Iterative Disposal Processes and the Lambert W Function

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ABSTRACT

Given ℕ, choose a number randomly. Evens are chosen without replacement and odds are chosen with replacement. Repeat this process for as many times as there are naturals. Assess the expected value for the probability even in the resultant set. Then consider this question for the same process instead iterating only as many times as there are even members. Solutions are proposed in terms of the Lambert W function.

CCS CONCEPTS

• **Mathematics of computing → Probabilistic algorithms**; *Permutations and combinations.*

KEYWORDS

probability, Lambert W function, unit interval, measure theory

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1 INTRODUCTION

1.1 A Rigorous Application of Infinitesimals in Probability Theory, by Example.

In many areas of mathematics, such as in calculus, we find ourselves able to successfully wield infinitesimals. In 1961, the first rigorous description of calculus in terms of infinitesimals was given by Abraham Robinson [13]. The application to other areas has been accumulating evidence though this is not without opposition; specifically, to the application of infinitesimals as probabilities [3]. A counter to the objections of Pruss, 2014 [11], is provided by the authors Benci, Horsten, and Wenmackers [1]. Bottazzi and Katz [3] provide counters for many critiques against the use of infinitesimals in probability, including critiques by Parker [10], Pruss, 2018 [12], and Williamson [15]. The authors Calude and Dumitrescu provide a counter [4] for a critique by Williamson [15].

The authors Calude and Dumitrescu also contribute what they claim to be an acceptable framework for infinitesimals in probability [4]. Wenmackers provides a review [14] of the book by Vieri Benci and Mauro Di Nasso, in which the authors develop their approach to infinitesimals and claim to develop a framework for measuring the size of countable sets [2]. Other applications of infinitesimals include the authors Fiaschi's and Cacoccioni's application to game theory [7], and the author Jacobs' application to computer science [9].

The authors Benci, Horsten, and Wenmackers [1] provide an extensive discussion on the de Finetti lottery, a problem that requires random sampling from the naturals. A random sampling from the naturals requires infinitesimals. This paper is important because it provides both a question requiring random sampling from ℕ and the methods for obtaining solutions. This demonstration of there existing solutions seems to provide strong evidence in favor of the rigorous applications of infinitesimals to probability theory, by example.

2 METHODS

2.1 Bottom-up Computation.

Given ℕ, choose a number randomly. Evens are chosen without replacement and odds are chosen with replacement. Repeat this process for as many times as there are naturals. Assess the expected value for the probability even in the resultant set. Then consider this question for the same process instead iterating only as many times as there are even members.

Algorithm 1 Bottom-up: An experimental approach.

1: Create an arrary of size n, beginning at 1.
2: Choose a number from the array randomly.
3: If the number chosen is even, remove it from the array.
4: Else return the number.
5: Repeat steps (2-4) n times.
6: Count the even members remaining and divide by array length; store this value in a variable (M).
7: Repeat steps (1-6) as many times as desired (X), each time adding the final value to (M).
8: Divide (M) by (X).
9: Output: expected probability of an even in a set that results from this process.

In the bottom-up approach taken below we consider the iterative process for only as many times are there are members in the set at hand, and then to scale it up such that the number of iterations for this process is equal to exactly the number of naturals.

Consider the set $\{1\}$. There is one member and this process iterates only once. We draw from the set, a "1" is drawn, the "1" is replaced, and the resultant set is the same with $P_{even} = 0$.

Consider the set $\{1, 2\}$. For two drawings with replacement, there are $2^2 = 4$ possible outcomes: 1,1; 1,2; 2,1; and 2,2. In our case, however, not all arrangements are valid: if a "2" is drawn on the

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Table 1: Resultant Sets and Likelihood an Even is Drawn if to Draw From the Resultant Set

r	R	P_{even}
1,1	$\{1,2\}$	$\frac{1}{2}$
1,2	$\{1\}$	0
2,1	$\{1\}$	0

Table 2: Probability of Occurrence for Each Arrangement

r	P_r
1,1	$(\frac{1}{2})^2 = \frac{1}{4}$
1,2	$(\frac{1}{2})^2 = \frac{1}{4}$
2,1	$(\frac{1}{2})(\frac{1}{2-1}) = \frac{1}{2}$

first turn, it is removed from the set, and cannot again be drawn on the second turn. Thus, these are the sets that result, R, along with their associated probability even (Table 1). The probability of a given arrangement, P_r, is calculated in Table 2. The expected value for the probability even for a set containing only $\{1,2\}$ is $\mu P_{even} = (\frac{1}{2} * \frac{1}{4}) + (0 * \frac{1}{4}) + (0 * \frac{1}{2}) = \frac{1}{8}$.

In general, for a set of size n, there are n^n total arrangments, r, though not all are valid. Exact values for the expected probability even for a set of size n that have been calculated are given below (Fig. 1). As can be seen, you can also start at 0 instead of 1 (*e.g.* $\{0\}$, and then $\{0,1\}$, and so forth.), giving a similar graph with values equal to the those when starting at 1, for even n. The mean probability even, μP_{even}, is given by taking the sum of the products of the probability even in a set that results from a given arrangement and the probability of occurrence for that arrangement.

$$\mu P_{even} = \sum P_{even} * P_r \tag{1}$$

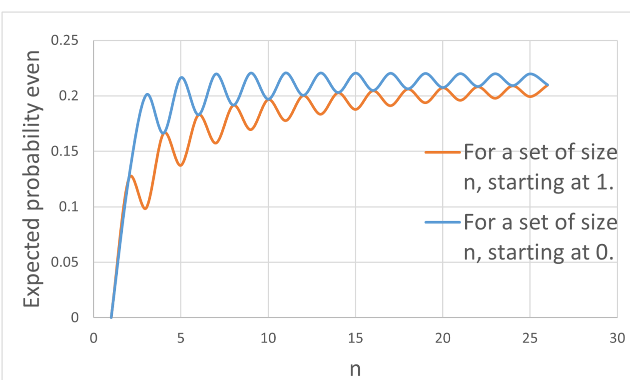

Figure 1: For a set of size n, starting at 0 or 1. Note that the smoothed lines between noninteger values are not precise, but serve to help visualize the behavior.

Instead of resolving this question for when the number of iterations are equal to the number of naturals, we can instead begin to resolve the question for when the number of iterations are equal to the number of evens. We do this by drawing instead $\frac{n}{2}$ times when n is even. For example, for a set $\{1,2\}$, we only draw once. Meaning that we will draw either a 1 or a 2. If we draw a 1 the set that results is $\{1,2\}$, with $P_r = \frac{1}{2}$ and $P_{even} = \frac{1}{2}$. If we draw a 2, then the set that results is $\{1\}$, with $P_r = \frac{1}{2}$ and $P_{even} = 0$. The expected value for the probability even then is $\mu P_{even} = (\frac{1}{2} * \frac{1}{2}) + (\frac{1}{2} * 0) = \frac{1}{4}$. A comprehensive graph is given below (Fig. 2).

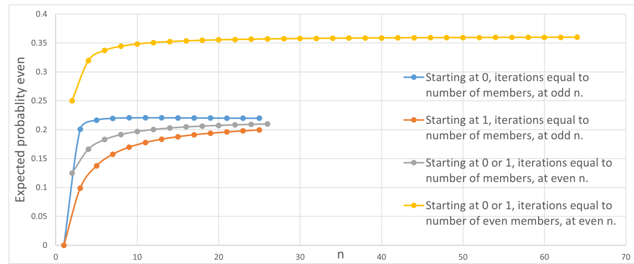

Figure 2: Bottom-up plots for when the number of iterations equal the numbers of members (lower) and when the number of iterations equal the number of even members (upper).

2.2 Top-down Computation.

Consider a variant of the iterative process described where we begin with the unit interval and this defines the fixed size of $\mathbb{N}$. Instead of taking our iterations one by one, let us take them at once and then separate off the evens. This reduces the interval by $\frac{1}{2}$ and our value for $P_{even} = 0$.

If instead we take our iterations over two turns, then on the first turn we will take an interval of size $\frac{1}{2}$ and $P_{even} = \frac{1}{2}$ so that $\frac{1}{2} * \frac{1}{2} = \frac{1}{4}$ even members will be removed. The interval remaining is $1 - \frac{1}{4} = \frac{3}{4}$. The number of even members remaining is $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. The P_{even} value of the remaining interval is $\frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$. For the second iteration we again take an interval of size $\frac{1}{2}$. This time $P_{even} = \frac{1}{3}$ so that $\frac{1}{2} * \frac{1}{3} = \frac{1}{6}$ even members will be removed. The interval remaining is $\frac{3}{4} - \frac{1}{6} = \frac{7}{12}$. The number of even members is $\frac{1}{4} - \frac{1}{6} = \frac{1}{12}$. Then P_{even} in the remaining interval is $\frac{\frac{1}{12}}{\frac{7}{12}} = \frac{1}{7} \approx .143$.

If we take our total number of interations over 3 turns this value becomes $\frac{22}{127} \approx .173$. As we take our total number of iterations over more and more x turns, the interval over which we are taking our turns grows smaller and smaller until the interval ceases to be and we are taking our turns over single values. For a number of iterations equal to the number of naturals, as $x \longrightarrow \inf$, this iterative sequence appears to become $\frac{W(\frac{1}{e})}{1+W(\frac{1}{e})} \approx 0.2178117$ where $W(x)$ is the productlog or Lambert W function[1]. If instead we take the

[1]This constant appears to arise as a minimum value that results in the value being an optimal choice for bin packing in the work of José Correa and Michael Goemnans [5, 6], and as a boundary point in chaotic systems in the work of A.C. Fowler and M.J. McGuinness describing the Lorenz Equations [8].

number of iterations to equal the number of evens, then as $x \to \inf$ this sequence appears to become $\frac{W(1)}{1+W(1)} \approx 0.361896$.[2]

The above iterative process is described as simultaneous, or mutual, recurrence relations[3] where $k = 1$ for $\mathbb{N}$ and $k = \frac{1}{2}$ for $\{2n : n \in \mathbb{N}\}$:

$$\begin{cases} a_0 = \frac{1}{2} - \frac{k}{2x} \\ b_0 = 1 - \frac{k}{2x} \\ a_{n+1} = a_n - \frac{a_n * k}{b_n * x} \\ b_{n+1} = b_n - \frac{a_n * k}{b_n * x} \end{cases} \tag{2}$$

At the first step we have:

$$\frac{a_0}{b_0} = \frac{\frac{1}{2} - \frac{k}{2x}}{1 - \frac{k}{2x}} \tag{3}$$

At the second step we have:

$$\frac{a_1}{b_1} = \frac{(\frac{1}{2} - \frac{k}{2x}) - \frac{(\frac{1}{2} - \frac{k}{2x}) * k}{(1 - \frac{k}{2x}) * x}}{(1 - \frac{k}{2x}) - \frac{(\frac{1}{2} - \frac{k}{2x}) * k}{(1 - \frac{k}{2x}) * x}} \tag{4}$$

This expression as continues at later steps can be reduced to:

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\frac{1}{2} - z}{1 - z} \tag{5}$$

where z is the limit of the continued expansion. Then we solve for the closed form of the limit of the expansion, which is the number of evens removed in the context of $k = 1$.

$$\frac{\frac{1}{2} - z}{1 - z} = \frac{W(\frac{1}{e})}{1 + W(\frac{1}{e})} \to z = \frac{1}{2} - \frac{W(\frac{1}{e})}{2} \approx 0.360768 \tag{6}$$

The number of evens remaining is: $\frac{W(\frac{1}{e})}{2} \approx 0.139232$. The expected value for the probability even is given as: $\frac{\frac{W(\frac{1}{e})}{2}}{\frac{1}{2} + \frac{W(\frac{1}{e})}{2}} = \frac{W(\frac{1}{e})}{1+W(\frac{1}{e})}$.

3 DISCUSSION

In the bottom-up approach taken we considered the iterative process for only as many times as there are members in the set at hand, and then to scaled it up such that the number of iterations for this process is equal to exactly the number of naturals. This can be achieved experimentally with Algorithm 1 or by Equation 1 resulting in exact solutions.

In Figure 1, the expected probability even is calculated for a set of size n, starting at 1 and also for a set of size n, instead starting at 0. This gives a similiar graph with values equal to those when starting at 1, for even n. As mentioned previously, the smoothed lines between noninteger values are not precise, but serve to help visualize the behavior. The topic of non-integer values of the iterative disposal processes examined is an area of future interest.

Figure 2 presents another look at the graph from Figure 1, highlighting the values that are equal, at even n, regardless of whether the set starts at 0 or 1. The graph also depicts the curve that results when instead choosing a number of iterations equal to the number of even members. A topic of odd integer values of the iterative disposal process examined when the number of iterations are equal to the number of even members is an area of future interest.

In the top-down computation, a variant of the iterative process was considered by starting with the unit interval and letting that define the fixed size of $\mathbb{N}$. Instead of taking the iterations one by one, we took them at once and then separated off the evens. In successive considerations, we take our iterations instead over two turns, and later over three turns, and so forth. As we take our total number of iterations over more and more turns, the interval over which we are taking our turns grows smaller and smaller until the interval ceases to be and we are taking our turns over single values. Doing so allows us to effectively transform this process into the desired iterative disposal process, and to address the questions of the expected probability of an even in a set that results from these processes.

The solutions arising from the top-down computation were referenced to a known constant value from publications in the topic of bin packing [5, 6] and the Lorenz Equations [8]. The top-down method can be expressed as simultaneous, or mutual, recurrence relations from which a direct solution is desirable.

Closer inspection of the recurrence relation allows consideration of this relation as a limit, which is set equal to the expected closed form when the number of iterations equal the number of naturals. This permits an expected closed form for the expected number of evens removed and consequently the expected number of evens remaining.

4 CONCLUSIONS

Many critiques to the application of infinitesimals in probability have been published, including those by: Pruss 2014 [11], and 2018 [12]; Parker [10]; and Williamson [15]. These objections have been met by many including: Benci, Horsten, and Wenmackers [1]; Bottazzi and Katz [3]; and Calude and Dumitrescu [4]. Benci, Horsten, and Wenmackers discuss another example that requires infinitesimals because it requires random sampling from $\mathbb{N}$, the de Finetti lottery [1].

This paper provides a question that requires a random sampling from $\mathbb{N}$, and because of this, also requires infinitesimals. This paper is interesting because it provides methods for obtaining the solutions to these problems, and by doing so seems to provide strong evidence in favor of the application of infinitesimals to probability.

In pursuit of direct solutions, one may wish to consider alternative iterative disposal processes. One such process requires drawing always from the original set, instead of the set that results after each iteration, while still drawing evens without replacement and odds with replacement.

The development of new frameworks for infinitesimals[2], including to probability theory [4], to game theory [7], and to computer science [9], are all promising developments. Iterative disposal processes are useful in that when they are applied to the set of naturals, and the parity of the remaining members is questioned, solutions exisiting seem to contribute to the growing body of knowledge supporting the successful applications of infinitesimals to probability.

[2]The constant $\frac{W(\frac{1}{e})}{1+W(\frac{1}{e})}$ is calculated to nine decimal places, 0.217811705, at $x = 10^9$ and the constant $\frac{W(1)}{1+W(1)}$ is calculated to ten decimal places, 0.3618962566, at the same x.

[3]A direct solution here is desirable.

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