This Is Driving Me Loopy: Efficient Loops in Arrowized Functional Reactive Programs

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Abstract

Arrowized Functional Reactive Programming (AFRP) is one approach to writing reactive programs declaratively, based on the arrows abstraction in Haskell. While AFRP elegantly expresses the relationships between inputs and outputs of a reactive system, naïve implementations suffer from poor performance. In particular, the *loop* combinator depends on lazy semantics: this inflicts the overheads of lazy evaluation and simultaneously prevents existing optimisation techniques from being applied to it.

We present a novel program transformation which utilises the Arrow and ArrowLoop laws to transform typical uses of *loop* into restricted forms that have an execution order that is known at compile-time and therefore can be executed strictly. We evaluate the performance gained from our transformations and prove that the transformations are correct.

CCS Concepts: • **Software and its engineering** → **Functional languages**; *Data flow languages.*

Keywords: Functional Reactive Programming, reactive programming, stream programming, arrows, program transformation

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1 Introduction

Arrowised Functional Reactive Programming (AFRP) [8] is a paradigm for writing reactive programs [24], which was popularised by the Haskell library Yampa. In AFRP, reactive programs are built using *signal functions*: functions which produce streams of outputs from streams of inputs. Execution of a program is broken up into *time steps*, in each of which signal functions get an input and produce a corresponding output. This means that the program effectively *reacts* to its inputs over time by producing outputs at the same rate. Signal functions can be combined with the arrow combinators [9] to form larger programs.

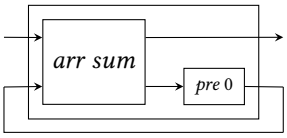

Figure 1. *loop* (*arr sum* ⋙ (*id* ⁂ *pre* 0))

As an example of AFRP in action, consider a reactive summing program which, at every time step, retrieves an input and adds it to a running total, which is also the output. This is implemented in Fig. 1 as a Yampa program and visualised as a box-and-wire diagram.

The overall program, which is itself a signal function, is built up from smaller signal functions. We have *arr sum*, which sums two inputs and returns the sum as both outputs; and *pre* 0, which returns 0 at the first time step and then the previous input at future time steps. Their inputs and outputs are routed as shown in the diagram using the ⁂ and ⋙ combinators. We finally enclose all this in *loop*, which connects the second output of its internal signal function to its second input.

loop seems to introduce a dependency cycle here, in which the second input of *arr sum* needs the second output of *arr sum* to be computed. Fortunately, *pre* can generate its output at a given time step only using its previous input, meaning that we get the output of *pre* before needing to compute its input. Therefore the above program works by retrieving the previous output of the program "stored" in *pre*, applying *arr sum* to that and the current input of the program to compute the new cumulative total, returning that as the output of the program and "storing" it in the *pre* for use at the next time step.

The above example shows how stateful programs are written in AFRP: *loop* is used in tandem with *pre* in order to use outputs from previous time steps as inputs at a subsequent time step. However, within this pattern lies a performance issue that has not yet been addressed by existing literature:

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in order to implement *pre* as shown, *loop*'s definition depends on lazy semantics. In pseudo-Haskell, we can define a simplified semantics of *loop* as follows, where `evalSF` takes a term of the form *loop f*, *f* is some signal function, and *a* is an input at the current time step. The definition is recursive and we can see that the second input *c* depends on the result of the recursive call:

```
evalSF :: SF a b -> a -> b
evalSF (loop f) a =
  let (b, c) = evalSF f (a, c) in b
```

When evaluating *loop f*, the order in which signal functions in *f* are evaluated to determine *c* is not always the same or obvious as the signal functions are not always run from left to right: in the example in Fig. 1 we have to first run *pre* 0 to get its output, which corresponds to the *c* that is the second input to *arr sum*. The execution order is determined at runtime by lazily evaluating *f*, which means that evaluating *loop f* suffers from the overheads needed for lazy evaluation.

This lazy evaluation does not happen just once at the start of the program, however. When Yampa evaluates a signal function for a single input, it also returns a possibly different signal function to run at the next time step. This is necessary for signal functions like *pre v* which need access to previous state: when we run *pre v* with input *a*, we get the output *v* and a new signal function *pre a*, which embeds the new state within the next signal function [17]. This approach to reactive programs that are essentially rewriting themselves at runtime might require us to re-evaluate the order of operations in *f* that needs to be performed at each time step.

Another issue with dependency resolution through lazy evaluation is that some well-typed *loop f* contain dependency cycles so cannot be run. Consider the program in Fig. 1 but with the *pre* term omitted: this contains a dependency cycle as *arr sum* needs its inputs to compute its outputs, but the second output of *arr sum* depends on its own second input. The presence of a dependency cycle is only noticed when trying to evaluate the *loop*, causing a runtime error.

We address the problems caused by evaluating *loop* lazily by introducing a program transformation which transforms *loop f* with no dependency cycles into alternative forms with known execution order, which can be executed strictly. More concretely, our contributions are as follows:

- We provide a program transformation for a subset of Yampa (Section 3) which transforms *loop f* without dependency cycles to use variants of *loop* with known execution orders. We accomplish this by applying the arrow laws [9] as well as novel rules to rewrite *loop f* so that uses of *pre* happen before everything else.
- We prove that our transformation works on programs without dependency cycles in them, and that this transformation does not affect program meaning (Section 4).
- We present a Haskell implementation, *Severn*, of this transformation (Section 5), which runs the resulting program strictly. We compare the performance of programs written in Severn to their equivalent Yampa programs (Section 6).

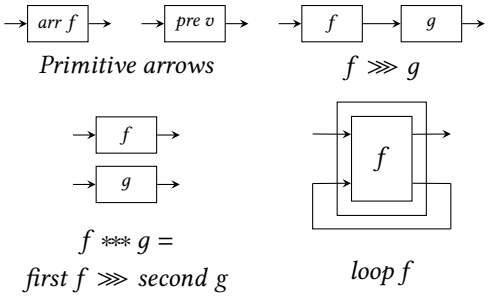


Figure 2. The minimal set of arrow operators used in this paper, presented as box-and-wire diagrams.

2 Yampa and Arrow Laws

For the benefit of readers unfamiliar with arrows and Yampa, we briefly introduce the arrow constructors and laws that we use throughout this paper.

Our work builds on Yampa, an AFRP implementation built around *signal functions* (SFs). We consider a minimal set of arrow operators[1] which are enough to define many useful Yampa programs: this minimal set is presented in Fig. 2. We also make use of *first f* and *second f* throughout as synonyms for f *** *id* and *id* *** f respectively.

We briefly describe each operator in turn: *arr f* allows pure functions to be turned into SFs where *f* transforms an input into an output. SFs can then be composed sequentially with ⋙ and in parallel with ***. These operators give us the ability to run SFs consisting of pure functions and compose them into larger programs.

pre v[2] introduces the effect of state by mirroring its input stream as output, delayed by one timestep. For example, the inputs 1, 2, 3 passed to *pre v* give us the outputs *v*, 1, 2.

Finally, *loop f* provides a way to introduce feedback into our arrow programs by directly connecting the second output of *f* to its own second input. This is where Yampa requires lazy evaluation, as we cannot run *f* strictly without its second output. In this work, we focus on its interactions with the *pre* operator: since *pre* can generate an output at a given timestep without its corresponding input, it can be used to generate the second output of *f* without needing the second input. We saw this in Fig. 1, where the second input of the *loop* is the second output of *loop* from the previous time step due to *pre*.

These operators are enough to define common Yampa programs. We discuss additional operators, such as *switch*, in Section 7.3.

[1] https://hackage.haskell.org/package/base/docs/Control-Arrow.html
[2] We use *pre* following Yampa's terminology, but Liu et. al [15] introduce it as the *init* operator, and it has also appeared elsewhere as *delay* or *iPre*.

Since SFs are instances of Haskell's `Arrow` and `ArrowLoop` type classes, SFs must obey their laws. These laws define required equivalences between programs, which we use throughout our work to prove that each step of our program transformation preserves program meaning. Throughout this paper we introduce necessary laws as they are needed, beginning with two keys ones below. Interested readers can consult Hughes [9, Section 7] for the full set.

$$first\ f \ggg first\ g = first\ (f \ggg g) \quad \text{(arrow functor)}$$
$$(f \ggg g) \ggg h = f \ggg (g \ggg h) \quad \text{(associativity)}$$

2.1 Commutative Causal Arrows (CCAs)

Liu et. al define CCAs, which extend arrows with two additional laws that hold for Yampa [14, 15]:

$$first\ f \ggg second\ g = second\ g \ggg first\ f \quad \text{(commutativity)}$$
$$pre\ i \ast\!\!\ast\!\!\ast pre\ j = pre\ (i, j) \quad \text{(product rule)}$$

With these additional laws, whole AFRP programs can be transformed into one of two forms: a single *arr*, or a loop of the form $LoopD\ f\ i = loop\ (f \ggg second\ (pre\ i))$ [30]. This transformation is performed using the `ArrowLoop` laws, which merge composed and nested loops together into a single loop using routing functions.

At first glance, this seems to solve the problem we are addressing in this paper: $LoopD\ f\ i$ can be executed strictly by first executing $pre\ i$ and then executing f. However, the routing functions used by the transformation still rely on lazy evaluation, and thus the *LoopD* created by the CCA transformation cannot be executed strictly.

3 Transforming *loop* Into Strict Variants

We claim that the lazy semantics required to evaluate $loop\ f$ is a cause of performance issues due to the involved overheads. Our goal is therefore to determine the execution order of f within $loop\ f$ at compile-time.

We achieve this by finding *decoupled* [27] parts of f: those which can produce outputs at time step t without any of their inputs at t, like *pre*. We define restricted forms of *loop* where those decoupled parts are separated out from the rest of f, with the aim of running those parts first. An example of this, and the main restricted form of *loop* we consider, is Yallop and Liu's $LoopD\ f'\ i$ construct[3] [30]. We define this as follows alongside an interpreter *runLoopD* which maps $LoopD\ f'\ i$ to a corresponding signal function.

```
data LoopD a b = LoopD (SF (a,c) (b, c)) c

runLoopD :: LoopD a b -> SF a b
runLoopD (LoopD f i) = loop (f >>> second (pre i))
```

A $loop\ f$ can be expressed as *LoopD* only if f contains a *pre* just before its second output. Since *pre* can produce an output at a given time step without the input at that time step, we know the execution order of $LoopD\ f'\ i$: evaluate the final $pre\ i$ to produce the second output, use it as second input, and run the rest of f' with both inputs. No lazy evaluation is required.

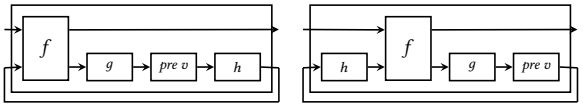


Figure 3. Right sliding of $loop\ (f \ggg (id \ast\!\!\ast\!\!\ast g) \ggg (id \ast\!\!\ast\!\!\ast pre\ v) \ggg (id \ast\!\!\ast\!\!\ast h))$

The question is then how to transform arbitrary $loop\ f$ into equivalents that can be expressed as $LoopD\ f'\ i$. Informally, given a $loop\ f$, our aim is to move a single *pre* within f to appear just before f's second output while preserving the semantics of that loop.

In the rest of this section we present the necessary transformation for *LoopD* and other restricted forms of *loop* with known execution orders. We do this in four parts, as follows:

1. In Section 3.1 we apply *ArrowLoop*'s *sliding* law to transform some $loop\ f$ into $LoopD\ f'\ i$ by moving $pre\ i$ within f to be just before the second output. We also discuss a variety of transformations that may need to be applied in order to allow sliding, and introduce *CCA composition form* to make sliding easier to apply.
2. Sometimes there are multiple looped values in a *loop*: a transformed $loop\ f$ will be of the form $LoopD\ f'\ (i, j)$. For this we slide a single $pre\ (i, j)$ to be before the second output of f. However, $pre\ (i, j)$ can be expressed in a few different ways, such as $pre\ i \ast\!\!\ast\!\!\ast pre\ j$. To make sure that we are able to work with these equivalent statements of $pre\ (i, j)$, we use CCA's *product rule* and a new *split* rule which finds nearby *pre* to combine them into a single $pre\ (i, j)$. (Section 3.2)
3. There are some loops where the *pre* is "trapped" between two non-*pre* arrows that we cannot slide and which therefore cannot be transformed by the above two steps. Fortunately, such loops also have a trivial execution order, for which we define another restricted form of *loop* called *LoopM* in Section 3.3.
4. We then look at the case where multiple *loop*s are present in a program, *e.g.* $loop\ (loop\ f)$, in Section 3.4. The inner *loop* is transformed using the above transformations, and then *LoopD* and *LoopM* are modified to allow nesting.
5. Finally, we combine these transformations into an algorithm in Section 3.5.

We justify that these steps cover all possible cases of a *loop* with no dependency cycles in Section 4.

3.1 Sliding

We start by looking at how to move a single *pre* to the rightmost position of a loop body. Examples of loops which can

[3] This is called *loopPre* in Yampa.

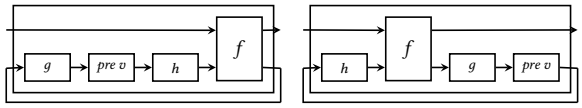


Figure 4. Left sliding of $loop\ ((id \mathbin{***} g) \ggg (id \mathbin{***} pre\ v) \ggg (id \mathbin{***} h) \ggg f)$

be transformed into *LoopD* in this way can be seen on the left sides of Fig. 3 and Fig. 4, along with their transformed versions on the corresponding right sides.

In these cases, we can employ *sliding* from the *ArrowLoop* laws, which allows parts of our program to be moved around inside the loop. Sliding is defined as follows:

$$loop\ (g \ggg arr\ (id \mathbin{***} k)) = loop\ (arr\ (id \mathbin{***} k) \ggg g)$$

If we have $loop\ f$, we can move a signal function k which appears just before the second output of f to be just after the second input of f, and vice versa. The equivalence holds because k still receives inputs from and gives outputs to the same signal functions as before. Figure 3 shows this at work with h, which is connected to the same signal functions before and after sliding. Note that, by design, this law does not permit g to be composed with an effectful computation. Since $id \mathbin{***} k$ is enclosed in *arr*, it is a signal function consisting of a pure function. In general, this is important, because changing the order of operations might lead to different results in the presence of implicit, computational effects.

However, we focus on a minimal subset of the arrow operators, which does *not* allow for effectful signal functions. Therefore we can generalise the sliding rule slightly:

$$loop\ (f \ggg (id \mathbin{***} k)) = loop\ ((id \mathbin{***} k) \ggg f)$$

We discuss the consequences of this decision in AFRP systems where signal functions can be effectful in Section 7.1.

We refer to transforming a program of the form on the left to the form on the right as *right sliding*, since we move the body of the loop to the *right*, causing k to fall off the end and reappear on the left side. We call the reverse direction *left sliding*. Sliding gives us the rules needed to justify the transformations in Fig. 3 and Fig. 4. The first example is solved through right sliding, moving h from the right of the loop to the left. In the second example we can left slide twice, moving g and then $pre\ v$ from the left of the loop to the right.

3.1.1 Distributivity of Composition. This presentation of sliding may not be applicable if programs are written in subtly different, but equivalent ways. An equivalent way to write Fig. 3 is $loop\ (f \ggg (id \mathbin{***} (g \ggg pre\ v \ggg h)))$. However, applying right sliding here moves all of $g \ggg pre\ v \ggg h$ over to the left side, preventing us from getting $pre\ v$ into the desired position. We solve this by noting that $\ggg$ distributes over $***$ for CCAs, proved as follows:

$$\begin{aligned}
& (f \ggg h) \mathbin{***} (g \ggg i) \\
= \quad & \{\quad first\ f \ggg second\ g = f \mathbin{***} g \quad\} \\
& first\ (f \ggg h) \ggg second\ (g \ggg i) \\
= \quad & \{\quad \text{by Arrow Functor law} \quad\} \\
& (first\ f \ggg first\ h) \ggg (second\ g \ggg second\ i) \\
= \quad & \{\quad \text{we ignore brackets as } \ggg \text{ is associative} \quad\} \\
& first\ f \ggg first\ h \ggg second\ g \ggg second\ i \\
= \quad & \{\quad \text{by CCA's commutativity law} \quad\} \\
& first\ f \ggg second\ g \ggg first\ h \ggg second\ i \\
= \quad & \{\quad first\ f \ggg second\ g = f \mathbin{***} g \quad\} \\
& (f \mathbin{***} g) \ggg (h \mathbin{***} i)
\end{aligned}$$

With this distributive law, we can rewrite $id \mathbin{***} (g \ggg pre\ v \ggg h)$ to $(id \mathbin{***} g) \ggg (id \mathbin{***} pre\ v) \ggg (id \mathbin{***} h)$. This is the same as the original definition of Fig. 3, allowing us to apply right sliding to get the $pre\ v$ into the correct position.

3.1.2 Sliding next to non-*id*. Another obstacle that can arise is when we have a term in parallel with the one we are trying to slide, as in the first diagram in Fig. 5. We are unable to apply left sliding here since it requires $id \mathbin{***} k$ at the start of the loop, but we have $y \mathbin{***} g$ instead of $id \mathbin{***} g$. We require a more general pair of program equivalences:

$$\begin{aligned}
& loop\ (f \ggg (g \mathbin{***} k)) && \text{(right sliding)} \\
= \quad & loop\ ((id \mathbin{***} k) \ggg f \ggg (g \mathbin{***} id)) \\
& loop\ ((g \mathbin{***} k) \ggg f) && \text{(left sliding)} \\
= \quad & loop\ ((g \mathbin{***} id) \ggg f \ggg (id \mathbin{***} k))
\end{aligned}$$

We prove the first of these below. The second is proved symmetrically.

$$\begin{aligned}
& loop\ (f \ggg (g \mathbin{***} k)) \\
= \quad & \{\quad \text{identity of } \ggg \quad\} \\
& loop\ (f \ggg ((g \ggg id) \mathbin{***} (id \ggg k))) \\
= \quad & \{\quad \text{distributive law} \quad\} \\
& loop\ (f \ggg ((g \mathbin{***} id) \ggg (id \mathbin{***} k))) \\
= \quad & \{\quad \text{ArrowLoop Laws' right sliding} \quad\} \\
& loop\ ((id \mathbin{***} k) \ggg f \ggg (g \mathbin{***} id))
\end{aligned}$$

With these, we can apply our new left sliding rule to reach $loop\ ((y \mathbin{***} id) \ggg (x \mathbin{***} pre\ v) \ggg f \ggg (id \mathbin{***} g))$ as shown in the second diagram in Fig. 5, which makes progress towards getting $pre\ v$ into the expected position. Unfortunately, we are now stuck – if we keep applying left sliding, all we do is keep sliding *id*, which does not help us move the $pre\ v$.

3.1.3 Pushing non-*id* terms through *id*. To avoid the problem of having *id* block non-*id* terms which we want to slide, we need rules to remove the offending *id*. We note that since *id* terms do not change program meaning, we can move them around and remove them as is needed. We therefore define some new rules to "push" a non-*id* term to take the place of an *id*, thus allowing it to be used by other rules. Our aim in this section is to take programs such as $(y \mathbin{***} id) \ggg (x \mathbin{***} pre\ v)$, shown in the second diagram

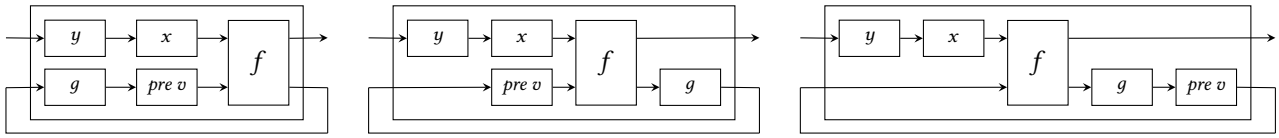


Figure 5. Left sliding of $loop\ ((y \mathbin{***} g) \ggg (x \mathbin{***} pre\ v) \ggg f)$

$$\frac{}{id \ggg i \overset{left\text{-}fill}{\Longrightarrow} i \ggg id}\ \textsc{LFill-Id}$$

$$\frac{k \neq id}{k \ggg i \overset{left\text{-}fill}{\Longrightarrow} k \ggg i}\ \textsc{LFill-NonId}$$

$$\frac{\begin{array}{l} a \ggg b \overset{left\text{-}fill}{\Longrightarrow} a' \ggg b' \\ c \ggg d \overset{left\text{-}fill}{\Longrightarrow} c' \ggg d' \end{array}}{\begin{array}{l}(a \mathbin{***} c) \ggg (b \mathbin{***} d) \\ \overset{left\text{-}fill}{\Longrightarrow} (a' \mathbin{***} c') \ggg (b' \mathbin{***} d')\end{array}}\ \textsc{LFill-}***$$

Figure 6. The three steps which define the *left-fill* rule.

in Fig. 5, and move the *pre v* to be in parallel with the *y* to give us *e.g.* $(y \mathbin{***} pre\ v) \ggg (x \mathbin{***} id)$.

We start by defining the *left fill* operation, which takes a composition of two terms and fills in any gaps (*id*) in the left term with parts of the right term. This is defined using the three rules shown in Fig. 6.

LFill-Id says that if we have an *id* as the left term, replace it with the right term in order to fill the gap within the left term. This does not change the meaning of the program since $id \ggg i = i = i \ggg id$. LFill-NonId says that if there is no *id* to fill, then do nothing.

LFill-*** considers parallel compositions. This transforms the input to $(a \ggg b) \mathbin{***} (c \ggg d)$ via our distributive law, uses the subordinate calls to left fill to transform $a \ggg b$ and $c \ggg d$ individually, and then uses the distributive law again to combine the results of those subordinate calls to the result of the main one.

Rather than using left fill just once when we have an *id* to slide, we need to apply it multiple times. This is needed to make sure that terms are propagated through multiple *id* if needed: for example, $(id \mathbin{***} f) \ggg (id \mathbin{***} g) \ggg (h \mathbin{***} i)$ requires a call to left fill on the last two terms and then on the first two terms if we want the *h* to be moved to the front of the program. We therefore define *left push* to be repeated application of left fill: given composition $a_1 \ggg a_2 \ggg \ldots \ggg a_n$, we first left fill a_{n-1} and a_n, then a_{n-2} and a_{n-1} and so on until we left fill a_1 and a_2.

We can now use this to finish transforming Fig. 5:

$$\begin{array}{ll} & loop\ ((y \mathbin{***} g) \ggg (x \mathbin{***} pre\ v) \ggg f) \\ = & \quad \{\quad \text{left sliding} \quad \} \\ & loop\ ((y \mathbin{***} id) \ggg (x \mathbin{***} pre\ v) \ggg f \ggg (id \mathbin{***} g)) \\ = & \quad \{\quad \text{left push} \quad \} \\ & loop\ ((y \mathbin{***} pre\ v) \ggg (x \mathbin{***} id) \ggg f \ggg (id \mathbin{***} g)) \\ = & \quad \{\quad \text{left sliding} \quad \} \\ & loop\ ((y \mathbin{***} id) \ggg (x \mathbin{***} id) \ggg f \ggg \\ & (id \mathbin{***} g) \ggg (id \mathbin{***} pre\ v)) \\ = & \quad \{\quad \text{definition of } LoopD \quad \} \\ & LoopD\ ((y \mathbin{***} id) \ggg (x \mathbin{***} id) \ggg f \ggg (id \mathbin{***} g))\ v \end{array}$$

With this set of new rules, we are now able to transform *loop*s which previously could not have left sliding applied to them. We also utilise equivalent *right fill* and *right push* laws to move non-*id* terms to the right for transforming *loop*s to have right sliding applied to them. We omit these definitions as they are symmetric to those for left fill and left push.

3.1.4 CCA Composition Form.

The issue of needing our program to be of a certain form in order to apply a rule is not unique to sliding. We also define *CCA composition form*, which forces *loop*s to have $\ggg$ at the top level only, in order to restrict the shape that a *loop* can take and thus make it easier to apply our rules.

We require that *pre* cannot contain a tuple value. This is because when we apply rules such as sliding, we need to have the $***$ to know that we can split the term in two: for example, if we had $loop\ (pre\ (i, j) \ggg f)$, we could not apply left sliding. Any $pre\ (i, j)$ can instead be written as $pre\ i \mathbin{***} pre\ j$ by CCA's product rule.

We now formally state the definition of CCA composition form.

Definition 3.1. An AFRP program is in *CCA composition form* if it can be parsed by the following grammar, where $\mathcal{L}$ is the start symbol, $\mathcal{F}$ is any pure function, and $\mathcal{V}$ is any non-tuple value.

$\mathcal{L}$	::=	$loop\ C$	*Loop*
C	::=	$C \ggg C$	*Composition*
	|	$\mathcal{P}$	*No composition*
$\mathcal{P}$	::=	$\mathcal{P} \mathbin{***} \mathcal{P}$	*Parallel composition*
	|	$arr\ \mathcal{F}$	*Lifted pure function* $\mathcal{F}$
	|	$pre\ \mathcal{V}$	*Pre with non-tuple argument* $\mathcal{V}$
	|	id	*Identity*
	|	$\mathcal{L}$	*Internal loop*

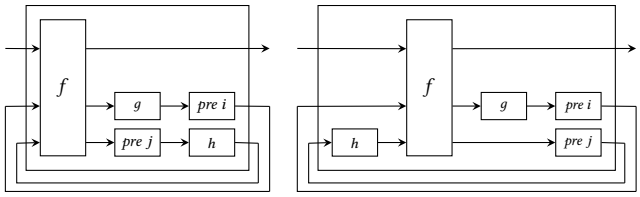


Figure 7. *loop* $(f \ggg second\ ((g \ast\!\!\ast\!\!\ast\ pre\ j) \ggg (pre\ i \ast\!\!\ast\!\!\ast\ h)))$

For the rest of this paper, we present rules assuming that our programs are in CCA composition form. This does not affect the expressiveness of our system as it is possible to transform any existing CCA into this form through application of distributive law as we did back in Section 3.1.1 and application of CCA product rule to avoid any *pre* (i, j).

3.2 Combining Smaller *pre* into Larger Ones

We have shown that using a combination of our new generalised sliding rule, distributive law and push, we can move a single *pre* within *loop* f to be before the second of output of f. However, we sometimes work with multiple *pre* rather than a single one, *e.g.* in *loop* $(g \ggg second\ (pre\ i \ast\!\!\ast\!\!\ast\ pre\ j))$. This *loop* still has a clear execution order: run the two *pre* to generate the second outputs of the *loop* body, which means we get the second inputs and can run g. We cannot currently transform it into *LoopD* however, since *LoopD* relies on there being a single *pre*.

To represent this as a *LoopD*, we need to merge the two occurrences of *pre* together into single use of *pre* using the previously discussed CCA product rule: $pre\ i \ast\!\!\ast\!\!\ast\ pre\ j = pre\ (i, j)$. This means that whenever we encounter two uses of *pre* in parallel, we can merge them and treat them as one. With this, we can transform our example to *loop* $(g \ggg second\ (pre\ (i, j)))$, which is equivalent to *LoopD* $g\ (i, j)$.

3.2.1 The *split* Rule.

The CCA product rule lets us combine *pre* which are in parallel, but the uses of *pre* we need to merge may not always be in parallel. Figure 7 shows an example with two uses of *pre* which cannot be solely solved by the product rule and sliding: the two halves of the second output each have a *pre* on them, but those uses of *pre* are not parallel to each other and the product rule therefore cannot be applied.

We therefore need a way of rearranging expressions such as $f \ggg (g \ast\!\!\ast\!\!\ast\ pre\ j) \ggg (pre\ i \ast\!\!\ast\!\!\ast\ h)$ to correctly group uses of *pre* together and merge them with the product rule. To do this, we define *split*, which attempts to split an input f into (f_l, f_d, f_r) where $f = f_l \ggg f_d \ggg f_r$ and f_d is a decoupled term containing no $\ggg$. We define this operation through a collection of rules, shown in Fig. 8.

Split-Pre dictates that if we have a *pre* v at the end of the composition, then we already have a trivial split with $f_d = pre\ v$. Split-$\ast\!\!\ast\!\!\ast$-R specifies that if we have two parallel paths given by some $g \ast\!\!\ast\!\!\ast\ h = (g_1 \ast\!\!\ast\!\!\ast\ h_1) \ggg \cdots \ggg (g_n \ast\!\!\ast\!\!\ast\ h_n)$ and we are able to split the two paths g and h, then we can split the two paths in parallel by aligning the g_d and h_d we get from the subordinate calls to *split*.

In any other case, we have not found a *pre* nor a $l \ast\!\!\ast\!\!\ast\ r$ where we can find a *pre* in each of l and r and thus combine them with CCA's product rule. Split-NonPre dictates that in this case, we can skip over this term as it will not lead to us finding the required *pre*. This covers *arr* f and *id*.

We present these rules in use with a derivation that correctly splits our earlier example of $f \ggg (g \ast\!\!\ast\!\!\ast\ pre\ j) \ggg (pre\ i \ast\!\!\ast\!\!\ast\ h)$ in Fig. 9. Running *split* is easy for a given $f = f_1 \ggg \ldots \ggg f_n$: find the rule matching $f' \ggg f_n$, or in the case of $f_n = a \ast\!\!\ast\!\!\ast\ b$ try each of the Split-$\ast\!\!\ast\!\!\ast$ rules in turn. We prove that running *split* always produces a valid split if it exists in Section 4.

3.2.2 Using *split* to Find *LoopD*.

The *split* rule now lets us find a *pre* that can be slid into position. Given *loop* $(g \ggg second\ f)$, we apply split to f to transform it into *loop* $(g \ggg second\ (f_l \ggg pre\ v \ggg f_r))$. We can then right slide f_r to get *LoopD* $(second\ f_r \ggg g \ggg second\ f_l)\ v$.

This only looks at the right side of the *loop* however: we also need to slide anything from the left side over to the right side so that it is considered by *split*. This is necessary for programs in which the *pre* we are looking for is on the opposite side, such as *loop* $(second\ (pre\ v) \ggg x)$. We therefore left slide as much as we can before applying *split*.

3.3 LoopM

While the sliding and split rules are enough to transform most *loop* f into *LoopD* $f'\ i$, there is one class of counterexamples for which this is not enough. In Fig. 10a, we are unable to slide a *pre* into position because neither f nor g can be slid, meaning that we cannot transform the *loop* into *LoopD*. We need to be able to transform this example however, as it can be executed by getting the outputs from the *pre*, then running g, and finally running f.

To transform *loop* f where f is split in two halves by a *pre*, we introduce a new restricted form of *loop* called *LoopM*. This is defined as follows alongside an interpreter *runLoopM* which maps *LoopM* $f\ i\ g$ to a corresponding signal function.

```
data LoopM a b = LoopM (SF (a,c) d) d (SF d (b, c))

runLoopM :: LoopM a b -> SF a b
runLoopM (LoopM f i g) = loop (f >>> pre i >>> g)
```

Checking whether we can transform *loop* x into *LoopM* only requires application of *split*. If we can split x to get $(f, pre\ i, g)$, then $x = f \ggg pre\ i \ggg g$ and therefore $loop\ x = LoopM\ f\ i\ g$. This can be applied to both of our examples in Fig. 10 to get *LoopM* $f\ (i, j)\ g$ and *LoopM* $(f \ggg second\ h)\ (x, y)\ (first\ i \ggg g)$ respectively.

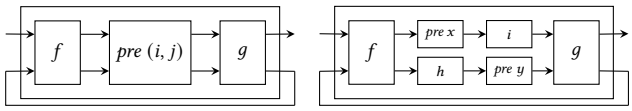
$$\frac{}{f \ggg pre\ v \overset{split}{\Longrightarrow} (f,\ pre\ v,\ id)}\ \textsc{Split-Pre} \qquad \frac{f \overset{split}{\Longrightarrow} (f_l, f_d, f_r)}{f \ggg g \overset{split}{\Longrightarrow} (f_l, f_d, f_r \ggg g)}\ \textsc{Split-NonPre}$$

$$\frac{g_1 \ggg \cdots \ggg g_n \overset{split}{\Longrightarrow} (g_l, g_d, g_r) \qquad h_1 \ggg \cdots \ggg h_n \overset{split}{\Longrightarrow} (h_l, h_d, h_r)}{f \ggg (g_1 \ast\!\!\ast\!\!\ast h_1) \ggg \cdots \ggg (g_n \ast\!\!\ast\!\!\ast h_n) \overset{split}{\Longrightarrow} (f \ggg (g_l \ast\!\!\ast\!\!\ast h_l), g_d \ast\!\!\ast\!\!\ast h_d, g_r \ast\!\!\ast\!\!\ast h_r)}\ \textsc{Split-}\ast\!\!\ast\!\!\ast\textsc{-R}$$

Figure 8. The *split* rules.

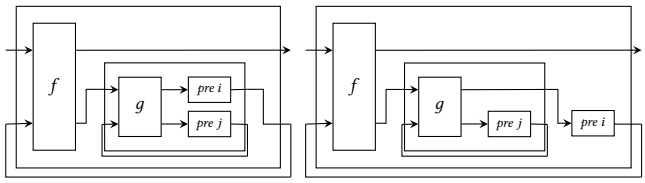
$$\frac{\textsc{Split-Pre}\ \dfrac{}{g \ggg pre\ i \overset{split}{\Longrightarrow} (g, pre\ i, id)} \qquad \dfrac{\dfrac{}{pre\ j \overset{split}{\Longrightarrow} (id, pre\ j, id)}\ \textsc{Split-Pre}}{pre\ j \ggg h \overset{split}{\Longrightarrow} (id, pre\ j, h)}\ \textsc{Split-NonPre}}{f \ggg (g \ast\!\!\ast\!\!\ast pre\ j) \ggg (pre\ i \ast\!\!\ast\!\!\ast h) \overset{split}{\Longrightarrow} (f \ggg (g \ast\!\!\ast\!\!\ast id), pre\ i \ast\!\!\ast\!\!\ast pre\ j, id \ast\!\!\ast\!\!\ast h)}\ \textsc{Split-}\ast\!\!\ast\!\!\ast\textsc{-R}$$

Figure 9. Derivation of splitting $f \ggg (g \ast\!\!\ast\!\!\ast pre\ j) \ggg (pre\ i \ast\!\!\ast\!\!\ast h)$



(a) *loop* $(f \ggg pre\ (i,j) \ggg g)$ **(b)** *loop* $(f \ggg (pre\ x \ast\!\!\ast\!\!\ast h) \ggg (i \ast\!\!\ast\!\!\ast pre\ y) \ggg g)$

Figure 10. Examples where *LoopM* is needed.



(a) *loop* $(f \ggg loop\ (g \ggg (pre\ i \ast\!\!\ast\!\!\ast pre\ j)))$ **(b)** *loop* $(f \ggg loop\ (g \ggg (id \ast\!\!\ast\!\!\ast pre\ j)) \ggg second\ (pre\ i))$

Figure 11. Example where we need to extract a *pre* from a *LoopD*, and a version with the *pre* extracted.

3.4 Multiple Loops

We can now transform a single *loop* f into its equivalent *LoopD* or *LoopM*. We now consider programs with multiple composed *loop*s or nested *loop*s, with the aim of being able to transform programs consisting of any number of *loop*s.

For composed *loop*s we note that the transformation of a given *loop* f relies on nothing except f. This means that we can transform compositions of *loop*s such as *loop* $f \ggg$ *loop* g by transforming each individual *loop*, giving us *e.g.* something of the form *LoopD* $f'\ i \ggg$ *LoopM* $g_1\ j\ g_2$.

Issues arise however when we introduce nested *loop*s such as *loop* (*loop* f). The inner *loop* f could contain the *pre* that is needed for the outer *loop* to be transformed. An example of this is presented in Fig. 11, where the *pre i* in the inner *loop* is needed by the outer *loop*. We therefore need a way to extract such a *pre* from an inner *loop* f.

In the rest of the section, we look at extracting *pre* from nested *loop*s in two cases: one where the inner *loop* can be transformed into *LoopD*, and another when it can be transformed into *LoopM*.

3.4.1 Extracting *pre* from *LoopD*.

When the inner *loop* is a *LoopD*, we need to get the *pre* out of that inner *loop* if we want to use it outside of that *loop*. We have already seen this in Fig. 11.

To achieve this, we turn again to the *ArrowLoop* laws, which given a *loop* f provide a way to extract unused *pre* from f. The two laws we need are stated below.

$$\begin{aligned} & loop\ ((x \ast\!\!\ast\!\!\ast z) \ggg y) && \text{(left tightening)} \\ = \ & x \ggg loop\ (second\ z \ggg y) \end{aligned}$$

$$\begin{aligned} & loop\ (x \ggg (y \ast\!\!\ast\!\!\ast z)) && \text{(right tightening)} \\ = \ & loop\ (x \ggg second\ z) \ggg y \end{aligned}$$

Right tightening takes a *loop* with $(y \ast\!\!\ast\!\!\ast z)$ as its last term and moves y outside of the *loop*. This preserves program meaning as y still receives the same inputs and produces the same outputs; it does not need to be within the *loop* to still be connected to the first output of x. This is shown in use in Fig. 11, where we use right tightening to move the

pre i outside of the *loop* while keeping it connected to the first output of g, meaning that we can use it to transform the outer *loop*. Left tightening is similar, but works with the front of the *loop* instead.

In most cases there is only one direction in which part of a *loop* can be tightened. Consider an arbitrary *loop* $((f_1 *** f_2) \ggg f_3 \ggg (f_4 *** f_5))$ where any f_i can be *id*. If f_3 is not *id*, then only f_1 can be moved outside of the *loop* via left tightening as f_3 "blocks" f_4 from being moved this way, and only f_4 can be moved outside of the *loop* via right tightening through a similar argument. In these cases, we apply left and right tightening to move f_1 and f_4 out of the *loop* once we have transformed it into *LoopD*.

In the case where $f_3 = id$, we end up with *loop* $((f_1 *** f_2) \ggg (f_4 *** f_5))$. f_1 and f_4 could be tightened out of the *loop* in either direction, but we do not know which way we need to tighten them to *e.g.* get the *pre* needed for an outer *loop*. The trick is to consider the more general case of *loop* $(f *** g)$, in which we only need to run f to get the output of the *loop*: g will never be run as its result is never needed. Thus we avoid having to decide which way to tighten by removing the *loop*, f_2 and f_5 to get $f_1 \ggg f_4$.

We can extract a *pre* from an inner *LoopD* by therefore either applying left and right tightening to move as much from inside the *LoopD* out as possible, or remove the *LoopD* entirely using *loop* $(f *** g)$. In either case, this allows *pre* that were part of the inner *loop* to be used when transforming the outer *loop*.

3.4.2 Extracting *pre* from *LoopM*. When we have an inner *LoopM*, things are simpler than for *LoopD*. The trick is that *LoopM* itself is decoupled like *pre*, as it can produce all of its outputs without any of its inputs by using its internal *pre*. To run a program like *loop* $(f \ggg second\ (LoopM\ g\ (i, j)\ h))$, we can first get the output of the inner *LoopM* by running its internal *pre*, then run h to get the second output of the outer *loop*, and then finally run f and g.

Since our aim with these restricted forms of *loop* is to fix the location of a decoupled part so that we know the execution order of the *loop* at compile time, we allow *LoopM* to take the place of a *pre* when transforming our *loop*s. This is implemented via a minor update to the *split* operation:

$$\frac{}{x \ggg LoopM\ f\ d\ g \xrightarrow{split} (x, LoopM\ f\ d\ g, id)}\ \textsc{Split-LoopM}$$

With this we can now use *LoopM* wherever we were aiming for a *pre*, allowing our earlier example to be expressed as *LoopD* f $(LoopM\ g\ (pre\ (i, j))\ h)$. Note that the definitions of *LoopD* and *LoopM* have changed slightly since they previously relied on specifically containing a *pre*, but can now contain arbitrary SFs consisting of *pre*, *LoopM* and $***$. The updated definitions can be seen in the code for `Decoupled` in Section 5.1.

3.5 Transformation Algorithm

We now combine the rules we have described for transforming different cases of *loop*s into an algorithm that we can run. The overall process that we present inspects *loop*s from innermost to outermost, transforming each one to a *LoopD* or *LoopM* until every *loop* is transformed. After transforming the program into *CCA composition form* (Section 3.1.4), perform the following for each *loop* from the innermost to the outermost:

1. Attempt to apply *loop* $(f_1 *** f_2) = f_1$ (Section 3.4.1) to remove the *loop* altogether. In CCA composition form, this is equivalent to checking whether each term a_i in *loop* $(a_1 \ggg a_2 \ggg \cdots \ggg a_n)$ consists of $a_{li} *** a_{ri}$.
2. If that does not work, attempt to transform the *loop* f to *LoopM* $f'\ d\ g'$ by using *split* to find $f = f' \ggg d \ggg g'$ for decoupled d (Section 3.3).
3. If that does not work, attempt to transform the *loop* f into *LoopD* $f'\ i$ in three stages:
   a. Slide left as much as possible, using left fill as needed, to get a program of the form *loop* $(g \ggg second\ x)$ (Section 3.1 and Section 3.2.2).
   b. Apply split to x to get $f = g \ggg second\ (x_l \ggg x_d \ggg x_r)$. Right slide x_r to get x_d in the right position, giving us *LoopD* $(second\ x_r \ggg g \ggg second\ x_l)\ x_d$ (Section 3.2).
   c. If this works, apply left and right tightening to extract any more *pre* or *LoopM* that could be used in outer *loop* from that *LoopD* (Section 3.4.1).

4 Completeness

We now prove that our transformation works on all *loop* without dependency cycles while preserving program meaning. In Section 3, we proved that program meaning is preserved by each individual operation through existing laws for arrows and CCAs. Since our transformation solely uses these operations, it preserves program meaning. We now prove that it is *complete* for any *loop* with no dependency cycles.

We first need to formalise what a dependency cycle is, in order to reason about them. Some output o depends on an input i if there is some path through the program from i to o: meaning that in order to get o, we need to know i. Decoupled functions like *pre* break this dependency as they are able to produce an output at a given time step without the input at that time step; the dependencies we therefore consider are represented by paths which do not go through a decoupled signal function. We define this as a direct dependency:

Definition 4.1. A *direct dependency* exists from input i to output o of some signal function f if there is a path from i to o through f that does not flow through a *pre* or *LoopM* (equivalently, a decoupled signal function).

A dependency cycle arises if there is a direct dependency from i and o and one from o to i. In *loop* f, dependency cycles

are created via *loop f*'s backedge from each part of its second output to each part of its second input. We define a direct dependency cycle within a *loop* as follows.

Definition 4.2. A *direct dependency cycle* within *loop f* exists if there is a direct dependency through f from a component of the second input of f to the same component of the second output of f.

Let us build an intuition for how paths, and thus direct dependencies, are built from each of the arrow constructors:

- *arr f*: Since we know nothing about f, we assume that there is a path from every input to every output, and therefore every output of *arr f* directly depends on every input. We show later that this also holds for *LoopD* generated by the transformation.
- *pre v*: If we have a *pre*, then there is a path from every input to every output which trivially goes through a decoupled signal function, so there is no direct dependency between *pre*'s inputs and outputs. The same holds for *LoopM*.
- $f \ggg g$: This sequentially composes the paths through f and g: if there is a path through f from i to m, and a path through g from m to o, then there is a path through $f \ggg g$ from i to o.
- $f \mathbin{***} g$: This composes two paths in parallel that do not interact. Therefore it will have a *pre* on every path if f and g each have a *pre* on every path between their inputs and outputs.

We now present some auxiliary lemmas used within our main proof. We first define three forms that a *loop* can take which allow us to perform case analysis in our other proofs:

Lemma 4.3. Any *loop f* can be represented as one of the following cases:

Case 1 *loop* $(a \mathbin{***} b)$

Case 2 *loop* $((a \mathbin{***} b) \ggg c \ggg (d \mathbin{***} e))$, where either:

Case 2a $c = r_1$, or

Case 2b $c = r_1 \ggg c' \ggg r_2$

where r_i is *arr x*, *pre v*, *LoopD f' d* or *LoopM f' d g'*.

Proof. Express f in CCA composition form: $f = f_1 \ggg f_2 \ggg \ldots \ggg f_n$. If every f_i is $x_i \mathbin{***} y_i$, then f is in the form denoted by Case 1, with $a = x_1 \ggg \ldots \ggg x_n$ and $b = y_1 \ggg \ldots \ggg y_n$. If exactly one f_i is not $x_i \mathbin{***} y_i$, then f is in the form denoted by Case 2a, with $a = x_1 \ggg \ldots \ggg x_{i-1}$, $b = y_1 \ggg \ldots \ggg y_{i-1}$, $c = f_i$, $d = x_{i+1} \ggg \ldots \ggg x_n$ and $e = y_{i+1} \ggg \ldots \ggg y_n$.

Otherwise, denote the first and last non-$***$ terms in f as f_i and f_j respectively. f is of the form in Case 2b, with $a = x_1 \ggg \ldots \ggg x_{i-1}$, $b = y_1 \ggg \ldots \ggg y_{i-1}$, $c = f_i \ggg \ldots \ggg f_j$, $d = x_{j+1} \ggg \ldots \ggg x_n$ and $e = y_{j+1} \ggg \ldots \ggg y_n$. □

We now prove that our *split* operation (Section 3.2.1) will transform every f with no direct dependencies between its inputs and outputs into $f_l \ggg f_d \ggg f_r$ where f_d is decoupled. We assume that every output of any *LoopD* within f directly depends on every input of it: this is necessary as it is possible to construct examples where a *LoopD* can have a decoupled signal function "hidden" inside it (discussed in Section 3.4.1). We show later that our transformation does not generate *LoopD* like that.

Lemma 4.4. Given f in CCA composition form for which:

1. f contains no *loop*, and
2. For every *LoopD* within f, all of its outputs directly depend on its inputs,

there are no direct dependencies between inputs and outputs of f if and only if we can apply split to f.

Proof. We consider the two directions of the equivalence in turn. The $\Longleftarrow$ direction is simple: all paths from the inputs of f to the outputs of f must go through f_d by definition of composition. f_d is decoupled so every path goes through a decoupled signal function, and thus there cannot be any direct dependencies.

We now turn to the $\Longrightarrow$ direction, which we prove by induction on the size of f. We define the size of f as follows: $f \mathbin{***} g$ and $f \ggg g$ each have size equal to the sum of the sizes of f and g, and all other terms have a size of 1.

We start with our base case: a program of size 1, meaning that f is one of *arr*, *pre*, *LoopD* or *LoopM*. We cannot have a *LoopD* or *arr* since there is at least one direct dependency in f, by definition and the second condition of our lemma respectively. Therefore, it must be *pre* or *LoopM*. In either of these cases, *split* applies Split-Pre or Split-LoopM and we are done.

We now prove our lemma for f of size $k+1$, assuming that it holds for f' of size k and smaller. We consider cases of g in $f = f' \ggg g$:

- $g = arr\ x$: All of *arr*'s outputs directly depend on its inputs. Therefore, we need f' to have no direct dependencies: if it had a direct dependency from f'_i to f'_o, then f would have a direct dependency from f'_i to f'_o to every output of g, contradicting the statement of the lemma. Therefore, we can apply split to f'. This is exactly what Split-NonPre does. The same applies for *id*, and *LoopD* by the second condition of the lemma.
- $g = pre\ v$ or *LoopM*: f matches the form needed by Split-Pre and Split-LoopM respectively.
- $g = (x_i \mathbin{***} y_i) \ggg \ldots \ggg (x_{k+1} \mathbin{***} y_{k+1})$, where f' does not end with $l \mathbin{***} r$: This means that $f' = w \ggg z$ and z is one of *arr*, *pre*, *LoopD* or *LoopM*. If z is *pre* or *LoopM*, we are done: use Split-NonPre to skip past g and then one of Split-Pre or Split-LoopM.
  If z is *arr* or *LoopD*, then there is a path from every input of z to every output. Consider cases of w.
  First, if w has no direct dependencies, we can apply split to it by the induction hypothesis and thus Split-NonPre can be applied to skip past g and z.
  Otherwise, if w has a direct dependency from w_i to w_o, there must be no direct dependencies in g for there to be

no direct dependencies in f. This is because if we had a direct dependency from g_i to g_o in g, we would have a direct dependency in f: from w_i to w_o, then through z to g_i, and finally to g_o.
This means that each of $x_i \ggg \ldots \ggg x_{k+1}$ and $y_i \ggg \ldots \ggg y_{k+1}$ have no direct dependencies by definition of $***$. We can therefore split each of x and y separately by the induction hypothesis and thus can apply Split-$***$-R.

Therefore, we have proven the property holds for the base and inductive cases, so it holds by induction. □

Finally, we now show that the transformation is complete.

Theorem 4.5. *Using our transformation, we can transform a loop f which either has no direct dependency cycles in it or has $f = a *** b$.*

Proof. By induction on *loop* nesting. Consider a *loop* f which contains no *loop*, *LoopD* or *LoopM*. By Lemma 4.3, we can express f in one of three forms, which we show can be transformed in turn.

Case 1. [$f = a *** b$.] Apply step 1 of the transformation to get $loop\ f = a$.

Case 2b. [$f = (a *** b) \ggg r_1 \ggg c' \ggg r_2 \ggg (d *** e)$, where r_1 and r_2 are *arr* or *pre*.] Step 1 does not apply, so we move onto step 2. If step 2 succeeds in splitting f, we finish with a *LoopM*.

If we cannot split f, then we must have that r_1 and r_2 are *arr*. This is because if either is *pre*, we are done since f has a decoupled signal function r_i on every path from the inputs of f to its outputs, meaning that it has no direct dependencies and thus can be split according to Lemma 4.4.

We know for the same reason that c' has at least one direct dependency from c'_i to c'_o, as otherwise we could split f: every path in f goes through c', and if c' has no direct dependencies then every path through c' has a decoupled signal function on it.

It follows that $c = arr\ x \ggg c' \ggg arr\ y$ has a direct dependency from all of its inputs to all of its outputs: from each input of $arr\ x$ to c'_i, to c'_o, and finally to each output of $arr\ y$.

Therefore, if we get to step 3 we are working with $(a *** b) \ggg c \ggg (d *** e)$. We apply step 3a to get $(a *** id) \ggg c \ggg (d *** (e \ggg b))$.

We require that it is possible to split $e \ggg b$ in the absence of dependency cycles for step 3b to be applicable. Assume for contradiction that $e \ggg b$ cannot be split, meaning that there exists a direct dependency within $e \ggg b$ from the jth input to the ith output. We then have a dependency cycle as follows: from the ith part of the second input of the *loop*, through c to the jth input of $e \ggg b$ to its ith output and thus completing the cycle. Therefore, there must be no direct dependencies in $e \ggg b$ and thus it can be split by Lemma 4.4.

We therefore get a *LoopD* of the form $LoopD\ ((a *** b) \ggg c \ggg (d *** e))\ y$. We apply tightening in step 3c to get $a \ggg LoopD\ ((id *** b) \ggg c \ggg (id *** e))\ z \ggg d$. Note that there is a direct dependency from every input to every output of this *LoopD*, as each input only goes through c to get to the output, and we showed earlier that c has a direct dependency from every input to every output.

Case 2a. [$f = (a *** b) \ggg r_1 \ggg (d *** e)$, where r_1 is *arr* or *pre*.] Case 2a is proved similarly to Case 2b. Step 1 does not apply, so we move onto step 2. If step 2 succeeds in splitting f, we finish with a *LoopM*.

If we cannot split f, then r_1 must be *arr* since if it were *pre* then f would have no direct dependencies as all paths go through that *pre*, meaning that f in that case can be split by Lemma 4.4.

We are therefore working with $f = (a *** b) \ggg arr\ x \ggg (d *** e)$. Apply step 3a of the transformation to get $(a *** id) \ggg arr\ x \ggg (d *** (e \ggg b))$. We know that every output of $arr\ x$ directly depends on every input, meaning that every input of $e \ggg b$ directly depends on every part of the second input of the *loop*, by the same logic as in Case 2b. We are guaranteed to be able to apply split in step 3b by the Lemma 4.4.

We end up with a *LoopD* of the form $LoopD\ ((a *** b) \ggg arr\ x \ggg (d *** e))\ y$. We apply tightening in step 3c to get $a \ggg LoopD\ ((id *** b) \ggg arr\ x \ggg (id *** e))\ y \ggg d$. Note that each output of this *LoopD* directly depends on each input, as each path only goes through $arr\ x$.

This concludes the base case. We also note that in every case where we create a *LoopD*, all of its outputs depend on its inputs. This means that we can use Lemma 4.4 in the inductive step as all *LoopD* considered will have this condition hold.

We now turn to the inductive step: proving the transformation works on *loop* f when all *loop* within f without dependency cycles can be transformed. We first apply our transformation to the inner *loop*. Then, the proof above also proves the inductive case, but with some minor modifications. We can now have r_i being *LoopD* or *LoopM*. Any *LoopD* can be treated identially to *arr*, as every *LoopD* created by this transformation has identical dependencies to an *arr*. Any *LoopM* can be treated identially to *pre* for the same reason. Therefore, the inductive step holds, and the proof holds. □

5 Implementation

In this section we describe the Haskell implementation of our transformation, *Severn*[4]. We start with a minimal AFRP implementation, then implement the transformation on top, and finally test that the implementation is correct.

[4]Available from https://github.com/finnbar/severn/tree/v1.0.0.0 and in the provided artefact [11].

5.1 Signal Descriptors and CCA Composition Form

We represent signals as in Chupin and Nilsson's SFRP [2] by using *signal descriptors*. These are defined by the `Desc` GADT, which is then lifted to the kind level via `DataKinds`.

```
data Desc x where
  V :: a -> Desc a
  P :: Desc a -> Desc b -> Desc (a, b)
```

This allows us to define signals which produce values of some type, and pairs of signals: `P (V Int) (V Int)` describes a pair of signals each containing `Int`s. Values produced by a signal with descriptor `d` are represented by the GADT `Val d`, used throughout the implementation.

Our arrow constructors are parameterised by these descriptors. This again mirrors SFRP, but rather than defining a single GADT with all of the arrow constructors, we enforce CCA composition form (Section 3.1.4) through the definition of multiple GADTs:

```
type CFSF :: Desc s -> Desc s' -> Type
data CFSF x y where
  (:>>>:) :: NoLoop a b -> NoLoop b c -> CFSF a c
  Single :: NoComp a b -> CFSF a b

data NoComp x y where
  LoopD :: CFSF (P a c) (P b d) -> Decoupled d c
    -> NoComp a b
  Arr :: (Val a -> Val b) -> NoComp a b
  Loop :: CFSF (P a c) (P b c) -> NoComp a b
  (:***:) :: NoComp a b -> NoComp c d
    -> NoComp (P a c) (P b d)
  Id :: NoComp (V a) (V a)
  Dec :: Decoupled a b -> NoComp a b

data Decoupled x y where
  LoopM :: CFSF (P a c) d -> Decoupled d e
    -> CFSF e (P b c) -> Decoupled a b
  Pre :: Val (V a) -> Decoupled (V a) (V a)
  BothDec :: Decoupled a b -> Decoupled a' b'
    -> Decoupled (P a a') (P b b')
```

By having `CFSF` (read *composed form signal function*) only introduce ⋙, and `NoComp` introduce the remaining combinators, ⋙ can only be added at the top level so that programs must be written in composition form. We also separate out decoupled terms into their own GADT, allowing us to enforce through the type system that a term is decoupled as shown in `LoopD`.

We provide smart constructors for each of the traditional arrow combinators that produce an equivalent `CFSF` using the laws discussed in Section 3, to avoid programmers having to directly use the above constructors to write their programs. This means that a programmer can write a `CFSF` in Severn in the same way that they would an `SF` in Yampa. We also implement a small optimisation pass which merges consecutive `Arr` together using the $arr\ f \ggg arr\ g = arr\ (g \cdot f)$ law, which is also an optimisation applied by Yampa [16].

5.2 The Transformation Algorithm

We now outline our implementation of our transformation on `CFSF`s. We focus on `transformLoop`, which transforms a given `Loop` using the steps outlined in Section 3.5; the transformation itself traverses the input `CFSF` by calling `transformLoop` on each `Loop` from innermost to outermost.

Each of the three cases outlined in our transformation are defined as a `CFSF a b -> Maybe (CFSF a b)` function, since a given case is not applicable to every `CFSF`. `transformLoop` is therefore defined as trying out each case using the alternative operator `<|>`.

The implementation of each case utilises the rules defined in Section 3, which are each implemented as Haskell functions. As an example, we present a slightly simplified implementation of `leftSlide` below:

```
data LoopBox a b where
  LB :: CFSF (P a c) (P b c) -> LoopBox a b

leftSlide :: LoopBox a b -> Maybe (LoopBox a b)
leftSlide (LB cfsf) =
  case headTail cfsf of
    Left _ -> Nothing -- Cannot slide if no :>>>:.
    Right (HT s ss) -> case s of
      s1 :***: s2 -> Just $ LB $ (Single s1 *** id)
        >>> ss >>> (id *** Single s2)
      _ -> Nothing -- Cannot slide non-:***: term.
```

We apply a few tricks here which are common throughout the implementation. We use auxiliary GADTs when we cannot determine the exact type of the output from the type of the input `CFSF`: here we require `LoopBox` since we cannot guarantee that sliding will lead to the same type `c`. We can also use this to guarantee that some part of the output is decoupled, which we do in our implementation of *split*.

Since `CFSF` allows arbitrary bracketing of ⋙, we cannot use pattern matching to get the first or last element of a given composition. We therefore provide `headTail x` to do this, which returns `Left x` if there are no ⋙, or the head and tail of the chain of ⋙ otherwise. The rest of the implementation follows from the definition of the rule: use our auxiliary functions to match the form of the rule, and if we can, return the result of applying it.

5.3 Running CFSFs

Once the transformation has been applied, we are left with a `CFSF` containing no `Loop`. Severn provides

```
runCFSF :: CFSF a b -> a -> (b, CFSF a b)
```

to run these transformed `CFSF`, taking an input value and producing the output at that time step along with the next `CFSF` to run. Since any `CFSF` applied to `runCFSF` no longer contains `Loop`, we define it strictly and thus avoid all of the overheads of lazy evaluation. The decoupled parts of `LoopD` and `LoopM` are run with

```
runDec :: Decoupled a b
  -> (Val b, Val a -> Decoupled a b)
```

Bench	Definition	Speedup
noloop	*arrs* = *arr f* ⋙ ... ⋙ *arr f*	1.65x
LoopD	*LoopD arrs* (*pre v*)	1.53x
LoopM	*LoopM arrs* (*pre v*) *arrs*	1.66x
Nested	*LoopD arrs* (*LoopM arrs* (*pre v*) *arrs*)	1.10x

Figure 12. Speedup compared to Yampa.

which produces the output without using any input, and the program to run at the next time step once it gets an input. Thus running `LoopD f d` consists of getting the output from `d`, running `f` and then using `d`'s input to get the next `CFSF`.

5.4 Testing via Arbitrary Program Generation

As well as our proof that the transformation is correct (Section 4), we also test arbitrary Severn programs without dependency cycles against their Yampa equivalents to make sure that the implementation is correct. We build test programs with a pair of mutually recursive generators using the Hedgehog library[5] which generate a Yampa `SF` and its equivalent Severn `CFSF`. One test generates decoupled programs, and one generates non-decoupled programs.

Programs are generated inductively: start with the smallest decoupled program `pre v` and the smallest non-decoupled program `arr f`, and then build larger programs by combining them. We use rules similar to those used by Sculthorpe and Nilsson [27] for their arrow combinator types indexed by decoupledness, with rules such as $f \ggg g$ being decoupled if one of f or g is. To generate a program of a given size and decoupledness, we generate two smaller programs and combine them using one of those rules.

These arbitrary decoupled and non-decoupled programs are then used to build *loop*s for testing. We use the same techniques as in Case 2b of Theorem 4.5 for building generic *loop*s without dependency cycles: start with *loop* $((a \lll b) \ggg c \ggg (d \lll e))$, generating *LoopM* by generating a decoupled c and generating *LoopD* by generating a decoupled $e \ggg b$.

We are able to test that if we have an `SF` and its equivalent `CFSF`, both programs produce the same results after transforming the `CFSF` using our transformation. Our implementation passes the tests for programs of arbitrary size.

6 Performance

To show the impact of our work on performance, we have two sets of benchmarks: on fixed networks to identify the improvements for specifics constructs, and on randomised networks. We use the Criterion library[6] to benchmark each program 100,000 times.

We first benchmarked four programs in order to test individual uses of *loop*. Fig. 12 shows the benchmarks, their definitions and average speedups compared to Yampa. Severn gives a speedup of between 1.5x and 1.7x for the first three benchmarks. The nested program gave a lower speedup of 1.1x, which we expect is due to the CFSF being allocated by Severn being larger in that case.

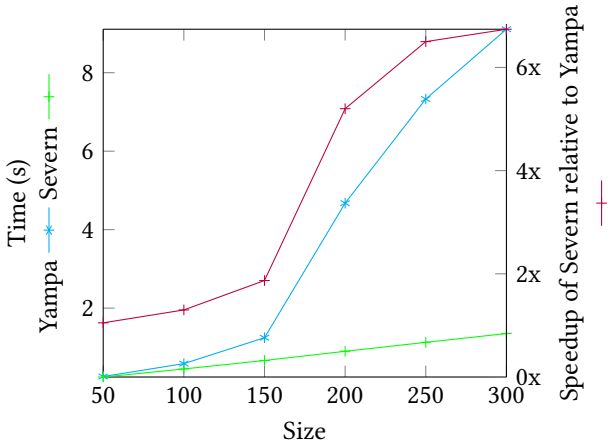


Figure 13. Time taken by Yampa and Severn on *loop* (*arr f* ⋙ *pre v* ⋙ ⋯ ⋙ *arr f* ⋙ *pre v*) for different numbers of primitives.

We also varied the number of *arr* in *arrs*, in order to test whether the speedup varied based on the amount of work done within the loop, but found no clear change. We expect this is due to the optimisations implemented by Yampa and in our optimisation pass (Section 5.1): the composition law allows composed *arr* to be merged into existing *arr*, so that the enlarged program is effectively the same but with larger pure functions to execute.

To avoid the effects of these optimisations we constructed a benchmark based on *LoopM* with *pre*s and *arr*s interleaved. By doing this, the composition law could not be applied. The results for this are shown in Fig. 13. We achieved speedups of between 1x and 2x for programs with 150 or fewer primitives, but performance improved significantly with 200 primitives.

For our randomised tests, we take a similar approach as was done for SFRP [2]. We varied two parameters: the size of the generated program, and the number of *loop*s within that program as a proportion of the size. We do not include the time it takes our transformation to run. We found that our speedups were always greater than 1x, and averaged 2.5x[7].

Since we achieve speedups in all benchmarks, we conclude that our transformation provides an effective improvement for *loop*s in AFRP. Further improvements may possible in the future by using `IORefs` to allocate *pre*s rather than returning an entirely new `CFSF`, by using stream fusion [3], or with the Pipes library[8] to get more performance out of `runCFSF`.

[5] https://hackage.haskell.org/package/hedgehog
[6] https://hackage.haskell.org/package/criterion
[7] The full results can be found alongside the artefact [11].
[8] https://hackage.haskell.org/package/pipes

7 Limitations and Future Work

While our transformation works on a large subset of Yampa programs, we make a few assumptions that may not hold for all AFRP programs. We discuss the consequences here and how our work could be extended in future work.

7.1 Effects and Monadic Signal Functions (MSFs)

We require that our arrows satisfy the laws needed for CCAs in order to apply the distributive law, which underpins most of operations we have defined. We also assume that there are no side-effects in the sliding law. These two points mean that our transformation is not applicable to effectful programs in general: it changes the execution order so that that a program can be run strictly, which may not be the same execution order as before. In the following example, lazy semantics will run f first since *pre x* produces its input immediately:

$$loop\ (g \ggg first\ (pre\ x \ggg f) \ggg i \ggg second\ (pre\ v))$$

However, our transformation will turn this into a *LoopD* which, when run strictly, runs g first. This does not pose an issue in Yampa as *pre* is the only "effectful" operation, and its effects are entirely local to itself. However, it is easy to add effects into such a system: Perez et al. embed monads into AFRP with Monadic Signal Functions (MSFs) [20]. Modifying our transformation to preserve execution order, and therefore support MSFs, is future work.

It is important to note however that the reordering does not change the meaning of programs when our effects are *commutative*: we eventually run every part of the program. Piponi [23] shows a number of monads whose effects are commutative, meaning their computations can be reordered without issues. MSFs built with these commutative monads, such as Reader and Writer with a commutative monoid, could therefore be safely transformed by our technique.

7.2 *arr* is a Black Box

We know nothing about f within *arr* f and must assume that all outputs of f depend on all of its inputs. However, *arr* is the only constructor we have for routing data and it permits programmers to write routing functions like $swap = arr\ (\lambda(x, y).(y, x))$ where that assumption is not required. Dealing with this would introduce complex dependencies between the inputs and outputs of an *arr*, but would allow additional ways to transform programs: for example, $first\ (pre\ x) \ggg swap = swap \ggg second\ (pre\ x)$. This poses a particular problem when working with *proc* notation [19], which introduces many additional *arr* during desugaring.

If we differentiated between *arr* for applying pure functions and *arr* for routing, we could modify our transformation to take these into account. Joseph's *generalised arrows* [10] introduces a variety of additional combinators such as `ga_assoc` and `ga_swap` with which routing can be implemented without using *arr* explicitly. SFRP [2] uses *routers* for arbitrary rearrangements of inputs into outputs.

7.3 Switching and Choice

Members of the `ArrowChoice` class allow for conditional execution of arrows. The key operator is `f +++ g`, which runs `f` if given a `Left` value and `g` otherwise. Since `f +++ g` depends on its input to decide which of `f` and `g` to run, we can never decouple it. It can therefore be treated in the same way as *arr* and thus should be easy to add.

Switching is harder to add: `switch f c` uses a continuation `c` to change the arrow being run (`f`) to a different one at runtime. This means that `switch` can change the structure of a program in a way that is unknown at compile-time. SFRP implements switching by running its transformation again once a switch occurs, but this can slow down the program temporarily as the entire transformation procedure is rerun. Winograd-Cort and Hudak transform some uses of `switch` into `+++`, which avoids these issues for them [29].

7.4 Well-typed *loop*

We proved that our transformation only fails if we are unable to run a program anyway (Section 4). Therefore, if our transformation succeeds, a *loop* is well-formed. A type system which guarantees that a *loop* contains no dependency cycles would be helpful to avoid running the transformation on a *loop* with dependency cycles. This also avoids the issue of a `switch f c` generating a *loop* that cannot be transformed, thus generating a runtime error.

A type system for checking for no dependency cycles in *loop* could build on some of the existing work by Sculthorpe and Nilsson [27] who label the decoupledness of signal functions at the type level, and Bahr [1] who introduces a modal type system that detects space leaks.

8 Related Work

FRP Applications. FRP sees significant use in a variety of domains where performance is important. It has been used in many embedded settings: the Juniper language for Arduino microcontrollers [7], the Hailstorm language for IoT [25] and the Emfrp language for embedded systems [26] are three examples which use a variant of FRP designed for restricted memory use, but could move to AFRP if it became performant enough. The original introduction to AFRP [8] discussed robots as its basis, which also tend to consist of programs run on embedded systems.

There has also been some research into making FRP safer in these contexts. Perez and Goodloe [22] incorporate fault tolerance into FRP, which could also be useful in domains like robotics. Copilot [21] allows users to write runtime verification systems in the style of FRP.

AFRP Optimisation. Beyond CCAs (Section 2.1), Scalable FRP (SFRP) [2] is another optimisation which transforms AFRP programs into IO operations on mutable memory cells to reduce the cost of routing data between signal functions.

Notably, SFRP does not currently support the *loop f* combinator at all since it needs to know the order in which the component signal functions of *f* are executed to order its IO operations. Our transformation would allow the *loop* combinator to be added.

Other projects have taken a similar approach to SFRP: Ultrametric FRP [12] implements FRP as an imperatively updated dataflow graph, and Patai's work on higher-order streams [18] translates FRP to an IO stepper action which runs with each new sample.

A common optimisation in the FRP world that may be applicable to AFRP is deciding whether a value needs to be computed at a given time step. Elliott [4] discusses how values that only sometimes change should only be recomputed when new values are *pushed*, but also that the results of FRP code should only be recomputed when they are *pulled* by whatever is utilising its results. Sculthorpe and Nilsson [28] define some temporal logic properties of FRP networks that could also be used to reason about change and thus whether a value needs recomputing.

All of the above optimisations could be combined with our work, and would likely produce improved speedups compared to those we presented in Section 6.

Synchronous Programming. Much of the work we have discussed aims to bring FRP's efficiency and safety closer to that of *synchronous dataflow languages* such as Lustre [6] which also permit writing reactive programs. While less expressive than FRP, they are simpler to efficiently implement [13]. They deal with dataflow cycles (*loop* in AFRP) via a syntactic check for a *delay* operator present in every cycle, which is similar to what we have done here. Digital circuits are similar: Ghica et al. [5] introduce a theory for rewriting dataflow categories with a *delay* operator, which is then used to talk about digital circuits that could also be applied to AFRP.

9 Conclusions

We showed that *loop*s in AFRP without dependency cycles can be transformed into more restrictive *LoopD* and *LoopM* forms which can be evaluated strictly, thus avoiding the overheads of lazy evaluation. This offers performance benefits and allows for easier compilation of bespoke AFRP-style languages in the future since such a language will no longer need lazy evaluation.

We proved that our transformation preserves program meaning, both theoretically using the Arrow and ArrowLoop laws, but also practically through a Haskell implementation whose tests ensure that programs before and after transformation behave equivalently. We also provided a proof that this transformation works on every *loop* expressible in our subset of AFRP that does not contain a dependency cycle.

While our implementation is a subset of Yampa, we believe it is large enough to support most useful programs. We also laid out how we could extend our transformation to be able to support even more programs in future. Finally, our benchmark shows that our implementation, Severn, provided a modest speedup for a variety of AFRP programs, and outlined potential further improvements through implementation of Yampa's optimisations.

Data Availability Statement

The implementation of Severn which is tested in Section 5.4 and benchmarked in Section 6 is available as an artefact on the ACM Digital Library [11]. It is also available on GitHub[9]. The full data from our benchmarks in Section 6 is also provided within the artefact.

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