_1+\cdots+\alpha_m=1}} e^{H(\alpha_1,\dots,\alpha_m)} \cdot \prod_{k=1}^m u_k^{\alpha_k} = \sum_{k=1}^m u_k.$$

Proof. Without loss of generality, assume that $u_k > 0$. Then, by Lagrange multipliers, the only maximum is at $\alpha_k = u_k/(u_1+\cdots+u_m)$. ◀

The third improvement is a special type of generating function that behaves well with regards to convex sums, allowing us to prove a matching upper bound for Theorem 33.

▶ **Definition 35.** *Let C be a chain of length n. The* triangulation generating function *is*

$$\phi_C(x) := T_C(x) - \left(\frac{x}{1-x}\right)^{n+1} T_C(1-x).$$

Note that $\phi_C(x)$ is a rational function. As a formal power series, $\phi_C(x) = T_C(x) + O(x^{n+1})$.

▶ **Theorem 36.** *For any two chains C_1 and C_2, we have*

$$\phi_{C_1\vee C_2}(x) = \phi_{C_1}(x)\cdot\phi_{C_2}(x)\cdot\frac{1-x}{1-2x}.$$

Proof. By Lemma 22, it suffices to prove this for $C_i = C_{\mathsf{ccv}}(n_i)$. We have

$$\phi_{C_{\mathsf{ccv}}(n)}(x) = 1 - \left(\frac{x}{1-x}\right)^{n+1}, \qquad T_{C_{\mathsf{ccv}}(n_1)\vee C_{\mathsf{ccv}}(n_2)}(x) = 1 + \sum_{l=1}^{n_1}\sum_{r=1}^{n_2}\binom{l+r-2}{l-1}x^{l+r-1}.$$

Then, induction on (n_1,n_2) and raw computations on power series suffice. ◀

▶ **Corollary 37.** *We have*

$$U(V(c_1,\dots,c_m)) \leq \prod_{k=1}^m (2^k(k+1))^{c_k}.$$

Proof. Let $n = c_1 + 2c_2 + \cdots + mc_m$ be the length of $V(c_1,\dots,c_m)$. Theorem 36 allows us to compute $\phi_{V(c_1,\dots,c_m)}$. We have $U(V(c_1,\dots,c_m)) = [x^{n-1}]\phi_{V(c_1,\dots,c_m)}(x)$, so we compute

$$[x^{n-1}]\phi_{V(c_1,\dots,c_m)}(x) = [x^{n-1}]\left(\frac{1-x}{1-2x}\right)^{c_1+\cdots+c_m-1}\cdot\prod_{k=1}^m\left(\phi_{C_{\mathsf{ccv}}(k)}(x)\right)^{c_k}$$
$$= [x^{n-1}]\frac{1-2x}{1-x}\prod_{k=1}^m\left(\sum_{i=0}^k\left(\frac{x}{1-x}\right)^i\right)^{c_k} \leq [x^{n-1}]\prod_{k=1}^m\left(\sum_{i=0}^k\left(\frac{x}{1-x}\right)^i\right)^{c_k} \leq 2^n\prod_{k=1}^m(k+1)^{c_k}$$

as expanding the second to last term gives us $\prod(k+1)^{c_k}$ summands, each some power of $\frac{x}{1-x}$, the x^{n-1}-coefficient of which is always less than 2^n. ◀

4.2 Proof of Theorem 28 (only Poly Chains)

Using Lemma 22, we can expand $T_{C_{\mathsf{poly}}(C_0,N)}(x)$ into an N-fold sum where each summand is a product of N triangulation numbers t_{k_i} and some $T_{V(a_1,\ldots,a_m)}(x)$. After grouping together summands with the same monomial of triangulation numbers, the leading coefficients are

$$U(C_{\mathsf{poly}}(C_0,N)) = \sum_{\substack{0\le a_1,\ldots,a_m\le N\\ a_1+\cdots+a_m=N}} \binom{N}{a_1,\ldots,a_m} \prod_{k=1}^{m} t_{m-k}(C_0)^{a_k}\cdot U(V(a_1,\ldots,a_m)).$$

Then, on one hand, by Corollary 37 and the multinomial theorem,

$$\begin{aligned} U(C_{\mathsf{poly}}(C_0,N)) &\le \sum_{\substack{0\le a_1,\ldots,a_m\le N\\ a_1+\cdots+a_m=N}} \binom{N}{a_1,\ldots,a_m} \prod_{k=1}^{m} t_{m-k}(C_0)^{a_k}\cdot \prod_{k=1}^{m}\left(2^k(k+1)\right)^{a_k}\\ &\le \left(\sum_{k=1}^{m} 2^k(k+1)\cdot t_{m-k}(C_0)\right)^N. \end{aligned}$$

On the other hand, by Theorem 33 and the entropy bound for multinomial coefficients,

$$U(C_{\mathsf{poly}}(C_0,N)) \ge \frac{1}{N^{c(m)}} \sum_{\substack{0\le a_1,\ldots,a_m\le N\\ a_1+\cdots+a_m=N}} e^{H(\frac{a_1}{N},\ldots,\frac{a_m}{N})} \prod_{k=1}^{m} t_{m-k}(C_0)^{a_k}\cdot \prod_{k=1}^{m}\left(2^k(k+1)\right)^{a_k}.$$

By picking the largest summand, given by Lemma 34, we get the lower bound

$$U(C_{\mathsf{poly}}(C_0,N)) \in \widetilde{\Omega}\left(\left(\sum_{k=1}^{m} t_{m-k}(C)\cdot 2^k(k+1)\right)^N\right).$$

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