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Recurrence Relations for the Fresnel Integral $\int_0^\infty \frac{\exp(-ct)dt}{\sqrt{t}(1+t^2)}$ and Similar Integrals

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The class of functions defined by $\int_0^\infty [\exp(-cX)dt/(1+Y)(\sqrt{t})^k]$ where X and Y are either t or t^2 and k is -1, 0, or 1 can be evaluated by recurrences for all but small values of the parameter c. These recurrences, given here, are more efficient than the usual asymptotic series.

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In this paper we give simple recurrence relations that evaluate integrals such as $\int_0^\infty [\exp(-ct^2)dt/(1+t)]$ and $\int_0^\infty [\exp(-ct)dt/(1+t^2)]$ for values of the argument $2 < c < \infty$. These recurrences are simple to program, are remarkably efficient for $c > 5$, and are especially suitable for the small programmable electronic desk calculators that are now appearing where storage for constants is limited. The recurrences provide an alternative to rational approximations (which are not always available) that are derived from asymptotic series via the Q-D algorithm, usually with some kind of economization. (Cody [3] gives some for our sample integral.)

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We emphasize that these recurrences, like asymptotic series, decrease in efficiency as the argument becomes smaller, but they remain viable until c is down to 3 or 2—i.e. well into the range for which a small-argument series is efficient, provided, of course, that any singular behavior at the origin is explicitly removed (see [2, Chap. 10]).

Specifically, the integrals we treat here are of the form

$$F(c) = \int_0^\infty [\exp(-cX)dt/(1+Y)(\sqrt{t})^k]$$

where X and Y are either t or t^2 (in all combinations) and k is ± 1 or zero. The recurrences are all given in Tables I and II. Here we continue our exposition via one example, the integral

$$J_0(c) = \int_0^\infty [\exp(-ct)dt/\sqrt{t}(1+t^2)]$$

that occurs as an asymptotic form of the Fresnel integrals (see 7.4.26 and 7.3.5 in [1]).

We define the three sequences of functions

$$J_n(c) = \int_0^\infty \frac{e^{-ct}}{\sqrt{t}(1+t^2)}\left[\frac{t^2}{1+t^2}\right]^n dt$$

$$c \geq 0$$

$$I_n(c) = \int_0^\infty \frac{e^{-ct}t}{\sqrt{t}(1+t^2)}\left[\frac{t^2}{1+t^2}\right]^n dt$$

$$n \geq 0$$

$$K_n(c) = \int_0^\infty \frac{e^{-ct}}{\sqrt{t}}\left[\frac{t^2}{1+t^2}\right]^n dt$$

and note that $J_0(c)$ is the integral we seek, $I_0(c)$ is essentially the other Fresnel asymptotic form, while $K_0(c)$ is elementary, being $\sqrt{\pi/c}$. Further, all three sequences of functions decrease monotonically with increasing n. The recurrences

$$I_{n-1} = \frac{4n\cdot I_n + 2c\cdot K_n}{4n-1}$$

$$J_{n-1} = \frac{4n\cdot J_n + 2c\cdot I_{n-1}}{4n-3}$$

$$K_{n-1} = K_n + J_{n-1}$$

may be derived through integration-by-parts with frequent use of the relation

$$\frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2}$$

and the observation that

$$\frac{d}{dt}\left[\frac{t^2}{1+t^2}\right] = \frac{2t}{(1+t^2)^2}.$$

The derivation of the first relation is typical:

$$J_n = \int_0^\infty \frac{e^{-ct}}{\sqrt{t}(1+t^2)}\left[\frac{t^2}{1+t^2}\right]^{n-1}\left(1-\frac{1}{1+t^2}\right)dt$$

$$= J_{n-1} - \frac{1}{2}\int_0^\infty \frac{e^{-ct}}{t^{3/2}}\left[\frac{t^2}{1+t^2}\right]^{n-1}\frac{2t\,dt}{(1+t^2)^2}$$

Table I. Integrals with $\exp(-ct)$ in Their Numerators

Integrals (all limits are $0-\infty$)		Recurrence	Normalization
$I_0 = \int \frac{e^{-ct}}{(1+t)} dt$	1	$I_{n-1} = I_n + \frac{c}{n} J_n$	5.1.28*
	0	$J_{n-1} = J_n + I_{n-1}$	$J_0 = \int e^{-ct} dt = \frac{1}{c}$
$I_0 = \int \frac{e^{-ct}}{\sqrt{t}(1+t)} dt$	1	$I_{n-1} = \frac{2n.I_n + 2c.J_n}{2n}$	7.4.9
$J_1 = \int \frac{e^{-ct}\sqrt{t}}{(1+t)} dt$	0	$J_{n-1} = J_n + I_{n-1}$	$J_0 = \int \frac{e^{-ct}}{\sqrt{t}} dt = \sqrt{\frac{\pi}{c}}$
$I_0 = \int \frac{e^{-ct} t}{1+t^2} dt$	1	$I_{n-1} = I_n + \frac{c}{2n} K_n$	
$J_0 = \int \frac{e^{-ct}}{1+t^2} dt$	0	$J_{n-1} = \frac{2nJ_n + cI_{n-1}}{2n-1}$	
	0	$K_{n-1} = K_n + J_{n-1}$	$K_0 = \int e^{-ct} dt = \frac{1}{c}$
$I_0 = \int \frac{e^{-ct}\sqrt{t}}{1+t^2} dt$	1	$I_{n-1} = \frac{4nI_n + 2cK_n}{4n-1}$	7.4.25
$J_0 = \int \frac{e^{-ct}}{\sqrt{t}(1+t^2)} dt$	0	$J_{n-1} = \frac{4nJ_n + 2cI_{n-1}}{4n-3}$	7.4.26
	0	$K_{n-1} = K_n + J_{n-1}$	$K_0 = \int \frac{e^{-ct}}{\sqrt{t}} dt = \sqrt{\frac{\pi}{c}}$

*These are the formula numbers in AMS-55 for the integrals.

Table II. Integrals with $\exp(-ct^2)$ in Their Numerators

Integrals		Recurrence	Normalization
$I_0 = \int \frac{e^{-ct^2}}{(1+t)} dt$	1	$I_{n-1} = I_n + \frac{2c}{n} K_n$	7.4.10
$J_1 = \int \frac{e^{-ct^2} t}{(1+t)} dt$	0	$J_{n-1} = J_n + I_{n-1}$	$J_0 = \int e^{-ct^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{c}}$
	1	$K_{n-1} = K_n + J_n$ (Not n-1)	$K_0 = \int e^{-ct^2} t dt = \frac{1}{2c}$
$I_0 = \int \frac{e^{-ct^2}}{1+t^2} dt$	1	$I_{n-1} = \frac{2n.I_n + 2c.J_n}{2n-1}$	7.4.11
	0	$J_{n-1} = J_n + I_{n-1}$	$J_0 = \int e^{-ct^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{c}}$
$I_0 = \int \frac{e^{-ct^2} dt}{\sqrt{t}(1+t^2)}$	1	$I_{n-1} = \frac{4n.I_n + 4c.J_n}{4n-3}$	
$J_1 = \int \frac{e^{-ct^2} t\sqrt{t} dt}{(1+t^2)}$	0	$J_{n-1} = J_n + I_{n-1}$	$J_0 = \int \frac{e^{-ct^2}}{\sqrt{t}} dt = \frac{\Gamma(1/4)}{2c^{1/4}}$
$I_0 = \int \frac{e^{-ct^2}\sqrt{t}}{(1+t^2)} dt$	1	$I_{n-1} = \frac{4n.I_n + 4c.J_n}{4n-1}$	
	0	$J_{n-1} = J_n + I_{n-1}$	$J_0 = \int e^{-ct^2}\sqrt{t} dt = \frac{\Gamma(3/4)}{2c^{3/4}}$
$I_0 = \int \frac{e^{-ct^2}}{\sqrt{t}(1+t)} dt$	0	$I_{n-1} = \frac{2n.I_n + 4c.K_n}{2n-1}$	
$J_0 = \int \frac{e^{-ct^2}\sqrt{t}}{(1+t)} dt$	1	$J_{n-1} = \frac{2n.J_n + 4c.L_n}{2n+1}$	
	0	$K_{n-1} = K_n + J_{n-1}$	$K_0 = \int e^{-ct^2}\sqrt{t} dt = \frac{\Gamma(3/4)}{2c^{3/4}}$
	0	$L_{n-1} = L_n + K_n$ (not n-1)	$L_0 = \int e^{-ct^2} t\sqrt{t} dt = \frac{\Gamma(5/4)}{2c^{5/4}}$

whence integration-by-parts gives

$$\begin{aligned} J_n &= J_{n-1} + \frac{1}{2n}\int_0^\infty e^{-ct}\left[\frac{t^2}{1+t^2}\right]^n\left(-\frac{c}{t^{3/2}} - \frac{3}{2t^{5/2}}\right) dt \\ &= J_{n-1} - \frac{c}{2n}\int_0^\infty \frac{e^{-ct}t}{\sqrt{t}(1+t^2)}\left[\frac{t^2}{1+t^2}\right]^{n-1} dt \\ &\quad - \frac{3}{4n}\int_0^\infty \frac{e^{-ct}t}{\sqrt{t}(1+t^2)}\left[\frac{t^2}{1+t^2}\right]^n dt \\ &= J_{n-1} - \frac{c}{2n}I_{n-1} - \frac{3}{4n}J_{n-1} \end{aligned}$$

so finally

$$J_n = \left(1 - \frac{3}{4n}\right) J_{n-1} - \frac{c}{2n} I_{n-1}.$$

The system of recurrences is homogeneous, and the familiar scheme of J.C.P. Miller [1, Sec.9.12] may be used in which canonical values of $I_n(c) = 1$, $J_n(c) = 0$, and $K_n(c) = 0$ are given for some suitably large n. The recurrences are then run down to $n = 0$ and the value of I_0 (or J_0) is finally normalized by multiplying by the ratio of $\sqrt{\pi/c}$ to the value of $K_0(c)$ that was computed —all values in the computation being erroneous by this common factor. For our example we take $c = 5$, $n = 10$ to produce the values in Table III and we see that we have achieved results correct to nearly seven significant figures in J_0 and five in I_0. (No intermediate values need be retained. They are given here merely to show the growth that is typical in these recurrences.) The larger the starting value of n, the more accurate the final results, but the accuracy is also a function of c. As a rough guide, $n_{\text{initial}} = 150/c$ will yield approximately ten significant figures in I_0 and J_0. Note that the recurrences involve no loss of precision through subtraction of imprecisely known quantities—all values of I, J, and K being positive. Again, this is typical.

Table III.

n	J_n	I_n	K_n
10	0	1.0	0
9	0.2772 002	1.0256 41	0.2772 00
8	0.6460 806	1.1341 451	0.9232 809
7	1.2193 176	1.4685 630	2.1425 985
6	2.2922 394	2.3165 093	4.4348 380
5	4.6889 486	4.3454 175	9.1237 866
4	11.0317 728	9.3761 166	20.1555 954
3	31.6069 748	23.4382 307	51.7625 342
2	122.8379 976	72.6258 282	174.6005 318
1	861.4013 517	332.4302 777	1036.0018 83
0	42411.4052 3	3896.5799 82	43447.4071 1
Normalized	0.77376 43797	.071090 1885	
Correct	0.77376 45665	.071089 873	

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