There is also only one zero (near b_1) if $\delta < 0$. Providing a correct zero for all cases entails a sharp distinction on δ at 0.

Our solution procedure to this problem employs a slight variation of the Dekker algorithm [4], by which the zero of $f(x)$ is enclosed in successive subintervals $[a_i, b_i]$ where $f(a_i)$ and $f(b_i)$ are always of opposite sign. When the width of the interval $[a_i, b_i]$ is sufficiently small, depending on the number of decimal places requested, an n decimal place approximation to the zero may be obtained from a_i (or b_i). However, when a point a_i or b_i is encountered such that its function value is so extremely small in absolute value that its sign can not readily be determined (as criterion we again employ the outcome of a floating point determination), then the procedure is halted. The value a_i or b_i is printed since this information may be useful, but no claim is made that a zero has been found.

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An Interactive Graphic Display for Region Partitioning by Linear Programming

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Using linear programming, an interactive graphic display system has been implemented to solve the region design problem of partitioning a region into N nonoverlapping subregions in such a way that their areas are in specified proportions and that the total cost of servicing them is a minimum. In a conversational manner, a user can easily obtain different partitionings by specifying and modifying the boundary, the service centers' locations, the area proportions, and the cost functions. Examples are included.

Key Words and Phrases: interactive graphic display, region partitioning, area specification, linear programming

CR Categories: 5.41, 8.2, 8.3

1. Introduction

In the study of region design problems such as urban planning, districting, warehousing, and facility allocation, the following problem often arises: Given a region R and N "service centers," it is required to partition R into N nonoverlapping subregions R_j, $j = 1, 2, \ldots, N$, in such a way that, R_j, being serviced by the jth center, has a specified proportion of the total area, and the total "cost" of providing these services is a minimum. For example, the service centers may be

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fire stations and the cost may be the response time to fire alarms. For operational or financial reasons, the area sizes of the subregions are often prespecified.

To date, there appears to be relatively little work published on this problem. A purely graphical technique was discussed by Keeney [3] for a very particular case where the cost function is the Euclidean distance and no area sizes are specified. Some theoretical results are given in [1].

In the system described in this paper an interactive display provides a user with the desirable features of the graphical description of the partitions and the possibility of obtaining different solutions with relative ease. Approximating the region's boundary by polygons, a user first types in the coordinates of their vertices. In response, the system (hereafter referred to as DESIGN) displays the polygons and the input data. The names and locations of the service centers, the discretization intervals, and the cost functions are then entered. Throughout the whole process a user may insert or delete some of the vertices or service centers by using the lightpen and the keyboard. On obtaining these data, DESIGN will determine and display the partitioning. The user may request that the partitioning be subject to specified ratios or that two partitionings be displayed at the same time.

As pointed out by Rosen [7], application of linear programming using interactive graphic display is a potentially promising area for future research. This paper describes one such contribution. Application to data fitting and the approximate solution of boundary value problems were discussed in [4, 5, 6].

In Section 2 we formulate the problem and discuss its mathematical foundation. In particular, the discretized problem with area specification will be shown to be equivalent to a transportation problem. Section 3 describes the interactive program DESIGN. Two examples are given in Section 4.

2. Formulation as a Transportation Problem

Let $A_1 : A_2 : \cdots : A_N$ be the ratios of the areas of the subregions. Set $A = \sum_{j=1}^{N} A_j$. Then A_j/A is the area proportion of subregion R_j. Let $f_j(x,y)$ denote the "cost" of servicing a location $(x,y) \in R$ by the jth center and $P(R)$ denote the set of all possible partitions of R into N nonoverlapping subregions with the area of R_j proportional to A_j. Our problem is to find a partition which minimizes the total cost, i.e.

$$\min_{(R_1, R_2, \cdots, R_N) \in P(R)} \left\{ \sum_{j=1}^{N} \int_{R_j} f_j(x, y) \right\} \tag{1}$$

A global minimum to problem (1) is called a minimal partition. Under the general assumptions: (i) A_j, $j = 1, 2, \cdots, N$, are positive real numbers; (ii) R and R_j, $j = 1, 2, \cdots, N$, are bounded and Lebesque-measurable; and (iii) f_j, $j = 1, 2, \ldots, N$, are real-valued and L-integrable over R; Corley and Robert [1] obtained some theoretical results—they proved the existence of a minimal partition to problem (1) and gave the necessary and sufficient conditions for such a solution.

A practical computational method for obtaining an approximate solution is now described. Let R lie completely within a rectangle. Discretize this rectangle by dividing its sides into equidistant intervals. The case without area specification is simple. A discrete point P is assigned to R_j if $f_j(P) < f_k(P)$ for all $k \neq j$, and to either R_j or R_k (but not both) if $f_j(P) = f_k(P)$. In the following, the more general case (1) will be formulated as a transportation problem.

Let D be the set of discrete points which lie within R and M denote the number of points of D. We shall determine a set of integers $\{a_j\}_{j=1}^{N}$ in the following way: Let a_j be the integer closest to MA_j/A (e.g. if $99.5 \leq MA_j/A < 100.5$, set $a_j = 100$). Set $k = M - \sum_{j=1}^{N} a_j$. If $k > 0$ ($k < 0$), add 1 to (subtract 1 from) k of the a_j's randomly. If $k = 0$, do nothing. In this way, we get $M = \sum_{j=1}^{N} a_j$ and $a_1 : a_2 : \cdots : a_N \approx A_1 : A_2 : \cdots : A_N$.

For any point $(x_i, y_i) \in D$, denote $c_{ij} = f_j(x_i, y_i)$. Problem (1) can then be approximated by the following linear program

$$\min_{t_{ij}} \left\{ \sum_{i=1}^{M} \sum_{j=1}^{N} t_{ij} c_{ij} \;\middle|\; \begin{array}{l} \sum_{j=1}^{N} t_{ij} = 1, \ i = 1, 2, \cdots, M \\ \sum_{i=1}^{M} t_{ij} = a_j, \ j = 1, 2, \cdots, N \\ \qquad t_{ij} = 0 \quad \text{or} \quad 1 \end{array} \right\} \tag{2}$$

where t_{ij} is equal to 1 if (x_i, y_i) is assigned to the jth center and is equal to 0 if not.

Problem (2) is the personnel classification problem, a particular case of the well-known transportation problem. The method we use for solving problem (2) is an implementation of the algorithm given by Dennis [2].

3. Interactive Program

The interactive program DESIGN is written in Fortran and implemented on an IBM 360/67 computer, using a CDC graphic terminal. At the beginning, DESIGN asks the user to type in the vertices of the polygons. The following types of constants are acceptable:

Type	*Example*
Integer	104, −115
Floating-point	2.1, −9.1
Exponential	2.8E-2, −1.01E30

After the polygons are displayed, a user may make modifications, or type in the sizes of the discretization intervals, the names and locations of the service centers and the area ratios. The cost functions are entered (as data) symbolically as follows. DESIGN accepts elementary functions, which, by definition, are obtained by addi-

Fig. 1(a). Regional division for helicopter emergency system at Eastern Canada (with area proportions 2:2:1:1:1).

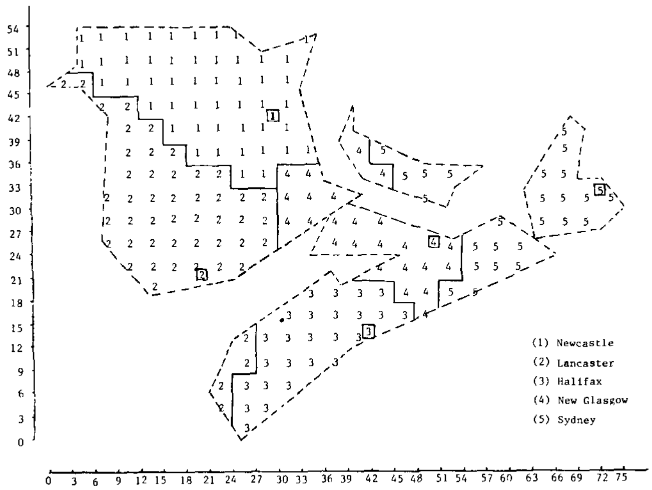


Fig. 1(b). Regional division for a helicopter emergency system at Eastern Canada (no area specification).

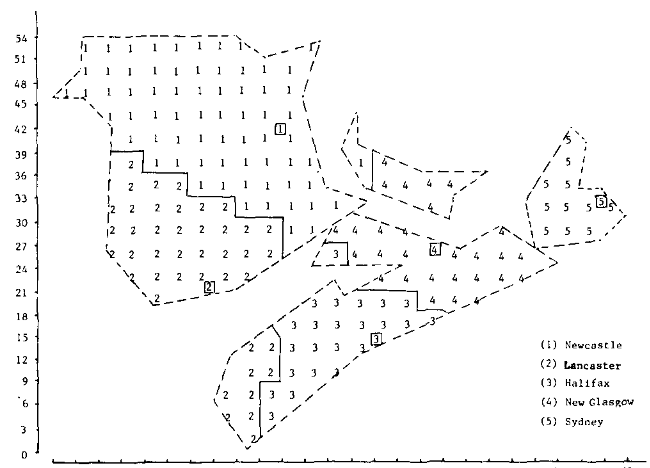


Fig. 1(c). Regional division for a helicopter emergency system at Eastern Canada (no area specification).

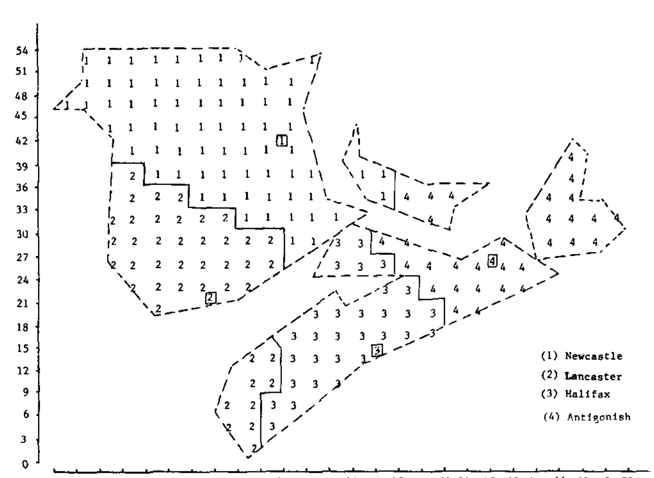


Table I(a). Vertices of Boundary Polygons

Polygon	*Vertices (in counterclockwise order)*
1	(4,49), (0,46), (4,46), (8,42), (7,26), (13,19) (24,21), (41,32), (36,34), (33,46), (35,53), (28,51), (24,54), (4,54)
2	(37,29.4), (34,24), (46,24), (38,20), (37,22), (24,13), (21,6), (25,0), (40,12), (66,24), (59,29), (53,26), (39,30.7)
3	(68,42), (62,32), (64,26), (72,27), (75,30), (72,34), (69,34), (70,40)
4	(40,44), (38,39), (41,34), (52,30), (53,33), (57,36), (49,36), (40,40)

Table I(b). Service Centers

Label	*Location*	*Cost Function*
1	(30.0,42.0)	((X−30) !2 + (Y−42) !2) !0.5
2	(20.0,21.0)	((X−20) !2 + (Y−21) !2) !0.5
3	(42.0,14.0)	((X−42) !2 + (Y−14) !2) !0.5
4	(50.0,26.0)	((X−50) !2 + (Y−26) !2) !0.5
5	(72.0,32.0)	((X−72) !2 + (Y−32) !2) !0.5

tion, subtraction, multiplication, division, exponentiation, and composition of the variables x,y and constants, and the functions e^v, $\sin(v)$, $\cos(v)$, $\arctan(v)$, $\log(v)$ and $|v|$, where v itself is an elementary function. The following symbols are used for operations:

Symbols	*Operations*
+, −	addition, subtraction (or negation)
*, /, !	multiplication, division, exponentiation
SIN, COS, ATAN	sin( ), cos( ), arctan( )
LN, EXP, ABS	log( ), exp( ), absolute value

Examples of elementary functions are:

((X−31.) ! 2 + (Y−15) ! 2) ! 0.5
−0.31 + (SIN(EXP(*X*+*Y*) + .5E4) * 4.) ! 0.1/2

When the user types in the instruction "PARTITION", DESIGN will start the partitioning process. It will discretize the region and evaluate the function values at the interior discrete points. Since the functions are also input data, they have to be evaluated interpretively, using a method similar to the Polish notation. (See, for example, LaFata and Rosen [4].) DESIGN will then partition these discrete points by the method as described in Section 2, plot the numeric labels of the service centers at the discrete points, and draw the subregion boundaries. These graphs are created in different display files and transmitted to the terminal for display. The user can then make any necessary modifications, using the lightpen and the keyboard, and continue the conversation until satisfied with the partitioning obtained.

4. Examples

The data used in the following examples are chosen arbitrarily, for the purpose of illustration only. During a normal load of the computing system, each problem takes about 3 minutes.

Table II(a). Vertices of Boundary Polygons

Polygon	Vertices (in counterclockwise order)
1	(2,28), (2.5,22), (0,14), (3,8), (10,5.5), (18,0), (18,2), (29.5,4), (31.5,0.5), (32.5,2), (31.5,5), (35.5,10), (34.5,10.9), (38.5,19.5), (39.18), (42,21), (41,21.5), (39.5,20), (38,23.5), (41,25), (41,28), (29,28), (28.5,26), (29,23), (28,22.5), (26.5,25.5), (22,28)
2	(8,21.5), (15,20.7), (14,14), (15.5,13.7), (15.4,13), (20,12.7), (20.5,12), (20.5,10), (10,11), (10.5,14), (9.5,14.5), (10.17.5), (9,17.6), (7.5,20.5)
3	(24,20.5), (26,19.5), (27,18.5), (22,18.5)
4	(27,18.5), (29,18), (29.5,17), (28.5,15.5), (26.5,17.5), (26,14.5), (25,13.5), (25,17.5)
5	(30.5, 16.5), (31.5,16.5), (29,14), (28,14)

Table II(b). Service Centers

Label	Location	Cost Function
1	(32.0,19.0)	((X−32) !2 + (Y−19) !2) !0.5 + 0.2*ABS(Y−19)
2	(10.0,25.0)	((X−10) !2 + (Y−25) !2) !0.5 + 0.15*ABS(Y−25)
3	(19.5,14.0)	((X−19.5) !2 + (Y − 14) !2) !0.5 + 0.1*ABS(Y−14)
4	(7.5,8.0)	((X−7.5) !2 + (Y−8) !2) !0.5
5	(28.5,7.5)	((X−28.5) !2 + (Y−7.5) !2) !0.5

Example 1. Suppose the Maritime provinces (except Newfoundland) of Eastern Canada want to set up a helicopter emergency system. The helicopter stations are located at Newcastle, Lancaster, Halifax, New Glasgow, and Sydney. Assume that the response time to an emergency call is proportional to the distance between the place where the accident occurs and a station. Two cases are considered, the first with area proportions: 2:2:1:1:1, and the second without area specification. The data shown in Table I(a) were first typed in. In less than one minute, the region's boundary (in dashed lines of Figure 1(a)) appeared. The area ratios 2:2:1:1:1 and the data shown in Table I(b) were then entered. After about 3.5 min, the graph in Figure 1(a) was displayed. Next, DESIGN was instructed to partition the same region without area specification. Figure 1(b) was then shown. Using the lightpen, the user eliminated the two service centers—New Glasgow and Sydney—and inserted a new one—Antigonish. Figure 1(c) shows the display after such modification.

Example. 2. Suppose the southern part of Canada and the mainland of the United States plan to establish a joint environmental control and development project. The states Montana, Wyoming, Colorado, and Kansas do not participate. (This assumption is purely for the purpose of making the problem more complex.) The centers are set up at Ottawa, Edmonton, Omaha, Phoenix, and Atlanta. The data for this problem are shown in Table II(a) and Table II(b). Area ratios are 2:2:2:3:3. The interactive process was similar to Example 1. Displayed graphs are shown in Figure 2(a) and Figure 2(b).

Fig. 2(a). Regional division for an environmental control and development project in Southern Canada and the United States (area proportions 2:2:2:3:3).

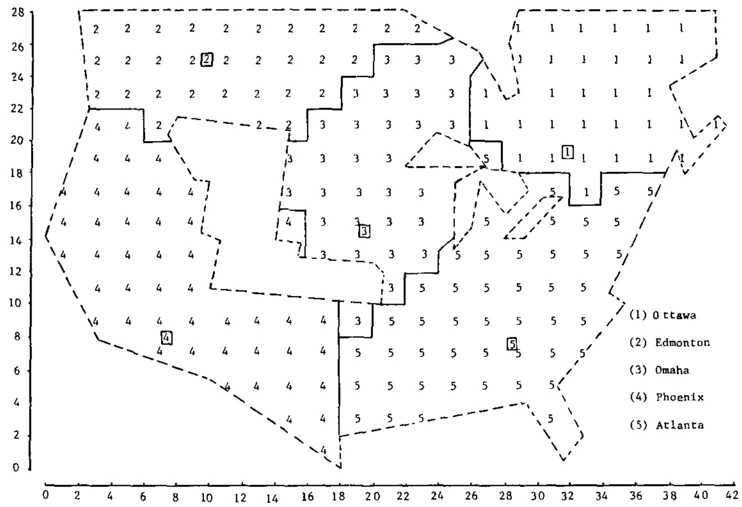
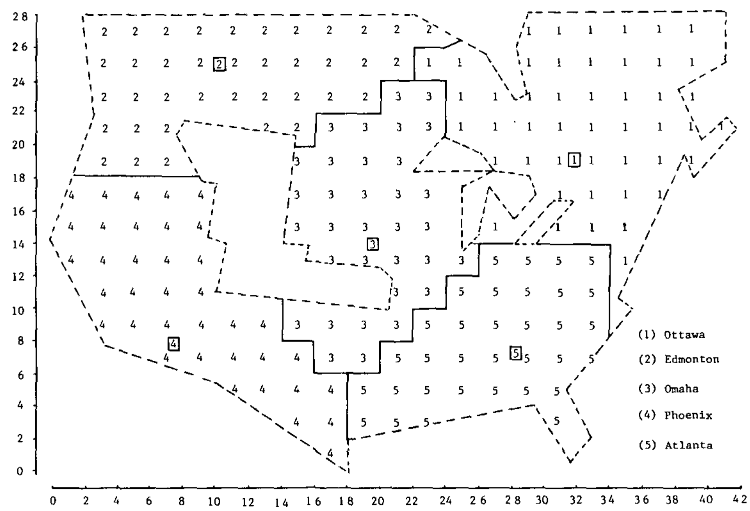


Fig. 2(b). Regional division for an environmental control and development project in Southern Canada and the United States (no area specification).



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